Packing Steiner Trees: Polyhedral Investigations

M. Grötschel A. Martin R. Weismantel

Abstract

Let $G = (V, E)$ be a graph and $T \subseteq V$ be a node set. We call an edge set S a Steiner tree with respect to T if S connects all pairs of nodes in T. In this paper we address the following problem, which we call the weighted Steiner tree packing problem. Given a graph $G = (V, E)$ with edge weights w_e, edge capacities $c_e, e \in E$, and node sets $T_1, \ldots, T_N$, find edge sets $S_1, \ldots, S_N$ such that each S_k is a Steiner tree with respect to T_k, at most c_e of these edge sets use edge e for each $e \in E$, and such that the sum of the weights of the edge sets is minimal. Our motivation for studying this problem arises from the routing problem in VLSI-design, where given sets of points have to be connected by wires. We consider the Steiner tree packing problem from a polyhedral point of view and define an appropriate polyhedron, called the Steiner tree packing polyhedron. The goal of this paper is to (partially) describe this polyhedron by means of inequalities. It turns out that, under mild assumptions, each inequality that defines a facet for the (single) Steiner tree polyhedron can be lifted to a facet-defining inequality for the Steiner tree packing polyhedron. The main emphasis of this paper lies on the presentation of so-called joint inequalities that are valid and facet-defining for this polyhedron. Inequalities of this kind involve at least two Steiner trees. The classes of inequalities we have found form the basis of a branch & cut algorithm. This algorithm is described in our companion paper [GMW92].

1 Introduction

Given a graph $G = (V, E)$ and a node set $T \subseteq V$, we call an edge set $S \subseteq E$ a *Steiner tree for* T if, for each pair of nodes $u, v \in T$, S contains a $[u, v]$-path. In this paper we investigate the following problem that we call the *Steiner tree packing problem*. Given an undirected graph $G = (V, E)$ with edge capacities $c_e \in \mathbb{N}$ for all $e \in E$ and a list of node sets $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \in \mathbb{N}$, find Steiner trees S_k for T_k, $k = 1, \ldots, N$ such that each edge $e \in E$ is contained in at most c_e of the edge sets $S_1, \ldots, S_N$. Every collection of Steiner trees $S_1, \ldots, S_N$

with this property is called a Steiner tree packing. If a weighting of the edges is given in addition and a (with respect to this weighting) minimal Steiner tree packing must be found, we call this the weighted Steiner tree packing problem.

This problem has important applications in the layout of electronic circuits. One of the major tasks in VLSI-design is the so-called routing problem. Roughly speaking, this problem can be stated as follows. Given an area (typically a rectangle with some "forbidden zones") and a list of point sets (so-called nets). The routing problem is to connect (route) the points of each net by wires on the area such that certain technical side constraints are satisfied and some objective function is minimized. The precise formulation of the routing problem depends on the used technology and the given design rules. Many variants of the routing problem, however, can be modelled as weighted Steiner tree packing problems. In a companion paper [GMW92] we are going to discuss such modelling issues and the relation between the routing and the Steiner tree packing problem in detail.

In this paper we consider the Steiner tree packing problem from a polyhedral point of view. We define a polyhedron whose vertices are in a one–to–one correspondence to the Steiner tree packings in the graph. The goal of the paper is to investigate this polyhedron, i. e., we try to describe it (partially) by means of equations and inequalities. The classes of inequalities we have found form the basis of a branch & cut algorithm for the (weighted) Steiner tree packing problem. This algorithm, the associated separation routines and computational results are described in our companion paper [GMW92].

This paper is organized as follows. In section 2 we list some graphtheoretic concepts and notation and give a formal definition of the (weighted) Steiner tree packing problem. In section 3 we introduce the Steiner tree packing polyhedron and investigate its trivial facet-defining inequalities. In section 4 we address the question how facet-defining inequalities change if the underlying graph is modified by operations such as edge deletion or node contraction. In section 5 we show that under certain conditions each facet-defining inequality for the Steiner tree polyhedron can be lifted to a facet-defining inequality for the packing polyhedron. Finally, we present several classes of so-called joint facets in section 6. Inequalities of this kind involve at least two Steiner trees.

2 Definitions and Notation

In this section we describe the problem that will be considered in this paper formally. We first sketch some graphtheoretic notation.
We denote graphs by $G = (V, E)$, where V is the node set and E the edge set. All graphs we consider are undirected and finite. For a given edge set $F \subseteq E$, we denote by $V(F)$ all nodes that are incident to an edge in F. Given two node sets $U, W \subseteq V$, we denote by $[U : W]$ the set of edges in G with one endnode in U

and the other in W. For a node set W, we also use $E(W)$ instead of $[W : W]$. A set of node sets $V_1, \ldots, V_p \subset V$, $p \geq 2$, is called a *partition of* V if all sets V_i are nonempty, the node sets are mutually disjoint and the union of these sets is V. (Note that we use "$\subset$" to denote strict set theoretic containment.) If $V_1, \ldots, V_p$ is a partition of V then $\delta(V_1, \ldots, V_p)$ denotes the set of edges in G whose endnodes are in different sets. For $W \subset V$, $W \neq \emptyset$, we write $\delta(W)$ instead of $\delta(W, V \setminus W)$ and call this set the *cut induced by* W. If $W = \{v\}$, we abbreviate $\delta(\{v\})$ by $\delta(v)$. For an edge set F, we define $d_F(v) = |\delta(v) \cap F|$, this is the degree of v in the subgraph (V, F) of G.

We call a sequence of nodes and edges $K = (v_0, e_1, v_1, e_2, \ldots, v_{l-1}, e_l, v_l)$, where each edge e_i is incident with the nodes v_{i-1} and v_i for $i = 1, \ldots, l$, and where the edges and nodes are pairwise disjoint (except possibly v_0 and v_l), a *path* (or a $[v_0, v_l]$*-path*), if $v_0 \neq v_l$, and a *cycle*, if $v_0 = v_l$ and $l \geq 2$. Each edge that connects two nodes of a cycle (path) K and that is not in K is called a *diagonal of* K. We say that two edges uv and $u'v'$ *cross with respect to* K if they appear in the sequence u, u', v, v' or u, v', v, u' by walking along the cycle (path). Similiarily, we call two sets of diagonals F_1 and F_2 *cross free* if, for all $e_1 \in F_1$ and $e_2 \in F_2$, e_1 and e_2 do not cross. Otherwise, F_1 and F_2 are *crossing*. For our purposes it is convenient to consider a path P or a cycle C, respectively, as a subset of the edge set. We call an edge set B a *tree* if $(V(B), B)$ is connected and contains no cycle. The *leaves* of B are the nodes that are incident to exactly one edge of B.

Finally, we call a graph G a *complete rectangular* $h \times b$ *grid graph*, if it can be embedded in the plane by h horizontal lines and b vertical lines such that the nodes of V are represented by the intersections of the lines and the edges are represented by the connections of the intersections. A *column* J (*row* J) of a complete rectangular $h \times b$ grid graph is a subset of the edge set that has cardinality $h-1$ $(b-1)$ and whose edges correspond to the same vertical (horizontal) line.

Definition 2.1 *Let $G = (V, E)$ be a graph and $T \subseteq V$ a node set of G. An edge set S is called a* **Steiner tree for** *T, if the subgraph $(V(S), S)$ contains a path from s to t for all pairs of nodes $s, t \in T, s \neq t$.*

Definition 2.1 differs from the terminology most frequently used in the literature. A Steiner tree is usually supposed to be a tree. For our purposes, however, Definition 2.1 simplyfies notation and is more convenient for the polyhedral investigations in the following. A Steiner tree that is a tree and whose leaves are terminals is called *edge-minimal*.

Using the above notation we define the *Steiner tree packing problem* as follows.

Problem 2.2 (The Steiner tree packing problem)

Instance:

A graph $G = (V, E)$ with positive, integer capacities $c_e \in \mathbb{N}$, $e \in E$.
A list of node sets $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \geq 1$, with $T_k \subseteq V$ for all $k = 1, \ldots, N$.

Problem:

Find edge sets $S_1, \ldots, S_N \subseteq E$ such that

(i) S_k is a Steiner tree in G for T_k for all $k = 1, \ldots, N$,

(ii) $\sum_{k=1}^{N} |S_k \cap \{e\}| \leq c_e$ for all $e \in E$.

In the application of Problem 2.2 we have in mind it is usual to call the list of node sets $\mathcal{N}$ a *net list*. We follow this custom. The number N denotes the cardinality of the net list. Any element $T_k \in \mathcal{N}$ is called a *set of terminals* or a *net* and the nodes $t \in T_k$ are called *terminals*. Instead of net T_k we will often simply say *net k*.

For notational reasons it is convenient to order the sets S_k, thus, we call an N-tupel $(S_1, \ldots, S_N)$ of edge sets a *Steiner tree packing* or *packing of Steiner trees* if the sets $S_1, \ldots, S_N$ form a solution of Problem 2.2. A Steiner tree packing $(S_1, \ldots, S_N)$ is called *edge-minimal*, if each S_k is edge-minimal.

We will also consider the following weighted variant of the Steiner tree packing problem.

Problem 2.3 (The weighted Steiner tree packing problem)

Instance:

A graph $G = (V, E)$ with positive, integer capacities $c_e \in \mathbb{N}$ and nonnegative weights $w_e \in \mathbb{R}_+$, $e \in E$.
A net list $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \geq 1$ with $T_k \subseteq V$ for all $k = 1, \ldots, N$.

Problem:

Find edge sets $S_1, \ldots, S_N \subseteq E$ such that

(i) $(S_1, \ldots, S_N)$ is a Steiner tree packing,

(ii) $\sum_{k=1}^{N} \sum_{e \in S_k} w_e$ is minimal.

In the following we will refer to an instance of the weighted Steiner tree packing problem by $(G, \mathcal{N}, c, w)$ and to an instance of the Steiner tree packing problem by $(G, \mathcal{N}, c)$.

It is not surprising that Problem 2.2 and Problem 2.3 are $\mathcal{NP}$-complete or $\mathcal{NP}$-hard, respectively, even in special cases. For example, the following variants are hard.

If we restrict Problem 2.3 to $N = 1$ and $c_e = 1$, for all $e \in E$, we obtain the problem of finding a minimal Steiner tree in G. This problem is $\mathcal{NP}$-hard even if G is restricted to be planar or a grid graph ([K72], [GJ77]). Futhermore, it is $\mathcal{NP}$-complete to decide whether there exists a feasible solution for Problem 2.2. Results here are due to Kramer and van Leeuwen [KL84], who proved that the problem of finding N edge-disjoint paths is $\mathcal{NP}$-complete. Similiarily, it was shown in [KPS90] that it is $\mathcal{NP}$-complete to decide whether a packing of two Steiner trees exists.

We close this section with some further definitions and notation frequently used throughout this paper.
Let $G = (V, E)$ be a graph and T a set of terminals. We call an edge e a *Steiner bridge with respect to* T, if every Steiner tree for T in G contains e. For a Steiner tree S for T in G, we define

$$\Upsilon(S) := \sum_{t\in T}(d_S(t) - 1) + \sum_{\substack{v\in V\setminus T\\ d_S(v)>2}} (d_S(v) - 2).$$

It is easy to see that if S is an edge-minimal Steiner tree the following equation holds,

$$\Upsilon(S) = |T| - 2. \tag{2.4}$$

Let S be an edge-minimal Steiner tree for T in G and let $uv \in S$ be an edge of S. If u and v are not leaves of S, then there exist edge sets $S_1, S_2 \subset S$, $S_1 \cap S_2 = \emptyset$ such that $S = S_1 \cup S_2 \cup \{uv\}$ with $u \in V(S_1), v \in V(S_2)$ and $(V(S_i), S_i)$, $i = 1, 2$ are connected. We call $S_1 \cup S_2 \cup \{uv\}$ an *edge-disjoint dissection of* S. If one of the endnodes of the edge uv, say u, is a leaf of S, we also call $S_1 \cup S_2 \cup \{uv\}$ an edge-disjoint dissection of S, where $S_1 = \emptyset$ and $S_2 = S \setminus \{uv\}$. It is particularly convenient in this case to set $V(S_1) := u$ and write $u \in V(S_1)$.
To make our unavoidably complicated notation a little less clumsy we slightly abuse standard notation and introduce the following technically useful operations on N-tupels of edge sets. Let $P = (F_1, \ldots, F_N), N \geq 1$, be an N-tupel of edge sets and $e \in E, F \subseteq E$. We define

$$\begin{aligned}
P \setminus_k e \; &:= (F_1, \ldots, F_k \setminus \{e\}, \ldots, F_N);\\
P \cup_k e \; &:= (F_1, \ldots, F_k \cup \{e\}, \ldots, F_N);\\
P \setminus e \; &:= (F_1 \setminus \{e\}, \ldots, F_N \setminus \{e\});\\
P \cup e \; &:= (F_1 \cup \{e\}, \ldots, F_N \cup \{e\});\\
e \in P \; &\Leftrightarrow e \in \cup_{k=1}^N F_k;\\
P \subseteq F \; &\Leftrightarrow \cup_{k=1}^N F_k \subseteq F.
\end{aligned}$$

If we have a Steiner tree packing $P = (S_1, \ldots, S_N)$ it is sometimes convenient to denote the k-th element S_k of the N-tuple by P_k. We call a net list $\mathcal{N} =$

$\{T_1, \ldots, T_N\}$ *disjoint*, if $T_i \cap T_j = \emptyset$ for all $i, j \in \{1, \ldots, N\}$, $i \neq j$. If W is a subset of V, we write $\mathcal{N} \subseteq W$, if $\cup_{k=1}^N T_k \subseteq W$.

To avoid the discussion of (trivial) special cases we assume from now on that every terminal set of a net list $\mathcal{N}$ has at least cardinality two and that $N \geq 1$.

3 The Steiner Tree Packing Polyhedron: Some Basic Results

In this section we introduce the polyhedron we are going to study. We assume the reader to be familiar with polyhedral theory, see, for instance [S86].

Suppose we are given a Steiner tree packing problem by a graph $G = (V, E)$ with edge capacities $c_e \in \mathbb{N}, e \in E$, and a net list $\mathcal{N} = \{T_1, \ldots, T_N\}$.

Vectors are considered as column vectors unless otherwise specified. The superscript "T" denotes transposition. We denote by $\mathbb{R}^E$ the vector space where the components of each vector are indexed by the elements of E, i. e., $x = (x_e)_{e \in E}$ for $x \in \mathbb{R}^E$. For an edge set $F \subseteq E$ we define the *incidence vector* $\chi^F \in \mathbb{R}^E$ *of* F by setting $\chi_e^F = 1$, if $e \in F$, and $\chi_e^F = 0$, otherwise. On the other hand, for each 0/1-vector $x \in \mathbb{R}^E$ the set $F_x := \{e \in E \mid x_e = 1\}$ denotes the *incidence set of* x. For an edge set $F \subseteq E$ and a vector $x \in \mathbb{R}^E$, we will often abbreviate $\sum_{e \in F} x_e$ by $x(F)$.
In addition, we will consider in this paper the $N \cdot |E|$ – dimensional vector space $\mathbb{R}^E \times \ldots \times \mathbb{R}^E$. We will denote this vector space by $\mathbb{R}^{\mathcal{N} \times E}$. The components of a vector $x \in \mathbb{R}^{\mathcal{N} \times E}$ are indexed by x_e^k for $k \in \{1, \ldots, N\}$, $e \in E$. For a vector $x \in \mathbb{R}^{\mathcal{N} \times E}$ and $k \in \{1, \ldots, N\}$ we denote by $x^k \in \mathbb{R}^E$ the vector $(x_e^k)_{e \in E}$. Instead of $x = ((x^1)^T, \ldots, (x^N)^T)^T \in \mathbb{R}^{\mathcal{N} \times E}$ we often write $x = (x^1, \ldots, x^N)$ if the meaning of the symbols is clear from the context. We define the *support of a vector* $a \in \mathbb{R}^{\mathcal{N} \times E}$ *in* E by $\operatorname{supp}(a) := \{e \in E \mid a_e^k \neq 0 \text{ for some } k \in \{1, \ldots, N\}\}$. For a subset $E' \subseteq E$ and a vector $a \in \mathbb{R}^{\mathcal{N} \times E}$ we define the vector $a|_{E'} \in \mathbb{R}^{\mathcal{N} \times E'}$ by $(a|_{E'})_e^k := a_e^k$ for all $k = 1, \ldots, N$ and $e \in E'$. Finally, for a subset S of a vector space, we denote by $\dim(S)$ the dimension of S and by $\operatorname{diff}(S) := \{x - y \mid x, y \in S\}$ the *difference set of* S.
We define now the *Steiner tree packing polyhedron* by

$$
\begin{aligned}
\operatorname{STP}(G, \mathcal{N}, c) := \operatorname{conv}\{\, (\chi^{S_1}, \ldots, \chi^{S_N}) \in \mathbb{R}^{\mathcal{N} \times E} \quad | \qquad & \\
(i) \quad & S_k \text{ is a Steiner tree for } T_k \text{ in } G \\
& \text{for } k = 1, \ldots, N; \\
(ii) \quad & \sum_{k=1}^{N} |S_k \cap \{e\}| \leq c_e, \text{ for all } e \in E\}.
\end{aligned}
\tag{3.1}
$$

If $N = 1$ and $c = \mathbb{1}$, i. e., $c_e = 1$ for all $e \in E$, we also refer to $\operatorname{STP}(G, \mathcal{N}, c)$

as the *Steiner tree polyhedron.* We call the vector $(\chi^{S_1}, \ldots, \chi^{S_N})$ the *incidence vector of a Steiner tree packing* $P = (S_1, \ldots, S_N)$. We will often abreviate the incidence vector of a Steiner tree packing P by χ^P. STP $(G, \mathcal{N}, c)$ is the convex hull of the incidence vectors of Steiner tree packings.

Alternatively, the Steiner tree packing polyhedron can be formulated as the solution set of an integer program as follows. Consider the following system of inequalities.

$$
(3.2)\quad
\begin{array}{lll}
\text{(i)} & \displaystyle\sum_{e\in\delta(W)} x_e^k \geq 1, & \text{for all } W \subset V,\ W \cap T_k \neq \emptyset,\ (V \setminus W) \cap T_k \neq \emptyset, \\
 & & k = 1, \ldots, N. \\
\text{(ii)} & \displaystyle\sum_{k=1}^{N} x_e^k \leq c_e, & \text{for all } e \in E. \\
\text{(iii)} & 0 \leq x_e^k \leq 1, & \text{for all } e \in E,\ k = 1, \ldots, N. \\
\text{(iv)} & x_e^k \in \{0, 1\}, & \text{for all } e \in E,\ k = 1, \ldots, N.
\end{array}
$$

Obviously, each incidence vector of a Steiner tree packing satisfies (3.2) (i) – (iv) and vice versa, each vector $x \in \mathbb{R}^{\mathcal{N} \times E}$ satisfying (3.2) (i) – (iv) is the incidence vector of a Steiner tree packing. Thus,

$$
(3.3)\qquad \text{STP}\,(G, \mathcal{N}, c) = \text{conv}\,\{x \in \mathbb{R}^{\mathcal{N} \times E} \mid x \text{ satisfies (3.2) (i),\ldots,(iv)}\}.
$$

holds. The inequalities (3.2) (ii) are called the *capacity inequalities* and the ones in (3.2) (iii) the *trivial inequalities.* The weighted Steiner tree packing problem can be solved — in principle — via the following linear programm:

$$
(3.4)\qquad \begin{array}{ll} \min & \displaystyle\sum_{k=1}^{N} w^T x^k \\ & x \in \text{STP}\,(G, \mathcal{N}, c). \end{array}
$$

In order to apply linear programming techniques, a "good" description of the Steiner tree packing polyhedron by means of equations and inequalities is indespensable. The aim of our paper is to study STP $(G, \mathcal{N}, c)$ and to describe this polyhedron partially by valid and facet-defining inequalities.

In the remainder of this section we investigate the dimension of the Steiner tree packing polyhedron and characterize the conditions under which the trivial and the capacity inequalities are facet-defining. Let us first consider the dimension problem.

Problem 3.5 (Dimension problem of the Steiner tree packing polyhedron)
Instance:
A Graph $G = (V, E)$ *with edge capacities* $c_e \in \mathbb{N}, e \in E$, *a net list* $\mathcal{N} = \{T_1, \ldots, T_N\}$ *with* $T_1, \ldots, T_N \subseteq V$ *and a nonnegeative number* l.
Problem:
Is the dimension of $STP(G, \mathcal{N}, c)$ *at least* l *?*

As we have mentioned above, the decision problem, "Does there exist a Steiner tree packing for a given instance $(G, \mathcal{N}, c)$?", is $\mathcal{NP}$-complete ([KL84], [KPS90]). Therefore Problem 3.5 is also $\mathcal{NP}$-complete even for the case $l = 0$.

Remark 3.6 *The dimension problem 3.5 is* $\mathcal{NP}$*-complete.*

This result does not give much hope for a successful study of Steiner tree packing polyhedra of general instances $(G, \mathcal{N}, c)$. Figure 1 shows some examples and the corresponding dimensions. The affine hull of the polytope of Figure 1 (b) is given by $x^1_{34} = 0$, $x^2_{34} = 1$; that of the polytope of Figure 1 (d) by $x^1_{12} = 1, x^2_{12} = 0, x^1_{23} = 0, x^2_{23} = 1$, for instance. The dimension jumps appear rather erratic.

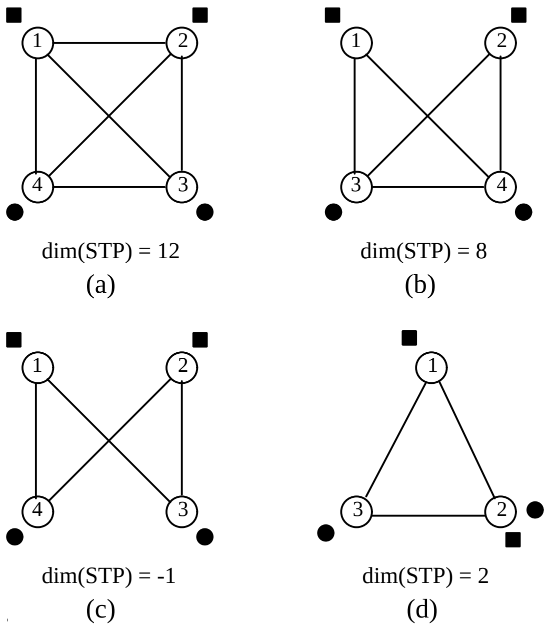
We have decided to study the Steiner tree packing polyhedron for special problem instances for which the dimension can be determined easily and to look for facet-defining inequalities for these special instances. Clearly, such an approach is only sensible if the results can be carried over (at least partially) to practically interesting instances as they occur, for example, in the design of electronic circuits.

It has turned out that an instance $(G, \mathcal{N}, c)$, where the graph G is complete, the net list $\mathcal{N} = \{T_1, \ldots, T_N\}$ is disjoint and the capacities are equal to one ($c = \mathbb{1}$), is an appropriate case. The following lemma shows that the Steiner tree packing polyhedron is fulldimensional in this case.

Lemma 3.7 *Let* $G = (V, E)$ *be the complete graph with node set* $V, |V| \geq 3$, *and edge capacities* $c_e = 1$, $e \in E$. *Furthermore let* $\mathcal{N} = \{T_1, \ldots, T_N\}$ *be a disjoint net list with* $T_1, \ldots, T_N \subseteq V$. *Then,*

$$dim(STP(G, \mathcal{N}, c)) = N \cdot |E|.$$

Proof. Let λ be a vector with $\lambda^T x = 0$ for all $x \in \text{diff}(\text{STP}\,(G, \mathcal{N}, c))$. We have to show that $\lambda^k_e = 0$ for all $e \in E$ und $k \in \{1, \ldots, N\}$. Let $e \in E$ be an arbitrary edge with endnodes u and v. We choose Steiner trees S_k, $k \in \{1, \ldots, N\}$, as follows. If $e \in E(T_k)$, set $S_k = [t : T_k]$ for some $t \in V \setminus \{u, v\}$. Such a node t exists since $|V| \geq 3$. Otherwise, set $S_k = E(T_k)$. Since $\mathcal{N}$ is a disjoint net list, $P = (S_1, \ldots, S_N)$ defines a packing of Steiner trees with $e \notin P$. Thus, $\lambda^k_e = \lambda^T(\chi^{P \cup_k e} - \chi^P) = 0$. $\square$



Figures (a) to (d) show some examples and the dimension of the corresponding polyhedron. The two terminal sets are drawn as rectangles or cycles respectively (T_1={1,2}, T_2={3,4} or T_2={2,3} resp.) and STP abbreviates STP(G,$\mathcal{N}$,$\mathbb{1}$). The polyhedron in (a) is fulldimensional. Deleting edge {1,2} (Figure (b)) decreases the dimension by 4. If additionally edge {3,4} (Figure (c)) is deleted, there even does not exist any feasible solution. Figure (d) shows an example in which the underlying graph is complete but the corresponding polyhedron is not fulldimensional.

Figure 1:

In the next section we will prove some lifting results. These theorems imply that results for the special instance described in Lemma 3.7 can be (partially) carried over to any problem instance. So, it seems reasonable to study this special case.

Let us close this section with the characterization of those conditions under which the trivial and the capacity inequalities are facet-defining. We will concentrate here on the case $N \geq 2$. The case $N = 1$ was solved in [GM90].

The proofs are easy, we only include them to make the reader familiar with the proof technique used throughout the paper, and in particular, with the unavoidably complicated notation.

Theorem 3.8 *Let $G = (V, E)$ be the complete graph with node set V and edge capacities $c_e \in \mathbb{N}$, $e \in E$. Furthermore, let $\mathcal{N} = \{T_1, \ldots, T_N\}, N \geq 2$, be a disjoint net list with $T_1, \ldots, T_N \subset V$. Let $e \in E$ be an arbitrary edge. Then, the following statements hold:*

(i) For all $k \in \{1, \ldots, N\}$, the inequality $x_e^k \geq 0$ defines a facet of $STP(G, \mathcal{N}, c)$ if and only if $|V| \geq 5$ or $e \notin E(T_k)$.

(ii) *For all* $k \in \{1, \ldots, N\}$, *the inequality* $x_e^k \leq 1$ *defines a facet of* $STP(G, \mathcal{N}, c)$ *if and only if* $c_e \geq 2$.

(iii) *The inequality*

$$\sum_{k=1}^{N} x_e^k \leq c_e$$

defines a facet of $STP(G, \mathcal{N}, c)$ *if and only if* $c_e \leq N - 1$.

Proof.
By definition all inequalities $0 \leq x_e^k \leq 1$ or $\sum_{k=1}^N x_e^k \leq c_e$, respectively, are valid.

(i). Since $\mathcal{N}$ contains at least two nets of size at least two, we have $|V| \geq 4$. The case $|V| = 4$ (see Figure 1 (a)) can be easily checked by hand. It turns out that $x_e^k \geq 0$ defines a facet if and only if $e \notin E(T_k)$. We can therefore assume that $|V| \geq 5$.
Let $e \in E$ be any edge and k be any net. Set $F := \{x \in \mathrm{STP}\,(G, \mathcal{N}, c) \mid x_e^k = 0\}$. To show that F is a facet of $\mathrm{STP}\,(G, \mathcal{N}, c)$ we assume that $b^T x \geq \beta$ defines a facet of $\mathrm{STP}\,(G, \mathcal{N}, c)$ and that $F \subseteq F_b := \{x \in \mathrm{STP}\,(G, \mathcal{N}, c) \mid b^T x = \beta\}$ holds. Our aim is to prove that $b^T x = \beta$ is a positive multiple of $x_e^k = 0$.
Suppose $e = uv$ and let $f = rs \in E \setminus \{e\}$ be an arbitrary edge. Choose, moreover, some node $w \in V \setminus \{u, v, r, s\}$. For $i = 1, \ldots, N$ we construct a Steiner tree S_i as follows. If at least three of the four nodes u, v, r, s are in T_i, set $S_i := E(T_i) \setminus \{e, f\}$. If $u, v \in T_i$ and $r, s \notin T_i$, set $S_i := (E(T_i) \cup \{uw, vw\}) \setminus \{e\}$. If $r, s \in T_i$ and $u, v \notin T_i$, set $S_i := (E(T_i) \cup \{rw, sw\}) \setminus \{f\}$. Otherwise set $S_i = E(T_i)$.
Since $\mathcal{N}$ is a disjoint net list, $P = (S_1, \ldots, S_N)$ is a Steiner tree packing with $e, f \notin P$. Therefore, the incidence vectors of the Steiner tree packings $P^i := P \cup_i e$, $i = 1, \ldots, N$ $(i \neq k)$ and $Q^i := P \cup_i f$, $i = 1, \ldots, N$ are in F and thus in F_b. This implies $0 = b^T \chi^{P^i} - b^T \chi^P = b_e^i$ $(i \neq k)$ and $0 = b^T \chi^{Q^i} - b^T \chi^P = b_f^i$ $(i = 1, \ldots, N)$ and we conclude that $b_g^i = 0$, unless $i = k$ and $g = e$. This observation yields (i).

(ii). Let $e \in E$ be any edge and k be any net. We will first prove that $x_e^k \leq 1$ defines a facet if $c_e \geq 2$. Set $F := \{x \in \mathrm{STP}\,(G, \mathcal{N}, c) \mid x_e^k = 1\}$ and suppose that $b^T x \geq \beta$ defines a facet of $\mathrm{STP}\,(G, \mathcal{N}, c)$ and that $F \subseteq F_b := \{x \in \mathrm{STP}\,(G, \mathcal{N}, c) \mid b^T x = \beta\}$ holds. Again, we want to show that $b^T x = \beta$ is a positive multiple of $x_e^k = 1$.
Let $f = rs \in E \setminus \{e\}$ be an arbitrary edge. In addition, choose any node $w \in V \setminus \{r, s\}$. For $i = 1, \ldots, N$ we construct Steiner trees as follows. If $r, s \in T_i$, set $S_i = E(T_i) \setminus \{f\} \cup \{rw, sw\}$. Otherwise, set $S_i = E(T_i)$. Since $\mathcal{N}$ is a disjoint net list, $P = (S_1, \ldots, S_N)$ defines a packing of Steiner trees for $(G, \mathcal{N}, c)$ satisfying $\sum_{i=1}^N \chi_e^{S_i} \leq 1$ and $f \notin P$. Since $c_e \geq 2$ holds, $P' = P \cup_k e$ is also a packing of Steiner trees for $(G, \mathcal{N}, c)$ with $f \notin P'$ and $\chi_e^{P'_k} = 1$. We conclude that $\chi^{P'}$ is in F and thus in F_b. Since $f \notin P'$, $Q^i = P' \cup_i f$ $(i = 1, \ldots, N)$ are

also Steiner tree packings and its incidence vectors are in F and thus in F_b. This implies $0 = b^T\chi^{Q^i} - b^T\chi^{P'} = b^i_f$ for $i = 1,\ldots,N$ and $f \in E \setminus \{e\}$.
It remains to be shown that $b^i_e = 0$ for all $i = 1,\ldots,N$, $i \neq k$. To this end let us choose Steiner trees S_i, $i = 1,\ldots,N$ as follows. Assume that u and v are the endnodes of e and select a node $w \in V \setminus \{u,v\}$. If $u,v \in T_i$, set $S_i = (E(T_i) \setminus \{e\}) \cup \{uw, vw\}$. Otherwise, set $S_i = E(T_i)$. Since $\mathcal{N}$ is a disjoint net list, $P = (S_1,\ldots,S_N)$ defines a packing of Steiner trees for $(G,\mathcal{N},c)$ where $e \notin P$. Then, $P' = P \cup_k e$ is a packing of Steiner trees for $(G,\mathcal{N},c)$ satisfying $\sum_{i=1}^N \chi_e^{P'_i} = 1$ and $\chi_e^{P'_k} = 1$. So, $\chi^{P'}$ is in F and thus in F_b. Since $c_e \geq 2$, $P' \cup_i e$ is also a Steiner tree packing for $i = 1,\ldots,N$, $i \neq k$. This implies $b^i_e = 0$ for $i = 1,\ldots,N$, $i \neq k$.

On the other hand, if $c_e = 1$ then the inequality $x^k_e \leq 1$ is the sum of the inequalities $\sum_{i=1}^N x^i_e \leq c_e = 1$ and $-x^i_e \leq 0$ for all $i = 1,\ldots,N$, $i \neq k$; therefore, $x^k_e \leq 1$ is not facet-defining.

(iii). Let $e \in E$ be any edge. Define $a \in \mathbb{R}^{\mathcal{N}\times E}$ such that $a^Tx = \sum_{k=1}^N x^k_e$.
Since $a^Tx \leq N$ is the sum of the valid inequalities $x^k_e \leq 1$, $k = 1,\ldots,N$, $a^Tx \leq c_e$ does not define a facet if $c_e \geq N$ holds.
We sketch the proof of the opposite direction. Set $F_a := \{x \in \mathrm{STP}\,(G,\mathcal{N},c) \mid a^Tx = c_e\}$ and let $b^Tx \geq \beta$ be valid such that $F_b := \{x \in \mathrm{STP}\,(G,\mathcal{N},c) \mid b^Tx = \beta\}$ defines a facet of $\mathrm{STP}\,(G,\mathcal{N},c)$ with $F_a \subseteq F_b$. If $c_e \leq N-1$, it is easy to construct, for each edge $f \neq e$ and each net $k \in \{1,\ldots,N\}$, a Steiner tree packing P such that the k-th Steiner tree does not contain f and such that c_e of the Steiner trees contain e. Set $P^k := P \cup_k f$, then $a^T\chi^P = a^T\chi^{P^k} = \alpha$ and hence $0 = b^T\chi^P - b^T\chi^{P^k} = b^k_f$. Let us now choose any two nets $i \neq k$. Since $c_e \leq N-1$ we can easily construct a Steiner tree packing $P = (S_1,\ldots,S_N)$ such that $a^T\chi^P = c_e$, $e \in S_i$ and $e \notin S_k$ and such that $Q := (P \cup_k e) \setminus_i e$ is also a Steiner tree packing. We therefore obtain $0 = b^T\chi^Q - b^T\chi^P$, which yields that $b^i_e = b^k_e$ for all $i,k \in \{1,\ldots,N\}$, $i \neq k$. Hence b is a (positive) multiple of a which proves that $a^Tx \leq c_e$ defines a facet. $\square$

4 Manipulating Facet-defining Inequalities

In this section we address the following question. Suppose we have a valid or facet-defining ineqality $a^Tx \geq \alpha$ of the Steiner tree packing polyhedron of some graph and suppose we manipulate the underlying graph using operations such as node splitting or addition, deletion or contraction of an edge, how do we have to modify the inequality $a^Tx \geq \alpha$ such that the resulting inequality is valid or defines a facet of the Steiner tree packing polyhedron of the new graph?

To be formally precise we should distinguish between incidence vectors, capacity vectors, net lists etc. taken with respect to the old and new graph. This formalism would make our notation even more clumsy. Thus, we have decided to drop the distinguishing super or subscripts and hope that it is clear from the context with respect to which of the two used graphs incidence vectors, inequalities etc. are considered.

Lemma 4.1 (Deleting an edge)
Let $(G, \mathcal{N}, c)$ be an instance of the Steiner tree packing problem. Let $a^T x \geq \alpha$ be a valid inequality of $STP(G, \mathcal{N}, c)$ and suppose $f \in E$ is deleted from G. Then $\hat{a}^T x \geq \alpha$ is a valid inequality of $STP(G \setminus f, \mathcal{N}, c)$ where $\hat{a}^k_e = a^k_e$ for all $e \in E \setminus \{f\}$, $k \in \{1, \ldots, N\}$ (where $G \setminus f$ denotes the graph that is obtained by deleting edge f).

Proof. This observation follows from the fact that every Steiner tree packing of $(G \setminus f, \mathcal{N}, c)$ is also a Steiner tree packing of $(G, \mathcal{N}, c)$. $\square$

Unfortunately, a facet-defining inequality for STP $(G, \mathcal{N}, c)$ is not always facet-defining for STP $(G \setminus f, \mathcal{N}, c)$ as the following example shows.

Example 4.2 *Consider the instance drawn in Figure 1 (a). It is easily checked that the inequality $x^2_{13} + x^2_{23} + x^2_{34} \geq 1$ defines a facet for $STP(G, \{T_1, T_2\}, \mathbb{1})$. By deleting edge $\{1, 2\}$ we obtain the picture shown in Figure 1 (b). However, the above inequality does not define a facet for $STP(G \setminus f, \{T_1, T_2\}, \mathbb{1})$, since it is a positive linear combination of the inequalities $x^2_{23} \geq 0$, $x^2_{13} \geq 0$ and the equation $x^2_{34} = 1$.*

The following is a typical sequential lifting result.

Lemma 4.3 (Adding an edge)
Let $(G, \mathcal{N}, c)$ be an instance of the Steiner tree packing polyhedron. Let $f \in E$ with $c_f = 1$ and $\hat{a}^T x \geq \alpha$ be a facet-defining inequality of $STP(G \setminus f, \mathcal{N}, c)$. Then, $a^T x \geq \alpha$ defines a facet for $STP(G, \mathcal{N}, c)$ with $a^k_e = \hat{a}^k_e$ for all $e \in E \setminus \{f\}$, $k \in \{1, \ldots, N\}$ and $a^k_f = \alpha - \min\{\hat{a}^T \chi^P \mid P$ is a packing of Steiner trees for $(G, \mathcal{N}, c)$ with $f \in P_k\}$ for all $k = 1, \ldots, N$.

Proof. Suppose P is a Steiner tree packing for $(G, \mathcal{N}, c)$. If $f \notin P$, P obviously defines a Steiner tree packing for STP $(G \setminus f, \mathcal{N}, c)$. Therefore, $a^T \chi^P \geq \alpha$. Otherwise let $f \in P_k$ for $k \in \{1, \ldots, N\}$. Since $c_f = 1$, $a^T \chi^P = a^k_f + a^T \chi^{P \setminus f} \geq a^k_f + \min\{\hat{a}^T \chi^{\widetilde{P} \setminus f} \mid \widetilde{P}$ is a packing of Steiner trees for $(G, \mathcal{N}, c)$ with $f \in \widetilde{P}_k\} = \alpha$. This implies that the above inequality is valid.

To prove that $a^T x \geq \alpha$ is also facet-defining let us assume that $b^T x \geq \beta$ defines a facet for STP $(G, \mathcal{N}, c)$ such that $F_a := \{x \in \text{STP}\,(G, \mathcal{N}, c) \mid a^T x = \alpha\} \subseteq F_b := \{x \in \text{STP}\,(G, \mathcal{N}, c) \mid b^T x = \beta\}$. Since $\hat{a}^T x \geq \alpha$ defines a facet for STP $(G \setminus f, \mathcal{N}, c)$ and $a^k_e = \hat{a}^k_e$ for all $e \in E \setminus f$, $k \in \{1, \ldots, N\}$ we know that there exists $\lambda > 0$ such that $b^k_e = \lambda a^k_e$ for all $e \in E \setminus \{f\}$, $k \in \{1, \ldots, N\}$ and $\alpha = \lambda \beta$. Let P' denote a Steiner tree packing for $(G, \mathcal{N}, c)$ with $f \in P'_{k_0}$ and $\hat{a}^T \chi^{P' \setminus f} = \min\{\hat{a}^T \chi^{P \setminus f}\} \mid P$ is a packing of Steiner trees for $(G, \mathcal{N}, c)$ with $f \in P_{k_0}\}$. Due to the choice of $a_f^{k_0}$ we obtain $\hat{a}^T \chi^{P'} = \alpha$. In addition we already know that $\lambda \hat{a}_e^k = \lambda a_e^k = b_e^k$ for all $e \in E \setminus \{f\}$, $k \in \{1, \ldots, N\}$. This implies $\lambda\alpha = \beta = b^T \chi^{P'} = b_f^{k_0} + \lambda \hat{a}^T \chi^{P' \setminus f}$. Therefore, $b_f^{k_0} = \lambda a_f^{k_0}$. This completes the proof. □

Lemma 4.4 (Splitting a node)
Let $(G, \mathcal{N}, c)$ be an instance of the Steiner tree packing polyhedron. Let $f \in E$ with $c_f = 1$ and let $\hat{a}^T x \geq \alpha$ be a valid inequality of STP $(G / f, \mathcal{N}, c)$ (note that G / f denotes the graph that is obtained by shrinking edge f). Then, $a^T x \geq \alpha$ defines a valid inequality for STP $(G, \mathcal{N}, c)$ with $a_e^k = \hat{a}_e^k$ for all $e \in E \setminus \{f\}$, $k \in \{1, \ldots, N\}$ and $a_f^k = 0$ for all $k = 1, \ldots, N$.

Proof. If STP $(G, \mathcal{N}, c) = \emptyset$, there is nothing to show. Otherwise, let P be a Steiner tree packing for $(G, \mathcal{N}, c)$. Obviously, $P \setminus f$ is a solution for $(G / f, \mathcal{N}, c)$. Thus, $a^T \chi^P \geq \hat{a}^T \chi^{P \setminus f} \geq \alpha$, since $a_f^k = 0$ for all $k = 1, \ldots, N$. This implies that $a^T x \geq \alpha$ is a valid inequality for $(G, \mathcal{N}, c)$. □

Unfortunately, again not every facet-defining inequality for STP $(G / f, \mathcal{N}, c)$ defines a facet for STP $(G, \mathcal{N}, c)$. Even worse, STP $(G, \mathcal{N}, c)$ may be empty, although STP $(G / f, \mathcal{N}, c)$ is not.

Example 4.5 *Consider the graph G' in Figure 2 (a) and the graph G in Figure 2 (b), with $T_1 = \{1, 4, 5, 7\}$ and $T_2 = \{2, 3, 6\}$. Let $G' = G / \{3, 3'\}$. Obviously, there does not exist any Steiner tree packing for $(G, \{T_1, T_2\}, \mathbb{1})$ and STP $(G, \{T_1, T_2\}, \mathbb{1}) = \emptyset$. However, STP $(G', \{T_1, T_2\}, \mathbb{1})$ is a fulldimensional polytope.*

Finally, let us consider the reverse operation of Lemma 4.4, i. e., the contraction of an edge. In this case a valid inequality $a^T x \geq \alpha$ for STP $(G, \mathcal{N}, c)$ is not always valid for STP $(G / f, \mathcal{N}, c)$. This remains true even in the case when $a_f^k = 0$ for all $k = 1, \ldots, N$.

Example 4.6 *Consider the graph G in Figure 3 (a) and the graph G' in Figure 3 (b) with $T_1 = \{1, 9\}$ and $T_2 = \{2, 10\}$. Let $G' = G / \{3, 6\}$. Obviously, $x_{47}^1 + x_{58}^1 + x_{47}^2 + x_{58}^2 \geq 1$ is a valid inequality for STP $(G, \{T_1, T_2\}, \mathbb{1})$, but this does not hold for STP $(G', \{T_1, T_2\}, \mathbb{1})$.*

5 Lifting Facets from Steiner Tree Polyhedra to Steiner Tree Packing Polyhedra

In this section we will make an important observation, namely, that, under mild assumptions, all nontrivial facets of the (single) Steiner tree polyhedron can

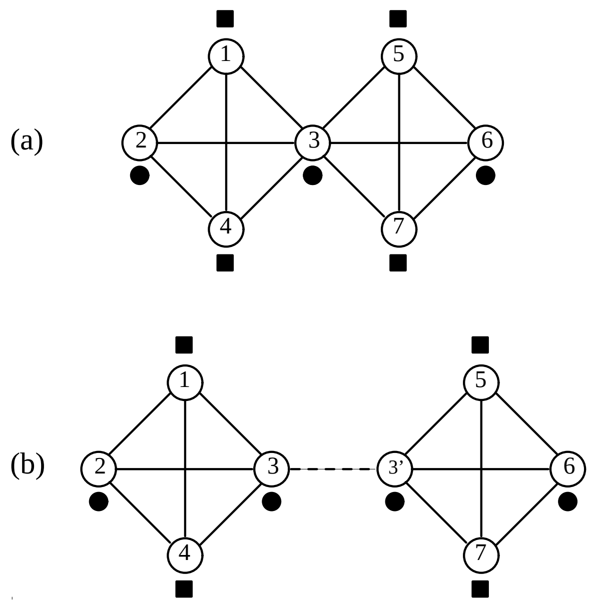


Figure 2:

be lifted to the Steiner tree packing polyhedron. This implies that, in order to obtain a complete characterization of some Steiner tree packing polyhedron STP $(G, \mathcal{N}, c)$, for all nets of the net list, all individual Steiner tree polyhedra STP $(G, \{T\}, c), T \in \mathcal{N}$, must be known completely.
We begin this investigation by stating a trivial lemma.

Lemma 5.1 *Let $G = (V, E)$ denote a connected graph that does not contain a Steiner bridge with respect to $T \subseteq V$. Let $a^T x \geq \alpha$ be a facet-defining inequality of $STP(G, \{T\}, 1\!\!1)$. Then, either $a^T x \geq \alpha$ is an inequality of the form $-x_e^1 \geq -1$ for some edge $e \in E$ or $a_e \geq 0$ holds for all $e \in E$.*

It is somewhat surprising that, for the Steiner tree packing polyhedron, even it is fulldimensional, a similiar statement is no longer true. In [M92] the interested reader can find an example of a facet-defining inequality with positive and negative coefficients.

Lemma 5.2 *Let $G = (V, E)$ be the complete graph on node set V and let $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \geq 2$, be a disjoint net list. Furthermore, let $a^T x \geq \alpha$ be a nontrivial facet-defining inequality of $STP(G, \{T_1\}, 1\!\!1)$ with $a \in \mathbb{R}^E$, $a \geq 0$. Then the following two statements are true:*

(i) For every edge $e \in E$ there exists a Steiner tree packing P for $(G, \mathcal{N}, 1\!\!1)$ with $e \notin P$ and $a^T \chi^{P_1} = \alpha$.

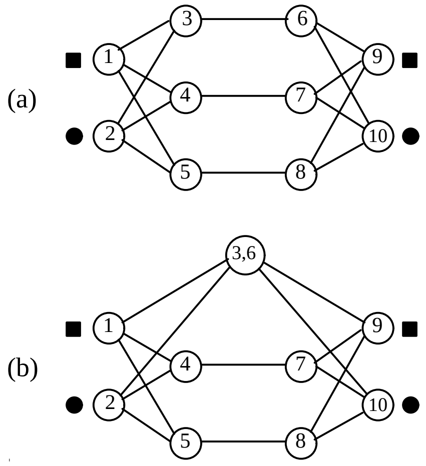


Figure 3:

(ii) For every edge-minimal Steiner tree S_1 for T_1 with $a^T\chi^{S_1} = \alpha$ there exist edge sets $S_2, \ldots, S_N$ such that $(S_1, \ldots, S_N)$ defines a Steiner tree packing for $(G, \mathcal{N}, \mathbb{1})$.

Proof. We start by proving (i). Let $e \in E$ and let S_1 be an edge-minimal Steiner tree with $a^T\chi^{S_1} = \alpha$ and $e \notin S_1$. Such a Steiner tree S_1 does exist, because $a^Tx \geq \alpha$ is a nontrivial facet-defining inequality with $a \geq 0$. Since S_1 is edge-minimal, it follows from (2.4) that

$$(*) \qquad \Upsilon(S_1) = |T_1| - 2.$$

We set $S_k = ([T_1 : T_k] \cup E(T_k)) \setminus \{S_1 \cup e\}$ for $k = 2, \ldots, N$. By $l(k)$ we denote the number of (connected) components in $(V(S_k) \cup T_k, S_k)$. Consider any $t \in T_1$. If $e \in [t : T_{k_0}]$ for some $k_0 \in \{2, \ldots, N\}$ we obtain $d_{S_1}(t) \geq \sum_{k=2}^{N}(l(k)-1) - 1$; otherwise we have $d_{S_1}(t) \geq \sum_{k=2}^{N}(l(k)-1)$. This is true, because otherwise two components of some net k could be connected via t. We get

$$\begin{aligned}
\Upsilon(S_1) &\geq \sum_{t \in T_1}\Big(\sum_{k=2}^{N}(l(k)-1) - 1\Big) - 1 \\
&= |T_1| \cdot \sum_{k=2}^{N}(l(k)-1) - |T_1| - 1 \\
&\geq |T_1| - 1, \quad \text{if } \sum_{k=2}^{N}(l(k)-1) \geq 2.
\end{aligned}$$

As a result, there exists at most one net k_1 with $l(k_1) = 2$. If there does not exist such a net k_1, then (i) is already shown. We extend S_{k_1} by the unused edges in $E(T_1)$ and set $S_{k_1} = S_{k_1} \cup (E(T_1) \setminus (S_1 \cup \{e\}))$. If $(V(S_{k_1}) \cup T_{k_1}, S_{k_1})$ is connected, we are finished. Otherwise let $(\bar{V}_1, \bar{E}_1)$ and $(\bar{V}_2, \bar{E}_2)$ denote the two (connected) components with $|\bar{V}_1| \leq |\bar{V}_2|$. Note that $t \in \bar{V}_1$ or $t \in \bar{V}_2$ for all $t \in T$, since otherwise S_1 would contain a cycle. We distinguish the following two cases.

(1) $e \notin [\bar{V}_1 : \bar{V}_2]$.

Here, $|\bar{V}_1| = 1$. Otherwise S_1 would contain a cycle, which contradicts the fact S_1 being edge-minimal. For $\{v\} = \bar{V}_1$ we obtain $d_{S_1}(v) \geq |T_1| + |T_{k_1}| - 1$, and so $\Upsilon(S_1) \geq |T_1| + |T_{k_1}| - 1 - 2 \geq |T_1| - 1$, which contradicts $(*)$. Therefore, $l(k) = 1$ for all $k = 2, \ldots, N$.

(2) $e \in [\bar{V}_1 : \bar{V}_2]$.

Again, we have to distinguish two subcases.

(a) $|T_1| \leq |V| - 3$.

If $|T_{k_1}| = 2$, there exists a node $v \notin T_{k_1} \cup T_1$, since $|T_1| \leq |V| - 3$. Then, $S_{k_1} \cup [v : T_{k_1}]$ defines a Steiner tree for T_{k_1} (note that $[v : T_{k_1}] \cap S_k = \emptyset$ for all $k \neq k_1$). So, the remaining case is $|T_{k_1}| \geq 3$. This however implies that $|\bar{V}_1| + |\bar{V}_2| \geq 5$ holds. Therefore, $|\bar{V}_1| = 1$, otherwise S_1 would contain a cycle. Let $\{v\} = \bar{V}_1$. Then $\Upsilon(S_1) \geq d_{S_1}(v) - 2 = (|T_{k_1}| - 1) + |T_1| - 2 - 1 \geq |T_1| - 1$, which contradicts $(*)$.

(b) $|T_1| = |V| - 2$.

Since $\mathcal{N}$ is disjoint, we know that $N = 2$, $|T_2| = 2$ (say $T_2 = \{t, t'\}$ and $T_1 = \{t_1, \ldots, t_{|T_1|}\}$). This case is inconvinient in the following sense. We can not show statement (i) for any Steiner tree S_1 being defined similiarily to above. For example for $S_1 = [t : T_1]$ and $e = tt'$ there does not exist a Steiner tree packing with $e \notin P$. In this case we prove the statement indirectly. Suppose, the statement

$(\star)$ "For each Steiner tree packing P with $a^T \chi^{P_1} = \alpha$, edge e is an element of P"

is correct. Let us first consider the case $|\bar{V}_1| = 1$. Without loss of generality we can assume that $\bar{V}_1 = \{t\}$.

First, suppose $e = tt'$. Then, $S_1 = [t : T_1]$. Due to the assumption $(\star)$ we know $(\diamond)$ $a_{tt_i} < a_{t_i t_j}$ for all $i, j \in \{1, \ldots, |T_1|\}, i \neq j$. Since $a^T x \geq \alpha$ defines a nontrivial facet with $a \geq 0$, there exists an edge-minimal Steiner tree S_1' for T_1 with $a^T \chi^{S_1'} = \alpha$, $e \notin S_1'$ and $S_1' \cap E(T_1) \neq \emptyset$, (suppose, there does not exist such a S_1'; due to the properties of $a^T x \geq \alpha$ there exists an edge-minimal Steiner tree S for T_1 with $a^T \chi^S = \alpha$ and $|S \cap E(T_1)| > 0$; the assumption implies that $e \in S$; thus, there exist i, j with $t't_i, t_i t_j \in S$; since $a^T \chi^S = \alpha$ and $t_j t \notin S$ we obtain $a_{t_i t_j} \leq a_{t_j t}$, which contradicts $(\diamond)$). W. l. o. g. let $t_1 t_2 \in S_1'$. The assumption $(\star)$ lets us conclude that $tt_1 \in S_1'$ or $t't_1 \in S_1'$ and $tt_2 \in S_1'$ or $t't_2 \in S_1'$. If $tt_1 \in S_1'$, then $tt_2 \notin S_1'$, and due to $a_{tt_2} < a_{t_1 t_2}$, $\bar{S}_1 = S_1' \setminus \{t_1 t_2\} \cup \{tt_2\}$ is a Steiner tree for T_1 with $a^T \chi^{\bar{S}_1} < \alpha$, a contradiction. Analogously, it can be shown that $tt_2 \notin S_1'$. Hence, $t't_2 \in S_1'$ and $t't_1 \in S_1'$, which is a contradiction to the property that S_1' is edge-minimal.

Now suppose $e = tt_i$ for some $i \in \{1, \dots, |T_1|\}$; without loss of generality say $i = 1$. Then, $S_1 = [t : T_1 \setminus t_1] \cup \{tt', t't_1\}$. The assumption $(\star)$ implies that $a_{tt_2} < a_{t_2t_1}$ and $a_{tt'} + a_{t't_1} < a_{t_2t_1}$. Since $a^T x \geq \alpha$ defines a nontrivial facet with $a \geq 0$, there exists an edge-minimal Steiner tree S_1' for T_1 with $a^T \chi^{S_1'} = \alpha$ and $t_1t_2 \in S_1'$. Due to the assumption $(\star)$ we can conclude that $tt' \in S_1'$ or $tt_1 \in S_1'$. Both alternatives however are leading to a contradiction, $tt' \in S_1'$ due to $a_{tt_2} < a_{t_2t_1}$ and $a_{t't_1} < a_{t_2t_1}$, and $tt_1 \in S_1'$ due to $a_{tt_2} < a_{t_2t_1}$.

Finally, the case $|\bar{V}_1| \geq 2$ must be investigated. Here we know that $|\bar{V}_1| = 2$ and $|\bar{V}_2| = 2$ (especially $|T_1| = 2$), otherwise S_1 would contain a cycle. Without loss of generality let $\hat{V}_1 = \{t, t_1\}$ and $\hat{V}_2 = \{t', t_2\}$. Since S_1 is edge-minimal, only the case $e = t_1t_2$ remains to be considered. Then, $S_1 = \{t_1t', t't, tt_2\}$. The assumption $(\star)$ implies, $a_{tt_2} + a_{tt'} < a_{t't_2}$ and $a_{t't_1} + a_{tt'} < a_{t't_1}$. Since $a^T x \geq \alpha$ defines a nontrivial facet with $a \geq 0$, there exists an edge-minimal Steiner tree S_1' for T_1 with $a^T \chi^{S_1'} = \alpha$ and $t't_2 \in S_1'$. Due to $(\star)$ we can conclude that $S_1' = \{t_2t', t't, tt_1\}$. Since $a_{tt_2} < a_{t't_2}$ and $a_{t't_1} < a_{t't_1}$ hold, we obtain the contradiction $a^T \chi^{S_1} < a^T \chi^{S_1'} = \alpha$.

Summing up it may be said that all cases in (b) lead to a contradiction. Therefore, the assumption $(\star)$ does not hold.

So we can conclude that there exists a Steiner tree packing with the properties in statement (i).

Statement (ii) can be shown similiarily. Since "$e \notin P$" is not required any more, we obtain $\Upsilon(S_1) \geq |T_1|$, if $\sum_{k=2}^{N}(l(k) - 1) \geq 2$, and only case (1) has to be considered. This completes the proof. □

The two lemmas enable us to prove the main result of this section.

Theorem 5.3 *Let $G = (V, E)$ be the complete graph with node set V and $\mathcal{N} = \{T_1, \dots, T_N\}$, $N \geq 2$, a disjoint net list. Let $\bar{a}^T x \geq \alpha$, $\bar{a} \in \mathbb{R}^E$, be a nontrivial facet-defining inequality for STP$(G, \{T_1\}, \mathbb{1})$. Then, $a^T x \geq \alpha$ defines a facet of STP$(G, \mathcal{N}, \mathbb{1})$, where $a \in \mathbb{R}^{\mathcal{N} \times E}$ denotes the vector with $a_e^1 = \bar{a}_e^1$, $a_e^k = 0$ for all $k = 2, \dots, N$, $e \in E$.*

Proof.
Since $\bar{a}^T x \geq \alpha$ defines a nontrivial facet for STP$(G, \{T_1\}, \mathbb{1})$ Lemma 5.1 implies $\bar{a} \geq 0$. Therefore, Lemma 5.2 can be applied.

The inequality $a^T x \geq \alpha$ is surely valid. Let $b^T x \geq \beta$ be a facet-defining inequality for STP$(G, \mathcal{N}, \mathbb{1})$ with $F_a := \{x \in \text{STP}(G, \mathcal{N}, \mathbb{1}) \mid a^T x = \alpha\} \subseteq F_b := \{x \in \text{STP}(G, \mathcal{N}, \mathbb{1}) \mid b^T x = \beta\}$. Statement (i) of Lemma 5.2 implies that for every edge $e \in E$ there exists a Steiner tree packing P with $e \notin P$ and $\chi^P \in F_a$. Therefore, $b_e^k = 0$ for all $k = 2, \dots, N$ and all $e \in E$. Since $\bar{a}^T x \geq \alpha$ defines a facet for STP $(G, \{T_1\}, \mathbb{1})$ and $\bar{a} \geq 0$, there exist edge-minimal packings of Steiner trees $S_1^1, \ldots, S_1^{|E|}$ for T_1, such that $\chi^{S_1^1}, \ldots, \chi^{S_1^{|E|}}$ are affinely independent and $\bar{a}^T \chi^{S_1^i} = \alpha$ for $i = 1, \ldots, |E|$. From Lemma 5.2 part (ii) follows that every S_1^i can be extended to a Steiner tree packing P^i with $P_1^i = S_1^i$. So, we can conclude that $a^T \chi^{P^i} = \alpha$ and $b^T \chi^{P^i} = \beta$ for $i = 1, \ldots, |E|$. Since $b_e^k = 0$ for all $k = 2, \ldots, N$ and all $e \in E$, we obtain ${b^1}^T \chi^{P^i} = \beta$. Due to the choice of $S_1^1, \ldots, S_1^{|E|}$ it follows that there exists $\lambda > 0$ such that $b_e^1 = \lambda \bar{a}_e^1 = \lambda a_e^1$ for all $e \in E$. Thus, we have shown that b and a are identical up to multiplication with a scalar. This completes the proof. $\square$

Remark 5.4 *The trivial facets of STP $(G, \{T_1\}, \mathbb{1})$ do not necessarily define facets of STP $(G, \mathcal{N}, \mathbb{1})$. As an example let us consider the instance of Figure 1 (a). From Theorem 3.8 we know that neither $x_{12}^1 \geq 0$ nor any of the inequalities $x_e^1 \leq 1$ defines a facet of STP $(G, \mathcal{N}, \mathbb{1})$. On the other hand, each of these inequalities defines a facet for STP $(G, \{T_1\}, \mathbb{1})$ (see [GM90]).*

6 Joint Facets

In this section we consider inequalities that combine two or more nets. We will proceed in the following way. First, we describe each inequality. All inequalities we are going to consider are of the form $a^T x \geq \alpha$, $a \geq 0$. The coefficients of some of the edges will turn out to be zero for all nets. We call these edges *zero edges* and the graph induced by the zero edges the *zero graph*. We will use the structure of the zero graph to name the inequalities. This has the following reasons. The zero graph is structured in such a way that there exists no Steiner tree packing for the nets involved in this graph. Therefore, each feasible solution must use edges whose coefficients are different from zero. This means that each inequality is in some sense (but not necessarily uniquely) determined by the zero graph.

We will always define the inequalities for an arbitrary instance without guaranteeing that the inequality is also valid for the corresponding polyhedron. In the succeeding theorem we characterize the instances for which the inequality defines a facet of the corresponding polyhedron. In addition, edges get value zero for some single nets (we typically denote these sets by $F_1, \ldots, F_N$). These edge sets $F_1, \ldots, F_N$ must usually satisfy very technical restrictions in order that the inequality defines a facet. The results can often be generalized, for example, by modifying the net list or by adding a node. Due to the rich variety of possibilities we typically only sketch the ideas and hint at possible extensions. In order to remain within the scope of the paper we have also decided, to concentrate on the validity of the corresponding inequalities at the expense of giving detailed proofs that the inequalities are facet-defining. In particular, to prove that the corresponding inequalities are facet-defining requires essentially the same scheme. We illustrate this scheme in one sample. For specific proofs of the remaining statements we refer the interested reader to [M92].

Alternating Cycle Inequalities

Definition 6.1 *Let $G = (V, E)$ be a graph and $\mathcal{N} = \{T_1, T_2\}$ a net list. We call a cycle F an alternating cycle with respect to T_1, T_2, if $F \subseteq [T_1 : T_2]$ and $V(F) \cap T_1 \cap T_2 = \emptyset$ (see Figure 4). Moreover, let $F_1 \subseteq E(T_2)$ and $F_2 \subseteq E(T_1)$ be two sets of diagonals of the alternating cycle F with respect to T_1, T_2. The inequality*

$$(\chi^{E\setminus(F\cup F_1)}, \chi^{E\setminus(F\cup F_2)})^T x \geq \tfrac{1}{2}|F| - 1$$

is called an alternating cycle inequality.

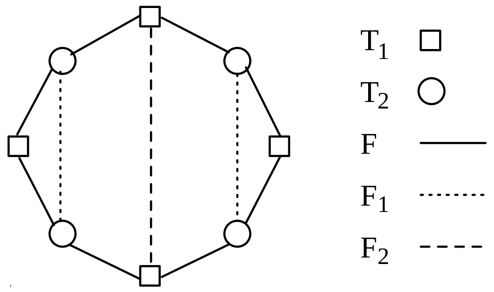


Figure 4:

It is not difficult to see that the basic form of an alternating cycle inequality, i. e., $F_1 = F_2 = \emptyset$, is valid for STP $(G, \mathcal{N}, 1\!\!1)$, but in general, it is not facet-defining. The sets F_1 and F_2 are used to strengthen the basic form; in fact, choosing them appropriately we can obtain facet-defining inequalities.

The sets of diagonals $F_1 \subseteq E(T_2)$ and $F_2 \subseteq E(T_1)$ are called *maximal cross free with respect to* F, if F_1 and F_2 are cross free, and each diagonal $e_1 \in E(T_1) \setminus F_2$ crosses F_1 and each diagonal $e_2 \in E(T_2) \setminus F_1$ crosses F_2 (see Figure 4). Then, the following theorem holds.

Theorem 6.2 *Let $G = (V, E)$ be the complete graph with node set V and let $\mathcal{N} = \{T_1, T_2\}$ be a disjoint net list with $T_1 \cup T_2 = V$ and $|T_1| = |T_2| = l$, $l \geq 2$. Furthermore, let F be an alternating cycle with respect to T_1, T_2 with $V(F) = V$ and $F_1 \subseteq E(T_2)$, $F_2 \subseteq E(T_1)$. Then the alternating cycle inequality*

$$(\chi^{E\setminus(F\cup F_1)}, \chi^{E\setminus(F\cup F_2)})^T x \geq l - 1$$

defines a facet of STP $(G, \mathcal{N}, 1\!\!1)$ if and only if F_1 and F_2 are maximal cross free.

Proof. Set $E_k = E \setminus (F \cup F_k),\ k = 1, 2$ and $a := (\chi^{E_1}, \chi^{E_2})$. First, we prove that $a^T x \geq l - 1$ is valid if F_1 and F_2 are cross free. It suffices to show that for every packing of Steiner trees (S_1, S_2), $|(S_1 \cap E_1) \cup (S_2 \cap E_2)| \geq l - 1$ holds (note that $c = 1\!\!1$).

Let (S_1, S_2) be any Steiner tree packing. W. l. o. g. S_1 and S_2 are edge-minimal. Set $T_1' := \{t \in T_1 \mid \delta(t) \cap F \subseteq S_2\}$ and $T_2' := \{t \in T_2 \mid \delta(t) \cap F \subseteq S_1\}$. Since S_1 and S_2 are edge-minimal and $|F| = 2l$, we have that $|T_1'| + |T_2'| \leq l - 1$. This implies that $T_1 \setminus T_1'$ and $T_2 \setminus T_2'$ are nonempty. Therefore, at least $|T_1'| + |T_2'|$ edges $e \in S_1 \cap E_1 \cup S_2 \cap E_2$ are necessary to connect T_1' with $T_1 \setminus T_1'$ and T_2' with $T_2 \setminus T_2'$. Consider the remaining terminals $T_1 \setminus T_1'$ and $T_2 \setminus T_2'$. Set $k_i := \kappa((V(S_i), S_i \setminus F_i))$ for $i = 1, 2$, where $\kappa(\hat{G})$ denotes the number of components of graph $\hat{G}$. Since F_1 and F_2 are cross free, we obtain $k_1 + k_2 \leq l + 1$. Thus,

$$\begin{array}{rl} a^T(\chi^{S_1}, \chi^{S_2}) & \geq (|T_1'| + |T_2'|) + (|T_1 \setminus T_1'| + |T_2 \setminus T_2'| - (k_1 + k_2)) \\ & \geq |T_1| + |T_2| - (k_1 + k_2) \geq l - 1. \end{array}$$

Let us now outline the proof that $a^T x \geq l - 1$ defines a facet of STP $(G, \mathcal{N}, 1\!\!1)$.

Suppose $b^T x \geq \beta$ is a facet-defining inequality of STP $(G, \mathcal{N}, 1\!\!1)$ such that $F_a := \{x \in \text{STP}\,(G, \mathcal{N}, 1\!\!1) \mid a^T x = l - 1\} \subseteq F_b := \{x \in \text{STP}\,(G, \mathcal{N}, 1\!\!1) \mid b^T x = \beta\}$. In the following we show that b is a multiple of a.

In the first two steps we show that for any coefficient $a^k_e = 0,\ k \in \{1, 2\}$ there exists a Steiner tree packing P with $a^T \chi^P = l - 1$ and $e \notin P$. This implies $b^k_e = 0$.

(1) $b^k_e = 0$ for $e \in F,\ k = 1, 2$.
Choose $S_1 = F \setminus \{e\}$ and $S_2 = [t : T_2],\ t \in T_2$. Furthermore set $S_1' = S_1 \cup \{e\}$. Then $P = (S_1, S_2)$ and $P' = (S_1', S_2)$ are Steiner tree packings with $\chi^P, \chi^{P'} \in F_a$ and $0 = b^T(\chi^{S_1'}, \chi^{S_2}) - b^T(\chi^{S_1}, \chi^{S_2}) = b^1_e$. Analogously we obtain $b^2_e = 0$.

(2) $b^k_e = 0$ for $e \in F_k,\ k = 1, 2$.
Choose $S_1 = F$ and $S_2 = [t : T_2],\ t \in T_2$. Furthermore set $S_1' = S_1 \cup \{e\}$. Then $P = (S_1, S_2)$ and $P' = (S_1', S_2)$ are Steiner tree packings with $\chi^P, \chi^{P'} \in F_a$ and $0 = b^T(\chi^{S_1'}, \chi^{S_2}) - b^T(\chi^{S_1}, \chi^{S_2}) = b^1_e$. Analogously we obtain $b^2_e = 0$.

Next, we prove that the coefficients of edges that connect terminals of the same net are equal. Typically this can be done by constructing two Steiner trees inside the subgraph induced by the corresponding terminal set that differ only in two edges.

(3) $b^k_e = b^k_{e'}$ for $e, e' \in E(T_k),\ k = 1, 2$.
Let $e = uv$ with $u, v \in T_1$. Set $S_2 = F$ and $S_1 = [v : T_1]$. Let $e' \in [u : T_1] \setminus \{e\}$ and $S_1' = S_1 \setminus \{e\} \cup \{e'\}$. Then $P = (S_1, S_2)$ and $P' = (S_1', S_2)$ are Steiner tree packings with $\chi^P, \chi^{P'} \in F_a$ and $0 = b^T(\chi^{S_1'}, \chi^{S_2}) - b^T(\chi^{S_1}, \chi^{S_2}) = b^1_{e'} - b^1_e$ for all $e, e' \in \delta(u),\ u \in T_1$. Analogously we obtain $b^2_e = b^2_{e'}$.

In the remainder of the proof set $\bar{k} := 1$, if $k = 2$, and $\bar{k} := 2$, if $k = 1$.

In steps (4) and (5) we fix the remaining coefficients of one net. To this end we use the structure of the zero graph, the properties fullfilled by F_1 and F_2 and the fact proved in (3).

(4) $b^k_e = b^k_{e'}$ for $e' \in E(T_k),\ e \in [T_k : T_{\bar{k}}],\ k = 1, 2.$

Let $e = uw$ with $u \in T_1,\ w \in T_2$ and $v \in T_1$ such that $vw \in F$. Choose $S_2 = F \setminus \delta(v),\ S_1 = [u : T_1]$ and $S'_1 = S_1 \setminus \{uv\} \cup \{uw\} \cup \{vw\}$. Then $P = (S_1, S_2)$ and $P' = (S'_1, S_2)$ are Steiner tree packings with $\chi^P, \chi^{P'} \in F_a$ and $0 = b^T(\chi^{S'_1}, \chi^{S_2}) - b^T(\chi^{S_1}, \chi^{S_2}) = b^1_{uw} + b^1_{vw} - b^1_{uv} = b^1_{uw} - b^1_{uv}$, because $b^1_{vw} = 0$ (see (1)). This together with (3) proves the statement. Analogously we obtain $b^2_e = b^2_{e'}$.

(5) $b^k_e = b^k_{e'}$ for $e \in E(T_{\bar{k}}) \setminus F_k,\ e' \in E(T_k),\ k = 1, 2.$

Let $e = uv \in E(T_2) \setminus F_1$. Since F_1 and F_2 are maximal cross free, there exists an edge $u_2v_2 \in F_2$ which crosses e. Let $u^-,\ v^+ \in T_1$ such that $u^-u,\ vv^+ \in F$ and uv crosses u^-v^+. Choose $S_1 = [u^- : E(T_1)]$ and $S_2 = F$. Furthermore set $S'_1 = S_1 \setminus \{u^-v^+\} \cup \{u^-u, uv, vv^+\}$ and $S'_2 = S_2 \setminus \{u^-u, vv^+\} \cup \{u_2v_2\}$. Then $P = (S_1, S_2)$ and $P' = (S'_1, S'_2)$ are Steiner tree packings with $\chi^P, \chi^{P'} \in F_a$ and $0 = b^T(\chi^{S_1}, \chi^{S_2}) - b^T(\chi^{S'_1}, \chi^{S'_2}) = b^1_{u^-v^+} - b^1_{uv}$. This together with (3) proves the statement. Analogously we obtain $b^2_e = b^2_{e'}$.

It remains to be shown that the coefficients of different nets are equal. This is typically done by constructing two Steiner tree packings; in the first solution the Steiner tree for net 1 uses only zero edges, whereas in the second solution zero edges are only used by net 2.

(6) $b^1_e = b^2_{e'}$ for $e \in E(T_1),\ e' \in E(T_2)$.

Let $e = uv \in E(T_1)$ and $e' = wx \in E(T_2)$. Choose $S_1 = [u : E(T_1)],\ S_2 = F,\ S'_1 = F$ and $S'_2 = [w : E(T_2)]$. Then $P = (S_1, S_2)$ and $P' = (S'_1, S'_2)$ are Steiner tree packings with $\chi^P, \chi^{P'} \in F_a$ and $0 = b^T(\chi^{S'_1}, \chi^{S'_2}) - b^T(\chi^{S_1}, \chi^{S_2}) = \sum_{i \in T_2 \setminus \{w\}} b^2_{iw} - \sum_{i \in T_1 \setminus \{u\}} b^1_{iu} = (l-1) \cdot b^2_{xw} - (l-1) \cdot b^1_{vu}$ because of (3). So we obtain $b^1_e = b^2_{e'}$.

(1) - (6) imply that b is a multiple of a.

It remains to be shown that F_1 and F_2 are maximal cross free if $a^T x \geq l - 1$ defines a facet of STP $(G, \mathcal{N}, \mathbb{1})$.

First, we show that F_1 and F_2 have to be cross free. Suppose, F_1 and F_2 are not cross free. Then, there exist two crossing diagonals $e_1 = u_1v_1 \in F_1$ and $e_2 = u_2v_2 \in F_2$. Let $u_1^-,\ v_1^+ \in T_1$ such that $u_1^-u_1,\ v_1v_1^+ \in F$ and u_1v_1 crosses $u_1^-v_1^+$. Choose $S_1 = [u_1^- : E(T_1)] \setminus \{u_1^-v_1^+\} \cup \{u_1^-u_1, u_1v_1, v_1v_1^+\}$ and $S_2 = F \setminus \{u_1^-u_1, v_1v_1^+\} \cup \{u_2v_2\}$. Then, (S_1, S_2) is a Steiner tree packing with $a^T(\chi^{S_1}, \chi^{S_2}) = l - 2$, a contradiction.

Finally, we show that F_1 and F_2 are maximal cross free. Suppose, this is not the case. Let $F'_1 \subseteq E(T_2)$ and $F'_2 \subseteq E(T_1)$ such that $F_1 \cup F_2 \subset F'_1 \cup F'_2$ and F'_1 and F'_2 are maximal cross free. Due to part 1 of this proof $(\chi^{E \setminus (F \cup F'_1)}, \chi^{E \setminus (F \cup F'_2)})^T x \geq l - 1$ defines a facet of STP $(G, \mathcal{N}, \mathbb{1})$. Summing up this facet-defining inequality together with the valid inequalities $x^1_e \geq 0$ for all $e \in F'_1 \setminus F_1$ and $x^2_e \geq 0$ for all $e \in F'_2 \setminus F_2$ we obtain $a^T x \geq l - 1$. Thus, $a^T x \geq l - 1$ does not define a facet of

STP $(G, \mathcal{N}, \mathbb{1})$, a contradiction. $\square$

A consequence of the proof is that the alternating cycle inequality is valid if and only if F_1 and F_2 are cross free. Let us now focus on some extensions of the alternating cycle inequalities.

First, we consider the case where parallel edges are added to the complete graph. The coefficients of the new edges that are not parallel to an edge of the alternating cycle F obtain the value of the coefficients of the "original edge". The coefficients of the edges that are parallel to an edge of F obtain the value 1.

Theorem 6.3 *Let $G = (V, E)$ be a graph that contains the complete graph on node set V as a subgraph and let $\mathcal{N} = \{T_1, T_2\}$ be a disjoint net list with $T_1 \cup T_2 = V$ and $|T_1| = |T_2| = l$, $l \geq 2$. Furthermore let F be an alternating cycle with respect to T_1, T_2 with $V(F) = V$ and $F_1 \subseteq E(T_2)$, $F_2 \subseteq E(T_1)$. Then the alternating cycle inequality*

$$(\chi^{E\setminus(F\cup F_1)}, \chi^{E\setminus(F\cup F_2)})^T x \geq l - 1$$

defines a facet of STP $(G, \mathcal{N}, \mathbb{1})$ if and only if F_1 and F_2 are maximal cross free.

The proof of Theorem 6.3 is very similiar to that of Theorem 6.2, so we omit it here. A complete proof can be found in [M92].

Next, let us consider the case where an additional node z is added to the complete graph in Theorem 6.2. We address the question how the coefficients of the edges that are incident to the extra node z must be chosen to obtain a facet-defining inequality for the corresponding Steiner tree packing polyhedron.

Suppose we have given a complete graph $G = (V \cup \{z\}, E)$ and a net list $\mathcal{N} = \{T_1, T_2\}$ such that $T_1 \cap V, T_2 \cap V$ is a partition of V with $|T_1 \cap V| = |T_2 \cap V| = \frac{|V|}{2}$. Note that we do not require $\mathcal{N}$ to be disjoint. Furthermore, let F be an alternating cycle with respect to T_1, T_2 with $V(F) = V$ and let $F_1 \subseteq E(V \cap T_2)$ and $F_2 \subseteq E(V \cap T_1)$ be maximal cross free.

Suppose $\hat{a} \in \mathbb{R}^{\mathcal{N}\times E}$ is a vector such that $\hat{a}|_{E(V)} = (\chi^{E(V)\setminus(F\cup F_1)}, \chi^{E(V)\setminus(F\cup F_2)})$ and the other coefficients are yet undetermined. It turns out that there are many ways to specify the coefficients such that the resulting inequality $\hat{a}^T x \geq \alpha$ is facet-defining for STP $(G, \mathcal{N}, \mathbb{1})$. In fact, the coefficients $\hat{a}^k_e$, $e \in \delta(z)$ can be independently chosen from the following list of alternatives for each net k.

Definition 6.4 (Possible choices for the new coefficients by adding an additional node)

Let $k \in \{1, 2\}$ with $\bar{k} = 1$, if $k = 2$, and $\bar{k} = 2$, if $k = 1$.

(1) If z is a terminal of net k ($z \in T_k$), all coefficients obtain value 1, that is

$$\hat{a}^k_e = 1, \quad \text{for all } e \in \delta(z).$$

(2) If z is not a terminal of net k ($z \notin T_k$), there are the following possibilities.

$$\text{(i)} \quad \hat{a}^k_e = \tfrac{|V|-2}{|V|}, \quad \text{for all } e \in \delta(z).$$

$$\text{(ii)} \quad \begin{aligned} \hat{a}^k_{zt} &= 0, && \text{for one } t \in T_k; \\ \hat{a}^k_e &= 1, && \text{for all } e \in \delta(z) \setminus \{zt\}. \end{aligned}$$

$$\text{(iii)} \quad \begin{aligned} \hat{a}^k_{zt} &= 0, && \text{for one } t \in T_{\bar{k}}; \\ \hat{a}^k_{zt'} &= 0, && \text{for all } t' \in T_{\bar{k}} \text{ with } tt' \in F_k; \\ \hat{a}^k_e &= 1, && \text{for all remaining edges } e \in \delta(z). \end{aligned}$$

Theorem 6.5 *Let $G = (V \cup \{z\}, E)$ be the complete graph with node set $V \cup \{z\}$ and let $\mathcal{N} = \{T_1, T_2\}$ be a net list such that $T_1 \cap V$, $T_2 \cap V$ is a partition of V with $|T_1 \cap V| = |T_2 \cap V| = l$, $l \geq 2$. Furthermore let F be an alternating cycle with respect to T_1, T_2 with $V(F) = V$ and let $F_1 \subseteq E(T_2 \cap V)$, $F_2 \subseteq E(T_1 \cap V)$ be maximal cross free. Let $\hat{a} \in \mathbb{R}^{\mathcal{N} \times E}$ be a vector such that $\hat{a}|_{\delta(z)}$ satisfies one of the alternatives of 6.4 and $\hat{a}|_{E \setminus \delta(z)} = (\chi^{E(V) \setminus (F \cup F_1)}, \chi^{E(V) \setminus (F \cup F_2)})$. Finally, let $\alpha_k = |\{z\} \cap T_k|$ for $k = 1, 2$. Then*

$$\hat{a}^T x \geq l - 1 + \alpha_1 + \alpha_2$$

defines a facet of $STP(G, \mathcal{N}, \mathbb{1})$.

We give a rough idea of the proof. The proof of validity can be reduced to that of Theorem 6.2, i. e., for an arbitrary Steiner tree packing P it can be shown that there exists a Steiner tree packing P' with $\hat{a}^T \chi^{P'} \leq \hat{a}^T \chi^P$ and $|\delta(z) \cap P'_k| = \alpha_k$ for $k = 1, 2$; this implies the validity. To show that the inequality is also facet-defining it remains to fix the new coefficients. This can easily be done, see [M92].

Obviously, in the same manner we can add an arbitrary number of additional nodes $z_1, \ldots, z_c$ to the complete graph $G = (V, E)$ of Theorem 6.2. For each of the nodes z_i we can independently choose the coefficients of the edges in $[z_i : V]$ according to Definition 6.4. The coefficients of the edges that connect two different nodes z_i and z_j can be determined by applying Lemma 4.3.

Grid Inequalities

Definition 6.6 *Let $G = (V, E)$ be a graph and $\mathcal{N} = \{T_1, T_2\}$ be a net list. Furthermore, let $\hat{G} = (\hat{V}, \hat{E})$ be a subgraph of G such that $\hat{G}$ is a complete rectangular $h \times 2$ grid graph with $h \geq 3$. Assume that the nodes of V are numbered such that $\hat{V} = \{(i,j) \mid i = 1, \ldots, h,\ j = 1, 2\}$. Moreover, let $(1,1), (h,2) \in T_1$ and $(1,2), (h,1) \in T_2$. We call the inequality*

$$(\chi^{E \setminus \hat{E}}, \chi^{E \setminus \hat{E}})^T x \geq 1$$

a $h \times 2$ grid inequality.

If we consider in the following a complete rectangular $h \times 2$ grid graph, which is a subgraph of a given graph $G = (V, E)$, we always assume for the ease of notation that the node set V is numbered such that the nodes of the grid graph have a numbering as assumed in Definition 6.6.

Let $G = (V, E)$ be a graph, $F \subset E$ and $u, v \in V$. We call a path $Q_F(u, v)$ from u to v in G a *quasi path from u to v with respect to F*, if there exists an edge $e \in Q_F(u, v)$ such that $e \in E \setminus F$ and $Q_F(u, v) \setminus \{e\} \subseteq F$. Then, the following theorem holds.

Theorem 6.7 *Let $\hat{G} = (\hat{V}, \hat{E})$ be a complete rectangular $h \times 2$ grid graph with $h \geq 3$. Let J_1 and J_2 be the two columns of $\hat{G}$. Let $\mathcal{N} = \{T_1, T_2\}$ be a net list where $T_1 = \{(1,1), (h,2)\}$ and $T_2 = \{(1,2), (h,1)\}$. Furthermore, let $G = (V, E)$ be a graph with $\hat{V} \subseteq V$, $\hat{E} \subseteq E$ such that $[V(J_1) : V(J_2)]$ is a cut in G. Set $F := \hat{E}$ and let $F_1, F_2 \subset E \setminus F$, then the following holds. $STP(G, \mathcal{N}, \mathbb{1})$ is fulldimensional and the inequality*

$$(\chi^{E \setminus (F \cup F_1)}, \chi^{E \setminus (F \cup F_2)})^T x \geq 1$$

defines a facet of $STP(G, \mathcal{N}, \mathbb{1})$ if and only if F_1 and F_2 satisfy the following properties (see Figure 5):

(i) For all $i \in \{1, \ldots, h\}$ (at least) one of the following conditions is fullfilled in $(V, E \setminus F)$:

(a) There exists a quasi path $Q_{F_k}((r, l), (s, l))$, $k, l \in \{1, 2\}$, $r, s \in \{1, \ldots, h\}$ with $r \leq i - |k - l|$ and $s \geq i + 2 - |k - l|$.

(b) There exists a quasi path
$Q_{F_{k_l}}((r_l, l), (s_l, l))$, $k_l \in \{1, 2\}$, $r_l, s_l \in \{1, \ldots, h\}$ with
$r_l \leq i < i + 1 \leq s_l$ for $l = 1, 2$
and a quasi path
$Q_{F_k}((r, l), (s, l))$, $k, l \in \{1, 2\}$, $r, s \in \{1, \ldots, h\}$, $r < s$ that satisfies:
$s - r \geq 2$, if $r \in \{i - 1, i\}$,
$r \neq i$, if $k \neq l$,
$s \neq i$, if $k = l$.

(ii) $\bigcap_{Q \in \mathcal{Q}} Q = \emptyset$, where $\mathcal{Q} = \{Q \subset E \setminus F \mid$ there exist $u, v \in V$ such that Q is a quasi path from u to v with respect to F_1 or F_2 in $(V, E \setminus F)\}$.

(iii) For all $u, v \in V(F)$, $u \neq v$ there does not exist a path from u to v in (V, F_k) for $k = 1, 2$.

(iv) F_1 and F_2 are maximal with respect to the properties (i) – (iii).

Proof. The validity of the inequality is easy to see. There obviously does not exist a Steiner tree packing in $(V(F), F)$, since all nodes of $V(F)$ have degree at most three (with respect to F) and the terminal nodes have degree two (with respect to F). This together with property (iii) implies that the inequality is valid. $\square$

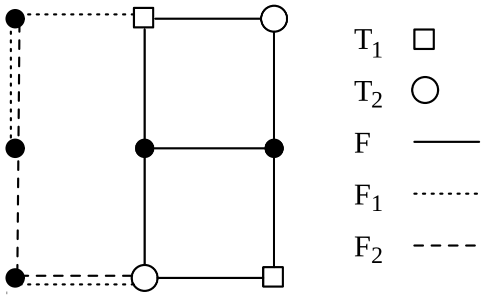


Figure 5:

Note, that the underlying graphs G in Theorem 6.7 need not be complete. In the following we give a formulation for the complete graph.

Theorem 6.8 *Let $G = (V, E)$ be the complete graph with node set V and let $E' \subset E$ be an edge set such that $(V, E \setminus E')$ is a complete rectangular $h \times 2$ grid graph with $h \geq 3$. Let $\mathcal{N} = \{T_1, T_2\}$ be the net list, where $T_1 = \{(1,1),(h,2)\}$ and $T_2 = \{(1,2),(h,1)\}$. Set $F := E'$ and let $F_1, F_2 \subset E \setminus F$. Finally, set $\bar{k} := 3 - k$ for $k = 1, 2$. Then, the inequality*

$$(\chi^{E\setminus(F\cup F_1)}, \chi^{E\setminus(F\cup F_2)})^T x \geq 1$$

defines a facet of $STP(G, \mathcal{N}, \mathbb{1})$ if and only if F_1 and F_2 satisfy the following properties (see Figure 6):
(i) $F_k \subseteq \mathcal{F}_k := \{[(i,\bar{k}),(i+1,k)] \mid i = 1,\ldots,h-1\}$ for $k = 1, 2$.
(ii) For all $[(i_k,\bar{k}),(i_k+1,k)] \in F_k$, $k = 1, 2$ holds $i_1 \neq i_2$.
(iii) F_1 and F_2 are maximal with respect to (i) and (ii).

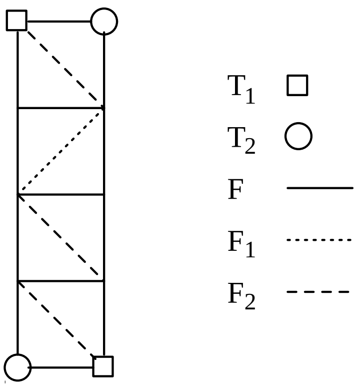


Figure 6:

Proof. Let $P = (S_1, S_2)$ be an arbitrary Steiner tree packing. W. l. o. g. S_1 and S_2 are paths. Suppose that $a^T\chi^P = 0$. For the same reason as in the proof of Theorem 6.7 there does not exist a Steiner tree packing in (V, F). This implies that $(S_1 \cap F_1) \cup (S_2 \cap F_2) \neq \emptyset$. Let $[(i_k, \bar{k}), (i_k + 1, k)] \in (S_1 \cap F_1) \cup (S_2 \cap F_2)$ such that i_k is minimal. We consider the case $k = 1$ (the case $k = 2$ can be shown analogously). Obviously, $J_k \subset S_k$ for $k = 1, 2$, where $J_k := \{[(i, k), (i + 1, k)] \mid i = 1, \ldots, i_1 - 1\}$. Since $[(i_k, \bar{k}), (i_k + 1, k)] \in S_1$ and S_1 is a path, we obtain that either $[(i_1, 1), (i_1, 2)], [(i_1 + 1, 1), (i_1 + 1, 2)] \in S_1$ or $[(i_1, 1), (i_1 + 1, 1)], [(i_1, 2), (i_1 + 1, 2)] \in S_1$. In the first case set $W = \{(i, j) \mid i = 1, \ldots, i_1 - 1, j = 1, 2\} \cup \bigcup_{i \in I}\{[(i, 2), (i + 1, 2)]\}$, where $I = \{i \in \{i_1, \ldots, h\} \mid [(i, 2), (i+1, 1)] \in F_1 \cap S_1$ and $[(i', 2), (i'+1, 1)] \in F_1 \cap S_1$ for all $i' = i_1, \ldots, i-1\}$. In the second case set $W = \{(i, j) \mid i = 1, \ldots, i_1, j = 1, 2\}$. Property (i) and (ii) imply that $(\delta(W) \cap (F \cup F_2)) \setminus S_1 = \emptyset$. Since $(1, 2) \in W$ and $(h, 1) \in V \setminus W$, it follows that $(a^2)^T\chi^{S_2} \geq 1$, a contradiction. $\square$

Critical Cut Inequalities

In this subsection we will describe a quite "small" class of facet-defining inequalities. Nevertheless, they turn out to be very helpful in solving practical problems.

Definition 6.9 *Let $G = (V, E)$ be a graph with edge capacities $c_e \in \mathbb{N}$, $e \in E$. Moreover, let $\mathcal{N} = \{T_1, \ldots, T_N\}$ be a net list. For a node set $W \subseteq V$ we define $S(W) := \{k \in \{1, \ldots, N\} \mid T_k \cap W \neq \emptyset,\ T_k \cap (V \setminus W) \neq \emptyset\}$.*

(a) We call a cut induced by a node set W critical for $(G, \mathcal{N}, c)$, if $s(W) := c(\delta(W)) - |S(W)| \leq 1$.

(b) If V_1, V_2, V_3 is a partition of V such that $\delta(V_1)$ is a critical cut and if $T_1 \cap V_1 = \emptyset$ and $T_1 \cap V_i \neq \emptyset$ for $i = 2, 3$, we call the inequality

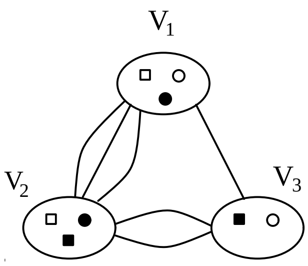
$$x^1([V_2 : V_3]) \geq 1$$

a critical cut inequality with respect to T_1. (See Figure 7.)

Theorem 6.10 *Let $G = (V, E)$ be a graph with $V = \{u, v, w\}$ and let $\mathcal{N} = \{T_1, \ldots, T_N\}$ be a net list with $T_1 = \{u, v\}$. Set $E_{ij} := \{e \in E \mid e$ is incident to u and $v\}$ for $i, j \in V$ and $N_i = \{k \in \{1, \ldots, N\} \mid i \in T_k\}$ for $i \in V$. Assume that $|E_{uv}| \geq 2$, $|N_w| = N - 1$, $|N_u| + |N_v| = N + 1$, $|E_{uw}| \geq |N_u| - 1$, $|E_{vw}| \geq |N_v| - 1$ and $|E_{uw}| + |E_{vw}| = |N_u| + |N_v| - 1$. Then, the inequalilty*

$$x^1(E_{uv}) \geq 1$$

defines a facet of $STP(G, \mathcal{N}, \mathbb{1})$.



Consider the partition V_1,V_2,V_3 in Figure 7. Suppose the capacities of the edges are equal to one. Then, $\delta(V_1)$ is a critical cut. The critical cut inequality says that the net depicted by black rectangles must use at least one of the edges of $[V_2{:}V_3]$.

Figure 7:

Proof. Let S_1 be a Steiner tree for T_1 with $S_1 \cap E_{uv} = \emptyset$. Then we know that $S_1 \cap E_{uw} \neq \emptyset$ and $S_1 \cap E_{vw} \neq \emptyset$. So we obtain that $|\delta(w) \setminus S_1| \leq |N_u| + |N_v| - 3 = N-2$. Since $|N_w| = N-1$ and $1 \notin N_w$ there cannot exist Steiner trees $S_2, \ldots, S_N$, such that $P = (S_1, \ldots, S_N)$ defines a Steiner tree packing. Thus, the inequality is valid. □

We now give a formulation of an inequality for an instance that is very similiar to the one described in Theorem 6.10. We reduce as many parallel edges as possible and require of the edge capacities the necessary conditions. But it is not possible to dispense with parallel edges at all. See also the succeeding remark.

Theorem 6.11 *Let $G = (V, E)$ be a graph with $V = \{u, v, w\}$ and $E = \{e_1, e_2, e_3, e_4\}$ where $e_1 = uw$, $e_2 = vw$ and e_3 and e_4 are both connecting u and v. Let $\mathcal{N} = \{T_1, \ldots, T_N\}$ be a net list with $T_1 = \{u, v\}$. Suppose that $|N_w| = N - 1$ and $|N_u| + |N_v| = N + 1$, where $N_i := \{k \in \{1, \ldots, N\} \mid i \in T_k\}$ for $i \in V$. Moreover, assume that $c_{e_3} + c_{e_4} \geq 3$, if $N_u \setminus \{1\} \neq \emptyset$, and $c_{e_1} \geq |N_u| - 1$, $c_{e_2} \geq |N_v| - 1$ such that $c_{e_1} + c_{e_2} = |N_u| + |N_v| - 1$. Then, the inequality*

$$x^1_{e_3} + x^1_{e_4} \geq 1$$

defines a facet of $STP(G, \mathcal{N}, c)$.

Proof. The proof that the inequality is valid is just the same as the one of Theorem 6.10. Let S_1 be a Steiner tree for T_1 with $e_3, e_4 \notin S_1$. Then, we have $S_1 = \{e_1, e_2\}$. So we obtain for the updated capacities $c'_{e_1} = c_{e_1} - 1$, $c'_{e_2} = c_{e_2} - 1$, $c'_{e_3} = c_{e_3}$ and $c'_{e_4} = c_{e_4}$. Thus, $c'(\delta(w)) = c(\delta(w)) - 2 = N - 2$. Since $|N_w| = N - 1$ and $1 \notin N_w$, there cannot exist Steiner trees $S_2, \ldots, S_N$ such that $P = (S_1, \ldots, S_N)$ is a Steiner tree packing. So, the inequality is valid. □

Conclusions

Research in polyhedral combinatorics in the past years concentrated on easily describable "primal" problems such as the matching, the traveling salesman, the spanning tree or the stable set problem. Very little work has been done on "dual" problems like the edge or node colouring or packing problems. One of the reasons for this is certainly that the latter problems may give rise to integer programming models with an exponential number of rows and columns, that they often combine the difficulties of various subproblems and that additional subtilities creep in by considering graph structures, capacities etc. jointly. The technicalities involved require enormous mathematical machinery and seem insurmountable in the general case.

We have encountered the Steiner tree packing problem in practice and considered it worthwhile to engage in a study of Steiner tree packing polytopes in order to get some experience in the investigation of packing problems from this point of view. As can be seen from the results of this paper, all expected difficulties turned up. The dimension of the Steiner tree packing polyhedron is hard to determine. Thus one has to resort to a "handy" but representative special case. The difficulties of the individual Steiner tree polyhedrons are inherited by the packing polytope. Finding rich classes of valid joint inequalities and describing them in an understandable manner is a considerable challenge; characterizing the facet-defining inequalities in these classes leads to tremendous technical difficulties. To keep this paper within acceptable space limits we have given only a few complete proofs and restricted ourselves to the presentation of a few classes of joint facets, namely those that have been used in [GMW92]. Further classes of facet-defining inequalities will be presesented in a forthcoming paper, see [GMW93].

The most important objective of our research project, however, was to see whether the machinery of polyhedral combinatorics can help solve practical instances of Steiner tree packing problems. Indeed, it can — to some extend. This topic will be discussed in our companion paper [GMW92].

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