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The Normal Graph Conjecture is true for Circulants

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Abstract

Normal graphs are defined in terms of cross-intersecting set families: a graph is normal if it admits a clique cover $\mathcal{Q}$ and a stable set cover $\mathcal{S}$ s.t. every clique in $\mathcal{Q}$ intersects every stable set in $\mathcal{S}$.

Normal graphs can be considered as closure of perfect graphs by means of co-normal products (Körner [6]) and graph entropy (Cziszár et al. [5]). Perfect graphs have been recently characterized as those graphs without odd holes and odd antiholes as induced subgraphs (Strong Perfect Graph Theorem, Chudnovsky et al. [3]). Körner and de Simone [9] observed that C_5, C_7, and $\overline{C}_7$ are minimal not normal and conjectured, as generalization of the Strong Perfect Graph Theorem, that every C_5, C_7, $\overline{C}_7$-free graph is normal (Normal Graph Conjecture, Körner and de Simone [9]).

We prove this conjecture for a first class of graphs that generalize both odd holes and odd antiholes, the circulants, by characterizing all the normal circulants.

1 Introduction

Normal graphs come up in a natural way in an information theoretic context [7, 5]. A graph G is called *normal* if it has the property of so-called "qualitative independence": G admits a clique cover $\mathcal{Q}$ and a stable set cover $\mathcal{S}$ s.t. every clique in $\mathcal{Q}$ intersects every stable set in $\mathcal{S}$. (A set is a clique (resp. stable set) if its nodes are mutually adjacent (resp. non-adjacent).)

The interest in normal graphs is caused by the fact that they form, in many ways, a closure of the well-known class of perfect graphs. Berge introduced the latter class in 1960, motivated from Shannon's information-theoretic problem of finding the zero-error capacity of a discrete memoryless channel [11]. Shannon's problem has a graph-theoretic formulation, regarding the asymptotic growth of the maximum cliques in the co-normal product G^n of $G = (V, E)$, where G^2 has $V \times V$ as node set and

$$\{(a_1, b_1), (a_2, b_2) : (a_1, a_2) \in E \text{ or } (b_1, b_2) \in E\}$$

as edge set. The Shannon capacity of G is

$$C(G) = \lim_{n \to \infty} \frac{1}{n} \log \omega(G^n)$$

where $\omega(G^n)$ denotes the size of a maximum clique in G^n. Shannon observed that $\omega(G^n) = (\omega(G))^n$ holds for graphs G with $\omega(G) = \chi(G)$ which makes the otherwise difficult problem of determining $C(G)$ tractable ($\chi(G)$ denotes the least number of stable sets covering V). Shannon observed further that the C_5 is the smallest graph G with $\omega(G) < \chi(G)$ and that computing $C(C_5)$ is a difficult task, as for all chordless odd cycles C_{2k+1} with $k \geq 2$, called *odd holes*, see Figure 1.

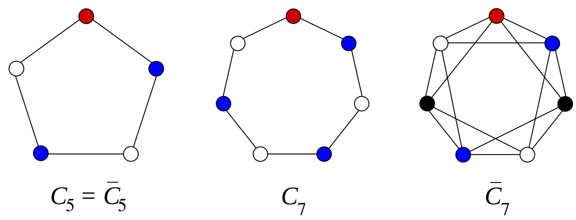


Figure 1: Small odd holes and odd antiholes

This led Berge [2] introduce *perfect graphs* as those graphs G, where $\omega(G')$ equals $\chi(G')$ for each induced subgraph $G' \subseteq G$. Berge observed that, besides odd holes, also their complements, the *odd antiholes* $\overline{C}_{2k+1}$ with $k \geq 2$, admit the property $\omega < \chi$, see Figure 1. (The complement $\overline{G}$ has the same node set as G but two nodes are adjacent in $\overline{G}$ iff they are non-adjacent in G.) This motivated Berge's Strong Perfect Graph Conjecture:

$$G \text{ is perfect} \Leftrightarrow G \text{ is odd hole- and odd antihole-free.}$$

In particular, Berge claimed that the class of perfect graphs is closed under taking complements; this weaker conjecture has been proved in 1972 by Lovász [10]. The Strong Perfect Graph Conjecture stimulated the study of perfect graphs. It turned out that they have many fascinating properties and interesting relationships to other fields of scientific enquiry, see e.g. [1]. In a sequence of remarkable results based on the work of many graph-theorists Chudnovsky et al. [3] finally verified the conjecture; thus the Strong Perfect Graph Conjecture turned into the Strong Perfect Graph Theorem.

Normal graphs are closed under taking complements by definition, thus they share this property with perfect graphs. In this context it is natural to ask whether perfect and normal graphs are also closed under taking co-normal products. Körner and Longo [7] showed that all co-normal products of G are perfect if and only if G is the union of disjoint cliques. However, all co-normal products of perfect and normal graphs are *normal* [6]; hence the class of normal graphs is the *closure of perfect graphs w.r.t. taking co-normal products.*

Another information-theoretic link between perfect and normal graphs has been established by means of graph entropy, see [12] for a survey. The *entropy* of a graph G w.r.t. a probability distribution p on its node set is

$$H(G,p) = \limsup_{k\to\infty} \min\{\frac{1}{k}\log_2 \chi(G^k[U]) : U \subseteq V(G^k), \sum_{x\in U} p^k(x) > 1-\varepsilon\}.$$

The most important property of the graph-entropy is its sub-additivity w.r.t. graphs on the same node set, in particular for complementary graphs:

$$H(p) \leq H(G,p) + H(\overline{G},p) \ \forall p$$

where $H(p)$ stands for the entropy of the complete graph, i.e., for the entropy of p itself. Körner raised the question for which graphs G the minimum $H(p)$ is attained, that means when equality holds rather than just sub-additivity. Körner and Longo [7] and Körner and Marton [8] proved

$$H(p) = H(G,p) + H(\overline{G},p) \text{ for at least one } p > 0 \Leftrightarrow G \text{ is normal}$$

and Cziszár et al. [5] obtained

$$H(p) = H(G,p) + H(\overline{G},p) \text{ for all } p \Leftrightarrow G \text{ is perfect}$$

after a sequence of deep results on entropies of so-called convex corners. Hence normal graphs form also a superclass of perfect graphs by means of splitting graph entropies.

Since normal graphs are "weaker" perfect graphs in several ways, Körner and de Simone [9] asked finally for a similarity of the two classes in terms of forbidden subgraphs. Körner [6] showed that an odd hole C_{2k+1} is normal iff $k \geq 4$. In particular, C_5 and C_7 are not normal, and so neither $\overline{C}_7$ is.

These three graphs are even *minimally* not normal since all of their proper induced subgraphs are perfect and, hence, normal. This led Körner and de Simone conjecture:

Conjecture 1 (Normal Graph Conjecture [9]) *Graphs without any C_5, C_7, or $\overline{C}_7$ as induced subgraph are normal.*

The validity of this conjecture would imply that the only minimally not normal graphs are precisely C_5, C_7, and $\overline{C}_7$ – as the only minimally imperfect graphs are precisely all odd holes and odd antiholes due to the Strong Perfect Graph Theorem.

Remark 2 In contrast to the Strong Perfect Graph Theorem, the validity of the Normal Graph Conjecture would provide us a *sufficient* condition for normality only, but no characterization: The non-existence of C_5, C_7, and $\overline{C}_7$ in a graph is not necessary for its normality. For example, the graphs G_1 and G_2 in Figure 2 are both normal (with the bold edges as clique covers and $\{\{1,3,5\},\{1,4,6\},\{2,4,5,7\}\}$ resp. $\{\{0,2,4,6\},\{0,3,5,7\},\{1,3,6\}\}$ as stable set covers) and so their complements are; but we have $C_5 \subset G_1$, $C_7 \subset G_2$, and $\overline{C}_7 \subset \overline{G}_2$.

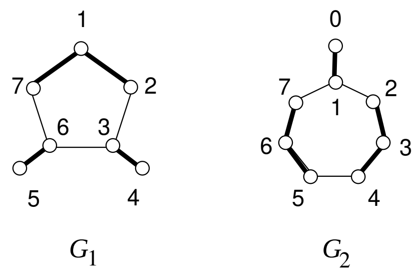


Figure 2: Normal graphs containing C_5 and C_7

Körner and de Simone [9] found a sufficient and necessary condition for the normality of connected triangle-free graphs: the existence of nice edge covers. (An edge cover $\mathcal{F}$ of a graph G is *nice* if it is minimal w.r.t. set inclusion and if every odd cycle C of G has at least three nodes which are incident to an even number of edges in $E(C) \cap \mathcal{F}$; the bold edges in Figure 2 form nice edge covers.) However, this result in [9] does *not* prove the Normal Graph Conjecture for triangle-free graphs. In this paper we verify the conjecture for a first graph class: the circulants which are a common generalization of odd holes and odd antiholes.

The paper is organized as follows: We introduce circulants in Section 2 and reformulate the Normal Graph Conjecture in an appropriate way in order to know for which circulants normality has to be shown. The main result is a characterization of all the normal circulants in Section 3. As a consequence we obtain that the Normal Graph Conjecture is true for circulant graphs. We conclude with some remarks and open problems.

2 The Normal Graph Conjecture for Circulants

We consider graphs with circular symmetry of their maximum cliques and stable sets, introduced by [4] as generalization of odd holes and odd antiholes. A *circulant* C_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge if i and j differ by at most k (mod n) and $i \neq j$.

We assume $k \geq 1$ and $n \geq 2(k+1)$ in the sequel in order to exclude the degenerated cases when C_n^k is a stable set or a clique. C_n^1 is a hole and C_{2k+1}^{k-1} an odd antihole for $k \geq 2$. The circulants C_9^k on nine nodes are depicted in Figure 3. The size of a maximum clique of the circulant C_n^k is $k+1$ and the size of a maximum stable set, called stability number α, is $\lfloor \frac{n}{k+1} \rfloor$. Unless stated otherwise, arithmetics are always performed modulo the number of nodes of the circulant involved in the computation.

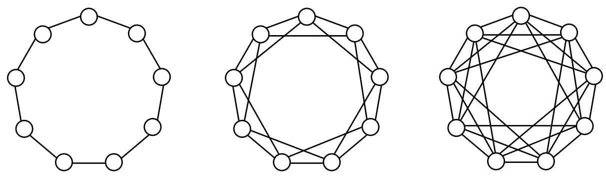


Figure 3: The circulants C_9^k

In order to prove the Normal Graph Conjecture for circulants, a first step is to figure out *which* circulants contain C_5, C_7, and $\overline{C}_7$ as induced subgraphs at all. This can be done with the help of the following result.

Theorem 3 (Trotter [13]) *$C_{n'}^{k'}$ is an induced subgraph of C_n^k if and only if $\frac{k+1}{k'+1}n' \leq n \leq \frac{k}{k'}n'$ holds.*

Remark 4 $C_{n'}^{k'} \subset C_n^k$ implies in particular $k' < k$ and $n' < n$ by [13].

Recalling $C_5 = C_5^1$, $C_7 = C_7^1$, and $\overline{C}_7 = C_7^2$, we immediately obtain from Theorem 3, for each fixed value of k, those numbers n of nodes s.t. C_n^k contains one of the subgraphs in question:

Lemma 5

(i) $C_5^1 \subseteq C_n^k$ *if and only if* $\frac{5(k+1)}{2} \leq n \leq 5k$.

(ii) $C_7^1 \subseteq C_n^k$ *if and only if* $\frac{7(k+1)}{2} \leq n \leq 7k$.

(iii) $C_7^2 \subseteq C_n^k$ *if and only if* $\frac{7(k+1)}{3} \leq n \leq \frac{7k}{2}$.

To illustrate the assertion of Lemma 5 look at the chart depicted in Figure 4. Every circle stands for a non-complete circulant C_n^k with the respective values of n and k. The dotted (resp. solid, resp. dashed) line indicates the range of circulants containing a $\overline{C}_7$ (resp. C_5, resp. C_7) as induced subgraph.

The overlap of the three ranges implies that, for each $k \geq 2$, the circulant C_n^k is $(C_5, C_7, \overline{C}_7)$-free whenever $n < \frac{7(k+1)}{3}$ (lower bound for $C_7^2 \subseteq C_n^k$) or $n > 7k$ (upper bound for $C_7^1 \subseteq C_n^k$), see the grey-shaded range in Figure 4. Since it is known from Körner [6] that all holes C_n^1 are normal if $n \neq 5, 7$ we obtain as reformulation of the Normal Graph Conjecture for circulants:

Conjecture 6 (Normal Graph Conjecture for Circulants)
All circulants C_n^k *with* $k \geq 2$ *are normal for* $n < \frac{7(k+1)}{3}$ *or* $n > 7k$.

Note that the circulants with $n < \frac{7(k+1)}{3}$ have stability number 2 by

$$\alpha(C_n^k) = \left\lfloor \frac{n}{k+1} \right\rfloor < \left\lfloor \frac{\frac{7(k+1)}{3}}{k+1} \right\rfloor = \left\lfloor \frac{7(k+1)}{3(k+1)} \right\rfloor = 2.$$

Hence we start in Subsection 3.1 with the case of circulants with $\alpha(C_n^k) = 2$ and generalize these ideas in Subsection 3.2 further for the general case. This provides us a characterization of all normal circulants which shows particularly that the non-normal circulants are not $(C_5, C_7, \overline{C}_7)$-free – and finally verifies the Normal Graph Conjecture for circulants.

3 The normal circulants

For the sequel of this section, we consider circulants C_n^k with $k \geq 2$ only (since the case of holes C_n^1 is trivial). In order to construct clique covers for normal graphs, we obviously need cliques maximal w.r.t set inclusion only. The following holds for maximal cliques of circulants:

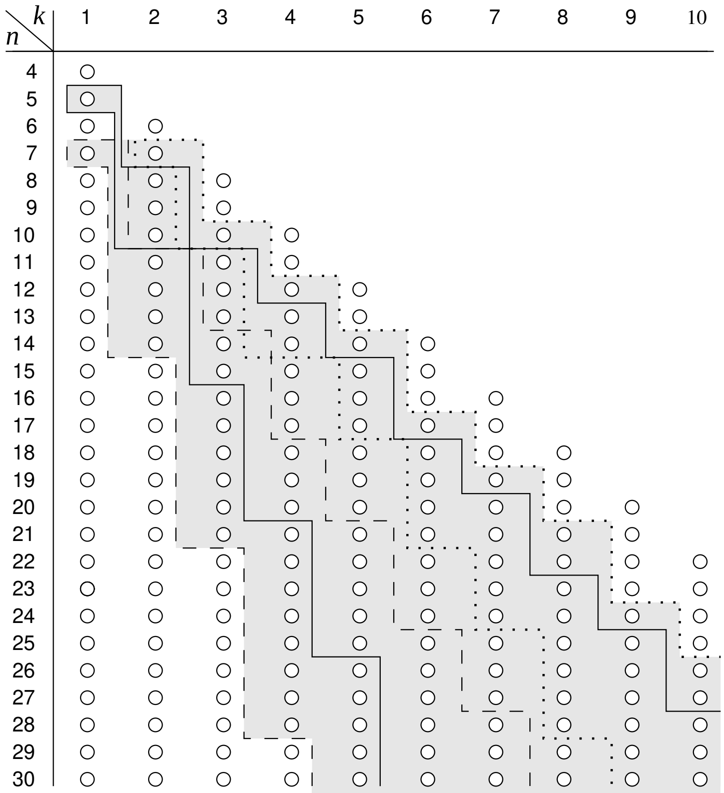


Figure 4: Circulants containing C_5, C_7, or $\overline{C}_7$

Lemma 7 *A circulant C_n^k with $k \geq 2$ has maximal cliques Q with $|Q| \leq k$ only if $n \leq 3k$.*

Proof. Consider a maximal clique $Q \subseteq C_n^k$ with $|Q| \leq k$. Then Q can neither be a single node nor a single edge, hence Q must contain a triangle x, y, z. Furthermore, Q is not contained in any maximum clique of C_n^k (by the maximality of Q), hence Q is not contained in any interval of $k+1 = \omega(C_n^k)$ consecutive nodes of C_n^k in particular. The neighborhood $N(i)$ of every node i of C_n^k splits into two cliques $N^-(i) = \{i-k, \ldots, i-1\}$ and $N^+(i) = \{i+1, \ldots, i+k\}$ where all indices are taken modulo n. This implies $y = x+d \in N^+(x)$ and $z = x-d' \in N^-(x)$ where $d+d'+1 > k+1$ holds, see Figure 5.

Thus $y, z \in Q$ implies $z \in N^+(y)$ and we conclude: $y \leq x+k$, $z \leq y+k \leq x+2k$, and $x \leq z+k \leq x+3k$ (where all indices are taken modulo

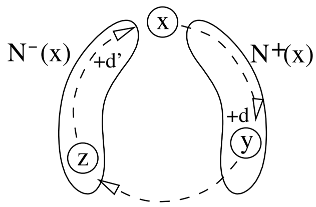


Figure 5: The neighborhood of x

n again). Consequently, $n \leq 3k$ is a necessary condition for x, y, z to form a triangle. This implies that maximal but non-maximum cliques can appear in circulants C_n^k with $n \leq 3k$ only. $\square$

3.1 The case $\alpha(C_n^k) = 2$

In this subsection, we characterize which circulants with stability number 2 are normal.

Lemma 8 *The circulants C_n^k with $n \leq 3k$ are normal.*

Proof. We show normality in this case by explicitely constructing a clique cover $\mathcal{Q}$ and a cross-intersecting stable set cover $\mathcal{S}$.

Choose a family of three pairwise intersecting maximum cliques, e.g. $\mathcal{Q} = \{Q^+(1), Q^+(k+1), Q^+(2k+1)\}$ where $Q^+(i) = \{i, \ldots, i+k\}$ and all indices are taken modulo n, see Figure 6 (recall $n \geq 2(k+1)$). By construction, we have

$$\begin{array}{rcl} Q^+(1) \cap Q^+(k+1) & = & \{k+1\}, \\ Q^+(k+1) \cap Q^+(2k+1) & = & \{2k+1\}, \\ Q^+(2k+1) \cap Q^+(1) & = & \{1, \ldots, 3k+1 \bmod n\} \end{array}$$

(thus the intersection contains in the latter case at least node 1 by $n \leq 3k$), see Figure 6. In particular, $\mathcal{Q}$ is a clique cover of C_n^k.

We construct a cross-intersecting stable set cover $\mathcal{S}$ by choosing as stable sets the nodes x belonging to one clique $Q \in \mathcal{Q}$ only together with the minimal non-neighbor of x in the intersection of the other two cliques in $\mathcal{Q} - Q$. Thus, we construct stable sets

$$S_x := \begin{cases} \{x, 2k+1\} & \text{if } x \in \{3k+2 \bmod n, \ldots, k\} \\ \{x, y\} & \text{if } x \in \{k+2, \ldots, 2k\} \\ \{x, k+1\} & \text{if } x \in \{2k+2, \ldots, n\} \end{cases}$$

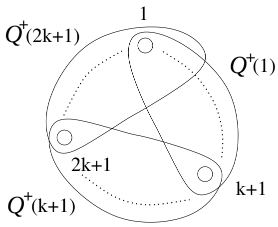


Figure 6: The clique cover of C_n^k with $n \leq 3k$

with $y \in \{1, \ldots, 3k+1 \bmod n\}$ minimal non-neighbor of x in $Q^+(2k+1) \cap Q^+(1)$ (i.e., $y = 1$ if $x \leq n-k$ up to $y = 3k+1 \bmod n$ if $x = 2k$), see Figure 6 again. By construction, every set S_x is stable and meets all three cliques in $\mathcal{Q}$. The union $\mathcal{S}$ of these sets S_x covers all nodes of C_n^k (this is obvious for the nodes $3k+2 \bmod n, \ldots, n$ and follows for $1, \ldots, 3k+1 \bmod n$ by choosing the minimal non-neighbor of x in $Q^+(2k+1) \cap Q^+(1)$). Hence $\mathcal{Q}$ and $\mathcal{S}$ are the set families showing normality of C_n^k with $n \leq 3k$. $\square$

Example 9 The set families $\mathcal{Q}$ and $\mathcal{S}$ for C_{11}^4 constructed as above are the following: $\mathcal{S} = \{\{3,9\},\{4,9\},\{6,1\},\{7,1\},\{8,2\},\{10,5\},\{11,5\}\}$ and $\mathcal{Q} = \{Q^+(1), Q^+(5), Q^+(9)\}$ (see Figure 7).

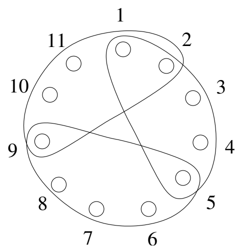


Figure 7: The clique cover of C_{11}^4

Due to $\frac{7(k+1)}{3} \leq 3k+1$, Lemma 8 proofs already one part of the Normal Graph Conjecture for circulants. The above construction, however, cannot be extended to the remaining cases of circulants with stability number 2:

Lemma 10 *For all $k \geq 2$, the circulants C_{3k+1}^k and C_{3k+2}^k are not normal.*

Proof. Assume in contrary C_n^k with $n \in \{3k+1, 3k+2\}$ has a clique cover $\mathcal{Q}$ and a cross-intersecting stable set cover $\mathcal{S}$. Lemma 7 implies that all maximal cliques of C_n^k have maximum size $k+1$ by $n > 3k$. Hence, $\mathcal{Q}$ consists of cliques $Q^+(i) = \{i, \ldots, i+k\}$ only.

In particular, there are two *disjoint* cliques Q and Q' in $\mathcal{Q}$: Otherwise, every two cliques of $\mathcal{Q}$ would intersect. For $|\mathcal{Q}| = 2$, we would obtain $n \leq 2k+1$, a contradiction to $k \geq 2$ and $n > 3k$. For $|\mathcal{Q}| \geq 3$, the cliques intersect pairwise only if $n \leq 3k$, a contradiction to $n > 3k$ again.

The two disjoint cliques Q and Q' cannot cover all nodes of C_n^k with $k \geq 2$ by $n \geq 3k+1 > 2(k+1) = |Q \cup Q'|$. Hence, there is a node $x \notin Q \cup Q'$. Due to $\alpha(C_n^k) = 2$, there is no stable set in $\mathcal{S}$ which contains x and meets both cliques Q and Q' (such a stable set had to contain three nodes). Thus, C_n^k with $n = 3k+1, 3k+2$ and $k \geq 2$ does not admit a clique cover and a cross-intersecting stable set cover and is, therefore, not normal. $\Box$

Remark 11 The complements $\overline{C}_n^k$ of the circulants with stability number 2 are triangle-free. Hence the complements are normal if and only if they admit a nice edge cover due to Körner and de Simone [9]. Lemma 8 constructs, therefore, stable set covers corresponding to nice edge covers in $\overline{C}_{\leq 3k}^k$ whereas Lemma 10 shows that $\overline{C}_{3k+1}^k$ and $\overline{C}_{3k+2}^k$ do not admit any nice edge cover. The proofs of the two lemmas are, however, easier than verifying directly whether nice edge covers exist in the complements.

3.2 The general case

We generalize the clique covers from the previous subsection to the case $n \geq 3k+3$. Note that in this case all maximal cliques of C_n^k have maximum size $k+1$ by Lemma 7. For $n \geq 3k+1$, there are no *pairwise* intersecting cliques covering all nodes, but *cyclic* intersecting cliques with that property.

We call a clique cover $\mathcal{Q} = \{Q_1, \ldots, Q_l\}$ of C_n^k *cyclic* if each clique Q_i has a non-empty intersection with precisely the cliques Q_{i-1} and Q_{i+1} (where the indices are taken modulo l).

A circulant C_n^k has in general several cyclic clique covers $\mathcal{Q} = \{Q_1, \ldots, Q_l\}$ of the same size consisting of maximum cliques, see Figure 8, namely as many as C_n^k has different holes C_l^1 as induced subgraphs: Denote by q_i the first node in Q_i, then q_i is adjacent to q_{i-1} and q_{i+1} but not to q_j with $i+1 < j < i-1 \bmod l$ by definition; thus $q_1, \ldots, q_l$ induce an l-hole in C_n^k. On the other hand, consider $C_l^1 \subseteq C_n^k$ with nodes $q_1, \ldots, q_l$ and the maximum cliques $Q^+(q_i) = \{q_i, \ldots, q_i + k\}$ of C_n^k starting in q_i. A result of Trotter [13] shows that $Q^+(q_i)$ contains precisely two nodes of C_l^1, namely q_i and q_{i+1}; thus $\mathcal{Q} = \{Q^+(q_1), \ldots, Q^+(q_l)\}$ is a cyclic clique cover. Hence

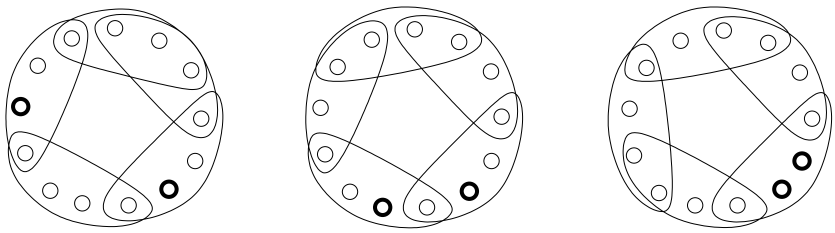


Figure 8: Different cyclic clique covers of size 5 in C_{13}^3

cyclic clique covers of size l and holes $C_l^1 \subseteq C_n^k$ correspond to each other and by Theorem 3 we have

$$C_l^1 \subseteq C_n^k \text{ iff } \frac{(k+1)}{2} l \leq n \leq \frac{k}{1} l$$

implying the following:

Lemma 12 *C_n^k with $k \geq 2$ admits, for each $t \geq 2$, a cyclic clique cover $\mathcal{Q}$ of size $2t-1$ iff*

$$\frac{(k+1)}{2}(2t-1) \leq n \leq \frac{k}{1}(2t-1).$$

Let $q(x, \mathcal{Q})$ stand for the number of cliques in $\mathcal{Q}$ containing node x. By the definition of $\mathcal{Q}$, we have $q(x, \mathcal{Q}) \in \{1, 2\}$ for all nodes x (since $\mathcal{Q}$ covers all nodes but no three cliques intersect). We call x a *1-node* (resp. *2-node*) w.r.t. $\mathcal{Q}$ if $q(x, \mathcal{Q}) = 1$ (resp. $q(x, \mathcal{Q}) = 2$) holds.

Note that, for a cyclic clique cover $\mathcal{Q}$ containing $2t-1$ cliques of size $k+1$ each, the lower bound $\frac{(k+1)}{2}(2t-1) \leq n$ in Lemma 12 is attained if there are 2-nodes only whereas the upper bound $n \leq \frac{k}{1}(2t-1)$ guarantees that there is at least one 2-node in the intersection of two consecutive cliques.

The next step is to construct cross-intersecting stable sets and to show that they cover all nodes. The idea goes as follows: $\mathcal{Q}$ contains $2t-1$ cliques. If a 1-node x belongs to $Q \in \mathcal{Q}$, then $\mathcal{Q} - Q$ consists of $2t-2$ cliques or, in other words, of $t-1$ pairs of intersecting cliques. We denote by $S(x, \mathcal{Q})$ a t-set containing x and one node from the intersection of the $t-1$ pairs of cliques, see Figure 9. Thus $S(x, \mathcal{Q})$ intersects all cliques in $\mathcal{Q}$ by construction; we have to show that there exist *stable* sets $S(x, \mathcal{Q})$ whose union covers all nodes of C_n^k.

Lemma 13 *Consider a cyclic clique cover $\mathcal{Q}$ of C_n^k of size $2t-1$ where $t(k+1) \le n \le (2t-1)k$ and $k,t \ge 2$. For each 1-node x w.r.t. $\mathcal{Q}$ of C_n^k there is a stable set $S(x,\mathcal{Q})$ of size t in C_n^k.*

Proof. C_n^k admits by $\frac{(2t-1)(k+1)}{2} < t(k+1)$ a cyclic clique cover $\mathcal{Q} = \{Q_1, \ldots, Q_{2t-1}\}$ due to Lemma 12. Furthermore, $t(k+1) \le n$ guarantees that C_n^k contains stable sets of size t by $t \le \alpha(C_n^k) = \lfloor \frac{n}{k+1} \rfloor$.

Consider a 1-node x of C_n^k and assume w.l.o.g. that x belongs to $Q_1 \in \mathcal{Q}$. We construct a stable set $S(x,\mathcal{Q}) = \{x, x_1, \ldots, x_{t-1}\}$ s.t. $x_i \in Q_{2i} \cap Q_{2i+1}$ for $1 \le i \le t-1$, see Figure 9.

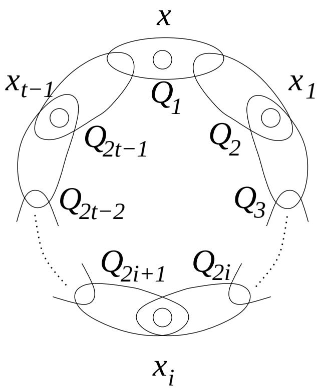


Figure 9: Constructing the stable set $S(x,\mathcal{Q})$ for $x \in Q_1$

Since $x \in Q_1 - Q_2$, there is a non-neighbor of x in $Q_2 \cap Q_3$ (at least the last node in Q_2 is not adjacent to x but belongs to Q_3). We choose $x_1 = x + (k+1) + d_1 \in Q_2 \cap Q_3$ with $d_1 \in \mathbb{N} \cup \{0\}$ minimal.

In order to construct x_i from x_{i-1} for $2 \le i \le t-1$, notice that we have $x_{i-1} \in Q_{2i-2} \cap Q_{2i-1}$, in particular $x_{i-1} \in Q_{2i-1} - Q_{2i}$. As before, there is a non-neighbor of x_{i-1} in $Q_{2i} \cap Q_{2i+1}$ and we choose $x_i = x_{i-1} + (k+1) + d_i \in Q_{2i} \cap Q_{2i+1}$ with $d_i \in \mathbb{N} \cup \{0\}$ minimal.

Then $S(x,\mathcal{Q})$ is a stable set if x_{t-1} and x are non-adjacent (all other nodes are non-adjacent by construction).

If $d_i = 0$ for $1 \le i \le t-1$, then $x_{t-1} = x + (t-1)(k+1)$. Hence, there are at least $k+1$ nodes between x_{t-1} and x (in increasing order modulo n) due to $n \ge t(k+1)$ and we are done. Otherwise, let j be the smallest index s.t. $d_j > 0$. Then x_j is the first node in Q_{2j+1} since we choose d_j minimal: By $x_{j-1} \notin Q_{2j}$, we have $x_{j-1} + (k+1) \in Q_{2j}$. The only reason for choosing $d_j > 0$ was, therefore, $x_{j-1} + (k+1) + d'_j \notin Q_{2j+1}$ for all $0 \le d'_j < d_j$ by the minimality of d_j. Hence, x_j is indeed the first node in Q_{2j+1}. This implies that its first non-neighbor is the node $x_j+(k+1)$ belonging to $Q_{2j+2}-Q_{2j+1}$ and $x_{j+1} = x_j + (k+1) + d_{j+1} \in Q_{2j+2} \cap Q_{2j+3}$ is, by the minimality of d_{j+1}, the first node of Q_{2j+3}. The same argumentation shows that *every* further x_i with $i > j+1$ is the first node in Q_{2i+1}; in particular, x_{t-1} is the first node of Q_{2t-1}. Hence, $x \in Q_1 - Q_{2t-1}$ shows that x_{t-1} and x are non-adjacent. Thus $S(x, \mathcal{Q}) = \{x, x_1, \ldots, x_{t-1}\}$ is a stable set of size t and intersects all cliques of $\mathcal{Q}$ by $x \in Q_1$ and $x_i \in Q_{2i} \cap Q_{2i+1}$ for $1 \leq i \leq t-1$. □

Remark 14 Lemma 13 implies that there is, for each 1-node x, *at least* one stable set $S(x, \mathcal{Q})$. In general, there may be several such sets: Consider, e.g., the cyclic clique covers $\mathcal{Q}$ of C_{13}^3 indicated in Figure 8. There are *two* stable sets $S(x, \mathcal{Q})$ for the bold 1-nodes x.

It is left to show that the union of *all* stable sets $S(x, \mathcal{Q})$ covers the circulant.

Lemma 15 *Consider a cyclic clique cover $\mathcal{Q}$ of C_n^k of size $2t-1$ where $t(k+1) \leq n \leq (2t-1)k$ and $k, t \geq 2$. Then the union $\mathcal{S}$ of the stable sets $S(x, \mathcal{Q})$, where x is a 1-node of C_n^k w.r.t $\mathcal{Q}$, covers all nodes of C_n^k.*

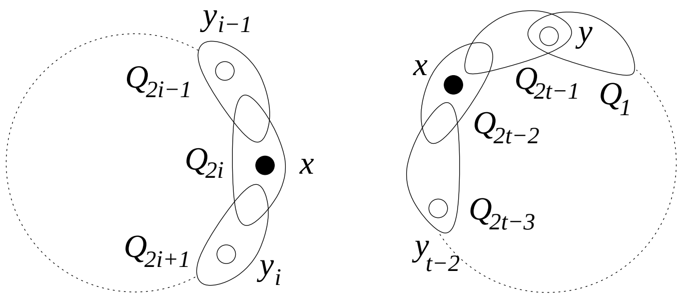
Proof. Assume in contrary that there is a node y in C_n^k not covered by $\mathcal{S}$. Then there is no stable set $S(x, \mathcal{Q})$ with $y \in S(x, \mathcal{Q})$. In particular, y is a 2-node w.r.t. $\mathcal{Q}$ by Lemma 13. W.l.o.g. let $y \in Q_1 \cap Q_{2t-1}$. We first show $y_l = y + l(k+1) \in Q_{2l+1}$ for $0 \leq l \leq t-2$. Clearly, we have $y = y_0 = y + 0(k+1) \in Q_1$ by assumption and prove that $y_{i-1} = y+(i-1)(k+1) \in Q_{2i-1}$ implies $y_i = y + i(k+1) \in Q_{2i+1}$ for $1 \leq i \leq t-2$.

If there is a 1-node x in $Q_{2i} \backslash (Q_{2i-1} \cup Q_{2i+1})$, then x is adjacent to $y_{i-1} = y + (i-1)(k+1)$, see Figure 10(a) (otherwise, there is a stable set $S(x, \mathcal{Q})$ containing x and $y_0, \ldots, y_{i-1}$ in contradiction to our assumption) and $x < y + i(k+1)$ yields $y_i = y + i(k+1) \in Q_{2i+1}$.

If $Q_{2i} \backslash (Q_{2i-1} \cup Q_{2i+1}) = \emptyset$, then $y_i = y + i(k+1)$ clearly belongs to Q_{2i+1} (since we have $y_{i-1} = y+(i-1)(k+1) \in Q_{2i-1}$ and $|Q_{2i-1}| = k+1$).

In particular, we have $y_{t-2} = y+(t-2)(k+1) \in Q_{2t-3}$. Any 1-node x in $Q_{2t-2} \backslash (Q_{2t-3} \cup Q_{2t-1})$ is adjacent to y_{t-2} or to y, see Figure 10(b) (otherwise, x together with $y_0, \ldots, y_{t-2}$ would be a set $S(x, \mathcal{Q}) \in \mathcal{S}$ in contradiction to our assumption). We distinguish three cases:

If x is adjacent to y, then $x > y - (k+1) = y_{-1}$ follows and y_{-1} is adjacent to y_{t-2}: either y_{-1} belongs to Q_{2t-3} or is as 1-node adjacent to y_{t-2}; thus, $y_{-1} = y - (k+1) \leq y + (t-2)(k+1) + k = y_{t-2} + k$ implies $y \leq y + (t-1)(k+1) + k$.



(a) (b)

Figure 10: Constructing the nodes $y_i \in Q_{2i+1}$

If x is adjacent to y_{t-2}, we obtain $x < y_{t-2} + (k+1) = y_{t-1}$ and y_{t-1} either belongs to Q_{2t-1} or is a 1-node adjacent to y; here, $y_{t-1} \geq y - k$ and, therefore, $y + (t-1)(k+1) \geq y - k$ holds.

The non-existence of a 1-node in Q_{2t-2} implies $y_{t-1} \in Q_{2t-2}$ and, therefore, $y_{t-1} \geq y - k$ follows again.

All three cases imply $n \leq (t-1)(k+1) + k$. By $n \geq t(k+1)$, we obtain

$$t(k+1) \leq (t-1)(k+1) + k$$

yielding the final contradiction. Hence the union $\mathcal{S}$ of the stable sets $S(x, \mathcal{Q})$, where x is a 1-node of C_n^k w.r.t $\mathcal{Q}$, covers all nodes of C_n^k. $\square$

Since each $S(x, \mathcal{Q})$ meets all cliques in $\mathcal{Q}$ by construction, $\mathcal{S}$ is the required cross-intersecting stable set cover. Thus Lemma 12, Lemma 13, and Lemma 15 together imply:

Theorem 16 *A circulant C_n^k with $k \geq 2$ admits for each $t \geq 2$*

- *a cyclic clique cover $\mathcal{Q}$ of size $2t-1$ and*
- *a cross-intersecting stable set cover $\mathcal{S}$ of stable t-sets*

if $t(k+1) \leq n \leq (2t-1)k$ holds.

Remark 17 Theorem 16 includes the case $t = 2$ from Lemma 8: every circulant C_n^k with $n \leq 3k$ has a cyclic clique cover of size 3 and a cross-intersecting cover of stable 2-sets.

Theorem 16 shows that circulants C_n^k are normal for certain numbers n of nodes – depending on the values of k and t. We finally have to check whether, for fixed k, there are gaps between the ranges $t(k+1) \le n \le (2t-1)k$ and $(t+1)(k+1) \le n \le (2t+1)k$ of normal circulants for different values of $t \ge 2$, see Figure 11. The dotted (resp. solid, resp. dashed) line indicates the range for $t = 2$ (resp. $t = 3$, resp. $t = 4$). There is *no* gap between the two ranges if

$$\begin{array}{rcl} (t+1)(k+1) & \le & (2t-1)k+1 \\ tk+t+k+1 & \le & 2tk-k+1 \\ \frac{2k}{k-1} & \le & t. \end{array}$$

This is true for $k = 2$ if $t \ge 4$ and for $k \ge 3$ if $t \ge 3$, hence we obtain:

Lemma 18 *The ranges $t(k+1) \le n \le (2t-1)k$ and $(t+1)(k+1) \le n \le (2t+1)k$ overlap for $k = 2$ if $t \ge 4$ and for $k \ge 3$ if $t \ge 3$.*

Thus Theorem 16 and Lemma 18 show the normality for all circulants C_n^k with $k \ge 2$ *except* the cases $n = 3k+1, 3k+2$ (gap between the ranges for $t = 2$ and $t = 3$) and C_{11}^2 (gap between the ranges for $t = 3$ and $t = 4$), see Figure 11. We proved in Lemma 10 that C_{3k+1}^k and C_{3k+2}^k are not normal for all $k \ge 2$; in order to characterize completely which circulants are normal and which are not, it is only left to check C_{11}^2.

Lemma 19 *The circulant C_{11}^2 is not normal.*

Proof. Assume in contrary C_{11}^2 admits a clique cover $\mathcal{Q}$ and a cross-intersecting stable set cover $\mathcal{S}$. Lemma 7 implies that all maximal cliques of C_{11}^2 are of the form $Q^+(i) = \{i, i+1, i+2\}$.

First, $\mathcal{Q}$ must not contain three *disjoint* cliques Q, Q', Q'': These three cliques cannot cover all nodes of C_{11}^2 by $11 > 9 = |Q \cup Q' \cup Q''|$. Hence, there is a node $x \notin Q \cup Q' \cup Q''$ but due to $\alpha(C_{11}^2) = 3$, there is no stable set in $\mathcal{S}$ containing x and meeting all three cliques Q, Q', Q''.

Thus $\mathcal{Q}$ contains at most two disjoint cliques Q and Q'. They must not be consecutive (see Figure 12(a)): If $Q = \{1, 2, 3\}$ and $Q' = \{4, 5, 6\}$ we need a clique Q'' among $\{7, 8, 9\}$, $\{8, 9, 10\}$, $\{9, 10, 11\}$ in order to cover node 9, but Q, Q', Q'' would be disjoint in any case.

There is no possibility for avoiding two consecutive disjoint cliques: Suppose $Q = \{1, 2, 3\}$. If $Q' = \{5, 6, 7\}$ (Figure 12(b)), then $\{8, 9, 10\}$, $\{9, 10, 11\}$ $\notin \mathcal{Q}$ follows. But in order to cover the nodes 9 and 10, we need the two consecutive cliques $\{7, 8, 9\}$ and $\{10, 11, 1\}$. If $Q' = \{6, 7, 8\}$ (Figure 12(c)), then $\{4, 5, 6\}$, $\{3, 4, 5\} \notin \mathcal{Q}$ follows. Thus we need the two consecutive cliques

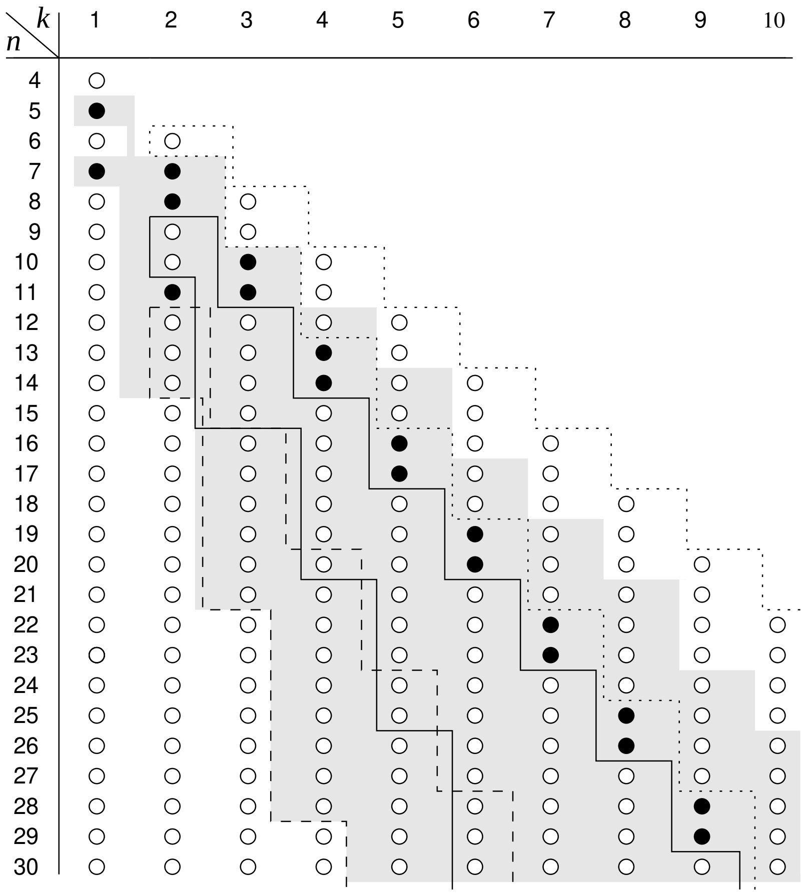


Figure 11: The ranges of normal circulants for $t = 2, 3, 4$

$\{2, 3, 4\}$ and $\{5, 6, 7\}$ in order to cover the nodes 4 and 5 which yields the final contradiction. $\square$

As a consequence, we are able to characterize all the normal circulants:

Theorem 20 *A circulant C_n^k is normal if and only if*

- $k = 1$ *and* $n \neq 5, 7$,
- $k = 2$ *and* $n \neq 7, 8, 11$,
- $k \geq 3$ *and* $n \neq 3k + 1, 3k + 2$.

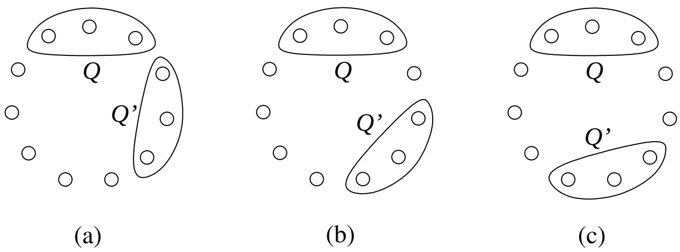


Figure 12: Two disjoint cliques Q and Q' in C_{11}^2

Since all the non-normal circulants are not $(C_5, C_7, \overline{C}_7)$-free (see the black circles in Figure 11), Theorem 20 finally verifies the Normal Graph Conjecture for circulants; since normality is closed under taking complements, we obtain the same assertion for the complementary class:

Corollary 21 *The Normal Graph Conjecture is true for circulants and their complements.*

4 Concluding Remarks

In this paper we verified the Normal Graph Conjecture for a first graph class, the circulants, and for their complements. Next steps could be to treat the conjecture for partitionable graphs (as a different generalization of odd holes and odd antiholes) and for circular-arc graphs (as generalization of circulants).

However, even proving the Normal Graph Conjecture in general would not yield a *characterization* of normal graphs, see Remark 2, since induced subgraphs of normal graphs are not necessarily normal. In order to get a better analogy with perfect graphs, Körner and de Simone [9] introduced a hereditary property by defining *strongly normal graphs* as those normal graphs whose induced subgraphs are all normal. In terms of strongly normal graphs the Normal Graph Conjecture is equivalent to the following:

Conjecture 22 (Strongly Normal Graph Conjecture [9]) *A graph G is strongly normal if and only if neither G nor its complement $\overline{G}$ contain any C_5 or C_7 as induced subgraph.*

The interest of this conjecture lies in the fact that it would immediately lead to a polynomial time recognition algorithm for strongly normal graphs.

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