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An Overview of Graph Covering and Partitioning

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An Overview of Graph Covering and Partitioning

Stephan Schwartz

Abstract

While graph covering is a fundamental and well-studied problem, this field lacks a broad and unified literature review. The holistic overview of graph covering given in this article attempts to close this gap. The focus lies on a characterization and classification of the different problems discussed in the literature. In addition, notable results and common approaches are also included. Whenever appropriate, our review extends to the corresponding partitioning problems.

Graph covering problems are among the most classical and central subjects in graph theory. They also play a huge role in many mathematical models for various real-world applications. There are two different variants that are concerned with covering the edges and, respectively, the vertices of a graph. Both draw a lot of scientific attention and are subject to prolific research. In this paper we attempt to give an overview of the field of graph covering problems.

In a graph covering problem we are given a graph G and a set of possible subgraphs of G . Following the terminology of Knauer and Ueckerdt [KU16], we call G the *host graph* while the set of possible subgraphs forms the *template class*. Elements of the template class are called template graphs or, for short, templates. An edge (resp. vertex) covering is a set of templates such that each edge (resp. vertex) of the host graph belongs to at least one template. In addition, there is a measure for the quality of a covering, e.g., the number of used templates. The graph covering problem is to find a covering that optimizes this quality measure among all coverings.

If the template graphs of a covering are pairwise disjoint, the covering is called a partition. In many cases the analogously defined graph partitioning problem is considered instead of the graph covering problem.

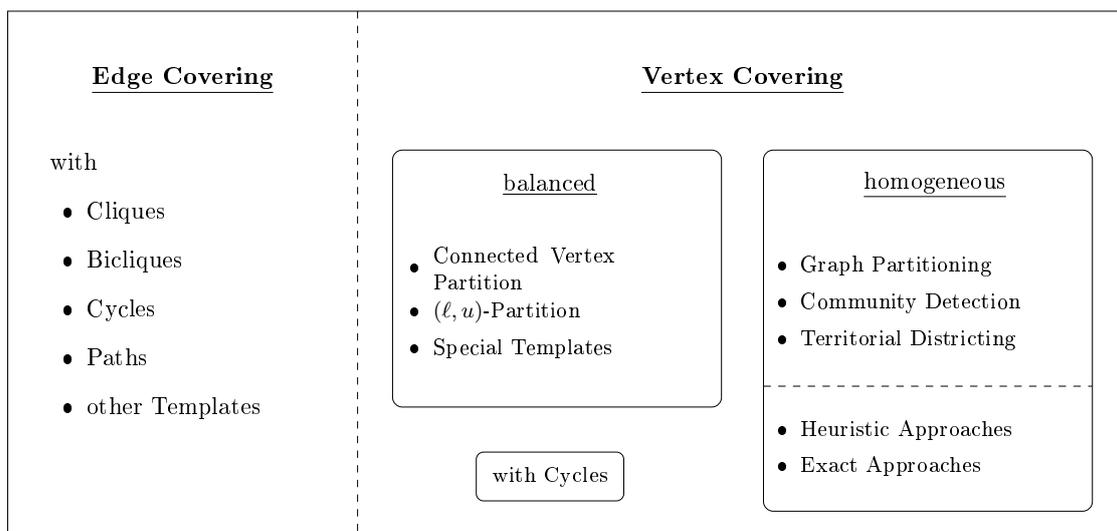


Figure 1: Structure of this survey article.

This paper intends to give an overview of studied problems, techniques, and results in graph covering. Whenever appropriate, we also include works on graph partitioning. We are not aware of a similar attempt to characterize and group covering problems to this extent and in such a general form. Concerning vertex covering or partitioning problems, this seems to be the first attempt of compiling and organizing the rich literature that is divided into multiple subproblems. Concerning edge covering problems, the only review article seems to be 30 years old [Pyb91]. Despite the wide range of problems and papers that we consider here, this survey does not claim to be exhaustive.

Figure 1 shows that we divide this overview in two practically independent parts: edge covering and vertex covering. The former is dominated by questions and contributions from the graph theory community and structured in a linear fashion. Vertex covering problems, on the other hand, are mostly motivated by applications and advances are mainly driven by the operations research community.

If not specified otherwise, we consider undirected and simple graphs. We adhere to standard notation and nomenclature of graph theory (see, e.g., [Die16]) and denote by n the number of vertices of the host graph.

1 Edge Covering

The investigation of edge covering problems in its present form was initiated by graph theory legends Paul Erdős [EGP66] and László Lovász [Lov68]. Since then, research is predominantly concerned with improving bounds on the covering number, i.e., the minimum number of templates necessary for an edge covering. The covering number is studied for different template classes and often, special host graphs or special coverings are considered.

As a special covering we single out the local covering number here. It is defined as the smallest number ℓ such that a covering exists where every vertex is contained in at most ℓ template graphs.

The only survey article on edge covering that we are aware of is the long-ago paper by Pyber [Pyb91]. Our approach for an overview is summarized in Table 1, organizing the contributions to various edge covering problems. In the following we will point out the considered problems of these works. While notable results are also included, for some papers we limit this overview to a categorization of the content. Our presentation is organized along the lines of different template classes.

Cliques: The clique covering problem is the oldest variant and has been studied in different types. A number of survey articles summarize earlier results on this topic as well as a few applications [Pul83; Rob85; MPR95; Cav05].

The problem was first studied by Erdős, Goodman, and Pósa [EGP66] in terms of decomposing a graph into edges and triangles. They showed that such a decomposition can be done with at most $\lfloor n^2/4 \rfloor$ template graphs and that this bound is met for the complete bipartite graph $K_{n/2, n/2}$. Recently, Král et al. [Král+19] extended this result and proved that any graph admits an edge partitioning into n_2 copies of K_2 and n_3 copies of K_3 such that $2n_2 + 3n_3 \leq (1/2 + o(1))n^2$, settling a conjecture of Győri and Tuza [GT87].

Bounds for the clique covering number of general host graphs are given by Lovász [Lov68] if the number of edges is known, and by Alon [Alo86] depending on the maximum degree. Other authors study the relation between covering and partitioning of the edges. Caccetta et al. [Cac+85] investigate the difference between clique covering number and clique partitioning number, while Erdős, Faudree, and Ordman [EFO88] focus on the ratio of the two numbers.

Concerning the complexity, Kou, Stockmeyer, and Wong [KSW78] and, independently, Orlin [Orl77] proved that clique covering is NP-hard. The former also proved that there is no 2-approximation, unless $P=NP$.

More results on clique covering can be found for specific host classes. For instance, the problem has been studied on line graphs [Orl77], on claw-free graphs [JMO16; JH19], on regular graphs [PC80; PC81], and on graphs with bounded maximum degree [Pul84]. Furthermore, the clique covering problem has been considered for complements of graphs. While Wallis [Wal82] derives bounds for complements of general graphs, other articles focus on complementary graphs of certain

Table 1: Overview on edge covering contributions for different template classes.

	survey	general bounds	special host graphs	special coverings	other	partition
cliques	[Pul83], [Rob85], [MPR95], [Cav05]	[EGP66], [Lov68], [Alo86]	[Orl77], [PC80]. [PC81], [PD81], [GP82], [Wal82], [Pul84], [CGP85], [EFO88], [Che+00], [CV08], [CFS14]. [JMO16], [JH19]	[PSW82], [JMO16]	[KSW78], [Cac+85], [EFO88]	[EGP66], [PSW82], [Cac+85], [EFO88], [Uiy03], [Krá+19]
bicliques	[MPR95]	[HHM77], [Ber78], [Chu80], [Tuz84], [JK09]	[GWS99], [Bez+08]	[FH96], [EP97], [DL07]	[Orl77], [Mül96], [AVJ98], [CF06], [Wat06] [Gün07].	[GP72], [Tve82], [CES83], [EP97], [GWS99], [DL07], [Pin14], [ABH17]
cycles	[Zha97]	[EGP66], [Pyb85], [Bon90], [Fan02]		[Ita+81], [BJJ83], [AT85], [Jae85], [Fan97], [Tho97], [Fan98], [BS01], [HO01]. [IMM05], [Cha09], [Zha16]		
paths		[Chu80], [Pyb96], [Fan02]				[Lov68], [Don80], [Yan98], [Fan05], [ZL06], [BJ17]

graph classes such as paths and cycles [CGP85], cliques [Orl77; PD81; EFO88], matchings [Orl77; EFO88; GP82], or forests [CV08; CFS14].

Instead of restricting the host class, one can also consider special covers. Pullman, Shank, and Wallis [PSW82], for instance, consider the edge partitioning into maximal cliques. With regard to local covering, Javadi, Maleki, and Omoomi [JMO16] examine the local clique covering number. In particular, they study the relationship between this number and other graph parameters. In addition, they give asymptotic bounds for the local clique covering number if the host graph is claw-free.

Bicliques: Another template class to consider are bicliques, i.e., complete bipartite graphs. It is easy to see that the edges of K_n can be covered with $n - 1$ stars. The same holds for the partition case, but here fewer bicliques do not suffice [GP72; Tve82]. For the covering, however, we can reduce the number of required template graphs to $\lceil \log_2 n \rceil$ [HHM77; Ber78]. Bounds for the general case are provided by Bermond [Ber78], Chung [Chu80], and Tuza [Tuz84]. In relation to the number of edges and the size of a matching in the host graph, Jukna and Kulikov [JK09] provide another lower bound.

The difference between biclique covering and biclique partitioning is explored in [Pin14]. A lower bound for the biclique partition number is proved in [GWS99]. Alon, Bohman, and Huang [ABH17] give a probabilistic upper bound for the partition case.

Just as the clique covering problem, the biclique covering problem is NP-hard, even if the host graph is bipartite [Orl77]. It remains NP-hard if the bipartite graph is chordal [Mül96], but is polynomial solvable for bipartite domino-free graphs [AVJ98].

If the host graph is the complement of a path or a cycle, Gregory, Watts, and Shader [GWS99] give exact values for the biclique partition number. Bezrukov et al. [Bez+08] give a tight bound on the biclique covering number if the host graph is a $K_{n,n}$ with a perfect matching removed.

The local biclique covering number is studied by Fishburn and Hammer [FH96] and Dong and Liu [DL07]. The latter, together with Erdős and Pyber [EP97] also examine the local biclique partition number.

Cornaz and Fonlupt [CF06] formulate an integer program to find a minimum biclique cover, while Watts [Wat06] considers the LP relaxation of this problem, i.e., fractional biclique covers.

An application for the biclique cover number is presented by Günlük [Gün07]. It is shown that this number is an upper bound on the min-cut max-flow ratio of a multicommodity flow problem.

Cycles: It is obvious that a cycle covering of the edges exists if and only if the host graph is 2-connected. For arbitrary host graphs, Erdős, Goodman, and Pósa [EGP66] conjectured that any graph can be covered with $n - 1$ cycles and edges. While this bound is clearly tight for trees, it was proven for general graphs by Pyber [Pyb85].

Below we summarize the results for 2-connected host graphs and different variants of the cycle cover problem. These variants are also discussed in the survey article by Jackson [Jac93] and in a book on this topic by Zhang [Zha97].

Bondy [Bon90] conjectured that $\lfloor \frac{2n-1}{3} \rfloor$ cycles always suffice for the covering and showed that this bound is best possible. His conjecture was confirmed by Fan [Fan02].

A famous variant of the present problem is the *cycle double cover conjecture*. As the name suggests, it asks whether every 2-connected graph has a cycle covering where each edge of the graph belongs to exactly two cycles. While this conjecture is still open, advances towards it are given in the surveys by Jaeger [Jae85], Chan [Cha09], and Zhang [Zha16].

Another well studied and practically relevant version of the cycle covering problem takes into account the length of the covering cycles. More specifically, the goal of this *minimum cost cycle-cover problem (MCCP)* is to find an edge covering with cycles such that the sum of the length of all cycles is minimal. For the unit weight case, Itai et al. [Ita+81] give a constructive upper bound on the total length. This bound is further improved by Bermond, Jackson, and Jaeger [BJJ83], Alon and Tarsi [AT85], Fraisse [Fra85], Fan [Fan97], and Fan [Fan98]. Determining the optimal value, however, is NP-hard [Tho97]. The case of metric edge weights is studied by Immorlica, Mahdian,

and Mirrokni [IMM05]. They give an approximation result for a natural greedy covering and show that imposing length bounds on the template cycles makes the problem APX-hard. The MCCP with length constraints is also studied by Bläser and Siebert [BS01], and Hochbaum and Olinick [HO01]. Finally, let us briefly describe the difference between MCCP and the well-known Chinese Postman Problem (CPP). The CPP is concerned with finding a closed walk of minimum length that contains every edge of a given edge-weighted graph. "Besides its obvious application to mail delivery in China, this problem finds application in a variety of routing problems such as robot navigation and city snow plowing planning" [IMM05]. Unlike the MCCP, the CPP is known to be polynomial solvable [EJ73]. Every solution of MCCP can easily be transformed into a solution of CPP. The only difference between the two is that a CPP solution can be the union of cycles and edges, whereas edges are forbidden templates for the MCCP. This small change, however, leads to the complexity increase from P to NP-hardness.

Paths: The Gallai conjecture states that any graph can be partitioned into $\lceil \frac{n}{2} \rceil$ edge-disjoint paths. Indeed, one can see that this bound is sharp by considering a graph where every vertex has odd degree. Then, n is even and every vertex must be an endpoint of a covering path, which leads to a minimum of $\frac{n}{2}$ covering paths (cf. [Lov68]). Gallai's conjecture is still open, but progress has been made for special cases, see [Lov68; Don80; Yan98; Fan05; ZL06; BJ17] and references therein.

How about the respective covering problem? Analogously to the partition case, Chung [Chu80] conjectured that any graph can be covered with at most $\lceil \frac{n}{2} \rceil$ paths. Pyber [Pyb96] was able to prove a weaker bound. In contrast to the partition case, however, the covering conjecture is now settled and was confirmed by Fan [Fan02].

Other templates: While Gallai conjectured that any graph can be partitioned into $\lceil \frac{n}{2} \rceil$ paths, Chung [Chu78] showed that it is indeed possible with the same number of **trees**. Earlier, Nash-Williams [N64] had already studied this problem, while Vidyasankar [Vid78] considers the respective covering problem on directed graphs.

More general, Harary, Hsu, and Miller [HHM77] derive a bound for the number of **bipartite graphs** required to cover all edges. In particular, they show that $\lceil \log_2 \chi(G) \rceil$ template graphs always suffice, where $\chi(G)$ is the chromatic number. A remarkable consequence of their result is that any planar graph can be covered with two bipartite graphs.

Motivated by an application for testing printed circuit boards, Bussieck [Bus94] studies the problem of covering the edges of a graph by **cuts**. The objective is to minimize the sum of the cardinalities of all cuts. He proves that finding an optimal cut cover is NP-hard and discusses approximation possibilities. Füredi and Kündgen [FK01] provide general bounds as well as sharp bounds for special host graphs. Kündgen and Spangler [KS05] give an improved bound on the size of the cut cover and describe a connection to cycle covers.

Another variant is the covering with **odd graphs**, i.e., graphs where each non-isolated vertex has odd degree. Due to Mátrai [Mát06], three odd subgraphs suffice to cover any simple graph. The same result also holds for loopless graphs [PŠ19].

Knauer and Ueckerdt [KU16] give results for various template classes on different host graphs. As templates they consider **star forests**, **caterpillar forests**, and **interval graphs**. As host graphs they consider, e.g., planar graphs or graphs with bounded tree-width. They overview existing results for these cases and provide new bounds, not only for the classical objective of minimizing the number of necessary templates but also for slightly different objectives.

Thite [Thi06] considers the covering with a given number of **induced subgraphs** and the objective to minimize the number of vertices in the largest template graph. Finding this minimum is NP-hard, but he proves lower bounds and presents approximation algorithms for certain host graph classes.

Finally, we have the template class of **connected subgraphs**. Again, the number of templates to use is given. Existing work is focused on the partition case and on the objective that the template graphs should have similar weight. The weight of a template graph is the sum of all edge weights in this graph. For the unit weight case, Jünger, Reinelt, and Pulleyblank [JRP85] work

towards an equipartition of the edges into connected subgraphs. Chu et al. [Chu+10] consider a similar problem for trees as host graphs. The general case with arbitrary graphs and edge weights is handled in [CWC13]. The authors present a linear time algorithm to find a partition into connected components such that the heaviest template has at most twice as much weight as the lightest template. A 2-approximation for this problem is independently derived by Borndörfer, Eljazyfer, and Schwartz [BES19] by transforming the problem into a vertex partition problem and exploiting the claw-freeness of line graphs. Finding a partition into connected subgraphs of similar weight is also the subject of the connected vertex partition problem that is discussed in the next section.

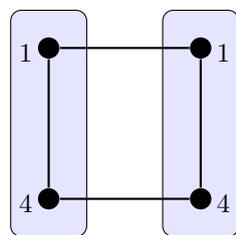
2 Vertex Covering

The literature concerning vertex covering has a fundamentally different focus compared to the edge covering. Motivated by a number of different applications, the works are often oriented towards constructive algorithms and solution techniques instead of theoretical bounds. Furthermore, research on vertex covering is dominated by the partition case. As a consequence, this survey is also mostly concerned with vertex partitioning problems. The presented optimization methods, however, can obviously be used or adapted for the respective and more general covering problems.

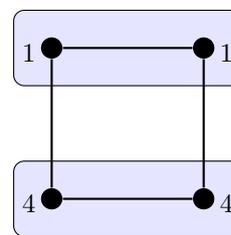
Note that the partition problem is closely related to the well-studied graph factors. A factor of a graph is a spanning subgraph and typically, further restrictions on the spanning subgraph are imposed. Most prominent are degree factors where every vertex must have a specified degree. For instance, 1-factors are perfect matchings, while 2-factors are a covering of the vertices with disjoint cycles. On the other hand, a component factor is described on a more general graphical concept, e.g., every component has to be a tree. The literature on graph factors is focused on existence results under certain conditions. We refrain from covering these works here and refer the interested reader to [Plu07; AK11].

Several applications require the host graphs to be weighted. The goal of a partition is usually to obtain either balanced parts or a clustering. For the balancing objective the template graphs shall be as similar as possible. A classical application is to distribute jobs evenly among machines. With the clustering goal, on the other hand, each template graph is supposed to be as homogeneous as possible. Applications for this range from image processing to the design of sales territories. Figure 2 illustrates solutions with respect to each objective. Note that homogeneity and balancing are not necessarily opposing objectives. In many cases both are pursued, e.g., if we are seeking a cover that is homogeneous with respect to given vertex weights but balanced with respect to the cardinality of the templates.

Following this distinction, our overview is divided into balanced covers and homogeneous covers. The latter also includes combinations of the two. As a third, and independent problem, we consider vertex covering with cycles as templates.



(a) balanced templates



(b) homogeneous templates

Figure 2: Example for balanced and homogenous template graphs. The numbers indicate vertex weights.

2.1 Balanced Covers

There are several important applications for this problem. While the task of political districting clearly requires a partition [BEL03; GW18], other applications such as terrain coverage with multiple robots admit overlapping solutions. The latter includes significant future tasks such as cleaning, harvesting, and security patrolling jobs performed by a number of coordinated robots [Zhe+05; YJC13]. In both applications it is desired that the resulting subregions have a similar weight, i.e., the number of voters or the time necessary for job completion.

There are several popular objectives for balanced covers. Among the most prominent variants are the max-min and min-max objectives. Given a weighted graph and a number of templates, the goal is to maximize the total weight of the minimum part or, respectively, to minimize the weight of the maximum part. The bounded covering problem, on the other hand, aims to minimize the number of templates given a maximum template weight.

Without constraints on the template graphs, a balanced partition problem becomes a multi-way number partitioning problem, which is a generalization of the classical NP-hard partition problem, see e.g. [Kor09]. Therefore, the minimum requirement that is generally set on template graphs is connectivity. We divide the literature review for balanced covers into three parts. First, we consider the basic problem with connected templates. Then, additional constraints on the template sizes are imposed. And finally, we consider several other classes of connected templates, namely trees, paths, stars, and cycles.

Connected Vertex Partition (CVP): For a vertex weighted graph and an input parameter $k \in \mathbb{N}$ the CVP asks for a partition of the vertices into k connected components. If k is fixed, the problem is denoted by CVP_k . The CVP is usually considered with max-min or min-max objective and also known as the *balanced connected partition problem* (BCP).

The min-max and max-min version of CVP are closely related. In fact, for CVP_2 the two versions obviously coincide. For larger k though, optimal solutions of the two problems *can* differ. A simple example of this is presented in Figure 3. While for this instance one can find a solution that simultaneously optimizes the min-max and max-min objective, some instances do not admit such a solution (cf. [LPS93]).

Concerning the complexity, already CVP_2 is NP-hard [CGM83], even on unit-weighted bipartite graphs as shown by Dyer and Frieze [DF85]. Moreover, it is also hard to approximate. Chataigner, Salgado, and Wakabayashi [CSW07] prove that CVP_2 does not admit a polynomial-time approximation scheme (PTAS) and that there is no α -approximation for CVP with $\alpha < \frac{6}{5}$, unless $\text{P} = \text{NP}$. Earlier, Chlebiková [Chl96] had given a $\frac{4}{3}$ -approximation for CVP_2 . An approximation for the general CVP is studied by Borndörfer, Elijazyfer, and Schwartz [BES19].

Focussing on the max-min CVP, more approximation results are known. For CVP_3 and CVP_4 the best known result is a 2-approximation for 3-connected or, respectively, 4-connected graphs

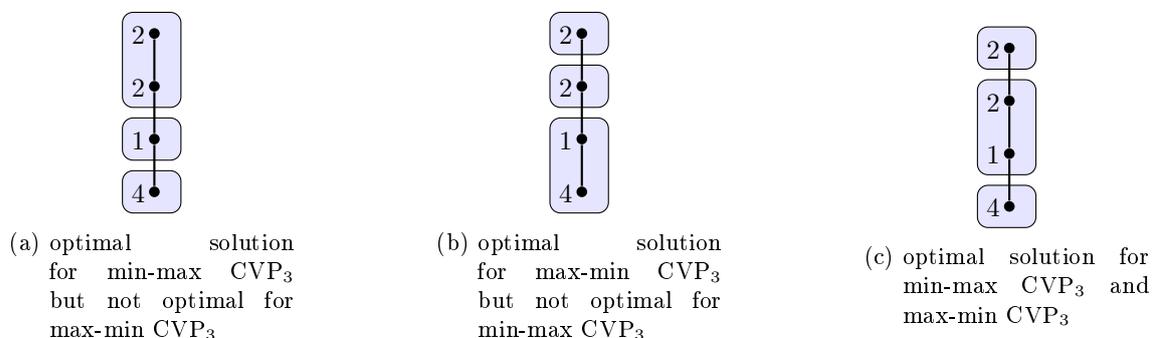


Figure 3: Comparison of solutions for min-max CVP_k and max-min CVP_k for an exemplary instance. The numbers indicate vertex weights.

[CSW07].

The max-min CVP for special host graphs allows for even better results. For instance, CVP_k is polynomially solvable for trees. Note that on a tree, any partition into k connected parts is uniquely determined by choosing $k - 1$ cut edges. This observation was used to give polynomial algorithms for the max-min CVP_k [PS81] and for the min-max CVP_k [BPS80]. Earlier, Kundu and Misra [KM77] had considered a closely related problem and provided an algorithm that can be employed to solve the min-max CVP_k in polynomial time. In 1991, Frederickson [Fre91] gave linear-time algorithms for both problems. These are particularly important, since different heuristics transform the original instance onto a tree to efficiently solve the problem there [CWC13; Zho+19].

Another host graph class for which CVP was investigated are grid graphs. While it was shown that CVP_k is NP-hard for arbitrary grid graphs [Bec+98], the max-min CVP_k can be solved in polynomial time for ladders, i.e., grid graphs with two rows and an arbitrary number of columns [Bec+01].

For the class of series-parallel graphs, CVP_k is also NP-hard. Ito, Zhou, and Nishizeki [IZN06] give a pseudo-polynomial-time algorithm for both, the min-max and max-min CVP. They also show that their algorithm can be extended to graphs with bounded tree-width.

Wu [Wu12] gives a fully polynomial-time approximation scheme for sufficiently connected (i.e., k -connected for CVP_k) interval graphs.

Finally, Miyazawa et al. [Miy+20] formulate the max-min CVP in terms of linear programs. Their formulations make use of vertex separators or flows, respectively, to model the connectivity. For the separator version the respective polyhedron is studied for the case of unit-weighted vertices.

The min-max CVP has received less attention. As mentioned above, Becker, Perl, and Schach [BPS80] give a polynomial algorithm for the min-max CVP_k if the host graph is a tree. For general host graphs, it was studied by Zhou et al. [Zho+19] who give a mixed integer linear program to solve the problem using a flow formulation to ensure the connection of the subgraphs. Since this approach is only feasible for small instances, they also propose a genetic algorithm as a heuristic approach. The main idea is to find a proper spanning tree whose optimal partition will also be optimal for the given graph. While the authors of [Zho+19] perform a computational study to show that their heuristic performs well, there is no theoretical result that guarantees the quality of this solution.

(l, u) -Partition We can also impose additional constraints on the size of each template. Given a graph with nonnegative vertex weights and numbers $0 \leq l \leq u$, a partition into connected components, each of which has a cumulated weight in the range $[l, u]$, is called an (l, u) -partition. Typical objectives are to maximize or to minimize the number of components of the (l, u) -partition. There is also a feasibility variant that asks if an (l, u) -partition into a given number k of components exists. Applications occur in political districting where each district should have a similar number of residents, but also in paging systems of operation systems and in image processing [Cun19].

Lucertini, Perl, and Simeone [LPS93] consider this problem when the host graph is a path. Exploiting the simple graph structure, they give a preprocessing routine and a greedy algorithm to solve the three mentioned variants of (l, u) -partition in linear time.

The problems are still polynomially solvable on trees [Ito+12]. The authors solve it with dynamic programming while preventing the corresponding table from expanding unreasonably.

For general host graphs the problem is studied by Ito, Zhou, and Nishizeki [IZN06]. They show that all variants are NP-hard, even for series-parallel graphs. For the class of series-parallel graphs, however, a pseudo-polynomial algorithm whose runtime depends on the upper bound u is given.

Cunha [Cun19] tackles the (l, u) -partition of minimum cardinality with integer programming. The problem is solved with a branch-and-cut algorithm that makes use of lifted subtour elimination constraints and other valid inequalities that prevent too small templates.

Special template classes Here, we will discuss several vertex covering problems with different template classes that are related to the *vehicle routing problem* (VRP). The VRP generalizes the travelling salesperson problem (TSP) and belongs to the most important and studied problems in

combinatorial optimization. It asks for an optimal set of routes for a number of vehicles, starting and ending at one or multiple depots, to visit a given set of customers. There is a number of popular variants for the VRP, e.g., with inclusion of time windows or with pickup and delivery constraints. The capacitated VRP imposes an upper bound on the length of each route. For details on VRP variants and a review of solution techniques see [Lap92; TV02; BRVN16].

In contrast to the CVP and to (l, u) -partitions, we consider edge weighted host graphs. The weights are usually assumed to be metric and in some cases one or several root vertices for the templates are specified. A rooted covering problem requires that every template contains the single root vertex or, in case of multiple given roots, that every template includes a distinct root vertex.

The first covering that we consider uses **trees** as templates. Even et al. [Eve+04] show that the min-max tree covering is NP-hard and derive a 4-approximation which is independently achieved by Arkin, Hassin, and Levin [AHL06]. Khani and Salavatipour [KS14] improve the approximation ratio to 3, and Farbstein and Levin [FL15] further improve the approximation ratio but their algorithm is only pseudo-polynomial in k . The problem has also gained notable attention if the host graph is a tree. As seen above, min-max partitioning a tree into k subtrees is polynomially solvable, whereas the respective covering problem is NP-hard already for $k = 2$ [AB96]. Approximations for this problems are found in [AB97; NO04; KMN11] and, for rooted versions, in [AB96; NO03]. Rooted cases for general host graphs are discussed in [Eve+04] and [Nag05]. An approximation for the min-max k tree cover with cardinality constraints on each template is due to Das, Jain, and Kumar [DJK20]. The bounded tree cover problem is NP-hard [AHL06] and there are approximation algorithms with factors of 3 [AHL06] and 2.5 [KS14], respectively. A linear programming formulation for this problem is studied in [YLB20].

A special tree covering is the covering with **paths**. This problem was the subject of the *Whizzkids '96* competition to determine optimal paths for newspaper delivery from a single depot in a specific instance ($n = 120, k = 4$). Applegate et al. [App+02] tackle this particular problem with a branch-and-bound approach. The general problem is studied in depth by Arkin, Hassin, and Levin [AHL06], establishing NP-hardness and providing a 4-approximation, and by Xu and Wen [XW10] who prove that no approximation within a factor of 1.5 is possible, unless $P=NP$. As a byproduct the problem is discussed in [FL15; YL19; YLB20]. A vertex covering using **stars** is considered in [AHL06] and also by Jorati [Jor13].

Finally, we have the covering with **cycles** which is closest to the VRP. Most papers explicitly or implicitly assume the host graph to be complete. In this case any tree covering α -approximation can easily be transformed into a cycle covering 2α -approximation. Exploiting the metric edge weights, the transformation mimics the classic TSP approximation with doubling the edges of the tree and using shortcuts to avoid repeated vertices. Improved approximations for min-max cycle covering are due to Jorati [Jor13] and Xu, Liang, and Lin [XLL13] for rooted and unrooted cases. Further improvements by Farbstein and Levin [FL15] are again only pseudo-polynomial in the number of templates. Yu, Liu, and Bao [YLB19] consider the bounded cycle cover and provide approximation algorithms. In [YLB20] the same authors derive lower bounds with relaxed LP formulations for this problem. Traub and Tröbst [TT20] consider a version of the problem where the number of cycles as well as the maximum weight of the cycles is part of the objective function. Rooted versions of the min-max cycle covering problem date back to Frederickson, Hecht, and Kim [FHK76] who studied this as a min-max k -TSP problem deriving a first approximation. A number of other papers also deal with approximation results for the rooted case [Nag05; Jor13; XLL13; FL15; YL19; LL20]. Due to the closeness to the TSP, there is more work on cycle coverings. It is, however, not concerned with a balancing of the templates, and therefore overviewed in Section 2.3

2.2 Homogeneous Covers

Grouping similar entities in a complex network is a common and essential task in many applications. Such homogeneous covering problems occur in social studies, political districting, and sales territory design. That is also the reason why the research on this topic is mainly driven by the operations research community. As many applications require disjoint template graphs, the partition variant is predominant in this field. Additionally, the unique structure of a partition often makes the

problem theoretically more amenable.

As mentioned above, homogeneity and balancing are not necessarily opposing objectives. If, for a homogeneous cover, the template size is bounded by upper or sometimes lower bounds, we call this a capacitated covering. The other case is referred to as uncapacitated covering. Uncapacitated partitioning is closely related to data clustering, where data points are to be grouped in several clusters of similar data points. Formulated as a vertex partitioning problem, however, the host graph is usually complete. This makes methods for general vertex partitioning unattractive for data clustering problems. On the other hand, methods for data clustering are usually inapplicable for general vertex partition problems on arbitrary host graphs or with constrained templates. Therefore, we will not discuss problems or techniques for data clustering in this overview, but refer to [JMF99; HPK11].

The main question for homogeneous covers is: How should we measure homogeneity? We follow the presentation of the survey article by Hansen and Jaumard [HJ97]. Assume that the dissimilarity between any two vertices is given by a symmetric, non-negative matrix (d_{uv}) with zeros on the diagonal. For given vertex weights (w_v) this could be $d_{uv} = |w_u - w_v|$. For given edge weights, on the other hand, d_{uv} could be the shortest path distance between u and v . Another possibility is that (d_{uv}) is the adjacency matrix with edge weights as entries.

Two ways to achieve homogeneous groups are to maximize the separation between different groups or the homogeneity within the single groups. Given the dissimilarities we first focus on measures for the separation or homogeneity of a single template T . Concerning separation-based measures we list the following two.

- split $s(T) := \min_{u \in T, v \notin T} d_{uv}$,
- cut $c(T) := \sum_{u \in T} \sum_{v \notin T} d_{uv}$.

For a fixed template both measures consider vertex pairs where exactly one vertex belongs to the template. While split is concerned with the dissimilarity of the worst case pair, the cut objective considers the sum over all dissimilarities. Hence, cut is a more robust measure than split. In order to obtain a homogeneous cover we maximize the value of the worst template or the sum over all templates, respectively.

In addition to separation-based measures we can also consider homogeneity-based measures. For these we consider dissimilarities of vertex pairs with both vertices in a fixed template. Here we discuss the following measures.

- diameter $d(T) := \max_{u, v \in T} d_{uv}$,
- radius $r(T) := \min_{u \in T} \max_{v \in T} d_{uv}$,
- star $st(T) := \min_{u \in T} \sum_{v \in T} d_{uv}$,
- clique $cl(T) := \sum_{u, v \in T} d_{uv}$,
- sum-of-squares $ss(T) = \sum_{v \in T} (w_v - \bar{w}_T)^2$ where $\bar{w}_T = \frac{1}{|T|} \sum_{v \in T} w_v$.

Again, we can optimize the value of the worst template or the sum over all templates. But for the mentioned measures we aim for a minimization. As described in Hansen and Jaumard [HJ97] these measures may also have normalized variants.

Let us take a closer look at the measures (cf. [HJ97]). Note that maximizing the sum of cut values is equivalent to minimizing the sum of clique values for any set of templates. This does only hold if we consider the sum of the template values instead of the value of the worst template. Another useful observation is that radius, star, and sum-of-squares make use of a center that can also be used as a representative of a template. While radius considers the maximum distance (or

dissimilarity) to the center, the star criterium takes into account all distances. In this sense, the measures are not only comparable to split and cut, but also have a relation to facility location. In a nutshell, given a number of points, a distance matrix, and a number k , the facility location problem is to determine locations for k facilities to minimize the distance to the given points. The two most prominent objectives for this problem are k -center and k -median. The former aims to minimize the maximum distance from a point to its closest center which is equivalent to the worst-case radius objective. On the other hand, the k -median aims to minimize the sum over all distances from a point to its closest center which is equivalent to the sum of stars objective. A special feature of facility location problems is that the host graph is usually complete and hence, connectivity of templates is not an issue. Finally, note that the sum-of-squares can easily be extended to multidimensional vertex weights by considering the (squared) Euclidean distance between a vertex weight and the mean vertex weight.

We stress that the literature covers even more than the presented measures for homogeneity. However, our selection is concentrated on the most prominent ones in the operations research community. Depending on the dissimilarity measure, most of the objective functions tend to favor connected templates. This is also desired in various applications and thus, many models incorporate such connectivity constraints. Before we address different optimization methods, we single out a special homogeneous vertex covering problem and describe two independent fields of application that drive the research on homogeneous covers: community detection and territorial districting.

Graph Partitioning A famous variant of vertex covering is the partitioning case where the host graph is edge-weighted and the dissimilarities are given by the weighted adjacency matrix. The problem is often considered under the name graph partitioning, and when referring to this particular version, in contrast to a general vertex covering, we will adopt this notion in this overview. The almost universally accepted measure for the quality of a partition is the cut objective, sometimes also in normalized form. In many cases a capacitated clustering is sought, i.e., additional balancing constraints hold the templates to similar sizes. Various books and survey articles (e.g. [Fj98; BS11; Bul+16]) detail problems, methods and applications for this particular vertex partitioning problem. In the optimization methods overview below we also discuss results on graph partitioning.

Community Detection The advances in this field are driven by the analysis of social and biological networks that are typically very large and in most cases unweighted. Problems of community detection are considered in the partitioning but also in the covering version. The name stems from the studying of social or citation graphs where one can observe that two neighbors of the same vertex are more frequently linked by an edge than two arbitrary vertices [GN02]. Accordingly, we can find subgroups of people with relatively dense internal connections. By identifying these communities we can better understand the structure and the interdependencies in these networks.

The problem is, however, ill-posed and various measures for the community analysis are considered. Let us single out the modularity maximization, defined by Newman and Girvan [NG04]. Informally, the modularity measures the fraction of edges inside of the communities minus this (expected) fraction in a random graph with identical degree distribution. For the specifics, different variants of modularity, and a plethora of other measures we refer the reader to the survey on community measures by Chakraborty et al. [Cha+17]. While the modularity maximization is among the most popular objectives in community detection, it has a notable resolution limit, i.e., it tends to ignore relatively small clusters even if they are well defined communities like cliques [FB07]. Furthermore, as most objectives, modularity maximization is NP-hard [Bra+07].

Due to the size of the host graphs the solving methods in this field are not exact but rely on heuristic approaches. Schaeffer [Sch07] gives an excellent survey over prominent solving techniques, but there are also more recent surveys [FH16; Jav+18].

As pointed out above, the applications of community detection go well beyond the analysis of social networks. Closely related problems arise in the analysis of different biological networks. Here, the clustering helps to uncover the mechanisms behind gene expression [XOX02] or protein-protein interaction [BH03; KPJ04].

Territorial Districting While community detection is concerned with abstract graphs, territorial districting problems are usually tied to graphs representing geographic maps. The predominant application is political districting but the same problem occurs in the planning of school districts [FG90], police patrol areas [Ami+02; CHQ10; CCY19], or in sales territory design [HS71; FP88; ZS05; SRC11]. Motivated from these applications the goals of the partition are threefold. Resulting districts should be contiguous, compact, and balanced with respect to given vertex weights (e.g., the number of voters, children, or customers). While the contiguity is usually ensured by constraints on the template graphs to be connected, the balance of the templates can be implemented as constraints or as part of the objective function. The compactness of districts is the most vague goal among these. Informally, "a round-shaped district is deemed to be acceptable, while an octopus- or an eel-like one is not" [RSS13]. The compactness can be formalized by using homogeneity measures such as star or radius with appropriate distance measures for every pair of nodes (e.g., bee-line distance, shortest path distance).

Two excellent survey articles on this topic are due to Ricca, Scozzari, and Simeone [RSS13] with an emphasis on optimization methods and by Kalcsics and Ríos-Mercado [KR19] with an emphasis on different variants and applications. A recent book summarizes latest results, models, and applications [Río20].

Let us now turn to different approaches for solving homogeneous cover and partition problems. We are focused on works of the operations research community with the presented homogeneity measures. These papers mostly relate to districting applications or graph partitioning while several (heuristic) methods are also used in community design. Table 2 categorizes contributions to homogeneous vertex covering problems.

Table 2: Overview of homogeneous vertex covering contributions. Entries with dashed underlines do not impose the connectivity of the template graphs as a constraint.

	heuristic	exact
split	[Han+03]	<u>[HJM90]</u> , [MSN97]
cut	<u>[KL70]</u> , <u>[DH72]</u> , <u>[Fie75]</u> , <u>[SM00]</u> , [Bic11], [Kim+11], <u>[LK13]</u> , <u>[SS13]</u>	<u>[GW89]</u> , <u>[BCR97]</u> , <u>[Sen01]</u> , <u>[Arm+08a]</u> , <u>[LÖ10]</u> , <u>[Sør17]</u>
diameter	[Ami+02], [RS11]	[GN70]
radius	[RS08], [RF09], [ERD14]	[Nyg88], [RF09], [SRC11], [Lar+16] [CGP19]
star	[FP88], [BLP05], [Seg+07]	[MJN98], [Seg+07], [Apo+08] [SRC11]
clique	<u>[ABR92]</u> , [MSN97], <u>[HFV99]</u> , [DAR12]	<u>[JMN93]</u> , <u>[Fer+96]</u> , <u>[Fer+98]</u>
sum-of-squares	[MS95]	<u>[Hes+65]</u>
survey	[Mur85], [Gor96], <u>[HJ97]</u> , <u>[SKK00]</u> , [HM03], [RSS13], <u>[Bul+16]</u>	

Heuristic Approaches Especially for large instances many common solving methods rely on hierarchical clustering that can be either agglomerative (*bottom-up*) or divisive (*top-down*). The former start with singleton templates that are successively merged, for instance in a greedy fashion. On the other hand, divisive methods start with the host graph as single template and repeatedly divide the current template graphs, for instance by computing minimum cuts or by spectral methods exploiting eigenspaces of the graph’s Laplacian [DH72; Fie75; SM00].

Similar to agglomerative approaches, graph coarsening is a method where multiple vertices are contracted to a single vertex. This process can either be repeated until a sought number of vertices remains, or other optimization methods can be applied to the coarsened graph. Bichot [Bic11] gives an overview on this topic and discusses different coarsening and uncoarsening techniques. This approach is also particularly well suited for parallelization [Pel11; LK13].

Another heuristic approach is an adaption of the k -means algorithm. Iteratively, template centers are chosen (*location*) and every vertex is then assigned to its closest center (*allocation*). Again, a new template center can be determined for every template and the process iterates until a fixed point, i.e., a local optimum, is found. This process was first used by Hess et al. [Hes+65]. Later, Segura-Ramiro et al. [Seg+07] present an adaption for the case that the templates should also be balanced.

Graph partitioning techniques from computational geometry are reviewed by Schloegel, Karypis, and Kumar [SKK00]. Geometric approaches are based on coordinates of the vertices instead of the underlying graph structure. If such coordinates are not given, they can be set based on the connectivity of the graph [Hal70]. Again, many solution methods are based on recursive bisection (see e.g. [HR95; GMT98]). Another approach, lying inbetween geometric and location-allocation methods, uses Voronoi diagrams to partition the space. After fixing the Voronoi centers, the resulting Voronoi cells give a partition of the space and therefore, of a graph embedded therein. Partitioning using Voronoi regions is applied for community detection [Der+14] and also for political districting [RSS08].

If the host graph is a tree, many homogeneous covering problems can be solved in polynomial time. While Maravalle and Simeone [MS95] prove this for the sum-of-squares measure, Hansen et al. [Han+03] consider the split version. Both exploit this fact for another heuristic approach where the problem is solved on a spanning tree of the host graph.

Apart from these constructive approaches several local improvement heuristics can be employed. We start with approaches specialized for graph bipartition. Note that the uncapacitated version of this problem can efficiently be solved using a max-flow min-cut algorithm [FF56]. Including balancing constraints, the first and most famous improvement heuristic is due to Kernighan and Lin [KL70]. The method iteratively swaps two vertex sets of same cardinality in different parts of the current bipartition. The sets are chosen so as to improve the objective value as much as possible. Many other heuristics for graph bipartitioning either refine or build on this algorithm (e.g. [FM82; KK98]). A more modern approach by Sanders and Schulz [SS13] makes local improvements by detecting negative cycles.

Regarding general vertex partition problems, greedy local search techniques are used in [HFV99], [RF09], and [DAR12] for example. Hansen and Mladenović [HM03] propose a variable neighborhood search that uses multiple neighborhood definitions. If a local optimum in a neighborhood is found one can simply change the neighborhood and continue the search. Other improvement approaches include tabu search [ABR92; BEL03; DAR12], simulated annealing [ABR92; Ami+02; DAR12], or genetic algorithms [BLP05; Kim+11]. A comparison of different local search methods for districting problems is done by Ricca and Simeone [RS08]. In particular, they compare the greedy local search, tabu search, simulated annealing, and another heuristic called *old bachelor acceptance* that is due to Hu, Kahng, and Tsao [HKT95]. Their computational results based on political districting instances for Italy show that the latter produces the best results in the majority of the cases.

Exact Approaches Exact methods in this field rely on integer programming (IP) formulations. Early formulations enumerate all possible templates [GN70; Nyg88]. Johnson, Mehrotra, and Nemhauser [JMN93] are the first to generate new templates only if necessary in a branch-and-price framework. Mehrotra and Trick [MT98] copy this approach but develop a new method to tackle the pricing problem. A column generation approach for connected templates is done by Mehrotra, Johnson, and Nemhauser [MJN98]. They solve the homogeneous partition problem with star objective in a case study concerning electoral districts in South Carolina. To reduce the complexity of the problem, the set of possible template graphs is further restricted. Namely, it is required that with any vertex in a template also all vertices on a shortest path to the template center have to belong to this template. This constraint dramatically reduces the complexity of the subproblem [SC98]. Clautiaux, Guillot, and Pesneau [CGP19] consider a similar problem motivated by an application in waste collection. They also use column generation and solve the NP-hard pricing problem with Lagrangean relaxation and a dynamic program. However, identical to [MJN98], they demand that each template is a subtree of the shortest path tree to its center.

Other approaches for exact solutions build on polyhedral studies. Starting out with a suitable IP formulation, one can derive strong valid inequalities or even facets of the corresponding polyhedron. These inequalities can then be used in a branch-and-cut scheme for solving the IP. However, polyhedral descriptions are only known for graph partitioning problems.

Faigle, Schrader, and Suletzki [FSS87] appear to have been the first to take a polyhedral perspective in graph partitioning, studying the capacitated clustering problem. Sørensen considers the same problem and finds various classes of facets for the related polytope [Sør04; Sør07; Sør17]. Ferreira et al. [Fer+96] study the capacitated version on a vertex-weighted host graph. They find different facets and valid inequalities for this polytope and employ these in a branch-and-cut framework [Fer+98]. Labbé and Özsoy [LÖ10] examine the polytope if the cardinality of the templates is bounded from below and above.

A special case of capacitated clustering is the equipartition or equicut problem. The goal is a partition of the vertices into two sets that differ by at most 1 in cardinality. The associated polytope is studied by Conforti, Rao, and Sassano [CRS90] and Souza and Laurent [SL95]; a branch-and-cut algorithm and a computational study is presented by Brunetta, Conforti, and Rinaldi [BCR97]. In order to limit the size of the branch-and-bound tree, Lisser and Rendl [LR03] and Anjos et al. [Anj+13] develop improved bounds for this problem. The capacitated bipartition without requiring the parts to be of nearly equal cardinality is studied by Armbruster et al. [Arm+08b] from a polyhedral standpoint with an accompanying computational study in [Arm+08a].

Concerning the uncapacitated partitioning problem, many works are focused on a complete host graph. Grötschel and Wakabayashi [GW90] carry out a polyhedral study and find various facets of the corresponding polytope. The resulting cutting plane algorithm [GW89] solves all considered instances in the root node, i.e., no branching is necessary. Oosten, Rutten, and Spieksma [ORS01] continue the polyhedral study and provide more facets. Polytopes for partitioning a clique into a bounded number of templates are also considered by several authors [DGL92; CR95; AKP16; AK20]. If the host graph is a tree and the template centers are fixed, Apollonio et al. [Apo+08] prove that the polytope of connected centered partitions is integral, and hence, optimal partitions can be found efficiently. While [Apo+08] is focused on the star objective, Lari et al. [Lar+16] discuss the radius objective. A polyhedral study for arbitrary host graphs is due to Chopra and Rao [CR93]. Note that for a fixed number of templates the uncapacitated graph partitioning problem can be solved in polynomial time [GH94].

Moving away from polyhedral aspects, other exact methods are used to tackle vertex coverings. Sensen [Sen01] solves the graph partitioning problem with a branch-and-bound algorithm where the bounds are based on multicommodity flow problems. Hansen, Jaumard, and Musitu [HJM90] present an exact approach for the split case without connectivity constraints in which a sequence of bin packing problems is solved.

Incorporating connectivity constraints in the IP formulations is challenging. There are approaches that use separator inequalities that are similar to subtour elimination constraints. Since there is an exponential number of such constraints they are particularly well suited for cutting

plane approaches. Segura-Ramiro et al. [Seg+07] give a formulation with separator inequalities but they do not implement a branch-and-cut framework and are not able to solve the model. A similar problem with the same result is presented in [RF09]. Salazar-Aguilar, Ríos-Mercado, and Cabrera-Ríos [SRC11] reformulate this IP as a quadratic program keeping the separator inequalities for connectivity. While the chosen solver for the quadratic program does not guarantee optimality, their computational results show that it usually gives a solution within a reasonable small gap to the optimum.

2.3 Cycle Coverings

The final case that we consider is a vertex covering with cycles as templates, complementing the cycle covering part with balancing objective on page 9. As for the edge covering case we assume the host graph to be 2-edge-connected, i.e., every node is contained in a cycle. The first problem to be considered in this context aims at minimizing the total number of edges used in the cycle cover. Bermond, Jackson, and Jaeger [BJJ83] give a first bound on this number and conjecture that a cycle cover with at most $2n - 2$ edges is always possible. While Fraisse [Fra85] improves their bound, the conjecture is proved by Fan [Fan98].

Again, a majority of previous work is concerned with the respective partition problem while using the misleading term cycle cover. To distinct the two cases we will use the name cycle partition here. As mentioned above, a vertex partition with cycles is identical to a 2-factor, see [Plu07] for an overview. Apart from the theoretical studies by the graph factor community, the interest in cycle partitions mainly stems from their role in the design of approximation algorithms for the travelling salesperson problem (TSP). With the method of subtour patching, i.e., by joining the distinct cycles, a Hamiltonian cycle can be constructed [GLS84; Man08]. In order to improve the approximation one can further restrict the cycle partitions to prevent short cycles. In addition, the cycle partition is studied for undirected and directed graphs in order to obtain approximations for the symmetric and asymmetric TSP, respectively.

Bläser and Siebert [BS01] consider the problem of partitioning a graph (directed or undirected) into cycles of given minimum length. However, they only consider edge weight values of one and two and the presented approximation algorithm is tailored for this case. The asymmetric TSP and its relation to the cycle partition problem on directed graphs is considered in [CR07]. A parametrized and asymptotic approximation ratio is given if the arc weights satisfy a special triangle inequality.

For the L -cycle partition problem the length of each cycle has to be contained in a given set L of possible lengths. Hell et al. [Hel+88] show that it is NP-hard to decide if such a partition exists in undirected graphs for almost all L . Building on this work, Bläser and Siebert [BS01] prove that no polynomial-time approximation scheme (PTAS) exists for this problem. In the same paper they provide approximation results with factors $\frac{7}{6}$ and $\frac{4}{3}$, respectively, on undirected and directed graphs.

Instead of restricting the length of the cycles, Khachay and Neznakhina [KN16] restrict the number of cycles in the partition to be exactly k . They show that minimizing the total weight of such a partition in directed graphs is strongly NP-hard. For the metric and Euclidean weighted versions they give a 2-approximation and a PTAS, respectively.

The k travelling salesperson problem asks for a cycle covering with k cycles sharing a common depot vertex. We already mentioned the min-max objective for this problem in the part concerning balanced covers. Here we are concerned with the more popular min-sum objective, i.e., to minimize the sum of circle weights. Frieze [Fri83] transfers the famous TSP approximation by Christofides [Chr76] to the k -TSP case.

Laporte, Desrochers, and Nobert [LDN84] study the k -TSP where each cycle length is bounded. They formulate the problem as an IP and use branch-and-cut methods to solve it. Similar approaches are pursued in [LN83; LND85]. Li, Simchi-Levi, and Desrochers [LSD92] discuss the relation between the min-sum objective and the goal to minimize the number of templates. Nagarajan and Ravi [NR12] derive a 2-approximation for the min-sum problem if the edge weights of the complete host graph are induced by shortest path lengths in an underlying weighted tree. They also show that no approximation within a factor of 1.5 is possible.

References

- [AB96] I. Averbakh and O. Berman. “A heuristic with worst-case analysis for minimax routing of two travelling salesmen on a tree”. In: *Discrete Applied Mathematics* 68.1-2 (1996), pp. 17–32.
- [AB97] I. Averbakh and O. Berman. “ $(p-1)(p+1)$ -approximate algorithms for p -traveling salesmen problems on a tree with minmax objective”. In: *Discrete Applied Mathematics* 75.3 (1997), pp. 201–216.
- [ABH17] N. Alon, T. Bohman, and H. Huang. “More on the bipartite decomposition of random graphs”. In: *Journal of Graph Theory* 84.1 (2017), pp. 45–52.
- [ABR92] S. G. de Amorim, J.-P. Barthélemy, and C. C. Ribeiro. “Clustering and clique partitioning: simulated annealing and tabu search approaches”. In: *Journal of Classification* 9.1 (1992), pp. 17–41.
- [AHL06] E. M. Arkin, R. Hassin, and A. Levin. “Approximations for minimum and min-max vehicle routing problems”. In: *Journal of Algorithms* 59.1 (2006), pp. 1–18.
- [AK11] J. Akiyama and M. Kano. *Factors and factorizations of graphs: Proof techniques in factor theory*. Vol. 2031. Springer, 2011.
- [AK20] Z. Ales and A. Knippel. “The K -partitioning problem: Formulations and branch-and-cut”. In: *Networks* (2020).
- [AKP16] Z. Ales, A. Knippel, and A. Pauchet. “Polyhedral combinatorics of the k -partitioning problem with representative variables”. In: *Discrete Applied Mathematics* 211 (2016), pp. 1–14.
- [AT85] N. Alon and M. Tarsi. “Covering multigraphs by simple circuits”. In: *SIAM Journal on Algebraic Discrete Methods* 6.3 (1985), pp. 345–350.
- [AVJ98] J. Amilhastre, M.-C. Vilarem, and P. Janssen. “Complexity of minimum biclique cover and minimum biclique decomposition for bipartite domino-free graphs”. In: *Discrete applied mathematics* 86.2-3 (1998), pp. 125–144.
- [Alo86] N. Alon. “Covering graphs by the minimum number of equivalence relations”. In: *Combinatorica* 6.3 (1986), pp. 201–206.
- [Ami+02] S. J. d’Amico, S.-J. Wang, R. Batta, and C. M. Rump. “A simulated annealing approach to police district design”. In: *Computers & Operations Research* 29.6 (2002), pp. 667–684.
- [Anj+13] M. F. Anjos, F. Liers, G. Pardella, and A. Schmutzer. “Engineering branch-and-cut algorithms for the equicut problem”. In: *Discrete Geometry and Optimization*. Springer, 2013, pp. 17–32.
- [Apo+08] N. Apollonio, I. Lari, F. Ricca, B. Simeone, and J. Puerto. “Polynomial algorithms for partitioning a tree into single-center subtrees to minimize flat service costs”. In: *Networks: An International Journal* 51.1 (2008), pp. 78–89.
- [App+02] D. Applegate, W. Cook, S. Dash, and A. Rohe. “Solution of a min-max vehicle routing problem”. In: *INFORMS Journal on computing* 14.2 (2002), pp. 132–143.
- [Arm+08a] M. Armbruster, M. Fügenschuh, C. Helmberg, and A. Martin. “A comparative study of linear and semidefinite branch-and-cut methods for solving the minimum graph bisection problem”. In: *International Conference on Integer Programming and Combinatorial Optimization*. Springer, 2008, pp. 112–124.
- [Arm+08b] M. Armbruster, C. Helmberg, M. Fügenschuh, and A. Martin. “On the graph bisection cut polytope”. In: *SIAM Journal on Discrete Mathematics* 22.3 (2008), pp. 1073–1098.
- [BCR97] L. Brunetta, M. Conforti, and G. Rinaldi. “A branch-and-cut algorithm for the equicut problem”. In: *Mathematical Programming* 78.2 (1997), pp. 243–263.
- [BEL03] B. Bozkaya, E. Erkut, and G. Laporte. “A tabu search heuristic and adaptive memory procedure for political districting”. In: *European Journal of Operational Research* 144.1 (2003), pp. 12–26.
- [BES19] R. Borndörfer, Z. Elijazyfer, and S. Schwartz. *Approximating Balanced Graph Partitions*. Tech. rep. 19-25. Zuse Institute Berlin, 2019.
- [BH03] G. D. Bader and C. W. Hogue. “An automated method for finding molecular complexes in large protein interaction networks”. In: *BMC bioinformatics* 4.1 (2003), p. 2.

- [BJ17] F. Botler and A. Jiménez. “On path decompositions of $2k$ -regular graphs”. In: *Discrete Mathematics* 340.6 (2017), pp. 1405–1411.
- [BJJ83] J. C. Bermond, B. Jackson, and F. Jaeger. “Shortest coverings of graphs with cycles”. In: *Journal of Combinatorial Theory, Series B* 35.3 (1983), pp. 297–308.
- [BLP05] F. Bacao, V. Lobo, and M. Painho. “Applying genetic algorithms to zone design”. In: *Soft Computing* 9.5 (2005), pp. 341–348.
- [BPS80] R. I. Becker, Y. Perl, and S. R. Schach. “A shifting algorithm for min-max tree partitioning”. In: *International Colloquium on Automata, Languages, and Programming*. Springer, 1980, pp. 64–75.
- [BRVN16] K. Braekers, K. Ramaekers, and I. Van Nieuwenhuysse. “The vehicle routing problem: State of the art classification and review”. In: *Computers & Industrial Engineering* 99 (2016), pp. 300–313.
- [BS01] M. Bläser and B. Siebert. “Computing cycle covers without short cycles”. In: *European Symposium on Algorithms*. Springer, 2001, pp. 368–379.
- [BS11] C.-E. Bichot and P. Siarry. *Graph partitioning*. John Wiley & Sons, 2011.
- [Bec+01] R. Becker, I. Lari, M. Lucertini, and B. Simeone. “A polynomial-time algorithm for max-min partitioning of ladders”. In: *Theory of Computing Systems* 34.4 (2001), pp. 353–374.
- [Bec+98] R. Becker, I. Lari, M. Lucertini, and B. Simeone. “Max-min partitioning of grid graphs into connected components”. In: *Networks: An International Journal* 32.2 (1998), pp. 115–125.
- [Ber78] J.-C. Bermond. *Couverture des arêtes d’un graphe par des graphes bipartis complets*. Tech. rep. 10. LRI - CNRS, University Paris-Sud, 1978, pp. 1–3.
- [Bez+08] S. Bezrukov, D. Fronček, S. J. Rosenberg, and P. Kovář. “On biclique coverings”. In: *Discrete Mathematics* 308.2-3 (2008), pp. 319–323.
- [Bic11] C.-E. Bichot. “A Partitioning Requiring Rapidity and Quality: The Multilevel Method and Partitions Refinement Algorithms”. In: *Graph partitioning*. John Wiley & Sons, 2011, pp. 29–63.
- [Bon90] J. Bondy. “Small cycle double covers of graphs”. In: *Cycles and rays*. Springer, 1990, pp. 21–40.
- [Bra+07] U. Brandes, D. Delling, M. Gaertler, R. Gorke, M. Hoefer, Z. Nikoloski, and D. Wagner. “On modularity clustering”. In: *IEEE transactions on knowledge and data engineering* 20.2 (2007), pp. 172–188.
- [Bul+16] A. Buluç, H. Meyerhenke, I. Safro, P. Sanders, and C. Schulz. “Recent advances in graph partitioning”. In: *Algorithm Engineering*. Springer, 2016, pp. 117–158.
- [Bus94] M. Bussieck. *The minimal cut cover of a graph*. Tech. rep. Technical Report TR-94-02, Pennsylvania State University, 1994.
- [CCY19] H. Chen, T. Cheng, and X. Ye. “Designing efficient and balanced police patrol districts on an urban street network”. In: *International Journal of Geographical Information Science* 33.2 (2019), pp. 269–290.
- [CES83] F. Chung, P. Erdős, and J. Spencer. “On the decomposition of graphs into complete bipartite subgraphs”. In: *Studies in pure mathematics*. Springer, 1983, pp. 95–101.
- [CF06] D. Cornaz and J. Fonlupt. “Chromatic characterization of biclique covers”. In: *Discrete Mathematics* 306.5 (2006), pp. 495–507.
- [CFS14] D. Conlon, J. Fox, and B. Sudakov. “Short proofs of some extremal results”. In: *Combinatorics, Probability and Computing* 23.1 (2014), pp. 8–28.
- [CGM83] P. M. Camerini, G. Galbiati, and F. Maffioli. “On the complexity of finding multi-constrained spanning trees”. In: *Discrete Applied Mathematics* 5.1 (1983), pp. 39–50.
- [CGP19] F. Clautiaux, J. Guillot, and P. Pesneau. “Exact approaches for solving a covering problem with capacitated subtrees”. In: *Computers & Operations Research* 105 (2019), pp. 85–101.
- [CGP85] D de Caen, D. A. Gregory, and N. Pullman. “Clique coverings of complements of paths and cycles”. In: *North-Holland Mathematics Studies*. Vol. 115. Elsevier, 1985, pp. 257–267.

- [CHQ10] K. M. Curtin, K. Hayslett-McCall, and F. Qiu. “Determining optimal police patrol areas with maximal covering and backup covering location models”. In: *Networks and Spatial Economics* 10.1 (2010), pp. 125–145.
- [CR07] L. S. Chandran and L. S. Ram. “On the relationship between ATSP and the cycle cover problem”. In: *Theoretical Computer Science* 370.1-3 (2007), pp. 218–228.
- [CR93] S. Chopra and M. R. Rao. “The partition problem”. In: *Mathematical Programming* 59.1-3 (1993), pp. 87–115.
- [CR95] S. Chopra and M. Rao. “Facets of the k -partition polytope”. In: *Discrete Applied Mathematics* 61.1 (1995), pp. 27–48.
- [CRS90] M. Conforti, M. Rao, and A. Sassano. “The equipartition polytope I, II”. In: *Mathematical Programming* 49 (1990), pp. 49–70, 71–90.
- [CSW07] F. Chataigner, L. R. Salgado, and Y. Wakabayashi. “Approximation and inapproximability results on balanced connected partitions of graphs”. In: *Discrete Mathematics and Theoretical Computer Science* 9.1 (2007), pp. 177–192.
- [CV08] M. Cavers and J. Verstraëte. “Clique partitions of complements of forests and bounded degree graphs”. In: *Discrete Mathematics* 308.10 (2008), pp. 2011–2017.
- [CWC13] A.-C. Chu, B. Y. Wu, and K.-M. Chao. “A linear-time algorithm for finding an edge-partition with max-min ratio at most two”. In: *Discrete Applied Mathematics* 161.7-8 (2013), pp. 932–943.
- [Cac+85] L. Caccetta, P. Erdős, E. T. Ordman, and N. J. Pullman. “The difference between the clique numbers of a graph”. In: *Ars Combinatoria A* 19 (1985), pp. 97–106.
- [Cav05] M. S. Cavers. “Clique partitions and coverings of graphs”. MA thesis. 2005.
- [Cha09] M. Chan. “A survey of the cycle double cover conjecture”. In: *Princeton University* (2009).
- [Cha+17] T. Chakraborty, A. Dalmia, A. Mukherjee, and N. Ganguly. “Metrics for community analysis: A survey”. In: *ACM Computing Surveys (CSUR)* 50.4 (2017), pp. 1–37.
- [Che+00] G. Chen, M. S. Jacobson, A. E. Kézdy, J. Lehel, E. R. Scheinerman, and C. Wang. “Clique covering the edges of a locally cobipartite graph”. In: *Discrete Mathematics* 219.1-3 (2000), pp. 17–26.
- [Chl96] J. Chlebíková. “Approximating the maximally balanced connected partition problem in graphs”. In: *Information Processing Letters* 60.5 (1996), pp. 225–230.
- [Chr76] N. Christofides. *Worst-case analysis of a new heuristic for the travelling salesman problem*. Tech. rep. Carnegie-Mellon Univ Pittsburgh Pa Management Sciences Research Group, 1976.
- [Chu+10] A.-C. Chu, B. Y. Wu, H.-L. Wang, and K.-M. Chao. “A tight bound on the min-ratio edge-partitioning problem of a tree”. In: *Discrete Applied Mathematics* 158.14 (2010), pp. 1471–1478.
- [Chu78] F. Chung. “On partitions of graphs into trees”. In: *Discrete Mathematics* 23.1 (1978), pp. 23–30.
- [Chu80] F. Chung. “On the coverings of graphs”. In: *Discrete Mathematics* 30.2 (1980), pp. 89–93.
- [Cun19] A. S. da Cunha. “Formulation and Branch-and-cut algorithm for the Minimum Cardinality Balanced and Connected Clustering Problem”. In: *9th International Network Optimization Conference Avignon, France, June 12–14, 2019*. 2019, p. 25.
- [DAR12] J. C. Duque, L. Anselin, and S. J. Rey. “The max- p -regions problem”. In: *Journal of Regional Science* 52.3 (2012), pp. 397–419.
- [DF85] M. E. Dyer and A. M. Frieze. “On the complexity of partitioning graphs into connected subgraphs”. In: *Discrete Applied Mathematics* 10.2 (1985), pp. 139–153.
- [DGL92] M. Deza, M. Grötschel, and M. Laurent. “Clique-web facets for multicut polytopes”. In: *Mathematics of Operations Research* 17.4 (1992), pp. 981–1000.
- [DH72] W. E. Donath and A. J. Hoffman. “Algorithms for partitioning of graphs and computer logic based on eigenvectors of connection matrices”. In: *IBM Technical Disclosure Bulletin* 15.3 (1972), pp. 938–944.

- [DJK20] S. Das, L. Jain, and N. Kumar. “A Constant Factor Approximation for Capacitated Min-Max Tree Cover”. In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*. Ed. by J. Byrka and R. Meka. Vol. 176. Leibniz International Proceedings in Informatics. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2020, 55:1–55:13.
- [DL07] J. Dong and Y. Liu. “On the decomposition of graphs into complete bipartite graphs”. In: *Graphs and Combinatorics* 23.3 (2007), pp. 255–262.
- [Der+14] D. Deritei, Z. I. Lazar, I. Papp, F. Jaraı-Szabo, R. Sumi, L. Varga, E. R. Regan, and M. Ercsey-Ravasz. “Community detection by graph Voronoi diagrams”. In: *New Journal of Physics* 16.6 (2014), p. 063007.
- [Die16] R. Diestel. *Graph theory*. 5th ed. Vol. 173. Graduate Texts in Mathematics. Springer, 2016.
- [Don80] A. Donald. “An upper bound for the path number of a graph”. In: *Journal of graph theory* 4.2 (1980), pp. 189–201.
- [EFO88] P. Erdős, R. Faudree, and E. T. Ordman. “Clique partitions and clique coverings”. In: *Discrete Mathematics* 72 (1988), pp. 93–101.
- [EGP66] P. Erdős, A. W. Goodman, and L. Pósa. “The representation of a graph by set intersections”. In: *Canadian Journal of Mathematics* 18 (1966), pp. 106–112.
- [EJ73] J. Edmonds and E. L. Johnson. “Matching, Euler tours and the Chinese postman”. In: *Mathematical programming* 5.1 (1973), pp. 88–124.
- [EP97] P. Erdős and L. Pyber. “Covering a graph by complete bipartite graphs”. In: *Discrete mathematics* 170.1-3 (1997), pp. 249–251.
- [ERD14] M. G. Elizondo-Amaya, R. Z. Ríos-Mercado, and J. A. Díaz. “A dual bounding scheme for a territory design problem”. In: *Computers & operations research* 44 (2014), pp. 193–205.
- [Eve+04] G. Even, N. Garg, J. Könemann, R. Ravi, and A. Sinha. “Min–max tree covers of graphs”. In: *Operations Research Letters* 32.4 (2004), pp. 309–315.
- [FB07] S. Fortunato and M. Barthélemy. “Resolution limit in community detection”. In: *Proceedings of the National Academy of Sciences* 104.1 (2007), pp. 36–41.
- [FF56] L. R. Ford and D. R. Fulkerson. “Maximal Flow Through a Network”. In: *Canadian Journal of Mathematics* 8 (1956), 399–404.
- [FG90] J. A. Ferland and G. Guénette. “Decision support system for the school districting problem”. In: *Operations Research* 38.1 (1990), pp. 15–21.
- [FH16] S. Fortunato and D. Hric. “Community detection in networks: A user guide”. In: *Physics reports* 659 (2016), pp. 1–44.
- [FH96] P. C. Fishburn and P. L. Hammer. “Bipartite dimensions and bipartite degrees of graphs”. In: *Discrete Mathematics* 160.1-3 (1996), pp. 127–148.
- [FHK76] G. N. Frederickson, M. S. Hecht, and C. E. Kim. “Approximation algorithms for some routing problems”. In: *17th annual symposium on foundations of computer science (sfcs 1976)*. IEEE, 1976, pp. 216–227.
- [FK01] Z. Füredi and A. Kündgen. “Covering a graph with cuts of minimum total size”. In: *Discrete Mathematics* 237.1-3 (2001), pp. 129–148.
- [FL15] B. Farbstein and A. Levin. “Min–max cover of a graph with a small number of parts”. In: *Discrete Optimization* 16 (2015), pp. 51–61.
- [FM82] C. M. Fiduccia and R. M. Mattheyses. “A linear-time heuristic for improving network partitions”. In: *19th design automation conference*. IEEE, 1982, pp. 175–181.
- [FP88] B. Fleischmann and J. N. Paraschis. “Solving a large scale districting problem: a case report”. In: *Computers & Operations Research* 15.6 (1988), pp. 521–533.
- [FSS87] U. Faigle, R. Schrader, and R. Suletzki. “A cutting plane algorithm for optimal graph partitioning”. In: *Methods of operations research* 57 (1987), pp. 109–116.
- [Fan02] G. Fan. “Subgraph coverings and edge switchings”. In: *Journal of Combinatorial Theory, Series B* 84.1 (2002), pp. 54–83.

- [Fan05] G. Fan. “Path decompositions and Gallai’s conjecture”. In: *Journal of Combinatorial Theory, Series B* 93.2 (2005), pp. 117–125.
- [Fan97] G. Fan. “Minimum cycle covers of graphs”. In: *Journal of Graph Theory* 25.3 (1997), pp. 229–242.
- [Fan98] G. Fan. “Proofs of two minimum circuit cover conjectures”. In: *Journal of Combinatorial Theory, Series B* 74.2 (1998), pp. 353–367.
- [Fer+96] C. E. Ferreira, A. Martin, C. C. de Souza, R. Weismantel, and L. A. Wolsey. “Formulations and valid inequalities for the node capacitated graph partitioning problem”. In: *Mathematical Programming* 74.3 (1996), pp. 247–266.
- [Fer+98] C. E. Ferreira, A. Martin, C. C. de Souza, R. Weismantel, and L. A. Wolsey. “The node capacitated graph partitioning problem: a computational study”. In: *Mathematical programming* 81.2 (1998), pp. 229–256.
- [Fie75] M. Fiedler. “A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory”. In: *Czechoslovak Mathematical Journal* 25.4 (1975), pp. 619–633.
- [Fjä98] P.-O. Fjällström. *Algorithms for graph partitioning: A survey*. Vol. 3. Linköping University Electronic Press Linköping, 1998.
- [Fra85] P. Fraisse. “Cycle covering in bridgeless graphs”. In: *Journal of Combinatorial Theory, Series B* 39.2 (1985), pp. 146–152.
- [Fre91] G. N. Frederickson. “Optimal algorithms for tree partitioning”. In: *SODA*. Vol. 91. 1991, pp. 168–177.
- [Fri83] A. M. Frieze. “An extension of Christofides heuristic to the k -person travelling salesman problem”. In: *Discrete Applied Mathematics* 6.1 (1983), pp. 79–83.
- [GH94] O. Goldschmidt and D. S. Hochbaum. “A polynomial algorithm for the k -cut problem for fixed k ”. In: *Mathematics of operations research* 19.1 (1994), pp. 24–37.
- [GLS84] P. C. Gilmore, E. L. Lawler, and D. B. Shmoys. “Well-solved special cases of the traveling salesman problem”. In: *The traveling salesman problem: a guided tour of combinatorial optimization*. John Wiley Sons, 1984, pp. 87–143.
- [GMT98] J. R. Gilbert, G. L. Miller, and S.-H. Teng. “Geometric mesh partitioning: Implementation and experiments”. In: *SIAM Journal on Scientific Computing* 19.6 (1998), pp. 2091–2110.
- [GN02] M. Girvan and M. E. Newman. “Community structure in social and biological networks”. In: *Proceedings of the national academy of sciences* 99.12 (2002), pp. 7821–7826.
- [GN70] R. S. Garfinkel and G. L. Nemhauser. “Optimal political districting by implicit enumeration techniques”. In: *Management Science* 16.8 (1970), B–495.
- [GP72] R. L. Graham and H. O. Pollak. “On embedding graphs in squashed cubes”. In: *Graph theory and applications*. Springer, 1972, pp. 99–110.
- [GP82] D. A. Gregory and N. J. Pullman. “On a clique covering problem of Orlin”. In: *Discrete Mathematics* 41.1 (1982), pp. 97–99.
- [GT87] E. Györi and Z. Tuza. “Decompositions of graphs into complete subgraphs of given order”. In: *Studia Sci. Math. Hung* 22.315-320 (1987), p. 24.
- [GW18] S. Goderbauer and J. Winandy. *Political Districting Problem: Literature Review and Discussion with regard to Federal Elections in Germany*. Tech. rep. RWTH Aachen, 2018.
- [GW89] M. Grötschel and Y. Wakabayashi. “A cutting plane algorithm for a clustering problem”. In: *Mathematical Programming* 45.1-3 (1989), pp. 59–96.
- [GW90] M. Grötschel and Y. Wakabayashi. “Facets of the clique partitioning polytope”. In: *Mathematical Programming* 47.1-3 (1990), pp. 367–387.
- [GWS99] D. A. Gregory, V. L. Watts, and B. L. Shader. “Biclique decompositions and Hermitian rank”. In: *Linear Algebra and its Applications* 292.1-3 (1999), pp. 267–280.
- [Gor96] A. Gordon. “A survey of constrained classification”. In: *Computational Statistics & Data Analysis* 21.1 (1996), pp. 17–29.
- [Gün07] O. Günlük. “A new min-cut max-flow ratio for multicommodity flows”. In: *SIAM Journal on Discrete Mathematics* 21.1 (2007), pp. 1–15.

- [HFV99] S. Hanafi, A. Freville, and P. Vaca. “Municipal solid waste collection: an effective data structure for solving the sectorization problem with local search methods”. In: *INFOR: Information Systems and Operational Research* 37.3 (1999), pp. 236–254.
- [HHM77] F. Harary, D. Hsu, and Z. Miller. “The biparticity of a graph”. In: *Journal of graph theory* 1.2 (1977), pp. 131–133.
- [HJ97] P. Hansen and B. Jaumard. “Cluster analysis and mathematical programming”. In: *Mathematical programming* 79.1-3 (1997), pp. 191–215.
- [HJM90] P. Hansen, B. Jaumard, and K. Musitu. “Weight constrained maximum split clustering”. In: *Journal of classification* 7.2 (1990), pp. 217–240.
- [HKT95] T. C. Hu, A. B. Kahng, and C.-W. A. Tsao. “Old bachelor acceptance: A new class of non-monotone threshold accepting methods”. In: *ORSA Journal on Computing* 7.4 (1995), pp. 417–425.
- [HM03] P. Hansen and N. Mladenović. “Variable neighborhood search”. In: *Handbook of metaheuristics*. Springer, 2003, pp. 145–184.
- [HO01] D. S. Hochbaum and E. V. Olinick. “The bounded cycle-cover problem”. In: *INFORMS Journal on Computing* 13.2 (2001), pp. 104–119.
- [HPK11] J. Han, J. Pei, and M. Kamber. *Data mining: concepts and techniques*. Elsevier, 2011.
- [HR95] M. T. Heath and P. Raghavan. “A Cartesian parallel nested dissection algorithm”. In: *SIAM Journal on Matrix Analysis and Applications* 16.1 (1995), pp. 235–253.
- [HS71] S. W. Hess and S. A. Samuels. “Experiences with a sales districting model: criteria and implementation”. In: *Management Science* 18.4-part-ii (1971), P–41.
- [Hal70] K. M. Hall. “An r-dimensional quadratic placement algorithm”. In: *Management science* 17.3 (1970), pp. 219–229.
- [Han+03] P. Hansen, B. Jaumard, C. Meyer, B. Simeone, and V. Doring. “Maximum split clustering under connectivity constraints”. In: *Journal of Classification* 20.2 (2003), pp. 143–180.
- [Hel+88] P. Hell, D. Kirkpatrick, J. Kratochvíl, and I. Kříž. “On restricted two-factors”. In: *SIAM Journal on Discrete Mathematics* 1.4 (1988), pp. 472–484.
- [Hes+65] S. W. Hess, J. Weaver, H. Siegfeldt, J. Whelan, and P. Zitlau. “Nonpartisan political redistricting by computer”. In: *Operations Research* 13.6 (1965), pp. 998–1006.
- [IMM05] N. Immorlica, M. Mahdian, and V. S. Mirrokni. “Cycle cover with short cycles”. In: *Annual Symposium on Theoretical Aspects of Computer Science*. Springer, 2005, pp. 641–653.
- [IZN06] T. Ito, X. Zhou, and T. Nishizeki. “Partitioning a graph of bounded tree-width to connected subgraphs of almost uniform size”. In: *Journal of discrete algorithms* 4.1 (2006), pp. 142–154.
- [Ita+81] A. Itai, R. J. Lipton, C. H. Papadimitriou, and M. Rodeh. “Covering graphs by simple circuits”. In: *SIAM Journal on Computing* 10.4 (1981), pp. 746–750.
- [Ito+12] T. Ito, T. Nishizeki, M. Schröder, T. Uno, and X. Zhou. “Partitioning a weighted tree into subtrees with weights in a given range”. In: *Algorithmica* 62.3-4 (2012), pp. 823–841.
- [JH19] R. Javadi and S. Hajebi. “Edge clique cover of claw-free graphs”. In: *Journal of Graph Theory* 90.3 (2019), pp. 311–405.
- [JK09] S. Jukna and A. S. Kulikov. “On covering graphs by complete bipartite subgraphs”. In: *Discrete Mathematics* 309.10 (2009), pp. 3399–3403.
- [JMF99] A. K. Jain, M. N. Murty, and P. J. Flynn. “Data clustering: a review”. In: *ACM computing surveys (CSUR)* 31.3 (1999), pp. 264–323.
- [JMN93] E. L. Johnson, A. Mehrotra, and G. L. Nemhauser. “Min-cut clustering”. In: *Mathematical programming* 62.1-3 (1993), pp. 133–151.
- [JMO16] R. Javadi, Z. Maleki, and B. Omoomi. “Local clique covering of claw-free graphs”. In: *Journal of Graph Theory* 81.1 (2016), pp. 92–104.
- [JRP85] M. Jünger, G. Reinelt, and W. R. Pulleyblank. “On partitioning the edges of graphs into connected subgraphs”. In: *Journal of Graph Theory* 9.4 (1985), pp. 539–549.

- [Jac93] B. Jackson. “On circuit covers, circuit decompositions and Euler tours of graphs”. In: *Surveys in combinatorics*. Vol. 187. Cambridge Univ. Press Cambridge, 1993, pp. 191–210.
- [Jae85] F. Jaeger. “A survey of the cycle double cover conjecture”. In: *North-Holland Mathematics Studies*. Vol. 115. Elsevier, 1985, pp. 1–12.
- [Jav+18] M. A. Javed, M. S. Younis, S. Latif, J. Qadir, and A. Baig. “Community detection in networks: A multidisciplinary review”. In: *Journal of Network and Computer Applications* 108 (2018), pp. 87–111.
- [Jor13] A. Jorati. “Approximation algorithms for some min-max vehicle routing problems”. MA thesis. University of Alberta, 2013.
- [KK98] G. Karypis and V. Kumar. “Multilevelk-way partitioning scheme for irregular graphs”. In: *Journal of Parallel and Distributed computing* 48.1 (1998), pp. 96–129.
- [KL70] B. W. Kernighan and S. Lin. “An efficient heuristic procedure for partitioning graphs”. In: *The Bell system technical journal* 49.2 (1970), pp. 291–307.
- [KM77] S. Kundu and J. Misra. “A linear tree partitioning algorithm”. In: *SIAM Journal on Computing* 6.1 (1977), pp. 151–154.
- [KMN11] S. Karakawa, E. Morsy, and H. Nagamochi. “Minmax Tree Cover in the Euclidean Space”. In: *Journal of Graph Algorithms and Applications* 15.3 (2011), pp. 345–371.
- [KN16] M. Khachay and K. Neznakhina. “Approximability of the minimum-weight k-size cycle cover problem”. In: *Journal of Global Optimization* 66.1 (2016), pp. 65–82.
- [KPJ04] A. D. King, N. Pržulj, and I. Jurisica. “Protein complex prediction via cost-based clustering”. In: *Bioinformatics* 20.17 (2004), pp. 3013–3020.
- [KR19] J. Kalcsics and R. Z. Ríos-Mercado. “Districting problems”. In: *Location science*. Springer, 2019, pp. 705–743.
- [KS05] A. Kündgen and M. Spangler. “A bound on the total size of a cut cover”. In: *Discrete mathematics* 296.1 (2005), pp. 121–128.
- [KS14] M. R. Khani and M. R. Salavatipour. “Improved approximation algorithms for the min-max tree cover and bounded tree cover problems”. In: *Algorithmica* 69.2 (2014), pp. 443–460.
- [KSW78] L. T. Kou, L. J. Stockmeyer, and C.-K. Wong. “Covering edges by cliques with regard to keyword conflicts and intersection graphs”. In: *Communications of the ACM* 21.2 (1978), pp. 135–139.
- [KU16] K. Knauer and T. Ueckerdt. “Three ways to cover a graph”. In: *Discrete Mathematics* 339.2 (2016), pp. 745–758.
- [Kim+11] J. Kim, I. Hwang, Y.-H. Kim, and B.-R. Moon. “Genetic approaches for graph partitioning: a survey”. In: *Proceedings of the 13th annual conference on Genetic and evolutionary computation*. 2011, pp. 473–480.
- [Kor09] R. E. Korf. “Multi-way number partitioning”. In: *Proceedings of the 21st international joint conference on Artificial intelligence*. 2009, pp. 538–543.
- [Krá+19] D. Král, B. Lidický, T. L. Martins, and Y. Pehova. “Decomposing graphs into edges and triangles”. In: *Combinatorics, Probability and Computing* 28.3 (2019), pp. 465–472.
- [LDN84] G. Laporte, M. Desrochers, and Y. Nobert. “Two exact algorithms for the distance-constrained vehicle routing problem”. In: *Networks* 14.1 (1984), pp. 161–172.
- [LK13] D. LaSalle and G. Karypis. “Multi-threaded graph partitioning”. In: *2013 IEEE 27th International Symposium on Parallel and Distributed Processing*. IEEE, 2013, pp. 225–236.
- [LL20] M. Lin and R. J. La. “Miniature Robot Path Planning for Bridge Inspection: Min-Max Cycle Cover-Based Approach”. In: *arXiv preprint arXiv:2003.12134* (2020).
- [LN83] G. Laporte and Y. Nobert. “A branch and bound algorithm for the capacitated vehicle routing problem”. In: *Operations-Research-Spektrum* 5.2 (1983), pp. 77–85.
- [LND85] G. Laporte, Y. Nobert, and M. Desrochers. “Optimal routing under capacity and distance restrictions”. In: *Operations research* 33.5 (1985), pp. 1050–1073.
- [LÖ10] M. Labbé and F. A. Özsoy. “Size-constrained graph partitioning polytopes”. In: *Discrete mathematics* 310.24 (2010), pp. 3473–3493.

- [LPS93] M. Lucertini, Y. Perl, and B. Simeone. “Most uniform path partitioning and its use in image processing”. In: *Discrete Applied Mathematics* 42.2-3 (1993), pp. 227–256.
- [LR03] A. Lisser and F. Rendl. “Graph partitioning using linear and semidefinite programming”. In: *Mathematical Programming* 95.1 (2003), pp. 91–101.
- [LSD92] C.-L. Li, D. Simchi-Levi, and M. Desrochers. “On the distance constrained vehicle routing problem”. In: *Operations research* 40.4 (1992), pp. 790–799.
- [Lap92] G. Laporte. “The vehicle routing problem: An overview of exact and approximate algorithms”. In: *European journal of operational research* 59.3 (1992), pp. 345–358.
- [Lar+16] I. Lari, F. Ricca, J. Puerto, and A. Scozzari. “Partitioning a graph into connected components with fixed centers and optimizing cost-based objective functions or equipartition criteria”. In: *Networks* 67.1 (2016), pp. 69–81.
- [Lov68] L. Lovász. “On covering of graphs”. In: *Theory of Graphs (Proc. Colloq., Tihany, 1966)*. Academic Press New York, 1968, pp. 231–236.
- [MJN98] A. Mehrotra, E. L. Johnson, and G. L. Nemhauser. “An optimization based heuristic for political districting”. In: *Management Science* 44.8 (1998), pp. 1100–1114.
- [MPR95] S. D. Monson, N. J. Pullman, and R. Rees. “A survey of clique and biclique coverings and factorizations of $(0, 1)$ -matrices”. In: *Bull. Inst. Combin. Appl* 14 (1995), pp. 17–86.
- [MS95] M. Maravalle and B. Simeone. “A spanning tree heuristic for regional clustering”. In: *Communications in Statistics - Theory and Methods* 24.3 (1995), pp. 625–639.
- [MSN97] M. Maravalle, B. Simeone, and R. Naldini. “Clustering on trees”. In: *Computational Statistics & Data Analysis* 24.2 (1997), pp. 217–234.
- [MT98] A. Mehrotra and M. A. Trick. “Cliques and clustering: A combinatorial approach”. In: *Operations Research Letters* 22.1 (1998), pp. 1–12.
- [Man08] B. Manthey. “On approximating restricted cycle covers”. In: *SIAM journal on computing* 38.1 (2008), pp. 181–206.
- [Mát06] T. Mátrai. “Covering the edges of a graph by three odd subgraphs”. In: *Journal of Graph Theory* 53.1 (2006), pp. 77–82.
- [Miy+20] F. K. Miyazawa, P. F. Moura, M. J. Ota, and Y. Wakabayashi. “Cut and Flow Formulations for the Balanced Connected k -Partition Problem”. In: *International Symposium on Combinatorial Optimization*. Springer, 2020, pp. 128–139.
- [Mül96] H. Müller. “On edge perfectness and classes of bipartite graphs”. In: *Discrete Mathematics* 149.1-3 (1996), pp. 159–187.
- [Mur85] F. Murtagh. “A survey of algorithms for contiguity-constrained clustering and related problems”. In: *The computer journal* 28.1 (1985), pp. 82–88.
- [N64] C. Nash-Williams. “Decomposition of finite graphs into forests”. In: *Journal of the London Mathematical Society* 1.1 (1964), pp. 12–12.
- [NG04] M. E. Newman and M. Girvan. “Finding and evaluating community structure in networks”. In: *Physical review E* 69.2 (2004), p. 026113.
- [NO03] H. Nagamochi and K. Okada. “Polynomial time 2-approximation algorithms for the minmax subtree cover problem”. In: *International Symposium on Algorithms and Computation*. Springer, 2003, pp. 138–147.
- [NO04] H. Nagamochi and K. Okada. “A faster 2-approximation algorithm for the minmax p -traveling salesmen problem on a tree”. In: *Discrete Applied Mathematics* 140.1-3 (2004), pp. 103–114.
- [NR12] V. Nagarajan and R Ravi. “Approximation algorithms for distance constrained vehicle routing problems”. In: *Networks* 59.2 (2012), pp. 209–214.
- [Nag05] H. Nagamochi. “Approximating the minmax rooted-subtree cover problem”. In: *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences* 88.5 (2005), pp. 1335–1338.
- [Nyg88] B. Nygreen. “European assembly constituencies for wales-comparing of methods for solving a political districting problem”. In: *Mathematical Programming* 42.1-3 (1988), pp. 159–169.

- [ORS01] M. Oosten, J. H. Rutten, and F. C. Spijksma. “The clique partitioning problem: facets and patching facets”. In: *Networks* 38.4 (2001), pp. 209–226.
- [Orl77] J. Orlin. “Contentment in graph theory: covering graphs with cliques”. In: *Indagationes Mathematicae (Proceedings)*. Vol. 80. 5. Elsevier, 1977, pp. 406–424.
- [PC80] N. J. Pullman and D. de Caen. “Clique coverings of graphs III: clique coverings of regular graphs”. In: *Congressus Numerantium* 29 (1980), pp. 795–808.
- [PC81] N. J. Pullman and D. de Caen. “Clique coverings of graphs I: clique partitions of regular graphs”. In: *Utilitas Mathematica* 19 (1981), pp. 177–205.
- [PD81] N. J. Pullman and A. Donald. “Clique coverings of graphs II: complements of cliques”. In: *Utilitas Mathematica* 19 (1981), pp. 207–213.
- [PŠ19] M. Petruševski and R. Škrekovski. “Coverability of graph by three odd subgraphs”. In: *Journal of Graph Theory* 92.3 (2019), pp. 304–321.
- [PS81] Y. Perl and S. R. Schach. “Max-Min Tree Partitioning”. In: *J. ACM* 28.1 (1981), pp. 5–15.
- [PSW82] N. J. Pullman, H. Shank, and W. Wallis. “Clique coverings of graphs V: Maximal-clique partitions”. In: *Bulletin of the Australian Mathematical Society* 25.3 (1982), pp. 337–356.
- [Pel11] F. Pellegrini. “Parallelization of Graph Partitioning”. In: *Graph partitioning*. John Wiley & Sons, 2011, pp. 81–114.
- [Pin14] T. Pinto. “Biclique covers and partitions”. In: *The Electronic Journal of Combinatorics* 21.1 (2014), P1–19.
- [Plu07] M. D. Plummer. “Graph factors and factorization: 1985–2003: a survey”. In: *Discrete Mathematics* 307.7-8 (2007), pp. 791–821.
- [Pul83] N. J. Pullman. “Clique coverings of graphs – a survey”. In: *Combinatorial Mathematics X*. Springer, 1983, pp. 72–85.
- [Pul84] N. J. Pullman. “Clique covering of graphs IV: Algorithms”. In: *SIAM Journal on Computing* 13.1 (1984), pp. 57–75.
- [Pyb85] L. Pyber. “An Erdős-Gallai conjecture”. In: *Combinatorica* 5.1 (1985), pp. 67–79.
- [Pyb91] L. Pyber. “Covering the edges of a graph by...” In: *Sets, Graphs and Numbers, Colloquia Mathematica Societatis János Bolyai*. Vol. 60. 1991, pp. 583–610.
- [Pyb96] L. Pyber. “Covering the Edges of a Connected Graph by Paths”. In: *Journal of Combinatorial Theory, Series B* 66.1 (1996), pp. 152–159.
- [RF09] R. Z. Ríos-Mercado and E. Fernández. “A reactive GRASP for a commercial territory design problem with multiple balancing requirements”. In: *Computers & Operations Research* 36.3 (2009), pp. 755–776.
- [RS08] F. Ricca and B. Simeone. “Local search algorithms for political districting”. In: *European Journal of Operational Research* 189.3 (2008), pp. 1409–1426.
- [RS11] R. Z. Ríos-Mercado and J. C. Salazar-Acosta. “A GRASP with strategic oscillation for a commercial territory design problem with a routing budget constraint”. In: *Mexican International Conference on Artificial Intelligence*. Springer, 2011, pp. 307–318.
- [RSS08] F. Ricca, A. Scozzari, and B. Simeone. “Weighted Voronoi region algorithms for political districting”. In: *Mathematical and Computer Modelling* 48.9-10 (2008), pp. 1468–1477.
- [RSS13] F. Ricca, A. Scozzari, and B. Simeone. “Political districting: from classical models to recent approaches”. In: *Annals of Operations Research* 204.1 (2013), pp. 271–299.
- [Río20] R. Z. Ríos-Mercado. *Optimal Districting and Territory Design*. Vol. 284. Springer, 2020.
- [Rob85] F. S. Roberts. “Applications of edge coverings by cliques”. In: *Discrete applied mathematics* 10.1 (1985), pp. 93–109.
- [SC98] D. X. Shaw and G. Cho. “The critical-item, upper bounds, and a branch-and-bound algorithm for the tree knapsack problem”. In: *Networks: An International Journal* 31.4 (1998), pp. 205–216.
- [SKK00] K. Schloegel, G. Karypis, and V. Kumar. *Graph partitioning for high performance scientific simulations*. Army High Performance Computing Research Center, 2000.

- [SL95] C. C. de Souza and M. Laurent. “Some new classes of facets for the equitable polytope”. In: *Discrete Applied Mathematics* 62.1-3 (1995), pp. 167–191.
- [SM00] J. Shi and J. Malik. “Normalized cuts and image segmentation”. In: *IEEE Transactions on pattern analysis and machine intelligence* 22.8 (2000), pp. 888–905.
- [SRC11] M. A. Salazar-Aguilar, R. Z. Ríos-Mercado, and M. Cabrera-Ríos. “New models for commercial territory design”. In: *Networks and Spatial Economics* 11.3 (2011), pp. 487–507.
- [SS13] P. Sanders and C. Schulz. “Think locally, act globally: Highly balanced graph partitioning”. In: *International Symposium on Experimental Algorithms*. Springer, 2013, pp. 164–175.
- [Sch07] S. E. Schaeffer. “Graph clustering”. In: *Computer science review* 1.1 (2007), pp. 27–64.
- [Seg+07] J. Segura-Ramiro, R. Z. Ríos-Mercado, A. M. Álvarez-Socarrás, and K. de Alba Romenus. “A location-allocation heuristic for a territory design problem in a beverage distribution firm”. In: *Proceedings of the 12th Annual International Conference on Industrial Engineering Theory, Applications, and Practice (IJIE)*. 2007, pp. 428–434.
- [Sen01] N. Senses. “Lower bounds and exact algorithms for the graph partitioning problem using multicommodity flows”. In: *European Symposium on Algorithms*. Springer, 2001, pp. 391–403.
- [Sør04] M. M. Sørensen. “b-Tree facets for the simple graph partitioning polytope”. In: *Journal of combinatorial optimization* 8.2 (2004), pp. 151–170.
- [Sør07] M. M. Sørensen. “Facet-defining inequalities for the simple graph partitioning polytope”. In: *Discrete Optimization* 4.2 (2007), pp. 221–231.
- [Sør17] M. M. Sørensen. “Facets for node-capacitated multicut polytopes from path-block cycles with two common nodes”. In: *Discrete Optimization* 25 (2017), pp. 120–140.
- [TT20] V. Traub and T. Tröbst. “A Fast $(2 + 2/7)$ -Approximation Algorithm for Capacitated Cycle Covering”. In: *International Conference on Integer Programming and Combinatorial Optimization*. Springer, 2020, pp. 391–404.
- [TV02] P. Toth and D. Vigo. *The vehicle routing problem*. SIAM, 2002.
- [Thi06] S. Thite. “On covering a graph optimally with induced subgraphs”. In: *arXiv preprint cs/0604013* (2006).
- [Tho97] C. Thomassen. “On the complexity of finding a minimum cycle cover of a graph”. In: *SIAM Journal on Computing* 26.3 (1997), pp. 675–677.
- [Tuz84] Z. Tuza. “Covering of graphs by complete bipartite subgraphs; complexity of 0–1 matrices”. In: *Combinatorica* 4.1 (1984), pp. 111–116.
- [Tve82] H. Tverberg. “On the decomposition of K_n into complete bipartite graphs”. In: *Journal of Graph Theory* 6.4 (1982), pp. 493–494.
- [Uiy03] C. Uiyasathain. “Maximal-clique partitions”. PhD thesis. University of Colorado at Denver, 2003.
- [Vid78] K. Vidyasankar. “Covering the edge set of a directed graph with trees”. In: *Discrete Mathematics* 24.1 (1978), pp. 79–85.
- [Wal82] W. Wallis. “Asymptotic values of clique partition numbers”. In: *Combinatorica* 2.1 (1982), pp. 99–101.
- [Wat06] V. L. Watts. “Fractional biclique covers and partitions of graphs”. In: *the electronic journal of combinatorics* 13.1 (2006), p. 74.
- [Wu12] B. Y. Wu. “Fully polynomial-time approximation schemes for the max–min connected partition problem on interval graphs”. In: *Discrete Mathematics, Algorithms and Applications* 4.01 (2012).
- [XLL13] W. Xu, W. Liang, and X. Lin. “Approximation algorithms for min-max cycle cover problems”. In: *IEEE Transactions on Computers* 64.3 (2013), pp. 600–613.
- [XOX02] Y. Xu, V. Olman, and D. Xu. “Clustering gene expression data using a graph-theoretic approach: an application of minimum spanning trees”. In: *Bioinformatics* 18.4 (2002), pp. 536–545.

- [XW10] Z. Xu and Q. Wen. “Approximation hardness of min–max tree covers”. In: *Operations Research Letters* 38.3 (2010), pp. 169–173.
- [YJC13] Z. Yan, N. Jouandeau, and A. A. Cherif. “A survey and analysis of multi-robot coordination”. In: *International Journal of Advanced Robotic Systems* 10.12 (2013), p. 399.
- [YL19] W. Yu and Z. Liu. “Better approximability results for min–max tree/cycle/path cover problems”. In: *Journal of Combinatorial Optimization* 37.2 (2019), pp. 563–578.
- [YLB19] W. Yu, Z. Liu, and X. Bao. “New approximation algorithms for the minimum cycle cover problem”. In: *Theoretical Computer Science* 793 (2019), pp. 44–58.
- [YLB20] W. Yu, Z. Liu, and X. Bao. “New LP relaxations for minimum cycle/path/tree cover problems”. In: *Theoretical Computer Science* 803 (2020), pp. 71–81.
- [Yan98] L. Yan. “On path decompositions of graphs”. PhD thesis. Arizona State University, 1998.
- [ZL06] M. Zhai and C. Lü. “Path decomposition of graphs with given path length”. In: *Acta Mathematicae Applicatae Sinica* 22.4 (2006), pp. 633–638.
- [ZS05] A. A. Zoltners and P. Sinha. “Sales territory design: Thirty years of modeling and implementation”. In: *Marketing Science* 24.3 (2005), pp. 313–331.
- [Zha16] C.-Q. Zhang. “Circuit Double Covers of Graphs”. In: *Graph Theory*. Springer, 2016, pp. 273–291.
- [Zha97] C.-Q. Zhang. *Integer flows and cycle covers of graphs*. Vol. 205. CRC Press, 1997.
- [Zhe+05] X. Zheng, S. Jain, S. Koenig, and D. Kempe. “Multi-robot forest coverage”. In: *2005 IEEE/RSJ International Conference on Intelligent Robots and Systems*. IEEE, 2005, pp. 3852–3857.
- [Zho+19] X. Zhou, H. Wang, B. Ding, T. Hu, and S. Shang. “Balanced connected task allocations for multi-robot systems: An exact flow-based integer program and an approximate tree-based genetic algorithm”. In: *Expert Systems with Applications* 116 (2019), pp. 10–20.