

REINHARD HOCHMUTH

# **Homogenization for a Nonlocal Coupling Model**

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REINHARD HOCHMUTH<sup>1</sup>

## Abstract

In [7, 8, 12] homogenization techniques are applied to derive an anisotropic variant of the bio-heat transfer equation as asymptotic result of boundary value problems providing a microscopic description for microvascular tissue. In view of a future application on treatment planning in hyperthermia, we investigate here the homogenization limit for a coupling model, which takes additionally into account the influence of convective heat transfer in medium size blood vessels. This leads to second order elliptic boundary value problems with nonlocal boundary conditions on parts of the boundary. Moreover, we present asymptotic estimates for first order correctors.

**Key words:** Homogenization, nonlocal boundary conditions, Robin boundary conditions, correctors, heat transfer, bio-heat equation, hyperthermia.

**Mathematics subject classification:** 35B27, 80A20, 92C50, 35J25.

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<sup>1</sup>Institute for Applied Analysis, University of Mining and Technology Freiberg, Agricolastraße 1, D-09596 Freiberg, Germany, E-mail: hochmuth@math.tu-freiberg.de

# 1 Introduction

Homogenization problems are typically studied in the context of local boundary conditions. Here, we investigate a family of second order elliptic partial differential equations on periodically perforated domains, where on the inner boundaries Robin boundary conditions and on part of the outer boundary nonlocal boundary conditions, which are related to convective heat transfer, are posed.

The present paper emerges to a certain extent from problems arising in the patient-specific therapy planning for the cancer therapy *regional hyperthermia* – see [9, Chap. 1] for a general survey or [10] for details concerning applied adaptive multilevel methods. The underlying medical and mathematical problem is to tune radiowave antennas optimally in such a way that heat is concentrated within the patient’s tumor, but nowhere else in healthy tissue, see [9, 10, 23] for further details. Up to now, for quantitatively modeling of the distribution of the temperature  $T$  within an individual human body, the rather simple so-called *bio-heat transfer (BHT) equation*

$$-\nabla\kappa\nabla T + \rho_t\rho_b c_b m(T - T_b) + S = 0 \quad (1.1)$$

is mostly used, where  $\kappa$  denotes the thermal conductivity,  $\rho_{t/b}$  the density of tissue/blood,  $c_b$  the specific heat capacity,  $m$  the perfusion,  $T_b$  the arterial temperature, and  $S$  some external thermal source – which in regional hyperthermia is the heating by radiowave absorption. This elliptic PDE dates back to an early suggestion by the neurologist Pennes [20] from 1948, see also [3].

Stepping back to the original problem, we are facing a true multiscale situation. As shown in Fig. 1, we will have to deal in parallel with large blood vessels (a), medium size blood vessels (b), and small capillaries (c).

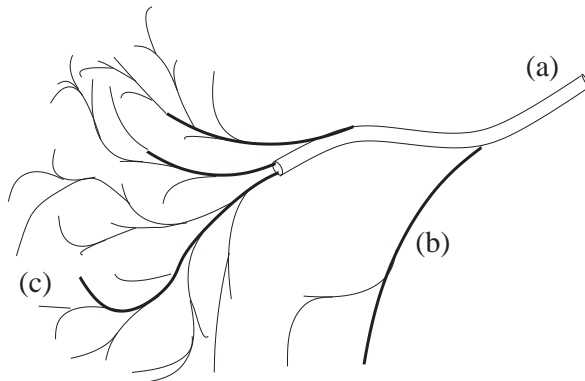


Figure 1: Multiscales in blood vessels: (a) large blood vessels, (b) medium size blood vessels, (c) capillaries.

The main topic of this paper is to present a first step towards a coupling of b) and c). Models for the temperature in medium size blood vessels are characterized by the assumption that the blood is locally thoroughly mixed to a uniform temperature and is dominated by convection. Consequently, see [4, 14, 17, 21, 22], the basic governing

equation for the mean blood temperature  $T_m$  at any position along a transiting medium size blood vessel is given by

$$M c_b \frac{dT_m}{dz} = hP(T_w - T_m) + Q_{ap}, \quad (1.2)$$

where  $M$  is the mass flow rate,  $c_b$  is the specific heat of blood,  $h$  is the convective heat transfer coefficient,  $P$  is the perimeter of the vessel,  $T_w$  is the wall temperature of the vessel and  $Q_{ap}$  is the external power deposition applied in the vessel, see [15].

In [4, 14, 17, 21] the influence of medium size blood vessels in vascularized tissue during hyperthermic treatment is modeled by combining Pennes BHT equation (1.1) with equation (1.2) via Newtons cooling law and by setting  $T_w = T$ . In [8, 12] it is shown that the BHT equation (1.1) might be understood as the homogenization limit of a certain class of micromodels. Coupling now those micromodells with equation (1.2) via Newton's cooling law, taking care of the influence of medium size vessels, gives micromodells with nonlocal boundary conditions. In the following we prove in particular, that the "ad hoc" coupling of the BHT equation (1.1) with (1.2) gives in fact the asymptotic limit.

The paper is organized as follows: In Section 2 we introduce microscale coupling model problems with nonlocal boundary conditions. After considering their unique solvability, we analyze their asymptotic limit via homogenization in Section 3. In Section 4, we prove asymptotic estimates with respect to first order corrector terms.

Throughout the paper, we use the notation  $a \lesssim b$  as an abbreviation of  $a \leq Cb$  with some constant  $C > 0$ ,  $a \gtrsim b$  equivalent to  $b \lesssim a$  and  $a \sim b$  for  $a \lesssim b \lesssim a$ .

## 2 Nonlocal Coupling Models

First, we introduce a specific periodic twodimensional setting, which will keep the technical problems as few as possible: Let  $Y \subset \mathbb{R}^2$  denote the bounded square  $(0, 1) \times (0, 1)$ . For a bounded domain  $B$  with  $\overline{B} \subset Y$  we define  $Y^* := Y \setminus \overline{B}$ ,  $\theta := \frac{|Y^*|}{|Y|} \in (0, 1)$  and  $\theta_B := \frac{|\partial B|}{|Y^*|} \in (0, 1)$ . The boundary of  $B$  is assumed to be at least  $C^2$ . For  $\epsilon > 0$  let  $\tau(\epsilon B) := \cup_{k \in \mathbb{Z}^3} \epsilon(k + B)$ . Then for  $\Omega := (0, 1) \times (0, 1) \subset \mathbb{R}^2$  and  $\epsilon \in \{1/n \mid n \in \mathbb{N}\}$  such that  $\partial\Omega \cap \tau(\epsilon B) = \emptyset$  we set  $B_\epsilon := \Omega \cap \tau(\epsilon B)$  and  $\Omega_\epsilon := \Omega \setminus \overline{B}_\epsilon$ .

With respect to the underlying problem of heat-transfer in microvascular tissue we think of  $B_\epsilon$  as small regions of blood of certain temperature. The domains  $\Omega_\epsilon$  describe solid tissue parts where heat-transfer by conduction takes place. For simplicity, we assume that heat transfer in the solid tissue is described by thermal conductivity with conductivity coefficient equal to 1. Taking into account some external thermal sources  $S_\epsilon$  the temperature distribution is then modeled by the diffusion equation  $-\Delta T_\epsilon = S_\epsilon$ . Furthermore, we shall assume that on the inner boundary  $\partial B_\epsilon$  the transition between blood regions and solid tissue is governed by Newtons cooling law with respect to an  $\epsilon$  dependent heat transfer coefficient. For a discussion of the  $\epsilon$  scaling in Newtons cooling law see [8].

On two sides of the outer boundary  $\partial\Omega$  we couple Newtons cooling law, thinking of a branching medium size vessel, with equations of type (1.2). More precisely, see Fig.

2, we consider on  $\Gamma_1 := (0, 1) \times \{0\}$  and  $\Gamma_2 := \{0\} \times (0, 1)$  for admissible  $\epsilon > 0$ ,  $\gamma_i > 0$  and for given thermal sources  $Q^i$  the ordinary differential equations

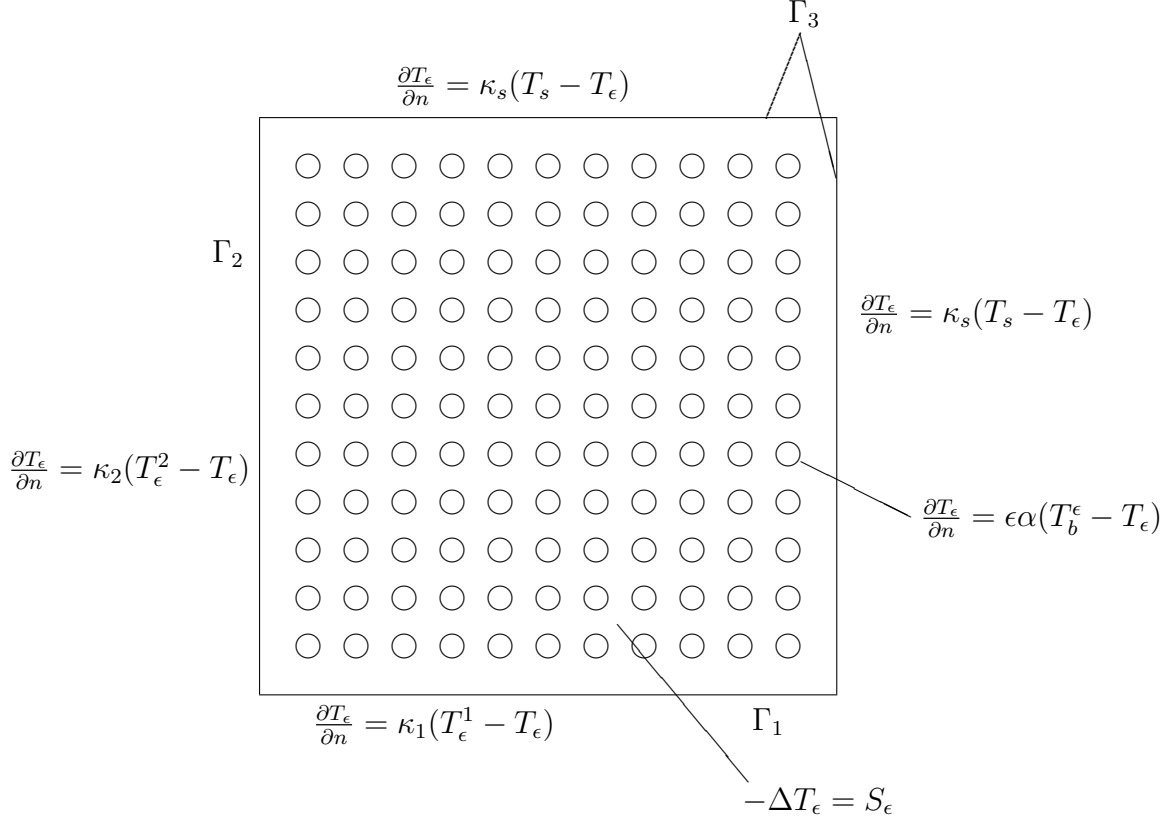


Figure 2: Geometry of the model problem

$$\begin{cases} \frac{dT_\epsilon^i}{dx_i} = \gamma_i(T_{\epsilon,i} - T_\epsilon^i) + Q^i, & \text{on } \Gamma_i, \\ T_\epsilon^i(0) = 0. \end{cases} \quad (2.1)$$

In (2.1)  $T_\epsilon^i$  represents the temperature of the blood in the medium size vessel  $\Gamma_i$  and  $T_{\epsilon,i}$  the temperature of the corresponding vessel wall  $\Gamma_i$ .

For the following, we introduce the bounded linear operators

$$\begin{aligned} P_i : H^{1/2}(\Gamma_i) &\rightarrow H^{1/2}(\Gamma_i) \\ f &\mapsto e^{-\gamma_i t} \int_0^t f(\tau) e^{\gamma_i \tau} d\tau. \end{aligned}$$

Then, for sufficiently smooth  $T_{\epsilon,i}$  and  $Q^i$  solutions to (2.1) are given by

$$T_\epsilon^i(t) = e^{-\gamma_i t} \int_0^t (\gamma_i T_{\epsilon,i}(\tau) + Q^i(\tau)) e^{\gamma_i \tau} d\tau = P_i(\gamma_i T_{\epsilon,i} + Q^i)(t). \quad (2.2)$$

Applying next Newtons cooling law and setting  $T_{\epsilon,i} = T_\epsilon|_{\Gamma_i}$  gives for the thermal problem in  $\Omega_\epsilon$  on  $\Gamma_i$  the nonlocal boundary conditions

$$\frac{\partial T_\epsilon}{\partial n} = \kappa_i(T_\epsilon^i - T_\epsilon) = \kappa_i(P_i(\gamma_i T_\epsilon|_{\Gamma_i} + Q^i) - T_\epsilon)$$

with respect to some  $\kappa_i > 0$ .

Thus, see again Fig. 2, we end up for given  $S_\epsilon, T_b^\epsilon, T_s, Q^i$  and  $\epsilon, \alpha, \kappa_i, \gamma_i, \kappa_s > 0$  with the boundary value problems

$$\begin{cases} -\Delta T_\epsilon = S_\epsilon & \text{in } \Omega_\epsilon, \\ \frac{\partial T_\epsilon}{\partial n} = \epsilon\alpha(T_b^\epsilon - T_\epsilon) & \text{on } \partial B_\epsilon, \\ \frac{\partial T_\epsilon}{\partial n} = \kappa_i(P_i(\gamma_i T_\epsilon|_{\Gamma_i} + Q^i) - T_\epsilon) & \text{on } \Gamma_i, \quad i = 1, 2, \\ \frac{\partial T_\epsilon}{\partial n} = \kappa_s(T_s - T_\epsilon) & \text{on } \Gamma_3. \end{cases} \quad (2.3)$$

The variational formulation of (2.3) looks as follows: Find  $T_\epsilon \in H^1(\Omega_\epsilon)$  such that for all  $v \in H^1(\Omega_\epsilon)$  holds

$$\begin{aligned} & \int_{\Omega_\epsilon} \nabla T_\epsilon \cdot \nabla v \, dx + \epsilon\alpha \int_{\partial B_\epsilon} T_\epsilon v \, d\sigma + \sum_{i=1}^2 \kappa_i \int_{\Gamma_i} (T_\epsilon - \gamma_i P_i(T_\epsilon|_{\Gamma_i}))v \, d\sigma + \kappa_s \int_{\Gamma_3} T_\epsilon v \, d\sigma \\ & = \int_{\Omega_\epsilon} S_\epsilon v \, dx + \epsilon\alpha \int_{\partial B_\epsilon} T_b^\epsilon v \, d\sigma + \kappa_s \int_{\Gamma_3} T_s v \, d\sigma + \sum_{i=1}^2 \kappa_i \int_{\Gamma_i} P_i(Q^i)v \, d\sigma. \end{aligned} \quad (2.4)$$

The corresponding bilinear form  $a_\epsilon(\cdot, \cdot)$  defined by

$$a_\epsilon(u, v) := \int_{\Omega_\epsilon} \nabla u \cdot \nabla v \, dx + \epsilon\alpha \int_{\partial B_\epsilon} uv \, d\sigma + \sum_{i=1}^2 \kappa_i \int_{\Gamma_i} (u - \gamma_i P_i(u|_{\Gamma_i}))v \, d\sigma + \kappa_s \int_{\Gamma_3} uv \, d\sigma \quad (2.5)$$

is  $H^1(\Omega_\epsilon)$ -coercive, i.e. up to compact perturbation  $H^1(\Omega_\epsilon)$ -elliptic. Therefore, notice that the right hand side in (2.4) gives a bounded linear form, the Fredholm alternative holds for the variational problem (2.4).

**Theorem 1** *The variational problems (2.4) are uniquely solvable in  $H^1(\Omega_\epsilon)$ .*

**Proof.** Let  $u \in H^1(\Omega_\epsilon)$  satisfy  $a_\epsilon(u, v) = 0$  for all  $v \in H^1(\Omega_\epsilon)$ . Then in particular holds  $\int_{\Omega_\epsilon} \nabla u \cdot \nabla \overline{v} \, dx = 0$  for all  $v \in C_0^\infty(\Omega_\epsilon)$ . Since  $\Omega_\epsilon$  is connected we have  $u = C \in \mathbb{R}$ . Take next  $\Gamma \subset \Gamma_3$  with  $|\Gamma| > 0$ ,  $\text{dist}(\Gamma, \Gamma_1 \cup \Gamma_2) > 0$  and  $\chi \in H^1(\Omega_\epsilon)$  with  $\chi \geq 0$ ,  $\chi|_\Gamma = 1, \chi|_{\Gamma_1 \cup \Gamma_2 \cup \partial B_\epsilon} = 0$ , then for  $v = C\chi$  holds

$$a_\epsilon(C, v) = \kappa_s \int_{\Gamma_3} C^2 \chi \, d\sigma > 0,$$

which gives  $u = C = 0$ . ■

Notice, that for given  $T_\epsilon^i \in L^2(\Gamma_i)$  the variational solutions  $T_\epsilon \in H^1(\Omega_\epsilon)$  to the boundary value problems described in Fig. 2 possess at least the regularity  $T_\epsilon \in H^{3/2}(\Omega_\epsilon)$ , i.e. (2.2) is satisfied.

### 3 The Asymptotic Limit

For proving the uniform boundedness of the variational solutions  $T_\epsilon \in H^1(\Omega_\epsilon)$ , we need a little bit more than the coerciveness of the bilinear form  $a_\epsilon(\cdot, \cdot)$  for each single  $\epsilon > 0$ . More precisely, we assume in the following, that the bilinear forms  $a_\epsilon(\cdot, \cdot)$  are uniformly elliptic, i.e., there is a constant  $c > 0$  independent of (the admissible)  $\epsilon > 0$  such that

$$a_\epsilon(u, u) \geq c \|u\|_{H^1(\Omega_\epsilon)}^2, \quad u \in H^1(\Omega_\epsilon). \quad (3.1)$$

For analyzing sufficient conditions for (3.1) let  $\gamma^*$  denote the root of the equation  $\gamma(1 - e^{-2\gamma}) = 2$ . Approximately, it is  $\gamma^* \approx 2.03476$ .

**Theorem 2** *If  $\gamma_i \leq \gamma^*$ , then (3.1) holds.*

**Proof.** For  $u \in H^1(\Omega_\epsilon)$  and  $i \in \{1, 2\}$ , we set  $f := u|_{\Gamma_i}$  and  $\gamma := \gamma_i$ . By the definition of  $P_i$  follows

$$\begin{aligned} |P_i(f)(t)| &= \left| \int_0^t f(\tau) e^{\gamma(\tau-t)} d\tau \right| \leq \left( \int_0^1 f^2(\tau) d\tau \right)^{1/2} \left( \int_0^t e^{2\gamma(\tau-t)} d\tau \right)^{1/2} \\ &\leq \left( \frac{1 - e^{-2\gamma}}{2\gamma} \right)^{1/2} \left( \int_0^1 f^2(\tau) d\tau \right)^{1/2}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{\Gamma_i} (f - \gamma P_i(f)) f d\sigma &\geq \int_{\Gamma_i} f^2 d\sigma - \gamma \left( \int_{\Gamma_i} P_i^2(f) d\sigma \right)^{1/2} \left( \int_{\Gamma_i} f^2 d\sigma \right)^{1/2} \\ &\geq \int_{\Gamma_i} f^2 d\sigma - \gamma \left( \frac{1 - e^{-2\gamma}}{2\gamma} \right)^{1/2} \int_{\Gamma_i} f^2 d\sigma \\ &\geq 0, \end{aligned}$$

since for  $\gamma \leq \gamma^*$  holds  $1 - \gamma \left( \frac{1 - e^{-2\gamma}}{2\gamma} \right)^{1/2} \geq 0$ . Thus (2.5) gives for  $\gamma_i \leq \gamma^*$

$$a_\epsilon(u, u) \geq \int_{\Omega_\epsilon} (\nabla u)^2 dx + \epsilon \alpha \int_{\partial B_\epsilon} u^2 d\sigma + \kappa_s \int_{\Gamma_3} u^2 d\sigma$$

and therefore for a suitable  $c > 0$  the ellipticity (3.1). ■

**Remark.** Let us shortly discuss the assumption  $\gamma_i \leq \gamma^* \approx 2.03476$ . If  $Q^i = 0$ ,  $T_\epsilon = T_{const}$ ,  $T(0) = 0$ , one gets

$$T(1) = \gamma e^{-\gamma} T_{const} \int_0^1 e^{\gamma\tau} d\tau = (1 - e^{-\gamma}) T_{const} \leq (1 - e^{-\gamma^*}) \cdot T_{const} \approx 0.869288 \cdot T_{const},$$

which means that there is an approximately 87% temperature transfer allowed. In the interesting applications, i.e. regional and local hyperthermia, this is sufficiently general. For further related remarks on the heat transfer from tissue to blood vessels, see [16].

**Theorem 3** *Under the assumption (3.1) holds*

$$\|T_\epsilon\|_{H^1(\Omega_\epsilon)} \lesssim 1$$

**Proof.** The coerciveness implies

$$\begin{aligned} \|T_\epsilon\|_{H^1(\Omega_\epsilon)}^2 &\lesssim a_\epsilon(T_\epsilon, T_\epsilon) \\ &= \int_{\Omega_\epsilon} S_\epsilon T_\epsilon dx + \epsilon\alpha \int_{\partial B_\epsilon} T_\epsilon^\epsilon T_\epsilon d\sigma + \kappa_s \int_{\Gamma_3} T_s T_\epsilon d\sigma + \sum_{i=1}^2 \kappa_i \int_{\Gamma_i} P_i(Q^i) T_\epsilon d\sigma. \end{aligned}$$

The only term which has to be considered in detail is the second one: However, proceeding analogously to [12] one gets

$$|\epsilon \int_{\partial B_\epsilon} T_b^\epsilon T_\epsilon d\sigma| \lesssim \|T_\epsilon\|_{H^1(\Omega_\epsilon)}.$$

■

Next we need for  $\epsilon > 0$  extension operators  $E^\epsilon : H^1(\Omega_\epsilon) \rightarrow H^1(\Omega)$  such that uniformly in  $\epsilon$  holds

$$\|T_\epsilon\|_{H^1(\Omega_\epsilon)} \lesssim 1 \implies \|E^\epsilon(T_\epsilon)\|_{H^1(\Omega)} \lesssim 1. \quad (3.2)$$

Such extension operators can easily be described in the given geometrical situation: First, take the usual bounded extension operator from  $H^1(Y^*)$  to  $H^1(Y)$ . Then take with respect to each  $Y_{\epsilon,k}^* = \epsilon(k + Y^*)$ ,  $k \in \mathbb{Z}^2$ , the related one from  $H^1(Y_{\epsilon,k}^*)$  to  $H^1(Y_{\epsilon,k})$ , where  $Y_{\epsilon,k} := \epsilon(k + Y)$ . Finally, the near by hand combination of those local operators gives extension operators  $E^\epsilon : H^1(\Omega_\epsilon) \rightarrow H^1(\Omega)$ , such that (3.2) holds. Consequently, under the assumption (3.1) follows for the extensions  $E^\epsilon(T_\epsilon)$

$$\|E^\epsilon(T_\epsilon)\|_{H^1(\Omega)}^2 \lesssim 1.$$

Consequently, there exists  $T \in H^1(\Omega)$  with

$$E^\epsilon(T_\epsilon) \rightharpoonup T \quad \text{in } H^1(\Omega).$$

In the following, we denote for an arbitrary function  $\varphi$  on  $\Omega_\epsilon$  by  $\tilde{\varphi}$  its zero-extension on  $\Omega$ , i.e., it is  $\tilde{\varphi} = \varphi$  in  $\Omega_\epsilon$  and  $\tilde{\varphi} = 0$  in  $\Omega \setminus \Omega_\epsilon$ .

**Theorem 4** *If  $S_\epsilon \in L^2(\Omega_\epsilon)$ ,  $S \in L^2(\Omega)$  with  $\tilde{S}_\epsilon \rightharpoonup \theta S$  in  $L^2(\Omega)$  and  $T_b^\epsilon \in W^{1,\infty}(\Omega)$ ,  $T_b \in H^1(\Omega)$  with  $\|T_b^\epsilon\|_{W^{1,\infty}(\Omega_\epsilon)} \lesssim 1$ ,  $\tilde{T}_b^\epsilon \rightharpoonup \theta T_b$  in  $L^2(\Omega)$ , then, under the assumption (3.1), there exists  $T \in H^1(\Omega)$  with*

$$E^\epsilon(T_\epsilon) \rightharpoonup T \quad \text{in } H^1(\Omega)$$

and  $T$  is the variational solution of the nonlocal boundary value problem

$$\begin{cases} -\operatorname{div}(\mathcal{A}\nabla T) + \alpha\theta_Q(T - T_b) = S & \text{in } \Omega, \\ \frac{\partial T}{\partial n_A} = \kappa_i(P_i(\gamma_i T|_{\Gamma_i} + Q^i) - T) & \text{on } \Gamma_i, \quad i = 1, 2, \\ \frac{\partial T}{\partial n_A} = \kappa_s(T_s - T) & \text{on } \Gamma_3. \end{cases} \quad (3.3)$$



The coefficients of the constant and positive definite matrix  $\mathcal{A} = (a_{ij})_{ij}$  are given by

$$a_{ij} = \delta_{ij} - \frac{1}{|Y^*|} \int_{Y^*} \frac{\partial \chi^j}{\partial y_i} dy \quad (3.4)$$

where the functions  $\chi^j$  are solutions to the cell problems

$$\begin{cases} -\Delta \chi^j = 0 & \text{in } Y^* \\ \frac{\partial(\chi^j - y_j)}{\partial n} = 0 & \text{on } \partial Q, \\ \chi^j \text{ } Y\text{-periodic.} \end{cases} \quad (3.5)$$

**Proof.** Setting  $\xi_\epsilon := \nabla T_\epsilon$ , there is  $\xi \in [L^2(\Omega)]^2$  with  $\tilde{\xi}_\epsilon \rightharpoonup \theta \xi$  in  $[L^2(\Omega)]^2$  for  $\epsilon \rightarrow 0$ . Proceeding as in [8], see also [2, 5, 11], we obtain in  $\Omega$

$$-\operatorname{div} \xi = S + \alpha \theta_Q (T_b - T) \quad (3.6)$$

as well as

$$\xi = \mathcal{A} \cdot \nabla T \quad \text{in } \Omega, \quad (3.7)$$

where  $\mathcal{A}$  is given by (3.4) and (3.5). Moreover, the identities (3.6) and (3.7) gives the first identity in (3.3).

Considering now the variational problem (2.4) with respect to  $v \in H^1(\Omega_\epsilon)$  and taking into account the identity (3.6), we obtain in the limit and by partial integration

$$\int_{\partial\Omega} v \xi \cdot n \, d\sigma + \sum_{i=1}^2 \kappa_i \int_{\Gamma_i} (T - \gamma_i P_i(T|_{\Gamma_i})) v \, d\sigma + \kappa_s \int_{\Gamma_3} T v \, d\sigma = \kappa_s \int_{\Gamma_3} T_s v \, d\sigma + \sum_{i=1}^2 \kappa_i \int_{\Gamma_i} P_i(Q^i) v \, d\sigma,$$

i.e. with (3.7)

$$\int_{\partial\Omega} \frac{\partial T}{\partial n_{\mathcal{A}}} v \, d\sigma = \kappa_s \int_{\Gamma_3} (T_s - T) v \, d\sigma + \sum_{i=1}^2 \int_{\Gamma_i} \kappa_i (P_i(\gamma_i T|_{\Gamma_i} + Q^i) - T) v \, d\sigma,$$

which gives the second and third identity in (3.3). ■

## 4 First Corrector Analysis

It is well-known from general homogenization theory, cf. [1, 6], that the solution to the homogenized equation is not the strong, but only the weak limit of the solutions to the periodic problems. Therefore correctors are introduced, i.e.  $\epsilon$ -dependent functions, such that the sum of the solutions to the periodic problems and those corrector functions converge strongly. Moreover, for many periodic problems it could be shown that the convergence is of some order  $\epsilon^s$  for  $s > 0$  with respect to appropriate norms. Generally, correctors can be understood as smoothing operators, because they suppress the fast oscillations in the gradient of  $T_\epsilon - T$ . For the role of correctors in numerical homogenization see [13].

In the following we additionally assume that  $T \in H^3(\Omega)$ ,  $S_\epsilon = S \in H^1(\Omega)$  and  $T_b^\epsilon = T_b \in H^1(\Omega)$ , which requires e.g.  $T_s \in H^{3/2}(\Gamma_3)$ .

The formal derivation of first and second order correctors follows the standard approach, cf. [1, 6, 18], and is completely analogous to [8]: Starting point is the asymptotic ansatz

$$T_\epsilon(x) = T_0(x, y) + \epsilon T_1(x, y) + \epsilon^2 T_2(x, y) + \dots,$$

with  $y = x/\epsilon$  and where  $T_i$  are  $Y$ -periodic functions defined on  $\Omega \times Y^*$ . Then, applying

$$\Delta = \epsilon^{-2} \Delta_y + 2\epsilon^{-1} \Delta_{xy} + \Delta_x,$$

where  $\Delta_y = \sum_{i=1}^3 \frac{\partial^2}{\partial y_i^2}$ ,  $\Delta_{xy} = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i \partial y_i}$ ,  $\Delta_x = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ , as well as

$$\frac{\partial}{\partial n} = \epsilon^{-1} \frac{\partial}{\partial n(y)} + n(y) \cdot \nabla_x,$$

we obtain in  $\Omega_\epsilon$

$$\begin{aligned} -\Delta T_\epsilon &= -\epsilon^2 \Delta_y T_0 - \epsilon^{-1} (\Delta_y T_1 + 2\Delta_{xy} T_0) - (\Delta_x T_0 + 2\Delta_{xy} T_1 + \Delta_y T_2) + \dots \\ &= S, \end{aligned} \tag{4.1}$$

and on the boundary  $\partial\Omega_\epsilon$

$$\begin{aligned} \frac{\partial T_\epsilon}{\partial n} &= \epsilon^{-1} \frac{\partial}{\partial n(y)} T_0 + \frac{\partial}{\partial n(y)} T_1 + n(y) \cdot \nabla_x T_0 + \epsilon \left( \frac{\partial}{\partial n(y)} T_2 + n(y) \cdot \nabla_x T_1 \right) + \dots \\ &= \begin{cases} \epsilon \alpha (T_b - T_0) - \epsilon^2 \alpha T_1 - \dots & \text{on } \partial B_\epsilon \\ \kappa_i (P_i(\gamma_i T_0|_{\Gamma_i} + Q^i) - T_0) + \epsilon \kappa_i P_i(\gamma_i T_1|_{\Gamma_i}) - \epsilon \kappa_i T_1 + \dots & \text{on } \Gamma_i, \quad i = 1, 2, \\ \kappa_s (T_s - T_0) - \epsilon \kappa_s T_1 - \dots & \text{on } \Gamma_3. \end{cases} \end{aligned}$$

Comparing the coefficients shows, that one can choose

$$T_0(x, y) = T(x)$$

and

$$T_1(x, y) = - \sum_{j=1}^3 \chi^j(y) \frac{\partial}{\partial x_j} T(x).$$

By  $\chi \in H^1(Y^*)$  we denote the unique solutions to

$$\begin{cases} -\Delta_y \chi = -\theta_Q \text{ in } Y^*, \\ \frac{\partial \chi}{\partial n} = 1 \text{ on } \partial Q, \\ \chi \text{ } Y\text{-periodic with vanishing mean value.} \end{cases} \tag{4.2}$$

Then the following Theorem holds.

**Theorem 5** *If  $\chi, \chi^j \in W^{1,\infty}(Y^*)$ , there is a constant  $C > 0$  independent of  $\epsilon > 0$  such that*

$$\|T_\epsilon - T + \epsilon \sum_{j=1}^3 \chi^j(\cdot) \frac{\partial T}{\partial x_j}\|_{H^1(\Omega_\epsilon)} \leq C\epsilon^{1/2}. \tag{4.3}$$

**Proof.** We proceed similar to [12]: Setting  $R_\epsilon := T_\epsilon - T - \epsilon T_1$  and taking into account (3.3), we get in  $\Omega_\epsilon$

$$-\Delta R_\epsilon = -\operatorname{div}(\mathcal{A}\nabla T) + \alpha\theta_Q(T - T_b) + \Delta_x T + 2\Delta_{xy}T_1 + \epsilon\Delta_x T_1$$

with  $\Delta_{xy}T_1 = \sum_{i,j=1}^3 \frac{\partial \chi^j}{\partial y_i} \frac{\partial^2 T}{\partial x_i \partial x_j}$ . It is  $\Delta_x T_1, \Delta_{xy}T_1 \in L^2(\Omega_\epsilon)$  and therefore  $\|\Delta R_\epsilon\|_{L^2(\Omega_\epsilon)} \lesssim 1$ .

On  $\partial B_\epsilon$  we have

$$\frac{\partial T_\epsilon}{\partial n} = \epsilon\alpha(T_b - T_\epsilon)$$

and

$$\frac{\partial(T + \epsilon T_1)}{\partial n} = n \cdot \nabla_x T + \frac{\partial}{\partial n(y)} T_1 + \epsilon n \cdot \nabla_x T_1 = \epsilon n \cdot \nabla_x T_1,$$

i.e., it is

$$\frac{\partial R_\epsilon}{\partial n} = \epsilon[\alpha(T_b - T_\epsilon) - n \cdot \nabla_x T_1].$$

Thus, we obtain on  $\partial B_\epsilon$

$$\frac{\partial R_\epsilon}{\partial n} + \epsilon\alpha R_\epsilon = \epsilon\alpha(T_b - T) - \epsilon n \cdot \nabla_x T_1 - \epsilon^2\alpha T_1. \quad (4.4)$$

On  $\Gamma_i, i = 1, 2$ , we have

$$\begin{aligned} \frac{\partial R_\epsilon}{\partial n} &= \kappa_i P_i(\gamma_i(T_\epsilon - T)|_{\Gamma_i}) - \kappa_i(T_\epsilon - T) + \frac{\partial T}{\partial n_{\mathcal{A}}} - \frac{\partial T}{\partial n} - \epsilon \frac{\partial T_1}{\partial n} \\ &= \kappa_i P_i(\gamma_i R_\epsilon|_{\Gamma_i}) - \kappa_i R_\epsilon + \kappa_i P_i(\gamma_i T_1|_{\Gamma_i}) - \kappa_i T_1 + \frac{\partial T}{\partial n_{\mathcal{A}}} - \frac{\partial T}{\partial n} - \epsilon \frac{\partial T_1}{\partial n} \end{aligned}$$

and on  $\Gamma_3$  holds

$$\frac{\partial R_\epsilon}{\partial n} = \kappa_s(T - T_\epsilon) + \frac{\partial T}{\partial n_{\mathcal{A}}} - \frac{\partial T}{\partial n} - \epsilon \frac{\partial T_1}{\partial n} = -\kappa_s T_\epsilon - \kappa_s \epsilon T_1 + \frac{\partial T}{\partial n_{\mathcal{A}}} - \frac{\partial T}{\partial n} - \epsilon \frac{\partial T_1}{\partial n}.$$

Integration by parts gives

$$\begin{aligned} \|\nabla R_\epsilon\|_{[L^2(\Omega_\epsilon)]^3}^2 &= -\int_{\Omega_\epsilon} \Delta R_\epsilon R_\epsilon dx + \int_{\partial B_\epsilon} R_\epsilon \frac{\partial R_\epsilon}{\partial n} d\sigma + \int_{\partial\Omega} R_\epsilon \frac{\partial R_\epsilon}{\partial n} d\sigma \\ &= -\int_{\Omega_\epsilon} \operatorname{div}(\mathcal{A}\nabla T) R_\epsilon dx + \alpha\theta_Q \int_{\Omega_\epsilon} (T - T_b) R_\epsilon dx + \int_{\Omega_\epsilon} \Delta_x T R_\epsilon dx \\ &\quad + 2 \int_{\Omega_\epsilon} \Delta_{xy} T_1 R_\epsilon dx + \epsilon \int_{\Omega_\epsilon} \Delta_x T_1 R_\epsilon dx - \epsilon\alpha \int_{\partial B_\epsilon} R_\epsilon^2 d\sigma \\ &\quad + \epsilon\alpha \int_{\partial B_\epsilon} (T_b - T) R_\epsilon d\sigma - \epsilon \int_{\partial B_\epsilon} n \cdot \nabla_x T_1 R_\epsilon d\sigma - \epsilon^2\alpha \int_{\partial B_\epsilon} T_1 R_\epsilon d\sigma \\ &\quad - \int_{\Gamma_3} \kappa_s R_\epsilon^2 d\sigma + \int_{\Gamma_3} [-\epsilon\kappa_s T_1 + \frac{\partial T}{\partial n_{\mathcal{A}}} - \frac{\partial T}{\partial n} - \epsilon \frac{\partial T_1}{\partial n}] R_\epsilon d\sigma \\ &\quad + \sum_{i=1}^2 \int_{\Gamma_i} \kappa_i (P_i(\gamma_i R_\epsilon|_{\Gamma_i}) - R_\epsilon) R_\epsilon d\sigma \\ &\quad + \sum_{i=1}^2 \int_{\Gamma_i} [\kappa_i (P_i(\gamma_i \epsilon T_1|_{\Gamma_i}) - \epsilon T_1) + \frac{\partial T}{\partial n_{\mathcal{A}}} - \frac{\partial T}{\partial n} - \epsilon \frac{\partial T_1}{\partial n}] R_\epsilon d\sigma. \end{aligned}$$

Because of (3.1) and

$$\int_{\partial\Omega} \epsilon \frac{\partial T_1}{\partial n} R_\epsilon d\sigma = \int_{\partial\Omega} \frac{\partial T_1}{\partial n_y} R_\epsilon d\sigma + \epsilon \int_{\partial\Omega} n \cdot \nabla_x T_1 R_\epsilon d\sigma$$

we have

$$\begin{aligned} & \|R_\epsilon\|_{H^1(\Omega_\epsilon)}^2 \\ & \lesssim a_\epsilon(R_\epsilon, R_\epsilon) \\ & = \|\nabla R_\epsilon\|_{[L^2(\Omega_\epsilon)]^3}^2 + \epsilon\alpha \int_{\partial B_\epsilon} R_\epsilon^2 d\sigma + \kappa_s \int_{\Gamma_3} R_\epsilon^2 d\sigma \\ & \quad - \sum_{i=1}^2 \kappa_i \int_{\Gamma_i} (P_i(\gamma_i R_\epsilon|_{\Gamma_i}) - R_\epsilon) R_\epsilon d\sigma \\ & = - \int_{\Omega_\epsilon} \operatorname{div}(\mathcal{A}\nabla T) R_\epsilon dx + \alpha\theta_Q \int_{\Omega_\epsilon} (T - T_b) R_\epsilon dx + \int_{\Omega_\epsilon} \Delta_x T R_\epsilon dx \\ & \quad + 2 \int_{\Omega_\epsilon} \Delta_{xy} T_1 R_\epsilon dx + \epsilon \int_{\Omega_\epsilon} \Delta_x T_1 R_\epsilon dx + \epsilon\alpha \int_{\partial B_\epsilon} (T_b - T) R_\epsilon d\sigma \\ & \quad - \epsilon \int_{\partial\Omega_\epsilon} n \cdot \nabla_x T_1 R_\epsilon d\sigma - \epsilon^2\alpha \int_{\partial B_\epsilon} T_1 R_\epsilon d\sigma \\ & \quad + \int_{\Gamma_3} [-\epsilon\kappa_s T_1 + \frac{\partial T}{\partial n_A} - \frac{\partial T}{\partial n} - \frac{\partial T_1}{\partial n_y}] R_\epsilon d\sigma \\ & \quad + \sum_{i=1}^2 \int_{\Gamma_i} [\kappa_i (P_i(\gamma_i \epsilon T_1|_{\Gamma_i}) - \epsilon T_1) + \frac{\partial T}{\partial n_A} - \frac{\partial T}{\partial n} - \frac{\partial T_1}{\partial n_y}] R_\epsilon d\sigma. \end{aligned} \tag{4.5}$$

For

$$\alpha_{i,j} := \frac{1}{|Y^*|} \int_{Y^*} \frac{\partial \chi^j}{\partial y_i} dy - \frac{\partial \chi^j}{\partial y_i}, \quad i, j = 1, 2, \tag{4.6}$$

holds

$$\int_{Y^*} \alpha_{ij}(y) dy = 0.$$

Thus we obtain

$$\begin{aligned} & \left| \int_{\Omega_\epsilon} (\operatorname{div}(\mathcal{A}\nabla T) - \Delta_x T - \Delta_{xy} T_1) R_\epsilon dx \right| \\ & = \left| \int_{\Omega_\epsilon} \left( \sum_{i,j=1}^2 \left( \frac{1}{|Y^*|} \int_{Y^*} \frac{\partial \chi^j}{\partial y_i} dy - \frac{\partial \chi^j}{\partial y_i} \right) \frac{\partial^2 T}{\partial x_i \partial x_j} \right) R_\epsilon dx \right| \\ & \lesssim \epsilon^{1/2} \|R_\epsilon\|_{H^1(\Omega_\epsilon)}, \end{aligned} \tag{4.7}$$

see [19].

Since

$$\Delta_{xy} T_1 = \epsilon \sum_{i=1}^2 \frac{\partial}{\partial x_i} \left( \frac{\partial T_1}{\partial x_i} \Big|_{y=x/\epsilon} \right) - \epsilon \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2} T_1,$$

it is

$$\begin{aligned} \int_{\Omega_\epsilon} \Delta_{xy} T_1 R_\epsilon dx &= \epsilon \sum_{i=1}^2 \int_{\Omega_\epsilon} \left( \frac{\partial}{\partial x_i} \left( \frac{\partial T_1}{\partial x_i} \Big|_{y=x/\epsilon} \right) \right) R_\epsilon dx - \epsilon \int_{\Omega_\epsilon} \Delta_x T_1 R_\epsilon dx \\ &= -\epsilon \int_{\Omega_\epsilon} \nabla_x T_1 \cdot \nabla R_\epsilon dx + \epsilon \int_{\partial\Omega_\epsilon} n \cdot \nabla_x T_1 R_\epsilon d\sigma - \epsilon \int_{\Omega_\epsilon} \Delta_x T_1 R_\epsilon dx. \end{aligned}$$

Therefore we get

$$\begin{aligned} \left| \int_{\Omega_\epsilon} \Delta_{xy} T_1 R_\epsilon dx + \epsilon \int_{\Omega_\epsilon} \Delta_x T_1 R_\epsilon dx - \epsilon \int_{\partial\Omega_\epsilon} n \cdot \nabla_x T_1 R_\epsilon d\sigma \right| &= \epsilon \left| \int_{\Omega_\epsilon} \nabla_x T_1 \cdot \nabla R_\epsilon dx \right| \\ &\lesssim \epsilon \|R_\epsilon\|_{H^1(\Omega_\epsilon)}. \end{aligned} \quad (4.8)$$

Furthermore, applying  $\chi \in W^{1,\infty}(Y^*)$  yields, see [12],

$$\begin{aligned} &\left| \epsilon \alpha \int_{\partial B_\epsilon} (T_b - T) R_\epsilon d\sigma - \alpha \theta_Q \int_{\Omega_\epsilon} (T_b - T) R_\epsilon dx \right| \\ &= \left| \epsilon \int_{\Omega_\epsilon} (\nabla_y \chi)(x/\epsilon) \nabla((T_b - T) R_\epsilon)(x) dx - \epsilon \int_{\partial\Omega} n \cdot (\nabla_y \chi)(x/\epsilon) (T_b - T) R_\epsilon(x) d\sigma \right| \\ &\lesssim \epsilon \|R_\epsilon\|_{H^1(\Omega_\epsilon)} \end{aligned} \quad (4.9)$$

Next we note that for  $u \in H^1(\Omega_\epsilon)$  holds

$$\|u\|_{L^2(\partial B_\epsilon)}^2 \lesssim \|u\|_{L^2(\partial\Omega)}^2 + \|u\|_{L^2(\Omega_\epsilon)} \|\nabla u\|_{[L^2(\Omega_\epsilon)]^3} + \epsilon^{-1} \|u\|_{L^2(\Omega_\epsilon)}^2 \quad (4.10)$$

see[12] and also [18]. Therefore, we get  $\|T_1\|_{L^2(\partial B_\epsilon)} \lesssim \epsilon^{-1/2}$ ,  $\|R_\epsilon\|_{L^2(\partial B_\epsilon)} \lesssim \epsilon^{-1/2}$  and

$$\left| \epsilon^2 \alpha \int_{\partial B_\epsilon} T_1 R_\epsilon d\sigma \right| \lesssim \epsilon. \quad (4.11)$$

Moreover, it is

$$\left| -\kappa_s \int_{\Gamma_3} \epsilon T_1 R_\epsilon d\sigma + \sum_{i=1}^2 \int_{\Gamma_i} \kappa_i (P_i(\gamma_i \epsilon T_1|_{\Gamma_i}) - \epsilon T_1) R_\epsilon d\sigma \right| \lesssim \epsilon \|R_\epsilon\|_{H^1(\Omega_\epsilon)}. \quad (4.12)$$

Because of

$$\frac{\partial T}{\partial n_{\mathcal{A}}} - \frac{\partial T}{\partial n} - \frac{\partial T_1}{\partial n_y} = \sum_{j=1}^2 \left[ \sum_{i=1}^2 n_i(y) \left( \frac{\chi^j(y)}{\partial y_i} - \frac{1}{|Y^*|} \int_{Y^*} \frac{\chi^j(y)}{\partial y_i} dy \right) \right] \frac{\partial T}{\partial x_j}$$

and, since with respect to  $\sigma_i := \{y \in \bar{Y} \mid y_i = 0 \text{ or } y_i = 1\}$ ,  $i = 1, 2$ , holds

$$\int_{\sigma_2} \alpha_{2,j}(y) dy_1 = \int_{\sigma_1} \alpha_{1,j}(y) dy_2 = 0, \quad j = 1, 2,$$

see the definitions (4.6) and e.g. the proof of Lemma 2.1 in [19], we have

$$\begin{aligned}
& \int_{\partial\Omega} \left( \frac{\partial T}{\partial n_{\mathcal{A}}} - \frac{\partial T}{\partial n} - \frac{\partial T_1}{\partial n_y} \right) R_\epsilon d\sigma \\
&= \int_{\partial\Omega} R_\epsilon \sum_{j=1}^2 \frac{\partial T}{\partial x_j} \left[ \sum_{i=1}^2 n_i(y) \left( \frac{\chi^j(y)}{\partial y_i} - \frac{1}{|Y^*|} \int_{Y^*} \frac{\chi^j(y)}{\partial y_i} dy \right) \right] d\sigma \\
&\lesssim \epsilon^{1/2} \|R_\epsilon\|_{H^1(\Omega_\epsilon)}. \tag{4.13}
\end{aligned}$$

Finally, combining the estimates (4.7)–(4.13), we are led to

$$\begin{aligned}
\|R_\epsilon\|_{H^1(\Omega_\epsilon)}^2 &\lesssim \epsilon + \epsilon^{1/2} \|R_\epsilon\|_{H^1(\Omega_\epsilon)} \\
&\lesssim \max\{\epsilon, \epsilon^{1/2} \|R_\epsilon\|_{H^1(\Omega_\epsilon)}\},
\end{aligned}$$

which implies

$$\|R_\epsilon\|_{H^1(\Omega_\epsilon)} \lesssim \epsilon^{1/2}.$$

■

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