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Integrable quadratic Hamiltonians with a linear Lie-Poisson bracket

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Abstract

Quadratic Hamiltonians with a linear Lie-Poisson bracket have a number of applications in mechanics. For example, the Lie-Poisson bracket $e(3)$ includes the Euler-Poinsot model describing motion of a rigid body around a fixed point under gravity and the Kirchhoff model describes the motion of a rigid body in ideal fluid. Advances in computer algebra algorithms, in implementations and hardware, together allow the computation of Hamiltonians with higher degree first integrals providing new results in the search for integrable models. A computer algebra module enabling related computations in a 3-dimensional vector formalism is described.

Key words: Hamiltonian systems, integrability, computer algebra

1 Overview

This contribution is based on a talk given at the one-day meeting held at Queen Mary, University of London, to celebrate Malcolm MacCallum's 60th birthday and his contributions to general relativity and computer algebra. This paper differs from the other ones in this GRG issue in that it has only very few links to general relativity. It deals with computer algebra applied to the solution of ordinary differential equations (ODEs) which is an area in which Malcolm made contributions as well. Together with his former PhD student Yiu-Kwong Man, Malcolm had worked successfully on single first order non-linear ODEs. Malcolm's computer program ODESOLVE which was revised in recent years by Francis Wright is an integral part of the REDUCE computer algebra system and much used, for example in my package CRACK [12].

Before coming to the more technical part of the talk it should not remain unmentioned that our computer algebra group at the Queen Mary and Westfield College (QMW) was integrated

in an ESPRIT Working Group together with groups from 5 other European countries. This was initiated by Malcolm and Evelyne Tournier/Paris and was running for 3 years 1992-95. Malcolm's guidance and integrating role were a key for its success and its extension for another 3 years from 1997 to 2000.

The computer algebra group at QMW was visited repeatedly by Vladimir Sokolov from the Landau Institute in Moscow. He introduced me to the problem that I am going to report on: integrable quadratic Hamiltonians. The talk at Queen Mary solely concentrated on the mathematics and the results of this collaboration with Sokolov as they are written up in the preprint [7]. Because this contribution is to relate to Malcolm's activity in computer algebra, here after an introduction to integrable Hamiltonians and Poisson brackets in the following section and an outline of results in section 3 we explain in more detail in section 4 a module of computer algebra routines that is able to do the required computations much more efficiently in a vector formalism and therefore will be useful as a tool for investigating other classes of Hamiltonians.

2 Poisson Brackets and Quadratic Hamiltonians

The Hamiltonian ODE-systems we investigated are formulated with the help of a Poisson bracket. In an m-dimensional space of dynamical variables $y_1, \ldots, y_m$, a Poisson bracket $\{\cdot\,,\cdot\}$ can be defined by a skew symmetric $m \times m$ structure matrix J through

$$\{F, G\} = \nabla F \cdot J \cdot \nabla G, \qquad \forall \text{ functions } F(y), G(y)$$

where J has to satisfy skew symmetry $J^{ij}(y) = -J^{ji}(y)$ and a Jacobi identity:
$\left(J^{il}\partial_l J^{jk} + J^{kl}\partial_l J^{ij} + J^{jl}\partial_l J^{ki}\right) = 0, \quad i, j, k = 1, .., m$. Equations of motion take the form

$$\frac{dy_i}{dt} = \{y_i, H(y)\}, \quad i = 1, .., m \tag{2.1}$$

where H is a given Hamiltonian function. Poisson brackets generalize the canonical form of the Hamiltonian formalism as seen in the following simple example. Taking for the structure matrix J the simple form $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the equations of motion of a point particle follow instantly: with $y_1 = p,\ y_2 = q,\ d/dt = \dot{},$ we get $\dot{p} = -\partial H/\partial q\ ,\ \ \dot{q} = \partial H/\partial p\ \ .$

An important special class of Poisson brackets is defined through a structure matrix J that is linear in the dynamical variables y_k. The Jacobi identity for $J_{ij} = c_{ij}^k y_k$ implies a Jacobi identity for the coefficients c_{ij}^k which have to be structure constants of a Lie algebra. For example, in the case of so(3) we have

$$J = \begin{pmatrix} 0 & -y^3 & y^2 \\ y^3 & 0 & -y^1 \\ -y^2 & y^1 & 0 \end{pmatrix}.$$

Hamilton's equations in this case take the form

$$\frac{dy}{dt} = y \times \nabla H(y)\ .$$

The quadratic Hamiltonian

$$H(y) = \frac{(y^1)^2}{2I_1} + \frac{(y^2)^2}{2I_2} + \frac{(y^3)^2}{2I_3}$$

where I_1, I_2, I_3 are constants gives Euler's equations of motion of a rigid body

$$\frac{dy^1}{dt} = \frac{I_2 - I_3}{I_2 I_3} y^2 y^3 \qquad \frac{dy^3}{dt} = \frac{I_3 - I_1}{I_3 I_1} y^3 y^1, \qquad \frac{dy^3}{dt} = \frac{I_1 - I_2}{I_1 I_2} y^1 y^2$$

with I_1, I_2, I_3 being the moments of inertia about the coordinate axis and y^1, y^2, y^3 the corresponding body angular momenta. H is the kinetic energy of the body and the angular velocities are $\omega^i = y^i / I_i$.

In the following we consider the family of Poisson brackets

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \qquad \{M_i, \gamma_j\} = \varepsilon_{ijk} \gamma_k, \qquad \{\gamma_i, \gamma_j\} = \kappa \varepsilon_{ijk} M_k \tag{2.2}$$

where the dynamical variables M_i and γ_i are components of 3-dimensional vectors $\mathbf{M}$ and $\mathbf{\Gamma}$, ε_{ijk} is the totally skew-symmetric tensor, κ is a constant parameter. The cases $\kappa = 0$, $\kappa > 0$ and $\kappa < 0$ correspond to the Lie algebras $e(3)$, $so(4)$ and $so(3,1)$. This bracket possesses two Casimir functions:

$$J_1 = (\mathbf{M}, \mathbf{\Gamma}), \qquad J_2 = \kappa |\mathbf{M}|^2 + |\mathbf{\Gamma}|^2,$$

i.e. integrals of motion for any Hamiltonian, with $(\cdot, \cdot)$ denoting the dot product. Each of the Casimirs can be used once to eliminate one of the 6 dynamical variables, the Hamiltonian H can be used twice and any other first integral can also be used twice. Therefore, for Liouville integrability we need one additional first integral I, functionally independent of the Hamiltonian H and J_1, J_2.

The object of our study are quadratic Hamiltonians. Their general form is

$$H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{M},\, B\mathbf{\Gamma}) + (\mathbf{\Gamma},\, C\mathbf{\Gamma}) + (P,\, \mathbf{M}) + (Q,\, \mathbf{\Gamma}),$$

where A, C are symmetric constant 3×3-matrices, B is an arbitrary constant matrix, P, Q are constant vectors.

There are two examples of classical problems with the $e(3)$-bracket $\kappa = 0$ and a Hamiltonian of the above form:

1. the Euler-Poinsot model describing the motion of a rigid body around a fixed point under gravity (with $H = (\mathbf{M},\, A\mathbf{M}) + (Q,\, \mathbf{\Gamma})$), and

2. the Kirchhoff model describing the motion of a rigid body in ideal fluid (with $H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{M},\, B\mathbf{\Gamma}) + (\mathbf{\Gamma},\, C\mathbf{\Gamma})$).

Hamiltonians of the above type with $\kappa \neq 0$ are related to the Poincare model of motion of a rigid body with an ellipsoidal hole filled with fluid. For $\kappa \neq 0$ the Kirchhoff model describes the motion of a rigid body in a space of constant curvature.

Most of the known Hamiltonians with an extra polynomial first integral, which we will call *integrable*, were found in the 19$^{\text{th}}$ century by Lagrange, Euler, Kirchhoff, Clebsch, Steklov. One example is the well known Kowalewski case

$$H = M_1^2 + M_2^2 + 2M_3^2 + q_1\gamma_1 + q_2\gamma_2;$$

with a 4$^{\text{th}}$ degree first integral (FI)

$$\begin{aligned} I &= G_1^2 + G_2^2, \qquad \text{where} \\ G_1 &= M_1^2 - M_2^2 - q_1\gamma_1 + q_2\gamma_2, \\ G_2 &= 2M_1M_2 - q_2\gamma_1 - q_1\gamma_2. \end{aligned}$$

This and the Hamiltonians of Lagrange (with a 1$^{\text{st}}$ FI) and Euler (with a 2$^{\text{nd}}$ FI) are all the integrable cases of the Euler-Poinsot model. For the Kirchhoff model the integrable cases of Frahm-Schottky, Steklov-Lyapunov and Adler-van Moerbeke-Reyman-Semenov-tian Shansky [1], [2] have a Hamiltonian with diagonal matrices A, B, C.

The first "non-diagonal" Hamiltonian with $\kappa = 0$ was found only recently by Sokolov ([4]). Shortly afterwards TW and Efimovskaya combined this and the Kowalewski Hamiltonian to a more general integrable Hamiltonian by determining all $e(3)$ Hamiltonians with a 4$^{\text{th}}$ degree first integral and a matrix A with two equal eigenvalues (cf. [11]).

A different generalization of Sokolov's Hamiltonian to $\kappa \neq 0$ can be written in the form

$$H = (\mathbf{M},\, A\mathbf{M}) + (\mathbf{b}, \mathbf{M} \times \mathbf{\Gamma}), \tag{2.3}$$

where A is a constant symmetric matrix, $\mathbf{b} \neq 0$ is a constant vector and $\times$ stands for the cross product. It turns out that this class is very rich in integrable cases. In the paper [6] all Hamiltonians (2.3) with a quartic additional integral were described. Moreover, it was mentioned in [8, 6, 9] that the general Sklyanin brackets [3] for the XXX-magnetic model lead to some integrable Hamiltonians of the same kind.

3 A Rich Class of Integrable Hamiltonians

In this section we summarize results of the collaboration with Vladimir Sokolov as it is described in more detail in [7] and as it was reported in the talk at Queen Mary.

Our starting point is the observation that known Hamiltonians of the form (2.3) have a linear partial integral P satisfying $\{H, P\} = QP$ for some (here necessarily linear) polynomial Q. A closer study reveals the existence of two cases. For one of both cases one can show that it contains exactly two integrable Hamiltonians, both Hamiltonians having an extra first integral of 4$^{\text{th}}$ degree. One Hamiltonian had been found earlier by Sokolov (cf. [4],[5]) and the other by Sklyanin and Tsiganov (cf. [3],[8]).

The second case requires the Hamiltonian to have the form

$$H = c_1\Big(\mathbf{a},\, \mathbf{b}\Big)|\mathbf{M}|^2 + c_2\Big(\mathbf{a},\, \mathbf{M}\Big)\Big(\mathbf{b},\, \mathbf{M}\Big) + \Big(\mathbf{b},\, \mathbf{M} \times \mathbf{\Gamma}\Big), \tag{3.4}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and c_i are constant scalars.

It is possible to formulate necessary integrability conditions on c_i using the Kowalewski-Lyapunov test as follows. The right hand side of the equations of motion (2.1) consists of homogeneous quadratic polynomials in y_i. In a first step one computes so-called *Kowalewski solutions* of the form

$$\mathbf{Y_0} = \frac{1}{t}\mathbf{K} \tag{3.5}$$

where $\mathbf{K}$ is a constant (6-component) vector computed from the algebraic system that arises when substituting (3.5) into (2.1). A linearization $\mathbf{Y} = \mathbf{Y_0} + \varepsilon\mathbf{\Psi}$ around each Kowalewski solution produces conditions

$$\frac{d\mathbf{\Psi}}{dt} = \frac{1}{t}S(\mathbf{\Psi}), \tag{3.6}$$

where S is a constant 6×6-matrix depending on the Kowalewski solution. Solutions have the form $\mathbf{\Psi} = \mathbf{s}\,t^{-k}$ where $\mathbf{s}$ is an eigenvector and k an eigenvalue of the matrix S.

The Kowalewski-Lyapunov test requires the so-called *Kowalewski exponent* $1-k$ either to be an integer, implying that the solutions of (2.1) are single valued, or at most to be rational, with the consequence that solutions of (2.1) have algebraic branching points. But k cannot occupy an open set of $\mathbb{C}$ or $\mathbb{R}$ if (2.1) is to be integrable. If we seek values of c_i such that H from (3.4) is integrable for any value of $\mathbf{a}$ and $\mathbf{b}$ then this has the consequence that k must not depend on arbitrary parameters involving $\mathbf{a}$ or $\mathbf{b}$. We set $\kappa = -|\mathbf{a}|^2$ which all the known integrable Hamiltonians of type (3.4) satisfy and scale the norm of $\mathbf{b}$ by scaling H. But the angle between $\mathbf{a}$ and $\mathbf{b}$ is a non-trivial parameter. Demanding all k from all Kowalewski solutions to be independent of this parameter narrows down c_i to the following 5 pairs:

a) c_1 is arbitrary, $c_2 = 0$;
b) $c_1 = 1$, $c_2 = -2$;
c) $c_1 = 1$, $c_2 = -1$;
d) $c_1 = 1$, $c_2 = -\frac{1}{2}$;
e) $c_1 = 1$, $c_2 = 1$.

The case a) describes a Hamiltonian that is a special case of

$$H = c_1|\mathbf{b}|^2\,|\mathbf{M}|^2 + c_2\Big(\mathbf{b},\,\mathbf{M}\Big)^2 + \Big(\mathbf{b},\,\mathbf{M}\times\mathbf{\Gamma}\Big), \tag{3.7}$$

which possesses the linear integral of motion $I = (\mathbf{b},\,\mathbf{M})$ without the additional restriction $\kappa = -|\mathbf{a}|^2$.

The Hamiltonian b) admits an additional cubic integral and had been found earlier by Tsiganov and Goremykin (cf. [9]). Hamiltonian c) is known too. It has an extra quartic integral (cf. [6]).

Case d) gives a new integrable Hamiltonian

$$H = \Big(\mathbf{a},\,\mathbf{b}\Big)|\mathbf{M}|^2 - \frac{1}{2}\Big(\mathbf{a},\,\mathbf{M}\Big)\Big(\mathbf{b},\,\mathbf{M}\Big) + \Big(\mathbf{b},\,\mathbf{M}\times\mathbf{\Gamma}\Big), \tag{3.8}$$

which under the condition $\kappa = -|\mathbf{a}|^2$ has a first integral of 6^{th} degree

$$\begin{aligned} I &= \Big(\mathbf{b},\mathbf{M}\Big)^2\Big[\Big(\mathbf{b}\times\mathbf{a},\mathbf{M}\times\mathbf{a}\Big)^2\mathbf{M}^2+2\Big(\mathbf{b}\times\mathbf{a},\mathbf{M}\times\mathbf{a}\Big)\Big(\mathbf{b}\times\mathbf{a},\mathbf{M}\times(\mathbf{M}\times\mathbf{\Gamma})\Big)\\ &\quad -\mathbf{\Gamma}^2\Big(\mathbf{M},\mathbf{b}\times\mathbf{a}\Big)^2-\Big(\mathbf{b}\times\mathbf{a},\mathbf{M}\times\mathbf{\Gamma}\Big)^2-\mathbf{M}^2\mathbf{\Gamma}^2\Big(\mathbf{b},\mathbf{a}\Big)^2-\kappa\mathbf{M}^2\mathbf{\Gamma}^2\mathbf{b}^2\Big]. \end{aligned} \tag{3.9}$$

Finally, it was shown recently by S.Yu. Sakovich that case e) does not pass the Painleve test (cf. [10]).

From the six integrable Hamiltonian of type (2.3) the Hamiltonian (3.7) includes a special case with *two* non-trivial first integrals. The Hamiltonian

$$H_{\text{hom}} = |\mathbf{b}|^2\,|\mathbf{M}|^2-\frac{1}{2}\Big(\mathbf{b},\,\mathbf{M}\Big)^2+\Big(\mathbf{b},\,\mathbf{M}\times\mathbf{\Gamma}\Big) \tag{3.10}$$

has under condition $\kappa = -|\mathbf{b}|^2$ apart from the linear first integral $(\mathbf{b},\mathbf{M})$ the additional integral of degree 4:

$$\begin{aligned} I &= (\mathbf{k},\,\mathbf{M})\left[(\mathbf{k},\,\mathbf{M})|\mathbf{b}|^2-2(\mathbf{k},\,\mathbf{b})(\mathbf{b},\,\mathbf{M})\right]\cdot\left[|\mathbf{\Gamma}|^2+|\mathbf{b}|^2|\mathbf{M}|^2\right]\\ &\quad +|\mathbf{M}|^2(\mathbf{b},\,\mathbf{M})^2\left[(\mathbf{k},\,\mathbf{b})^2+|\mathbf{k}|^2|\mathbf{b}|^2\right]-\Big(\mathbf{k}\times\mathbf{b},\,\mathbf{M}\times\mathbf{\Gamma}\Big)^2\\ &\quad +2\Big(\mathbf{k}\times\mathbf{b},\,\mathbf{b}\times\mathbf{M}\Big)\cdot\left[|\mathbf{M}|^2(\mathbf{k},\mathbf{b}\times\mathbf{\Gamma})-(\mathbf{M},\,\mathbf{\Gamma})(\mathbf{k},\mathbf{b}\times\mathbf{M})\right], \end{aligned}$$

where $\mathbf{k}$ is an arbitrary constant vector. With two integrals of motion, both functionally independent of H and the Casimirs, the Hamiltonian (3.10) is superintegrable.

All 7 homogeneous integrable Hamiltonians of type (2.3) (2 from case 1, 4 from case 2 and the super integrable one) can be generalized to inhomogeneous form by adding to H suitable terms linear in $\mathbf{M},\mathbf{\Gamma}$ and appropriate lower degree terms to the first integrals. The inhomogeneous version of case 2d) is shown below in (4.16),(4.17).

4 A 3-dimensional Vector Formalism

In this section we look into the computer algebra aspects of determining integrable Hamiltonians.

4.1 Motivation

To write down the Hamiltonian and first integrals in vector notation provides a geometric interpretation but it is not the way how they were originally obtained. Computations were performed for a Hamiltonian of the form

$$\begin{aligned} H &= a_1M_1^2+a_2M_2^2+a_3M_3^2+a_4M_1M_3+a_5M_2M_3\\ &\quad +M_1\gamma_2-M_2\gamma_1+(\mathbf{k},\,\mathbf{M})+(\mathbf{n},\,\mathbf{\Gamma}). \end{aligned} \tag{4.11}$$

with undetermined a_i and $\mathbf{k}, \mathbf{n}$. Its quadratic part can be reached by orthogonal transformations from any (real) Hamiltonian (2.3). The ansatz for the first integral I is a general polynomial of some degree d in the six variables M_i, γ_i also with undetermined coefficients. The commutator condition $\{I, H\} = 0$ results in a bi-linear system of algebraic equations for the coefficients of H and I. For homogeneous first integrals of degree $d = 6$ the bi-linear system involves already 791 equations with 8938 terms for 458 unknowns and for $d = 8$ already 2001 equations with 28158 terms for 1278 unknowns. Inhomogeneous versions of these problems have 2 to 2.5 times as many unknowns and conditions. Such systems can currently not be solved by standard computer algebra packages and also all attempts to use specialized packages for solving polynomial systems failed. Although being extremely large, the systems are also sparse and overdetermined, i.e, the kind of problems the package CRACK is designed for (cf. [12]). Using this package, degrees $d < 6$ could be analysed automatically and degrees $6 \leq d \leq 8$ partially interactively.

Once the first integral is computed the challenge is to rewrite it in a human readable form, not involving hundreds or thousands of terms. This was the initial aim of a package V3TOOLS of routines for facilitating computations with 3-dimensional vectors.

4.2 Converting Representations of Scalar Vector Expressions

The idea is to use products $(\mathbf{a}, (\mathbf{b} \times (\mathbf{c} \times (\mathbf{d} \times \ldots))))$ (in the following denoted as $(\mathbf{abcd}..)$) to compactify large expressions of scalar vector products. In total, routines `c2s, s2e, e2s, s2c` convert scalar vector expressions from the following three forms into each other:

- "*C*omponent form": polynomials in terms of components $a_i, b_j, ..$ of vectors $\mathbf{a}, \mathbf{b}, \ldots,$
- "*S*tandard vector form": polynomials in terms of dot products $(\mathbf{a}, \mathbf{b})$ and triple products $(\mathbf{a}, (\mathbf{b} \times \mathbf{c}))$ (in the following denoted $(\mathbf{ab})$ and $(\mathbf{abc})$)
- "*E*xtended vector form": polynomials in terms of dot and triple products but also products $(\mathbf{abcd}..)$.

The conversion from standard vector form to component form as implemented in `s2c` is obvious. Also the conversion from extended to standard vector form as implemented in `e2s` poses no problem when using recursively the identity

$$(\mathbf{abcd}..) = (\mathbf{ac})(\mathbf{bd}..) - (\mathbf{bc})(\mathbf{ad}..). \tag{4.12}$$

To rewrite a scalar vector expression K from component form into standard vector form (using `c2s`) one starts by partitioning $K = \sum_\alpha K_\alpha$ where all terms in one K_α have the same total degree with respect to all the components of the occuring vectors. For example, the two terms $(a_1)^2 b_2$ and $a_2 a_3 b_1$ would be in one K_α. Then each K_α is converted by

- formulating the most general homogeneous polynomial P_α in standard vector form with undetermined coefficients that has the same respective degree in all the vectors as K_α,
- converting this polynomial to component form and

- determining the coefficients so that $P_\alpha \equiv K_\alpha$.

If three vectors are involved and the polynomial is at least of 6th degree then the solution of the linear algebraic system for the coefficients will have an arbitrary parameter resulting from the vector identity

$$\begin{aligned} 0 &= (\mathbf{abc})^2 + (\mathbf{aa})(\mathbf{bc})^2 + (\mathbf{bb})(\mathbf{ac})^2 + (\mathbf{cc})(\mathbf{ab})^2 \\ &\quad -(\mathbf{aa})(\mathbf{bb})(\mathbf{cc}) - 2(\mathbf{ab})(\mathbf{bc})(\mathbf{ac}) \end{aligned} \tag{4.13}$$

for any vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. In the case of four vectors identities exist already with degree 5, like

$$0 = (\mathbf{abc})(\mathbf{dd}) - (\mathbf{abd})(\mathbf{cd}) + (\mathbf{acd})(\mathbf{bd}) - (\mathbf{bcd})(\mathbf{ad}). \tag{4.14}$$

The conversion from standard to extended vector form proceeds analogously but is more involved. Again the polynomial is partitioned and each part K_α is converted on its own. To convert one K_α,

- all permutations of all vectors with their multiplicity as they occur in each term of K_α (for example **abccd**) are listed, like the permutation **bccad**,
- each permutation is interpreted as a generalized product, like (**bccad**) and is converted using (4.12) to create an identity, here
  $0 = (\mathbf{bccad}) - (\mathbf{abd})(\mathbf{cc}) + (\mathbf{acd})(\mathbf{bc})$,
- these identities are used to reduce the number of terms in K_α.

When trying to convert, for example (3.9), the terms in there have between 12 and 14 vectorial factors. That means for each K_α to be converted several 100,000 permutations are possible. This number reduces if only those permutations are generated for which the left two vectors differ and are sorted, for example lexicographically and for which the right two vectors differ and are sorted. When writing each permutation as an extended product and expanding it into standard vector form only a small fraction of those turn out to be at least pairwise linearly independent. To utilize the resulting identities for a length reduction of expressions the related procedure `s2e` applies an algorithm described in [13] which is implemented in the package CRACK.

To give an example: the first integral (3.9) written in component form involves typically around 2000 terms. It is unique only up to the addition of trivial first integrals depending functionally exclusively on the Hamiltonian and the two Casimirs. In one of these equivalent versions the first integral is factorizable and reads in standard vector form

$$\begin{aligned} I_{6\mathrm{h}} &= (\mathbf{bM})^2 \big[(\mathbf{aa})^2(\mathbf{bM})^2(\mathbf{MM}) - 2(\mathbf{aa})(\mathbf{ab})(\mathbf{aM})(\mathbf{bM})(\mathbf{MM}) \\ &\quad -2(\mathbf{aa})(\mathbf{abM})(\mathbf{bM})(\mathbf{M\Gamma}) + 2(\mathbf{aa})(\mathbf{ab\Gamma})(\mathbf{bM})(\mathbf{MM}) \\ &\quad +(\mathbf{aa})(\mathbf{bM})^2(\mathbf{\Gamma\Gamma}) + (\mathbf{ab})^2(\mathbf{aM})^2(\mathbf{MM}) \\ &\quad +2(\mathbf{ab})(\mathbf{abM})(\mathbf{aM})(\mathbf{M\Gamma}) - 2(\mathbf{ab})(\mathbf{ab\Gamma})(\mathbf{aM})(\mathbf{MM}) \\ &\quad -2(\mathbf{ab})(\mathbf{aM})(\mathbf{bM})(\mathbf{\Gamma\Gamma}) + (\mathbf{aM})^2(\mathbf{bb})(\mathbf{\Gamma\Gamma}) \\ &\quad -(\mathbf{aM})^2(\mathbf{b\Gamma})^2 + 2(\mathbf{aM})(\mathbf{a\Gamma})(\mathbf{bM})(\mathbf{b\Gamma}) - (\mathbf{a\Gamma})^2(\mathbf{bM})^2\big] \end{aligned}$$

which is slightly longer than the "hand-compactified" form (3.9). The second factor of this expression partitions into 3 sums K_α. For example, K_1 contains 4 terms, each a product of vectors $\mathbf{aabbMMM\Gamma}$, giving 3280 permutations, leading to 273 pairwise independent identities of which one fits perfectly to simplify K_1 to $2(\mathbf{abM\Gamma aMMab})$. Together with the other two compressed K_α the first integral takes the compact form

$$I_{6\mathrm{h}} = (\mathbf{bM})^2\left[2(\mathbf{abM\Gamma aMMab}) + (\mathbf{MM})(\mathbf{abaM})^2 - (\mathbf{abM\Gamma\Gamma Mba})\right] \tag{4.15}$$

computed in 140 sec[1]. The inhomogeneous generalization of (3.8) reads

$$H = (\mathbf{ab})(\mathbf{MM}) - \frac{1}{2}(\mathbf{aM})(\mathbf{bM}) + (\mathbf{bM\Gamma}) + p(\mathbf{abM}) \tag{4.16}$$

with the compactified first integral

$$\begin{aligned}
I = \; & I_{6\mathrm{h}} \quad \text{from (4.15)} \\
& +4p(\mathbf{bM})[(\mathbf{abM\Gamma\Gamma Mbab}) - (\mathbf{aMababbMMaM}) - (\mathbf{aMaabbbM\Gamma M}) \\
& \qquad +2(\mathbf{aMababbM\Gamma M}) - (\mathbf{aMbMbaba\Gamma M})] \\
& +p^2[4(\mathbf{bM})(\mathbf{MM})(\mathbf{ab\Gamma})\left((\mathbf{aa})(\mathbf{bb}) - 3(\mathbf{ab})^2\right) + 4(\mathbf{ab})(\mathbf{abM})(\mathbf{a\Gamma})(\mathbf{bM})^2 \\
& \quad +4(\mathbf{abba})(\mathbf{bb})(\mathbf{M\Gamma})^2 + 4(\mathbf{abbM\Gamma\Gamma Mbba}) + 4(\mathbf{abbMM\Gamma bMaab}) \\
& \quad +4(\mathbf{bb})(\mathbf{MM})\left((\mathbf{aMaMabba}) - 2(\mathbf{aMaabMba})\right) \\
& \quad -4(\mathbf{aMababababMMMb}) + 4(\mathbf{aMbbaabbMMMa}) \\
& \quad +8(\mathbf{aMbaMbb\Gamma Mab}) - 4(\mathbf{bMMab\Gamma aMbab}) - 8(\mathbf{b\Gamma aMbbMMaab})] \\
& +8p^3[(\mathbf{abaMMMbbaab}) + (\mathbf{abbMMba\Gamma ba}) - (\mathbf{abb\Gamma aMMbba})] \\
& +4p^4[(\mathbf{ab})^2(\mathbf{MM}) - (\mathbf{bb})(\mathbf{\Gamma\Gamma})](\mathbf{abba})
\end{aligned} \tag{4.17}$$

4.3 The algebra of dot and triple products

As shown above the compression rate going from component form to only standard vector form is already enormous. To utilize this compression not only for display but for computations one has to work with dot and triple products as independent variables v_μ, has to find all identities between the v_μ, to compute a Gröbner basis of all identities and to perform any relevant computation modulo this Gröbner basis. For 3 vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ there are only 1 triple product, 6 dot products and 1 identity (4.13) between them. For 4 vectors there are already 4 triple products, 10 dot products and the Gröbner basis involves 25 identities[2].

To classify integrable Hamiltonians by working in the algebra of scalar vector products v_μ the key abilities are:

- the generation of an ansatz for H and I with specific homogeneity weights,
- the computation of the Poisson bracket,

[1] All run times refer to computations done in REDUCE under Linux on a 3GHz Pentium 4.

[2] when based on a lexicographical ordering giving triple products highest priority

- the solution of the algebraic system for the undetermined coefficients,
- the analysis of whether any found first integral is functionally independent of known trivial and non-trivial first integrals.

In the following we will discuss the first two and the last step.

4.4 Multiple Homogeneity

To generate a finite degree polynomial ansatz for the Hamiltonian and the first integral in terms of v_μ one has to introduce a list of homogeneity weights for each vector. We abbreviate 'weight list of ..' in the following by [..].

Inspecting the Poisson bracket (2.2) and Casimir $\kappa|\mathbf{M}|^2 + |\mathbf{\Gamma}|^2$ one is led to weight lists $[\mathbf{M}] = (1,0), [\mathbf{\Gamma}] = (1,1)$ and $[\kappa] = (0,2)$ with the meaning that the Casimir is scaling invariant under the group $M_i \to e^\varepsilon M_i, \gamma_i \to e^\varepsilon \gamma_i, \kappa \to \kappa$ *and* the group $M_i \to M_i, \gamma_i \to e^\varepsilon \gamma_i, \kappa \to e^{2\varepsilon}\kappa, \varepsilon \in \mathbb{R}$

Known vector formulations of integrable homogeneous Hamiltonians and their vector form involve two constant vectors. A natural choice is to give one constant vector $\mathbf{a}$ the weight list $[\mathbf{a}] = (0,1)$ to have $[\kappa] = [|\mathbf{a}|^2]$ and the other constant vector $\mathbf{b}$ a separate weight which is zero for all the other vectors and κ. Consequently one arrives at weight lists $[\mathbf{M}] = (1,0,0), [\mathbf{\Gamma}] = (1,1,0), [\mathbf{a}] = (0,1,0), [\mathbf{b}] = (0,0,1), [\kappa] = (0,2,0)$. Based on these weights the Hamiltonian (3.4) is homogeneous with weights $(2,1,1)$ and the first integral (3.9) is homogeneous with weights $(6,4,4)$.

From given weight lists for $\mathbf{M}, \mathbf{\Gamma}, \mathbf{a}, \mathbf{b}$ follow weight lists for their dot and triple products v_μ and it is no problem to computer generate a polynomial ansatz for a first integral I in terms of v_μ and undetermined coefficients such that I has a given list of weight. For example, using the above weight lists for $\mathbf{M}, \mathbf{\Gamma}, \mathbf{a}, \mathbf{b}, \kappa$ the ansatz for I with the weight list $(6,4,4)$ involves 136 terms.

4.5 Computing Poisson Brackets

After formulating an ansatz for H and I the next step is to compute Poisson brackets between both through

$$\{H, I\} = \sum_{\mu,\nu} \frac{\partial H}{\partial v_\mu} \frac{\partial I}{\partial v_\nu} \{v_\mu, v_\nu\} \tag{4.18}$$

working in the free algebra of dot and triple products v_μ modulo a set of identities. This is accomplished by computing the Poisson bracket structure constants $c^\rho_{\mu\nu}$ from

$$\{v_\mu, v_\nu\} = c^\rho_{\mu\nu}\, v_\rho \tag{4.19}$$

based on the component form of v_μ in terms of the dynamical variables M_i, γ_j, the Poisson bracket (2.2) and a vanishing Poisson bracket if one of their arguments is a component of a vector other than $\mathbf{M}, \mathbf{\Gamma}$. For example, when working with 4 vectors ($\mathbf{M}, \mathbf{\Gamma}$ + 2 constant vectors) then computing $c^\rho_{\mu\nu}$ from (4.19) for the 14 dot and vector products v_μ takes about one second.

With known $c^{\rho}_{\mu\nu}$ the computation of $\{H, I\}$ based on (4.18) with H from (3.4) and I with weight list (6,4,4) and 136 terms is done in milliseconds.

4.6 Functional Independence

The remaining task in an attempt to compute integrable Hamiltonians in vector form lies in dealing with functional dependence of first integrals. For our purpose, only Hamiltonians that have first integrals that are functionally independent of the two Casimirs and the Hamiltonian itself are of interest. Instead of computing at first the Hamiltonian and first integrals and investigating their functional dependence afterwards it is more elegant and efficient to prevent functional dependence already in the ansatz for the first integrals. Such a restricted ansatz will have fewer undetermined coefficients and, more importantly, will have fewer solutions (no trivial solutions) and will thus be much easier to be solved completely. The problem to overcome is that in a vector formalism we want to work modulo vector identities and at the same time develop an ansatz modulo the functional dependence on trivial first integrals. What is given initially is the weight list W of the first integral I for which an ansatz is to be computed. The following algorithm does the job.

- With each known first integral I_k one associates a symbol s_k and the weight list w_k of I_k.
- One generates the most general polynomial S with weight list W in terms of variables s_k, each having a weight list w_k.
- For each term $t_j(s_k)$ of S one
  - replaces s_k by $I_k(v_\mu)$ to obtain the *polynomial* $p_j(v_\mu) = t_j(s_k)|_{s_k \to I_k}$,
  - computes $\hat{p_j}(v_\mu) = p_j(v_\mu)$ modulo the Gröbner basis of vector identities, and
  - adds the leading term T_j of $\hat{p_j}(v_\mu)$ to a list L of such leading terms.
- After generating the most general polynomial ansatz I with weight list W for the first integral in terms of variables v_μ, based on a weight list for each v_μ, the ansatz for I is reduced modulo the Gröbner basis of vector identities to obtain $\hat{I}$.
- Finally, all terms of the list L are dropped from $\hat{I}$.

What remains is an ansatz for the first integral that can not be functionally dependent on known first integrals and is reduced modulo all vector identities but otherwise is as general as possible.

4.7 Comparison

Working in terms of v_μ has strengths and weaknesses. The advantage is that computations in vector notation are much faster. With the weights as described in section 4.4 the computation of integrable Hamiltonians with a 6^{th} degree first integral involves solving a system of 175 equations with 1861 terms for 143 unknowns (including $\kappa, (\mathbf{aa}), (\mathbf{ab}), (\mathbf{bb})$ to increase generality) and

takes less than 3 minutes compared to a partially interactive computation in component form taking about 1/2 day (see section 4.1). But the vector computation has also disadvantages.

- It requires to determine in advance how many constant vectors in addition to the two vectors $\mathbf{M}, \mathbf{\Gamma}$ are to be involved. For computing all the known homogeneous Hamiltonians of type (2.3) and their first integrals two extra constant vectors are enough.

- All the known integrable Hamiltonians and their first integrals listed in [7] have low weights and at least their homogeneous parts can be found quickly with the vector formalism if one guesses the right weight list for all vectors beforehand. The advantage of a computation in component form is to obtain a complete list of integrable cases. To reach the same completeness statement when computing in vector form would, for example, require to investigate all weights $(2, m, n)$ for H and weights $(6, p, q)$ for I with $m, n, p, q \to \infty$ to find all homogeneous quadratic Hamiltonians with a 6^{th} degree first integral. Therefore vector computations are more appropriate when searching for new integrable Hamiltonians whereas computations in component form are more suitable for formulating a completeness statement.

- If the Hamiltonian H has not only terms of second degree in $\mathbf{M}, \mathbf{\Gamma}$ but also of first degree then it is called inhomogeneous and if this has an additional first integral I then the homogeneous part I_{hom} of I, i.e. the terms of highest degree in $\mathbf{M}, \mathbf{\Gamma}$ in I are a first integral of the quadratic part H_{hom} of H. Therefore inhomogeneous integrable Hamiltonians can be found by first computing integrable homogeneous Hamiltonians and generalizing them afterwards. In vector notation this generalization need not be straight forward as it becomes apparent in case 4 of proposition 1 in [7]. In that case I_{inhom} is rational in the two constant vectors $\mathbf{a}, \mathbf{b}$ with denominator $(\mathbf{abab})$. One can write I as a polynomial by multiplying it with the constant denominator but this denominator still has to be found first, if necessary by doing a computation in component form. Another potential problem that becomes apparent in the above case is that the inhomogeneous generalization may allow an extra arbitrary constant vector that would not be found if one would try to generalize $H_{\text{hom}}, I_{\text{hom}}$ only with the constant vectors appearing in there.

5 Conclusion

A set of algorithms and programs has been developed to do integrability investigations of Hamiltonians in a vector formalism. This has been used so far to verify known integrable Hamiltonians and their first integrals as reported in [7] and to find a compact representation of those first integrals. Because of the large speed up that this vector representation allows, it is well suited for further integrability studies of other classes of Hamiltonians.

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