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Holes, Antiholes, and Maximal Cliques in a Railway Model for a Single Track

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Abstract

We present a graph theoretical model for scheduling trains on a single unidirectional track between two stations.

The set of departures of all possible train types at all possible (discrete) points of time is turned into an undirected graph G by joining two nodes if the corresponding departures are in conflict. This graph G has no odd antiholes and no k -holes for any integer $k \geq 5$. In particular, any finite, node induced subgraph of G is perfect.

For any integer $r \geq 2$ we construct minimal headways for r train types so that the resulting graph G has $2r$ -antiholes and 4-holes at the same time. Hence, G is neither a chordal graph nor the complement of a chordal graph, in general. At the end we analyse the maximal cliques in G .

1 Introduction

1.1 Motivation

In the project *Trassenbörse* a research group at the Zuse Institute Berlin is developing together with economists and railway experts from the Technical University Berlin and consultants a concept for the allocation of railway slots.

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Each slot has a monetary value which depends on how close it comes to the wishes of the train operating company that bids for the slot. We are looking for timetables that maximise the sum of the monetary values of the simultaneously realisable slots.

We formulated this problem as a multi commodity flow problem in a time-expanded graph F with additional constraints like safety restrictions, stopovers, and combinatorial bids (see [6]).

A core problem in a railway network is the small number of opportunities to overtake a slower train. Many nodes in the above graph F are therefore in conflict with each other due to safety restrictions on the tracks. A graph theoretical understanding of the conflict graph derived from F is therefore of special interest.

The conflict graph for a network decomposes more or less into conflicts for single unidirectional tracks. There are other conflicts like bidirectional traffic on a single track or the intersection of lines outside of stations. But, they occur less often. In this paper we therefore consider the conflict graph for a single unidirectional track.

1.2 Background on trains

We are given a unidirectional track s between two stations, say, A-town and B-town. Any train enters the track at A-town and leaves the track at B-town.

A *train type* is a specific dynamic for running a specific train along the track s . For example, there may be train types ICE_1 , ICE_2 , ICE_3 , RE_1 , and RE_2 for running an ICE (InterCityExpress) or an RE (RegionalExpress) with different dynamics along the track s . We only consider a finite set \mathbf{Y} of train types. In what follows we consider a discrete time horizon, typically in steps of one minute, which is modelled as the set of integers \mathbb{Z} .

Consider two trains running behind each other along the track s . Denote the train type of the first and the second train by y_1 and y_2 , respectively. Then, the safety restrictions on the whole track lead to the minimal headway time $\mathbf{H}(y_1, y_2)$ that the second train has to obey when it enters the track s . This means that all the safety conditions on the whole track are reduced to a single check of the departure times of the two trains (see [4, section 5.3.3.]). We remark that the headway time $H(y_1, y_2)$ is not necessarily the technically minimal headway time. It may be increased by some safety buffer time.

It is clear that a slow train in front of a fast train needs a more substantial headway than a fast train in front of a slow train. Therefore, the headway matrix H is not symmetric, in general.

The only property of the headway matrix that will be used in Sections 2, 3, and 4 is the validity of the triangle inequality. We shall call this property

triangle-linearity. The triangle inequality is clearly satisfied for the technically minimal headways. But, the safety buffer times may pose problems. Nevertheless, given a technically minimal headway matrix H and a positive integer m we can add an arbitrary safety buffer time $b(y_1, y_2)$ to the headway $H(y_1, y_2)$ without violating the triangle inequality provided that $m \leq b(y_1, y_2) \leq 2m$ for any $y_1, y_2 \in Y$.

Our research project *Trassenbörse* has data sets for both the technically minimal headways and the headways with additional safety buffer times. They satisfy the triangle inequality.

2 The graph G

2.1 Definition and Notation

We shall only consider simple graphs without loops.

Definition An $(n \times n)$ -matrix H of positive integers is called *triangle-linear* if

$$H(i, j) + H(j, k) \geq H(i, k)$$

for any $1 \leq i, j, k \leq n$.

These matrices are sometimes called distance matrices in the literature. It is irrelevant for our graph theoretical argumentations whether the matrix H is actually derived from some railway context or not. The index set $\{1, 2, \dots, n\}$ of the matrix H is denoted by Y and its elements are called *train types* even if there is no relation with a railway context.

Any triangle-linear matrix H defines an undirected graph G_H in the following way. The set V of nodes is the Cartesian product $Y \times \mathbb{Z}$. The set E of edges is

$$\begin{aligned} \{((y_1, t_1), (y_2, t_2)) \in V \times V : & \quad (y_1, t_1) \neq (y_2, t_2) \quad \text{and} \\ & \quad 1 - H(y_2, y_1) \leq t_2 - t_1 \leq H(y_1, y_2) - 1\}. \end{aligned}$$

Two nodes that are adjacent are said to be *in conflict*.

Example The matrix $\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$ for train types y_1 and y_2 and the restriction of \mathbb{Z} to the time interval $I = \{1, 2, 3, 4, 5\}$ lead to the graph G_H as depicted in Figure 1. Here, and in what follows, the second coordinate of G_H , namely the time-axis \mathbb{Z} , is displayed horizontally. The first coordinate, namely the set Y of train types, is displayed vertically.

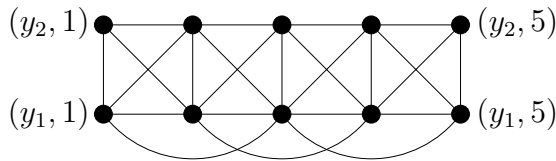


Figure 1: The graph defined by $H = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$ and $I = [1, 5]$.

In the train model interpretation, each node (y, t) stands for a train of type y with departure time t . An edge $((y_1, t_1), (y_2, t_2))$ corresponds to a conflict due to a too short headway between a train of type y_1 departing at time t_1 and a train of type y_2 departing at time t_2 . The graph G_H is the conflict graph for all possible departures of all possible train types.

Also, the graph G_H can be looked at as the conflict graph for the production of items on a single machine. Then, $H(i, j)$ is the minimal waiting time for the production of an item of type $j \in Y$ after the production of an item of type $i \in Y$. The time to produce an item would be looked at as being equal to zero.

In what follows, the complement of a graph will be denoted by an overline. We shall abbreviate G_H by G if this does not lead to ambiguities. All the subgraphs considered by us are node induced. For simplicity, the coordinates of the train types are not always depicted correctly in the following figures. Sometimes, nodes with different train types are depicted with the same value on the vertical axis. This has no effect on the graph theoretical argumentations, though.

A k -hole, $k \geq 4$, is a connected graph with k nodes and k edges so that each node is adjacent to exactly two other nodes. We remark that an embedding of a k -hole into the plane is homeomorphic to a circle. A graph that is the complement of a k -hole is called a k -antihole. A hole or antihole is *odd* (*even*) if k is odd (even). Our main result is

Theorem 2.1 *G has no k -holes for any integer $k \geq 5$ and no odd antiholes.*

The recent verification of the strong perfect graph conjecture (see [1, 2, 3] for the concept of perfectness and the proof of the conjecture) implies that

Corollary 2.2 *Any finite subgraph of G is perfect.*

The proof of Theorem 2.1 is split into Lemma 3.1 and Lemma 3.3. Both the Lemmas are based on two special subgraphs of $Y \times \mathbb{Z}$, denoted by $\mathfrak{3}_1$ and $\mathfrak{3}_2$, which we introduce now.

2.2 Types of subgraphs

Let M be a graph whose node set is $Y \times \mathbb{Z}$. The graph M is not necessarily isomorphic to a graph G_H . We define a pre-order on the node set of M by

$$(y_1, t_1) \preceq (y_2, t_2)$$

if $t_1 \leq t_2$. This pre-order is reflexive and transitive. It is not antisymmetric, though, if Y has more than one element. We introduce another relation on $Y \times \mathbb{Z}$ that is defined by

$$(y_1, t_1) \prec (y_2, t_2)$$

if $t_1 < t_2$.

We define an equivalence relation on the set of subgraphs of M . Two subgraphs of M , say, N_1 and N_2 , are equivalent if there exists a graph isomorphism from N_1 to N_2 that respects the pre-order \preceq . The resulting equivalence classes are called *types*. Two types will be especially important in what follows.

A subgraph of M consisting of three nodes $p_1 \preceq p_2 \preceq p_3$ is said to be of type $\mathfrak{3}_1$ if p_1 and p_3 are adjacent in M but p_1 and p_2 as well as p_2 and p_3 are not adjacent in M , respectively.

A subgraph of M consisting of three nodes $p_1 \preceq p_2 \preceq p_3$ is said to be of type $\mathfrak{3}_2$ if p_1 and p_3 are not adjacent in M but p_1 and p_2 as well as p_2 and p_3 are adjacent in M , respectively.

Obviously, M has a subgraph of type $\mathfrak{3}_1$ if and only if the complement \overline{M} of M has a subgraph of type $\mathfrak{3}_2$.

Lemma 2.3 *G has no subgraphs of type $\mathfrak{3}_1$.*

Proof Let H be a triangle-linear matrix. We assume that G_H has a subgraph of type $\mathfrak{3}_1$ which is induced by the three nodes $p_i = (y_i, t_i)$ for $i \in \{1, 2, 3\}$ with $t_1 \leq t_2 \leq t_3$. The two nodes p_1 and p_3 are adjacent in G_H but p_1 and p_2 as well as p_2 and p_3 are not adjacent, respectively. Then, by the definition of a conflict,

$$\begin{aligned} t_3 - t_1 &\leq H(y_1, y_3) - 1, \\ t_2 - t_1 &\geq H(y_1, y_2), \quad \text{and} \\ t_3 - t_2 &\geq H(y_2, y_3). \end{aligned}$$

Summation of the second and the third inequality yields

$$t_3 - t_1 \geq H(y_1, y_2) + H(y_2, y_3).$$

It follows that $t_3 - t_1 \geq H(y_1, y_3)$ because H is triangle-linear. This contradicts the above inequality $t_3 - t_1 \leq H(y_1, y_3) - 1$. Hence, G_H has no subgraphs of type $\mathfrak{3}_1$. \diamond

Clearly, it follows for the complement \overline{G} of G that

Corollary 2.4 \overline{G} has no subgraphs of type 3_2 .

Let C_k be a k -hole in the graph M and let p be a node in C_k . There are two nodes in C_k adjacent to p , say, q_1 and q_2 . We call p a *right spike* if $q_1 \prec p$ and $q_2 \prec p$. Similarly, we call p a *left spike* if $p \prec q_1$ and $p \prec q_2$.

A node of C_k that is neither a left spike nor a right spike has to be of type 3_2 . Corollary 2.4 implies

Lemma 2.5 Let C_k be a k -hole in \overline{G} . Then, any node of C_k is either a left spike or a right spike.

The nodes of C_k have a further property.

Lemma 2.6 Let C_k be a k -hole in \overline{G} . Then, the nodes of C_k appear alternatingly along this circle as right spikes and left spikes.

Proof Obviously, two nodes of C_k of the same spike type (right or left) cannot be adjacent. Hence, right spikes and left spikes appear alternatingly along C_k . \diamond

3 Holes and antiholes that do not exist in G

Lemma 3.1 G has no odd antiholes.

Proof A k -antihole in G gives rise to a k -hole C_k in \overline{G} . The nodes along C_k appear alternatingly as right spikes and left spikes. Hence, in C_k , the number of left spikes is equal to the number of right spikes and therefore k has to be even. \diamond

We call a path with k consecutive, pairwise different nodes v_1, v_2, \dots, v_k a *k -path* and denote it by (v_1, v_2, \dots, v_k) . Of course, this notation is not unique because the nodes along a path can be read forward and backward.

Lemma 3.2 Let M be a graph whose node set is $Y \times \mathbb{Z}$. Then, there are only two types of 4-subpaths of M that have neither subgraphs of type 3_1 nor of type 3_2 . These two types, 4_1 and 4_2 , are depicted in Figure 2.

Proof Let $q_1 \preceq q_2 \preceq q_3 \preceq q_4$ be four nodes of M . There are $4!/2$ ways to join the four nodes by 3 edges in order to produce a 4-subpath. Ten out of these twelve types do have a subgraph of type 3_1 or of type 3_2 . The remaining two types are depicted in Figure 2. \diamond

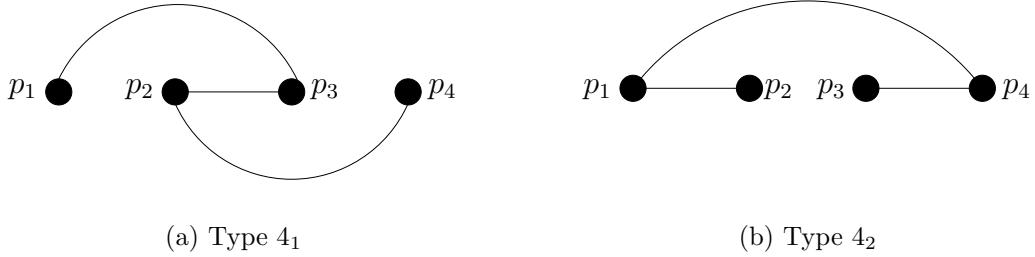


Figure 2: The only two types of 4-subpaths that have neither subgraphs of type 3₁ nor of type 3₂.

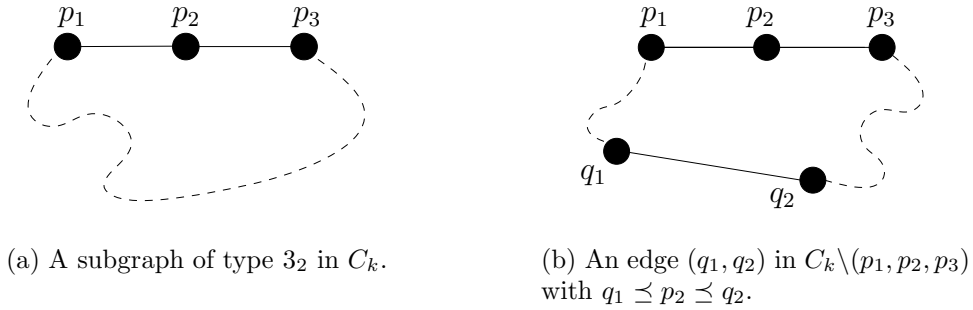


Figure 3: Case that C_k has a subgraph of type 3₂.

We now have enough tools to prove

Lemma 3.3 G has no k -holes for any integer $k \geq 5$.

Proof Let C_k be a k -hole in G for some integer $k \geq 5$. We shall derive a contradiction from this.

First, we assume that C_k has a subgraph of type 3₂ as shown in Figure 3(a). We remark that some of the nodes are depicted as if they had equal first coordinates though this is not necessarily the case.

The path $C_k \setminus (p_1, p_2, p_3)$ can be interpreted as the piecewise-linear image of an interval. Hence, there exists by (a combinatorial version of) the mean value theorem an edge $e = (q_1, q_2)$ in $C_k \setminus (p_1, p_2, p_3)$ so that $q_1 \preceq p_2 \preceq q_2$ as depicted in Figure 3(b). The three nodes q_1, p_2 , and q_2 are a subgraph of type 3₁ unless p_2 is adjacent to either q_1 or to q_2 . Since subgraphs of type 3₁ do not appear in C_k by Lemma 2.3, we have either $q_1 = p_1$ or $q_2 = p_3$. In what follows, we consider the case that $q_1 = p_1$. The other case is similar.

We have $p_2 \preceq q_2$ by construction and we know that q_2 and p_3 are neither equal nor adjacent because $k \geq 5$. Hence, $p_2 \preceq q_2 \preceq p_3$ or $p_3 \preceq q_2$. In

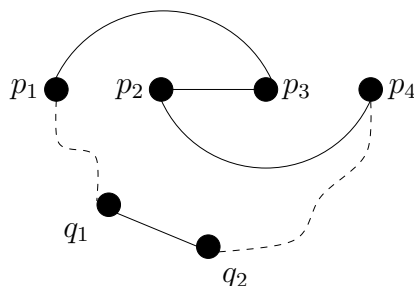


Figure 4: An edge (q_1, q_2) in $C_k \setminus (p_1, p_3, p_2, p_4)$ that satisfies $q_1 \preceq p_2 \preceq q_2$.

the first case, the three nodes p_2, q_2 , and p_3 were a subgraph of type 3_1 . In the second case, the three nodes q_1, p_3 , and q_2 were a subgraph of type 3_1 . But, subgraphs of type 3_1 do not exist. Hence, our assumption about the existence of a subgraph of C_k of type 3_2 led us to a contradiction. Therefore, C_k has no subgraphs of type 3_2 for any integer $k \geq 5$.

In what follows, we can apply Lemma 3.2. Every 4-subpath of C_k is either of type 4_1 or of type 4_2 . If C_k has a 4-subpath of type 4_2 as depicted in Figure 2(b) then the next node, say, p_5 , along C_k that is adjacent to p_3 has to lie to the right of p_4 , i.e. $p_4 \preceq p_5$, in order to avoid subgraphs of type 3_1 . Then, (p_2, p_3, p_4, p_5) is a 4-subpath of type 4_1 . Hence, any C_k has a 4-subpath of type 4_1 provided that $k \geq 5$.

We fix a 4-subpath (p_1, p_3, p_2, p_4) of C_k of type 4_1 . By the same argument as above, there exists an edge $e = (q_1, q_2)$ in $C_k \setminus (p_1, p_3, p_2, p_4)$ such that $q_1 \preceq p_2 \preceq q_2$ as depicted in Figure 4. The three nodes q_1, p_2 , and p_3 are a subgraph of type 3_1 unless $q_2 = p_4$. But, for $q_2 = p_4$, we observe that the three nodes q_1, p_3 , and q_2 are a subgraph of type 3_1 unless $q_1 = p_1$. But, $k \geq 5$ and therefore $q_1 \neq p_1$. Hence, we derived a contradiction. Therefore, G has no k -holes for any $k \geq 5$. \diamond

4 Holes and antiholes that do exist in G

We shall give examples for holes and antiholes of any possible cardinality. These examples will be minimal with respect to the number of train types involved. We show that G can have $2k$ -holes and 4-antiholes simultaneously for any $k \geq 2$. Hence, in general, G is neither a chordal graph nor the complement of a chordal graph. Therefore, G is neither an interval graph nor the complement of an interval graph, in general.

4.1 4-holes and 4-antiholes

A 4-antihole already appears for a single train type y if $H(y, y) \geq 2$. The four nodes $(y, 0)$, $(y, 1)$, $(y, H(y, y) + 1)$, and $(y, H(y, y) + 2)$ induce a 4-antihole.

On the other hand, it is easy to see that a 4-hole cannot exist if only a single train type is used. Here is an example of a 4-hole based on two train types. The matrix

$$H = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$$

indexed by train types x and y is triangle-linear. A 4-hole is induced by the nodes $(x, 0)$, $(x, 1)$, $(y, 2)$, and $(y, 3)$. A second example is given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and the nodes $(x, 0)$, $(x, 1)$, $(y, 0)$, and $(y, 1)$. This matrix gives rise to 4-antiholes, too, as explained above.

4.2 Even antiholes

We shall prove in Theorem 4.2 that large antiholes require a lot of train types.

Lemma 4.1 *Let C_{2k} be a $2k$ -hole in $\overline{G_H}$ with $k \geq 3$. Let p_1 and p_2 be two nodes of C_{2k} that have the same train type and that satisfy $p_1 \prec p_2$. Then, p_1 is a left spike and p_2 is a right spike in C_{2k} .*

Proof Let p_1 and p_2 be two nodes in $C_{2k} \subset \overline{G}$ that have the same train type, say, x , and that satisfy $p_1 \prec p_2$. Then $p_1 = (x, t_1)$ and $p_2 = (x, t_2)$ with $t_1 < t_2$.

If p_1 is a right spike then the two nodes adjacent to p_1 , say, r and s , satisfy $r \prec p_1$ and $s \prec p_1$ by definition. The two nodes r and s are adjacent to p_1 in \overline{G} and therefore both the nodes are not adjacent to p_1 in G . It follows from the definition of the graph G that r and s are not adjacent to any (x, t) with $t \geq t_1$. In particular, r and s are not adjacent to p_2 in G . Hence, the two nodes r and s are adjacent to p_2 in \overline{G} . This implies that the node set of C_{2k} is equal to $\{r, s, p_1, p_2\}$ and therefore $k = 2$.

By a similar argumentation it follows that if p_2 is a left spike then $k = 2$. Therefore, the assumption $k \geq 3$ implies that p_1 is a left spike and p_2 is a right spike. \diamond

Theorem 4.2 *Let H be a triangle-linear $(r \times r)$ -matrix. For $r = 1$ or $r = 2$, the only antiholes in G_H are 4-antiholes. For $r \geq 3$, the only antiholes are $2k$ -antiholes with $2 \leq k \leq r$.*

Proof Let H be a triangle-linear $(r \times r)$ -matrix. We know by Lemma 3.1 that the only antiholes in G_H are even antiholes. Hence, it is sufficient to prove that a $2k$ -antihole, $k \geq 3$, can only appear in G_H if $k \leq r$. Equivalently, we prove that a $2k$ -hole, $k \geq 3$, can only appear in $\overline{G_H}$ if $k \leq r$.

Let C_{2k} be a $2k$ -hole in $\overline{G_H}$ for some integer $k \geq 3$. Then, C_{2k} has k left spikes by Lemma 2.6 and these left spikes have pairwise different train types by Lemma 4.1. Hence, $k \leq r$. \diamond

We complete the result of Theorem 4.2. There exist $(r \times r)$ -matrices H that lead to $2r$ -antiholes in G_H , indeed.

Lemma 4.3 *Let $r \geq 3$. The following $(r \times r)$ -matrix H_r leads to a graph G_{H_r} that has $2r$ -antiholes. The train types are denoted by $z, y_1, y_2, \dots, y_{r-1}$. The matrix H_r is defined by*

$$\begin{aligned} H_r(z, z) &= 1, \\ H_r(z, y_i) &= 3i, \\ H_r(y_i, z) &= 3(r - i), \quad \text{and} \\ H_r(y_i, y_j) &= \begin{cases} 1 & \text{if } j \leq i \\ 5 & \text{if } j = i + 1 \\ 3(j - i) + 3 & \text{if } j \geq i + 2 \end{cases} \end{aligned}$$

where $1 \leq i, j \leq r - 1$.

Proof We fix an integer $r \geq 3$. First, we prove that the above matrix H_r is triangle-linear. Then, we present a $2r$ -antihole in G_{H_r} . We shall abbreviate H_r by H .

We prove that the triangle inequality is satisfied strictly. There are 2^3 types of the strict triangle inequality

$$H(a, b) + H(b, c) > H(a, c)$$

for $a, b, c \in \{z, y_1, \dots, y_{r-1}\}$ depending on whether the train type z or a y_i appears at some index. If $a = b = z$ or $b = c = z$ then the triangle inequality is clearly satisfied. Here are the five remaining cases,

$$\begin{aligned} H(z, y_j) + H(y_j, z) &> H(z, z), \\ H(z, y_j) + H(y_j, y_k) &> H(z, y_k), \\ H(y_i, y_j) + H(y_j, z) &> H(y_i, z), \\ H(y_i, z) + H(z, y_k) &> H(y_i, y_k), \quad \text{and} \\ H(y_i, y_j) + H(y_j, y_k) &> H(y_i, y_k) \end{aligned}$$

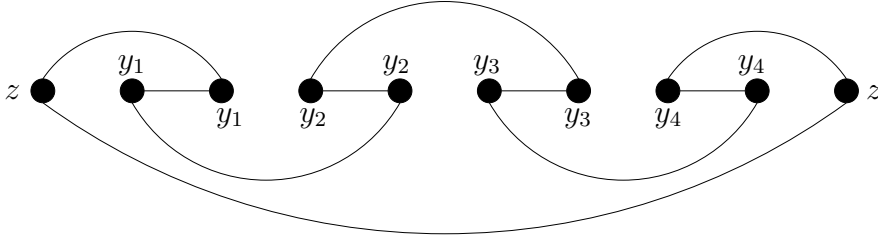


Figure 5: A 10-hole in $\overline{G_{H_5}}$ and the train types of its nodes.

for any $1 \leq i, j, k \leq r - 1$.

The first inequality is true because the entries of H are positive and the right hand side is equal to 1. For the second inequality we have

$$H(z, y_j) + H(y_j, y_k) \geq 3j + 3(k - j) + 1 > 3k = H(z, y_k).$$

The third inequality can be deduced from the second inequality by using $H(y_i, y_j) = H(y_{r-j}, y_{r-i})$ and $H(z, y_j) = H(y_{r-j}, z)$ for any $1 \leq i, j \leq r - 1$.

The fourth and the fifth inequality are true if $i \geq k$ because then the right hand side is equal to 1. We consider the remaining case $i \leq k + 1$. The fourth inequality is true because

$$H(y_i, z) + H(z, y_k) = 3(r - i) + 3k > 3(k - i) + 3 \geq H(y_i, y_k).$$

The fifth inequality is obviously true if either $i = j$ or $j = k$. For the remaining case $i \neq j$ and $j \neq k$ we have

$$H(y_i, y_j) + H(y_j, y_k) \geq 3(j - i) + 2 + 3(k - j) + 2 > 3(k - j) + 3 \geq H(y_i, y_k).$$

Hence, H_r is strictly triangle-linear.

We now present the announced $2r$ -antihole as a $2r$ -hole in $\overline{G_{H_r}}$ because it is easier to depict the $2r$ edges of a $2r$ -hole rather than the $r(2r - 3)$ edges of a $2r$ -antihole. The $2r$ -hole is the subgraph of $\overline{G_{H_r}}$ induced by the nodes $(y_i, 3i - 2)$ and $(y_i, 3i)$ for $i = 1, 2, \dots, r - 1$, and the nodes $(z, 0)$ and $(z, 3r - 2)$.

This graph is depicted in Figure 5 in the case $r = 5$. For simplicity, the coordinates of the train types z, y_1, \dots, y_{r-1} are not depicted correctly because all of them have the same value in the drawing. \diamond

For completeness, we mention that the above matrix H_r leads to 4-holes in G_{H_r} . As an example, restrict the index set of the matrix H_r to the train types z and y_1 . This leads to the matrix $\begin{pmatrix} 1 & 3 \\ 3r-3 & 1 \end{pmatrix}$. A 4-hole is induced by the four nodes $(y_1, 0), (z, 1), (y_1, 2),$ and $(z, 3)$.

5 Maximal Cliques in G

A *clique* in G is a complete subgraph of G , i.e. any two nodes of the subgraph are adjacent. A clique is called *maximal* if it is not included in some larger clique in G . We shall analyse the maximal cliques in G_H for a triangle-linear matrix H . Rather than using y_1 as the typical notation for an element of Y , we shall use letters like h, i, j , and k from now on. For mnemonic reasons we have avoided this kind of notation in the previous sections in order to visually distinguish between train types on the one side and the indices for the holes and antiholes on the other side.

5.1 Further basic results about G

In Lemmas 5.1 to 5.5 we consider the graph G_H for a triangle-linear matrix H . Let $i, j \in Y$ and $t_1, t_2, t_3, u_1, u_2 \in \mathbb{Z}$.

Lemma 5.1 *If (i, t_1) and (j, t_2) are in conflict then (i, t_1) is in conflict with any node in $\{j\} \times [\min(t_1, t_2), \max(t_1, t_2)]$.*

Proof Trivial. ◇

Lemma 5.2 (Convexity) *Let $t_2 \leq t_3$. If (i, t_1) is in conflict with both of (j, t_2) and (j, t_3) then (i, t_1) is in conflict with any node in $\{j\} \times [t_2, t_3]$.*

Proof Follows directly from Lemma 5.1. ◇

The set of nodes described in Lemma 5.2 can be visualised as a triangle. One side is parallel to the \mathbb{Z} -axis. If the third node, which is opposite to this side, is in conflict with both the ends of this side then it is in conflict with any node of this side.

Lemma 5.3 *For any maximal clique C in G and any train type $j \in Y$ we have $C \cap (\{j\} \times \mathbb{Z}) = \{j\} \times I$ for some (possibly empty) interval I .*

Proof If C is a maximal clique in G then $C \cap (\{j\} \times \mathbb{Z})$ has no holes because of the convexity property. ◇

Lemma 5.4 (Cross-convexity) *Let $t_1 \leq t_2$ and $u_1 \leq u_2$. Let (i, t_1) and (j, u_2) be in conflict and let (i, t_2) and (j, u_1) be in conflict. Then, any node in $\{i\} \times [t_1, t_2]$ is in conflict with any node in $\{j\} \times [u_1, u_2]$.*

Proof By the convexity property it is sufficient to prove that (i, t_1) and (j, u_1) are in conflict and that (i, t_2) and (j, u_2) are in conflict.

We show that (i, t_1) and (j, u_1) are in conflict. The other conflict is proved similarly. Without loss of generality we may assume that $t_1 \leq u_1$. The node (i, t_1) is of course in conflict with (j, t_1) and by assumption it is in conflict with (j, u_2) , too. Since $t_1 \leq u_1 \leq u_2$ we can apply the convexity property and deduce that (i, t_1) is in conflict with (j, u_1) . \diamond

The set of nodes described in Lemma 5.4 can be visualised as a rectangle with two sides, say, a and b , which are parallel to the \mathbb{Z} -axis. Lemma 5.4 states that if both of the two diagonals of the rectangle are edges in G then any node in a is adjacent to any node in b .

Lemma 5.5 *Let $t_1 \leq t_2$. The set of nodes in $\{j\} \times \mathbb{Z}$ that are in conflict with any node in $\{i\} \times [t_1, t_2]$ is given by*

$$\{j\} \times [t_2 - H(j, i) + 1, t_1 + H(i, j) - 1].$$

Proof Set $a = t_2 - H(j, i) + 1$ and $b = t_1 + H(i, j) - 1$. By definition, the nodes in $\{j\} \times (-\infty, a - 1]$ are not in conflict with (i, t_2) and the nodes in $\{j\} \times [b + 1, \infty)$ are not in conflict with (i, t_1) .

It remains to prove that any node in $\{j\} \times [a, b]$ is in conflict with any node in $\{i\} \times [t_1, t_2]$. If $a \leq b$ then this follows from cross-convexity. The remaining case $a \geq b + 1$ is trivial because it implies that $[a, b]$ is the empty set. \diamond

Remark The elements of the set $\{j\} \times [t_2 - H(j, i) + 1, t_1 + H(i, j) - 1]$ are not necessarily pairwise in conflict.

Lemma 5.6 *Let H be a triangle-linear matrix. Let C be a maximal clique in G_H . Then, $C \cap (\{j\} \times \mathbb{Z})$ contains at least $\min_{i, k \in Y} (H(i, j) + H(j, k) - H(i, k))$ elements for any $j \in Y$.*

Proof We fix a maximal clique C in G_H and we fix $j \in Y$. We denote $\min_{i, k \in Y} (H(i, j) + H(j, k) - H(i, k))$ by a_j . It follows from Lemma 5.3 that C can be written as

$$C = \bigcup_{i \in I} \{i\} \times [t_i, u_i]$$

for a subset $I \subset Y$ and integers $t_i \leq u_i$. The set I is not empty because C is not empty.

The set of nodes in $\{j\} \times \mathbb{Z}$ that are in conflict with any node of C is equal to $C \cap (\{j\} \times \mathbb{Z})$ because C is a maximal clique. This observation implies together with Lemma 5.5 that

$$\begin{aligned} C \cap (\{j\} \times \mathbb{Z}) &= \bigcap_{i \in I} \{j\} \times [u_i - H(j, i) + 1, t_i + H(i, j) - 1] \\ &= \{j\} \times [\max_{i \in I} (u_i - H(j, i) + 1), \min_{i \in I} (t_i + H(i, j) - 1)] \\ &= \{j\} \times [u_k - H(j, k) + 1, t_i + H(i, j) - 1] \end{aligned}$$

for some train types $k, i \in I$. Hence, the cardinality e_j of the set $C \cap (\{j\} \times \mathbb{Z})$ is given by $e_j = \max(d_j, 0)$ where

$$d_j = 1 + (t_i + H(i, j) - 1) - (u_k - H(j, k) + 1).$$

By the definition of a_j , we have

$$H(i, j) + H(j, k) \geq H(i, k) + a_j$$

and therefore

$$d_j \geq t_i - u_k + H(i, k) - 1 + a_j.$$

The two elements (k, u_k) and (i, t_i) of C are in conflict. This implies $t_i - u_k \geq 1 - H(i, k)$ by definition. When we use this in the above inequality we get $d_j \geq a_j$. Hence, $e_j = \max(d_j, 0) \geq a_j$ as we claimed. \diamond

5.2 Quadrangle-linear Matrices

Lemma 5.6 motivates the following definition.

Definition An $(n \times n)$ -matrix H of positive integers is called *quadrangle-linear* if

$$H(i, j) + H(j, k) \geq H(i, k) + H(j, j)$$

for any $1 \leq i, j, k \leq n$.

Clearly, quadrangle-linear matrices are triangle-linear. We remark that a matrix H is called weak Monge in the literature if the above inequality holds for any $1 \leq j \leq i \leq n$ and for any $1 \leq j \leq k \leq n$. The matrix H_r from Lemma 4.3 is quadrangle-linear because the triangle-inequality is satisfied strictly and all the diagonal elements of H_r are equal to 1.

Theorem 5.7 *Let the matrix H be quadrangle-linear. Let C be a maximal clique in G_H . Then, $C \cap (\{i\} \times \mathbb{Z})$ contains exactly $H(i, i)$ elements for any $i \in Y$.*

Proof The set $C \cap (\{i\} \times \mathbb{Z})$ contains at least $H(i, i)$ elements by Lemma 5.6. It cannot have more than $H(i, i)$ elements because at most $H(i, i)$ consecutive nodes in $\{i\} \times \mathbb{Z}$ can be pairwise in conflict. Hence, $C \cap (\{i\} \times \mathbb{Z})$ contains exactly $H(i, i)$ elements. \diamond

5.3 Characterisation of the maximal cliques in G

Theorem 5.7 implies that any maximal clique in G_H can be written as

$$\bigcup_{i \in Y} \{i\} \times [t_i, t_i + H(i, i) - 1] \quad (1)$$

for integers $(t_i)_{i \in Y}$ if H is quadrangle-linear.

It is of special graph theoretical interest to identify all the maximal cliques in G_H . To do this, we have to answer the following question. Given a quadrangle-linear matrix H : which integers $(t_i)_{i \in Y}$ lead in formula (1) to maximal cliques?

By cross-convexity, the intervals in (1) are pairwise in conflict if and only if each left end of an interval is in conflict with each right end of an interval. By the definition of a conflict, this is equivalent to

$$-H(j, i) + 1 \leq (t_j + H(j, j) - 1) - t_i \leq H(i, j) - 1$$

for any $i, j \in Y$. This is equivalent to

$$-H(j, i) - H(j, j) + 2 \leq t_j - t_i \leq H(i, j) - H(j, j). \quad (2)$$

We shall formulate condition (2) in a slightly nicer way. After the interchange of i and j and multiplication with (-1) we get

$$H(i, i) - H(j, i) \leq t_j - t_i \leq H(i, j) + H(i, i) - 2.$$

Since $H(i, i) \geq -H(j, j) + 2$ and $-H(j, j) \leq H(i, i) - 2$ we are able to combine the above two conditions. The set described in (1) is a maximal clique if and only if

$$H(i, i) - H(j, i) \leq t_j - t_i \leq H(i, j) - H(j, j) \quad (3)$$

for any $i, j \in Y$. The condition (3) is equivalent to condition (2) but it has the advantage that is symmetric in i and j . The case $i = j$ is trivially satisfied. The remaining $\binom{n}{2}$ cases of condition (3) for any unordered pair $\{i, j\} \subset Y$ with $i \neq j$ are linearly independent from each other, in general.

The \mathbb{Z} -translations operate freely on the set of maximal cliques because the definition of a conflict is invariant under time-translation. There are only finitely many classes with respect to this operation. When we fix some element $i_0 \in Y$ then each equivalence class contains exactly one representative with $t_{i_0} = 0$. We summarise our considerations.

Lemma 5.8 *Let H be a quadrangle-linear matrix. Up to \mathbb{Z} -translations, the maximal cliques in G are given by formula (1) where the integers $(t_i)_{i \in Y}$ satisfy condition (3) for any unordered pair $\{i, j\} \subset Y$ with $i \neq j$ after setting $t_{i_0} = 0$ for some fixed element $i_0 \in Y$.*

5.4 Are headways quadrangle-linear in reality?

The railway experts in our project have access to infrastructure data of the German railway only in a very limited way. They have provided us with data for the technically minimal headways for 45 tracks and 6 train types. This amounts to 9720, namely $45 \times 6 \times 6 \times 6$, cases of the quadrangle-linearity inequality on the tracks.

93.2% of these cases are satisfied by the data. For the remaining cases we have

$$H(i, j) + H(j, k) \geq H(i, k) + H(j, j) - \varepsilon$$

where $\varepsilon = 20$ sec. This is not far off quadrangle-linearity when we take into account that the headways from our data sets range from 64 sec to 2531 sec.

After the addition of standard safety buffer times, the above inequality holds for $\varepsilon = 66$ sec. After rounding up the data on full minutes, the above inequality holds for $\varepsilon = 2$ min. We remark that the process of rounding up does preserve triangle-linearity but does not necessarily preserve quadrangle-linearity. The headways from our data sets do not satisfy all the cases of the quadrangle-linearity inequality. But, they nearly do.

It seems that the addition of extra safety buffer times allows enough adjustment to make the headways quadrangle-linear. Furthermore, an analysis of the technically minimal headways based on a more detailed information about the infrastructure should shed some light on the 6.8% of constellations that do not satisfy the quadrangle-linearity inequality.

Anyway, quadrangle-linear headways may be produced in the following straightforward way. Denote the travel time of train type i by tt_i and define the headway matrix by $H(i, j) = \max(tt_i - tt_j, 0) + \varepsilon$ for some fixed integer ε . These headways are quickly verified to be quadrangle-linear. They have been used, e.g., in [5, section 3.4.7] with $\varepsilon = 3$ min.

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