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KKT Systems in Operative Planning for Gas Distribution Networks

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KKT SYSTEMS IN OPERATIVE PLANNING FOR GAS DISTRIBUTION NETWORKS

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ABSTRACT. Operative planning in gas networks with prescribed binary decisions yields large scale nonlinear programs defined on graphs. We study the structure of the KKT systems arising in interior methods and present a customized direct solution algorithm. Computational results indicate that the algorithm is suitable for optimization in small and medium-sized gas networks.

0. INTRODUCTION

In [3] we have introduced a nonlinear programming model for short-term planning in gas networks under prescribed combinatorial decisions (switching of compressors and valves); for related work see, e.g., [2, 4, 5]. Our model is based on a directed graph whose nodes (providers, customers, and junctions) are linked via active elements (compressors, valves, control valves) and passive elements (pipes, connections); the arc directions indicate the known direction of flow. Implicit Euler schemes in space and time yield a fully discretized model of the hyperbolic PDE governing the gas flow. The objective is to minimize the total energy consumption of the compressors, which accounts for most of the variable operating costs. Optimization results obtained with SNOPT have been presented in [3]. The present paper aims at novel solution algorithms for the large-scale NLP based on interior methods in combination with specialized KKT solvers. In what follows we focus on the latter.

1. STRUCTURE AND SOLUTION OF KKT SYSTEMS

In the model [3] we distinguish state variables z and (linearly entering) control variables u. All inequalities are simple bounds. In a primal-dual interior method, the Newton step KKT system for T time intervals then takes the general form

$$\begin{bmatrix} H(z,\lambda)+\Phi_z & & A(z)^* & F^* \\ & \Phi_u & B^* & \\ A(z) & B & & \\ F & & & \end{bmatrix} \begin{pmatrix} \Delta z \\ \Delta u \\ -\Delta\lambda \\ -\Delta\eta \end{pmatrix} = - \begin{pmatrix} f(z,\lambda,\eta) \\ d(u,\lambda) \\ h(z,u) \\ e(z) \end{pmatrix} \tag{1}$$

with $(f,d,h,e) \in \mathbf{R}^{(n_z+n_u+n_z+1)T}$. Here the Hessian H and control operator B are block-diagonal, and F corresponds to a single linear terminal constraint,

$$H = \mathrm{Diag}(H_1,\dots,H_T), \tag{2}$$

$$B = \mathrm{Diag}(B_1,\dots,B_T), \tag{3}$$

$$F = (0,\dots,0,F_T). \tag{4}$$

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The state operator A is block-bidiagonal due to the implicit Euler scheme in time (with fixed initial values z_0 at $t = 0$),

$$A = \begin{bmatrix} A_1 & & & \\ L_2 & A_2 & & \\ & \ddots & \ddots & \\ & & L_T & A_T \end{bmatrix}, \tag{5}$$

where A_t, L_t possess a substructure whose block columns correspond to pressure, flow rate, and density variables, $z_t = (p_t, q_t, \rho_t)$,

$$(L_t, A_t, B_t) = \left(\begin{array}{ccc|ccc|c} \cdot & \cdot & * & * & * & * & \cdot \\ \cdot & \cdot & \cdot & * & * & \cdot & * \\ \cdot & \cdot & \cdot & \cdot & * & \cdot & \cdot \end{array}\right) \begin{array}{l} \text{pipe flow,} \\ \text{other constraints,} \\ \text{node flow balances.} \end{array} \tag{6}$$

The sparsity pattern of the '$*$' blocks depends on the topology of the particular gas network. This is also true for the Hessian blocks H_t whose entries stem exclusively from pipes and compressors; all other constraints and the objective are linear.

Two key observations permit a direct solution of the KKT system at reasonable cost: (a) the invertible state operators A_t admit inexpensive factorizations, and (b) the control space dimension $n_u T$ is independent of the space discretization and comparatively small. As to (a), it can be shown that any effects of a space discretization can be pre-eliminated with linear cost by a spatial condensing, which in effect reduces pipe sequences to single pipe segments. Here, for simplicity and to avoid potential instabilities, we factorize A_t by the public domain sparse solver MA28. Direct factorizations of A and A^* are then readily constructed as forward and backward recursions in time.

Returning to system (1), the inverse state operator A^{-1} and its adjoint A^{-*} are employed to eliminate Δz and $\Delta\lambda$; the resulting reduced KKT system is then solved for Δu and $\Delta\eta$. That is, we use block pivots (3,1), (1,3), (2,2), and (4,4) in that order, where the respective (symmetric) pivot blocks (2,2) and (4,4) obtained from the previous eliminations are

$$S = \Phi_u + \bar{B}^*(H + \Phi_z)\bar{B}, \tag{7}$$

$$M = F\bar{B}S^{-1}\bar{B}^*F^* \tag{8}$$

with

$$\bar{B} = A^{-1}B. \tag{9}$$

Here $\bar{B}$ is block lower triangular and S is dense. The practical computation proceeds as follows: let $S^{(0)} = \Phi_u$, evaluate

$$(\bar{B}_{t1}, \dots, \bar{B}_{tt}) = A_t^{-1}\left[(0, \dots, 0, B_t) - L_t(\bar{B}_{t-1,1}, \dots, \bar{B}_{t-1,t-1}, 0)\right], \tag{10}$$

$$\begin{bmatrix} S_{11}^{(t)} & & \\ \vdots & \ddots & \\ S_{t1}^{(t)} & \dots & S_{tt}^{(t)} \end{bmatrix} = \begin{bmatrix} S_{11}^{(t-1)} & & \\ \vdots & \ddots & \\ S_{t1}^{(t-1)} & \dots & S_{tt}^{(t-1)} \end{bmatrix} + \begin{pmatrix} \bar{B}_{t1}^* \\ \vdots \\ \bar{B}_{tt}^* \end{pmatrix} H_t(\bar{B}_{t1}, \dots, \bar{B}_{tt}) \tag{11}$$

in a forward recursion $t = 1, \dots, T$, then set $S = S^{(T)}$, calculate the Cholesky factorization $S = LL^*$, and finally form

$$\bar{F} = F\bar{B} = F_T(\bar{B}_{T1}, \dots, \bar{B}_{TT}), \tag{12}$$

$$\tilde{F} = \bar{F}L^{-*}, \tag{13}$$

$$M = \tilde{F}\tilde{F}^* \in \mathbf{R}. \tag{14}$$

Except for the implicit state equation, $L_t\Delta z_{t-1} + A_t\Delta z_t + B_t\Delta u_t = -h_t$, this procedure is similar to a classical condensing recursion [1].

TABLE 1. Solution statistics for the test problem: T, number of time steps; Δt, length of time steps (min); Δx, maximal pipe length (km); n_z, n_u, dimensions of z_t, u_t; ne_A, ne_H, ne_{KKT}, numbers of entries of A_t, H_t, and entire system; $t(\ldots)$, CPU times (s) on a 3 GHz PC.

T	Δt	Δx	n_z n_u	ne_A ne_H	$n_z T$ $n_u T$	ne_{KKT}	$t(A^{-1})$	$t(S)$	$t(S^{-1})$	t_{total}
48	60	235	233 19	592 181	11184 912	39379	0.03	14.92	0.72	15.78
96	30	235	233 19	592 181	22368 1824	78787	0.07	116.81	5.66	122.96
144	20	235	233 19	592 181	33552 2736	118195	0.10	383.89	18.26	403.25
48	60	80	289 19	746 237	13872 912	50117	0.04	18.74	0.73	19.64
96	30	80	289 19	746 237	27744 1824	100277	0.09	144.36	5.60	150.61
144	20	80	289 19	746 237	41616 2736	150437	0.14	478.46	18.14	497.91
48	60	40	393 19	1032 341	18864 912	70059	0.07	25.80	0.70	26.76
96	30	40	393 19	1032 341	37728 1824	140187	0.16	195.28	5.62	201.79
144	20	40	393 19	1032 341	56592 2736	210315	0.23	647.13	18.24	667.21

2. COMPUTATIONAL RESULTS

The KKT solver just described has been implemented in C++ and tested on a medium-sized network representing the backbone transport network of Ruhrgas AG. It consists of 13 compressor stations, 6 regulators, 4 valves and, for the chosen space discretizations, between 63 and 103 nodes and between 29 and 69 pipe segments with a total length of roughly 2500 km. Table 2 gives solution statistics for the longest relevant planning horizon, 48 h, with selected space and time discretizations. Not surprisingly, the computational effort is clearly dominated by assembling the reduced Hessian S (with cubic complexity in $n_u T$) whereas the Cholesky factorization of S (also cubic in $n_u T$) and the factorization of $A_1, \ldots, A_t$ are much cheaper. In particular, the CPU time increases only moderately with the number of pipe segments. Since 30-minute steps and 40 km pipe segments suffice for practical purposes, the solution algorithm can be considered suitable for networks up to medium size.

3. CONCLUSION

We have presented a direct solution algorithm for KKT systems arising in gas management, based on sparse factorizations of the state operators A_t in combination with forward and backward recursions in time. The performance is satisfactory up to a few thousand degrees of freedom (i.e., control variables $n_u T$). Future research will be aimed at developing preconditioned iterative methods suitable for real-time optimization in large networks.

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