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Chromatic Scheduling Polytopes coming from the Bandwidth Allocation Problem in Point-to-Multipoint Radio Access Systems

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Abstract

Point-to-Multipoint systems are one kind of radio systems supplying wireless access to voice/data communication networks. Such systems have to be run using a certain frequency spectrum, which typically causes capacity problems. Hence it is, on the one hand, necessary to reuse frequencies but, on the other hand, no interference must be caused thereby. This leads to the bandwidth allocation problem, a special case of so-called chromatic scheduling problems. Both problems are NP-hard, and there exist no polynomial time approximation algorithms with a guaranteed quality. One kind of algorithms which turned out to be successful for many other combinatorial optimization problems uses cutting plane methods. In order to apply such methods, knowledge on the associated polytopes is required. The present paper contributes to this issue, exploring basic properties of chromatic scheduling polytopes and several classes of facet-defining inequalities.

Key Words: Bandwidth allocation, polyhedral combinatorics

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1 Introduction

Point-to-Multipoint Radio Access Systems (PMP-Systems) are one kind of radio systems supplying wireless access to voice/data communication networks. Base stations form the access points to the backbone network. Customer terminals are linked to base stations by means of radio signals. Some specific part of the radio frequency spectrum has to be used only to maintain the links. This typically causes capacity problems and, therefore, it is necessary to reuse frequencies.

There are two main differences between PMP-Systems and cellular phone networks. Firstly, each customer is provided a fixed antenna and is assigned to a certain sector of a base station (see Figure 1a). Secondly, the customers do not have a unique communication demand, but each customer has an individual one, hence the task is to assign frequency intervals instead of single channels (see Figure 1c).

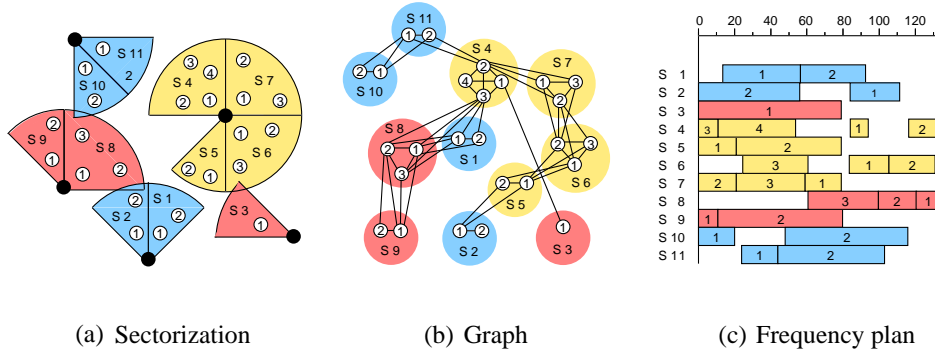


Figure 1: Bandwidth allocation in Point-to-Multipoint radio access systems.

A central issue is that a link connecting a customer terminal and a base station may be subject to interference from another link that uses the same frequency. In particular, links to customers of the same sector must not use the same frequency, since they are served by the same antenna of a base station. In addition, some links of customers in different sectors may also cause interferences (see Figure 1b). The task of reusing frequencies due to capacity constraints but providing an interference-free communication leads to the *bandwidth allocation problem*. The goal is to assign a frequency interval within the available radio frequency spectrum to each customer (see Figure 1c), taking into account the individual communication demands, possible interference, and several technical and legal restrictions.

This kind of problems is known in the literature in slightly different versions as *chromatic scheduling problem* [4] or, in some particular cases, as *consecutive coloring problem* [5] and *interval coloring problem* [7, 8]. Such problems are NP-hard in general [8] and there are no polynomial time approximation algorithms with a guaranteed quality. Small instances of the

bandwidth allocation problem could be solved by greedy-like heuristics [1], but in order to tackle problem sizes of real world applications, algorithms have to be designed that rely on a deeper insight of the problem structure. One kind of algorithms which turned out to be successful for many other applications uses *cutting plane methods* [3]. To apply such methods to the bandwidth allocation problem, the polytope representing the convex hull of the incidence vectors of all feasible solutions of the problem has to be studied. The present paper contributes to this issue.

This paper is organized as follows: Section 2 gives a precise description of the bandwidth allocation problem and presents an integer programming formulation. Section 3 starts with the initial study of the associated polytopes. Section 4 presents several facet-defining inequalities. We end up with some concluding remarks and open problems.

2 Problem description

In this section we discuss the details of the bandwidth allocation problem and provide an integer programming formulation. We interpret this problem as a scheduling problem and introduce the associated polytopes.

The input of this problem is given as follows. Let $\mathcal{T} = \{t_1, \dots, t_n\}$ be the set of all customer terminals, and $\mathcal{S} = \{S_1, \dots, S_k\}$ be a partition of \mathcal{T} into sectors, providing the information to which sector S_j the terminal $t_i \in \mathcal{T}$ belongs. Let $d = (d_1, \dots, d_n)$ be the vector of communication demands associated with the customer terminals, indicating that customer $t_i \in \mathcal{T}$ has demand $d_i \in \mathbf{Z}$. Additionally, we have a set \mathcal{E}_X of unordered pairs (t_i, t_j) of terminals in *different* sectors that must not use the same frequency due to possible interference.

This setting can be viewed as a weighted graph $(G, d) = (V, E, d)$, where

- $V = \{i : t_i \in \mathcal{T}\}$ is the node set,
- $E = E_X \cup E_I$ is the edge set with

$$\begin{aligned} E_I &= \{ij : t_i, t_j \text{ in the same sector } S_l \in \mathcal{S}\}, \\ E_X &= \{ij : (t_i, t_j) \in \mathcal{E}_X\}, \end{aligned}$$

- $d = (d_1, \dots, d_n)$ is the node weighting.

Thus, the nodes represent customer terminals, the node weights reflect the communication demands, and the edges indicate potential interference between the customer terminals. The latter is given by the set of external interferers \mathcal{E}_X and the partition of the node set V corresponding to the sectorization of \mathcal{T} . In graph theoretical terms, the partition of \mathcal{T} into sectors $\mathcal{S} = \{S_1, \dots, S_k\}$ corresponds to a clique covering of G , i.e., to a partition

of V into k subsets V_1, \dots, V_k such that the nodes in every V_i are pairwise adjacent.

Moreover, a guard distance g is given that must be kept between intervals of terminals $(t_i, t_j) \in \mathcal{E}_X$. Finally, we have the available radio frequency spectrum $[0, s]$ with $s \in \mathbf{Z}$ where all the frequency intervals have to be placed in.

The task is to provide, for each customer $t_i \in \mathcal{T}$, a certain part of the available frequency spectrum meeting the following two conditions. Firstly, the individual communication demand d_i is satisfied. Secondly, the assignment does not cause interference, i.e., no terminal within the same sector uses the same frequencies, and the guard distance is obeyed for each external interferer t_j , $(t_i, t_j) \in \mathcal{E}_X$.

The desired output is, therefore, an assignment of an interval $I(i) = [l_i, r_i]$ with $l_i, r_i \in \mathbf{Z}$ to each customer $t_i \in \mathcal{T}$ such that:

- $r_i - l_i \geq d_i$,
- $[l_i, r_i] \subseteq [0, s]$,
- $\max\{l_i, l_j\} - \min\{r_i, r_j\} \geq \begin{cases} 0 & \text{if } t_i \text{ and } t_j \text{ belong to the same sector} \\ g & \text{if } (t_i, t_j) \in \mathcal{E}_X. \end{cases}$

To represent a solution, we use besides the interval bounds l_i and r_i for all $i \in V$, in addition the ordering variables

$$x_{ij} = \begin{cases} 1 & \text{if } r_i \leq l_j, i < j \\ 0 & \text{otherwise} \end{cases}$$

for all $ij \in E$, indicating a precedence relation on the set of intervals, due to the following reason. In order to avoid interference, certain pairs of intervals must not overlap. The antiparallelity requirements for intervals of potential interferers are realized, in every feasible solution, by a precedence relation (i.e. a partial order) on the set of intervals. A valid schedule is, therefore, an assignment of values to $l_i, r_i \forall i \in V$ and $x_{ij} \forall ij \in E$ such that the following inequalities are satisfied:

$$d_i \leq r_i - l_i \quad \forall i \in V \quad (1)$$

$$0 \leq l_i \leq r_i \leq s \quad \forall i \in V \quad (2)$$

$$r_i \leq l_j + s(1 - x_{ij}) \quad \forall ij \in E_I, i < j \quad (3)$$

$$r_i + g \leq l_j + s(1 - x_{ij}) \quad \forall ij \in E_X, i < j \quad (4)$$

$$r_j \leq l_i + s x_{ij} \quad \forall ij \in E_I, i < j \quad (5)$$

$$r_j + g \leq l_i + s x_{ij} \quad \forall ij \in E_X, i < j \quad (6)$$

$$x_{ij} \in \{0, 1\} \quad \forall ij \in E, i < j \quad (7)$$

$$l_i, r_i \in \mathbf{Z} \quad \forall i \in V \quad (8)$$

The *demand constraints* (1) and the *bound constraints* (2) assert that the interval $I(i) = [l_i, r_i]$ must satisfy the demand d_i and fit within the available frequency spectrum $[0, s]$. Inequalities (3) to (6) realize the *antiparallelity constraints*, which prevent interfering pairs of intervals from overlapping. Finally, the *integrality constraints* (7) resp. (8) force the x -variables to be binary resp. the interval bounds to be integral.

Remark. This setting may be interpreted as *k-machine scheduling problem* where the k sectors correspond to the k machines, and the customer terminals to the jobs. In our case, the assignment of jobs to machines is fixed. The processing time of a job corresponds to the communication demand of the customer terminal. That no machine can process two jobs at the same time is given by E_I (recall that \mathcal{S} corresponds to a clique covering of G by k cliques), where E_X gives antiparallelity requirements between jobs processed on different machines. Moreover, g can be interpreted as changeover time, and s as upper bound on the allowed makespan $\max\{r_i : i \in V\} - \min\{l_j : j \in V\}$ with respect to a schedule (for more information on general scheduling problems see, e.g., [2]).

Moreover, if $d = \mathbf{1}$, i.e., $d_i = 1 \ \forall i \in V$, and $g = 0$, then we obtain the ordinary graph coloring problem as special case of the bandwidth allocation problem, where s corresponds to the maximal number of available colors.

One is mainly interested in finding *feasible* solutions satisfying all the constraints above. If this is possible, the task is, e.g., to find a span-minimal solution among all feasible ones (there are several other possible objective functions which we won't discuss here). In order to apply cutting plane methods to solve the problem, we are interested in investigating the convex hull of all feasible solutions satisfying these constraints.

Definition 1 (Chromatic scheduling polytope) Let $(G, d) = (V, E, d)$ be a graph with node weights d , and let s be the allowed makespan. The chromatic scheduling polytope $P(G, d, s) \subseteq \mathbf{R}^{2|V|+|E|}$ is defined as the convex hull of all integer solutions $(l, r, x) \in \mathbf{R}^{2|V|+|E|}$ satisfying constraints (1)-(8).

A special case of the bandwidth allocation problem is of particular interest, namely the case where each customer receives an interval $I(i) = [l_i, r_i]$ which has *precisely* the length of its demand, i.e., $r_i - l_i = d_i$. This case is in practice easier to solve and the solution space has lower dimension since the right interval bounds can be dropped for representing solutions. Thus, only the l - and x -variables are required, and every solution vector has only $|V| + |E|$ entries instead of the $2|V| + |E|$ entries in the general case.

Definition 2 (Fixed length chromatic scheduling polytope) Let us denote by $R(G, d, s) \subseteq \mathbf{R}^{|V|+|E|}$ the fixed-length chromatic scheduling poly-

tope, defined as the convex hull of all feasible solutions $(l, x) \in \mathbf{R}^{|V|+|E|}$ satisfying $r_i = l_i + d_i$ and constraints (2)-(8).

3 General facts on chromatic scheduling polytopes

In this section we explore a special property of chromatic scheduling polytopes, namely their symmetry, and we provide some partial results on the dimension of these polytopes.

Recall that we only have antiparallelity constraints for potential interferers $ij \in E$. Hence, in a feasible solution either the interval $I(i)$ of i has to be scheduled before the interval $I(j)$ of j or $I(j)$ comes before $I(i)$. Thus, for every feasible schedule S , there is a feasible schedule symmetric to S w.r.t. the available spectrum $[0, s]$, obtained by swapping all intervals of S . This is obviously not true for scheduling problems in general. The reason is that we have only antiparallelity requirements for chromatic scheduling problems, but no precedence relation on the jobs given in advance. Clearly, the polytopes $P(G, d, s)$ and $R(G, d, s)$ reflect the symmetry of the schedules. The polytope $R(G, d, s)$ is even symmetric to a certain point [6], namely,

$$p = \frac{1}{2}(s\mathbf{1} - d, \mathbf{1}) = \left(\underbrace{\frac{s-d_1}{2}, \dots, \frac{s-d_n}{2}}_{i \in V}, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{ij \in E} \right),$$

where $\mathbf{1} = (1, \dots, 1)$.

Definition 3 If $y \in R(G, d, s)$ is a feasible integer solution, then $\text{sym}(y) = 2p - y$ denotes its symmetrical solution, i.e.,

$$\text{sym} \begin{pmatrix} l \\ x \end{pmatrix} = \begin{pmatrix} s\mathbf{1} - d \\ \mathbf{1} \end{pmatrix} - \begin{pmatrix} l \\ x \end{pmatrix}.$$

Due to the symmetry of the polytope $R(G, d, s)$, to every face exists a parallel face of the same dimension and there is a simple formula to compute this parallel face.

Theorem 1 Let $b \leq a^T x$ be a valid (facet-inducing) inequality of $R(G, d, s)$. Then $a^T x \leq 2a^T p - b$ is also valid (facet-inducing) for $R(G, d, s)$.

Proof. Let S be a feasible fixed-length schedule and let $y^S \in R(G, d, s)$ be its associated vector. Consider the hyperplane $H = \{x \in \mathbf{R}^{|V|+|E|} : a^T x = b\}$. The straight line through y^S and the symmetry point p of $R(G, d, s)$ meets H in a point, say y_H^S . Then $\text{sym}(y_H^S)$ lies on a hyperplane $\text{sym}(H) = \{x \in \mathbf{R}^{|V|+|E|} : a^T x = b'\}$ and $R(G, d, s) \subseteq \{x \in \mathbf{R}^{|V|+|E|} : -b' \leq -a^T x\}$ follows by $R(G, d, s) \subseteq \{x \in \mathbf{R}^{|V|+|E|} : b \leq a^T x\}$ and the symmetry of the polytope, i.e., $a^T x \leq b'$ is valid for $R(G, d, s)$. It is left to determine b' . Since

p is the symmetry point, $\text{sym}(y_H^S) = 2p - y_H^S$ holds. Thus $a^T y_H^S = b$ and $a^T \text{sym}(y_H^S) = b'$ imply

$$b' = a^T \text{sym}(x_H^S) = a^T (2p - y_H^S) = 2a^T p - a^T y_H^S = 2a^T p - b$$

and $a^T x \leq 2a^T p - b$ is, therefore, the valid inequality of $R(G, d, s)$ symmetric to $b \leq a^T x$. If there are k affinely independent points in $H \cap R(G, d, s)$, there are obviously k affinely independent points in $H' \cap R(G, d, s)$. Thus, if $b \leq a^T x$ is facet-inducing for $R(G, d, s)$, then $a^T x \leq 2a^T p - b$ is facet-defining too. \square

Moreover, this symmetry provides us some tools for identifying facet-defining inequalities.

Theorem 2 *Let F be a face of $R(G, d, s)$ such that $y \in F \Leftrightarrow \text{sym}(y) \notin F$ for every $y \in R(G, d, s) \cap \mathbf{Z}^{|V|+|E|}$. Then F is a facet of $R(G, d, s)$.*

Proof. Assume that $\dim(F) = k$, and let y_0, \dots, y_k be a maximal set of affinely independent points in F . Let $y_{k+1} \notin F$ be any feasible solution outside F . Then, y_0, \dots, y_k, y_{k+1} are affinely independent, because y_0, \dots, y_k satisfy the equation which defines F and y_{k+1} does not.

Now let $y_{k+2} \notin F$ be some other feasible solution not in F . Note that $\text{sym}(y_{k+1})$ and $\text{sym}(y_{k+2})$ are in F , and thus they can be written as affine combinations of y_0, \dots, y_k . Then,

$$\begin{aligned} y_{k+2} - y_{k+1} &= \begin{pmatrix} s & \mathbf{1} - d \\ & \mathbf{1} \end{pmatrix} y_{k+1} - \begin{pmatrix} s & \mathbf{1} - d \\ & \mathbf{1} \end{pmatrix} y_{k+2} \\ &= \text{sym}(y_{k+1}) - \text{sym}(y_{k+2}) \\ &= \sum_{i=0}^k \alpha_i y_i - \sum_{i=0}^k \beta_i y_i \\ &= \sum_{i=0}^k (\alpha_i - \beta_i) y_i, \end{aligned}$$

where $\sum_i \alpha_i = \sum_i \beta_i = 1$. But then

$$y_{k+2} = y_{k+1} + \sum_{i=0}^k (\alpha_i - \beta_i) y_i$$

implies that y_{k+2} is an affine combination of the points y_0, \dots, y_k, y_{k+1} . This proves $\dim(R(G, d, s)) = \dim(F) + 1$, and thus F is a facet of $R(G, d, s)$. \square

There is also a symmetrical solution S' of every feasible schedule S in the general case. Here, the swapping of the intervals maps the left interval bounds of the intervals of S to the right interval bounds of the intervals of S' , and vice versa.

Definition 4 The symmetrical point of a solution $y \in P(G, d, s)$ is $\text{sym}(y)$ with

$$\text{sym} \begin{pmatrix} l \\ r \\ x \end{pmatrix} = \begin{pmatrix} s & \mathbf{1} - r \\ s & \mathbf{1} - l \\ \mathbf{1} - x \end{pmatrix} = \begin{pmatrix} s & \mathbf{1} \\ s & \mathbf{1} \\ \mathbf{1} \end{pmatrix} - \begin{pmatrix} r \\ l \\ x \end{pmatrix}.$$

The symmetry for the general case also provides some tools for identifying facet-defining inequalities. In order to state these results, we need to introduce some notations.

Definition 5 Define $F_s(G)$ as the set of nodes i such that there exists some feasible solution of $P(G, d, s)$ in which interval $I(i)$ has length strictly greater than d_i . That is, $F_s(G) := \{i \in V_G : y_{r_i} - y_{l_i} > d_i \text{ for some } y \in P(G, d, s)\}$.

Definition 6 Let $y \in R(G, d, s)$. The extension of y is $\text{ext}(y) \in P(G, d, s)$ with

$$\begin{aligned} \text{ext}(y)_{l_i} &= y_{l_i} & i &= 1, \dots, n \\ \text{ext}(y)_{r_i} &= y_{l_i} + d_i & i &= 1, \dots, n \\ \text{ext}(y)_{x_{ij}} &= y_{x_{ij}} & ij &\in E_G \end{aligned}$$

Conversely, the reduction of a point $z \in P(G, d, s)$ is $\text{red}(z) \in R(G, d, s)$ defined by:

$$\begin{aligned} \text{red}(z)_{l_i} &= z_{l_i} & i &= 1, \dots, n \\ \text{red}(z)_{x_{ij}} &= z_{x_{ij}} & ij &\in E_G. \end{aligned}$$

Remark. Note that $\text{red}(\text{ext}(y)) = y$ holds but $\text{ext}(\text{red}(y))$ differs from y if $y_{r_j} - y_{l_j} > d_j$ for some $j \in V$.

Lemma 1 $\dim(P(G, d, s)) = \dim(R(G, d, s)) + |F_s(G)|$.

Proof. For each $i \in F_s(G)$ let $y^i \in P(G, d, s)$ be a solution such that $y_{r_i}^i - y_{l_i}^i > d_i$ and $y_{r_j}^i - y_{l_j}^i = d_j$ for $j \neq i$ (such a solution exists by the definition of $F_s(G)$). Now, if $w^0, \dots, w^k \in R(G, d, s)$ is a set of affinely independent points, then $\text{ext}(w^0), \dots, \text{ext}(w^k)$ are also affinely independent, and moreover $\text{ext}(w)_{r_i} = \text{ext}(w)_{l_i} + d_i$ for $i = 1, \dots, n$. Then, $w^0, \dots, w^k, y^1, \dots, y^{|F_s(G)|}$ are affinely independent points in $P(G, d, s)$, and thus $\dim(R(G, d, s)) + |F_s(G)| \leq \dim(P(G, d, s))$.

For the reverse inequality, let z^0, \dots, z^k be a maximal set of affinely independent points in $P(G, d, s)$. We have that $\text{red}(z^0), \dots, \text{red}(z^k)$ are points in $R(G, d, s)$ and

$$\dim\{\text{red}(z^0), \dots, \text{red}(z^k)\} = \dim(R(G, d, s)) = \dim(P(G, d, s)) - |F_s(G)|,$$

since $y_{r_j}^i = y_{l_j}^i + d_i$ for every $i \notin F_s(G)$. We constructed $\dim(P(G, d, s)) - |F_s(G)|$ affinely independent points in $R(G, d, s)$ and thus $\dim(R(G, d, s)) \geq \dim(P(G, d, s)) - |F_s(G)|$. \square

Theorem 3 Let $F = \{y \in R(G, d, s) : a^T y = b\}$ be a face of $R(G, d, s)$ such that $\text{red}(z) \in F \Leftrightarrow \text{red}(\text{sym}(z)) \notin F$ for every $z \in P(G, d, s) \cap \mathbf{Z}^{2|V|+|E|}$. Then $F' = \{z \in P(G, d, s) : a^T \text{red}(z) = b\}$ is a facet of $P(G, d, s)$.

Proof. If $y \in R(G, d, s)$, then $\text{ext}(y) \in P(G, d, s)$. By the hypothesis, we have that either $\text{red}(\text{ext}(y)) \in F$ or $\text{red}(\text{sym}(\text{ext}(y))) \in F$ (but not both). But $\text{red}(\text{ext}(y)) = y$ and $\text{red}(\text{sym}(\text{ext}(y))) = \text{sym}(y)$ imply $y \in F \Leftrightarrow \text{sym}(y) \notin F$. F is a facet of $R(G, d, s)$ by Theorem 2. Let $r = \dim(R(G, d, s))$, then there exist r affinely independent vectors y^1, \dots, y^r in the facet F (i.e., with $a^T y^k = b$). Then, $\text{ext}(y^1), \dots, \text{ext}(y^r)$ are affinely independent points satisfying $a^T \text{red}(\text{ext}(y^k)) = b$ by definition.

Now, for each $k \in F_s(G)$ let $z^k \in P(G, d, s)$ be a solution such that $z_{r_k}^k - z_{l_k}^k > d_k$ and $z_{r_l}^k - z_{l_l}^k = d_l$ for $l \neq k$. We can assume that $\text{red}(z^k) \in F'$ (otherwise, consider the reduction of its symmetrical point $\text{sym}(z^k)$). Define the following set of feasible solutions:

$$A = \{\text{ext}(y^1), \dots, \text{ext}(y^r)\} \cup \{z^k : k \in F_s(G)\}.$$

For every $k \in F_s(G)$, z^k is affinely independent w.r.t. the points in $A \setminus \{z^k\}$, since all the points in $A \setminus \{z^k\}$ satisfy $r_k - l_k = d_k$, but z_k does not. This way we have by Lemma 1 $|A| = \dim(R(G, d, s)) + |F_s(G)| = \dim(P(G, d, s))$ affinely independent points in $P(G, d, s)$ which satisfy $a^T \text{red}(z) \leq b$ at equality and this inequality defines, therefore, a facet of $P(G, d, s)$. \square

Corollary 1 Let $F = \{y \in R(G, d, s) : a^T y = b\}$ be a face of $R(G, d, s)$ such that $y \in F \Leftrightarrow \text{sym}(y) \notin F$ for every $y \in R(G, d, s) \cap \mathbf{Z}^{2|V|+|E|}$ and $\text{proj}_l(a) = 0$ (i.e. only x -variables have nonnegative coefficients in $a^T y \leq b$). Then $F' = \{z \in P(G, d, s) : a^T \text{red}(z) = b\}$ is a facet of $P(G, d, s)$.

Proof. We verify that the assumptions of Theorem 3 are satisfied. Consider any feasible solution $z \in P(G, d, s)$. By the hypothesis, we know that $\text{red}(z) \in F \Leftrightarrow \text{sym}(\text{red}(z)) \notin F$. Moreover,

$$\begin{aligned} a^T \text{red}(\text{sym}(z)) &= \text{proj}_x(a) \text{proj}_x(\text{red}(\text{sym}(z))) \\ &= \text{proj}_x(a) \text{proj}_x(\text{sym}(\text{red}(z))) \\ &= a^T \text{sym}(\text{red}(z)). \end{aligned}$$

Then, we have that

$$\begin{aligned} \text{red}(z) \in F &\Leftrightarrow a^T \text{red}(z) = b \\ &\Leftrightarrow a^T \text{sym}(\text{red}(z)) < b \\ &\Leftrightarrow a^T \text{red}(\text{sym}(z)) < b \\ &\Leftrightarrow \text{red}(\text{sym}(z)) \notin F. \end{aligned}$$

So, the hypotheses of Theorem 3 are satisfied, and thus F' is a facet of $P(G, d, s)$. \square

G	Variables	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$	$s = 6$	$s = 7$
C_3	6	—	—	3	6	6	6	...
C_4	8	—	1	7	8	8	8	...
C_{2n+3}	$4n + 6$	—	—	$4n + 6$	$4n + 6$	$4n + 6$...	
C_{2n+4}	$4n + 8$	—	1	$4n + 8$	$4n + 8$	$4n + 8$...	
P_n	$2n - 1$	—	1	$2n - 1$	$2n - 1$	$2n - 1$...	
$K_{1,n}$	$2n + 1$	—	1	$2n + 1$	$2n + 1$	$2n + 1$...	
$K_4 - e$	9	—	—	3	9	9	9	...
$K_5 - e$	14	—	—	—	6	14	14	14
$K_6 - e$	20	—	—	—	—	10	20	20

Table 1: Dimension of $R(G, \mathbf{1}, s)$ as a function of s .

A common way of proving that a valid inequality is facet-defining for a certain polytope is to construct as many affinely independent points in the particular hyperplane as the dimension of the polytope is. However, determining the dimension of chromatic scheduling polytopes turns out to be a difficult task. It is easy to verify that $\dim(P(G, d, s))$ and $\dim(R(G, d, s))$ are nondecreasing functions of s . Obviously both polytopes are empty as long as s is strictly less than the weighted clique number $\omega(G, d) = \max\{\sum_{i \in Q} d_i : Q \subseteq G \text{ is a clique}\}$. As s increases, more feasible solutions exist. From a certain value of s on, all possible orderings among the intervals are feasible and the polytopes are full-dimensional. Table 1 shows some examples. P_n , C_n and K_n denote a chordless path, cycle and complete graph, respectively, on n nodes. $K_{n,m}$ stands for a complete bipartite graph with color classes A , B and $n = |A|$, $m = |B|$.

A lower bound on s such that both polytopes $P(G, d, s)$ and $R(G, d, s)$ are full-dimensional if s increases this bound has been obtained in [9]. Further partial results can be found in [9] showing some cases in which these polytopes are not full-dimensional.

4 Facets of $P(G, d, s)$ and $R(G, d, s)$

We now turn into devising facet-defining inequalities for chromatic scheduling polytopes. A first issue to tackle is determining which of the model constraints introduced in Section 2 induce facets of the polytopes. In this section, we give a characterization of the cases where the demand constraints (1) define facets of $P(G, d, s)$. We also show that the lower and upper bounds on the ordering variables $0 \leq x_{ij} \leq 1 \ \forall ij \in E$ implied by constraints (7) are always facet-defining whenever the polytopes are nonempty, and we present a further class of valid inequalities which admits the same property. In this section, we assume the polytopes to be nonempty. We start with the bounds on the ordering variables.

Theorem 4 *If $ij \in E$, then $x_{ij} \geq 0$ and $x_{ij} \leq 1$ define facets of $R(G, d, s)$ and $P(G, d, s)$.*

Proof. Let $F = \{y \in R(G, d, s) : y_{x_{ij}} = 1\}$ be the face defined by $x_{ij} \leq 1$, i.e., the convex hull of the set of points having $I(i)$ before $I(j)$. A point has $I(i)$ before $I(j)$ if and only if its symmetrical point has $I(j)$ before $I(i)$, and thus $y \in F \Leftrightarrow \text{sym}(y) \notin F$. Theorem 2 shows that F is a facet of $R(G, d, s)$, and Corollary 1 implies that $F' = \{z \in P(G, d, s) : z_{x_{ij}} = 1\}$ is a facet of $P(G, d, s)$. The same argumentation applies to $x_{ij} \geq 0$. \square

Definition 7 (Triangle inequalities) *Consider a triangle $T = i, j, k$ of G , i.e., a set of three pairwise adjacent nodes of G . We define*

$$x_{ij} + x_{jk} + x_{ki} \leq 2 \quad (9)$$

to be the triangle inequality associated with T .

Theorem 5 *Triangle inequalities define facets of $R(G, d, s)$ and $P(G, d, s)$.*

Proof. It is easy to verify that triangle inequalities are valid for both polytopes, since $x_{ij} = x_{jk} = x_{ki} = 1$ is obviously not possible in any feasible solution.

Let $y \in R(G, d, s)$ be an integer solution. Since $\{i, j, k\}$ is a complete subgraph, the intervals $I(i)$, $I(j)$ and $I(k)$ cannot overlap in y . Thus y contains one of the six configurations depicted in Figure 2. Note that the cases (a), (b), and (c) satisfy (9) at equality, whereas the cases (d), (e), and (f) do not. Moreover, the cases (a), (b), resp. (c) are the symmetric cases of (d), (e), resp. (f). Thus, if F is the face defined by (9), then $y \in F \Leftrightarrow \text{sym}(y) \notin F$ holds. Theorem 2 resp. Corollary 1 implies that F is a facet of $R(G, d, s)$ resp. $P(G, d, s)$. \square

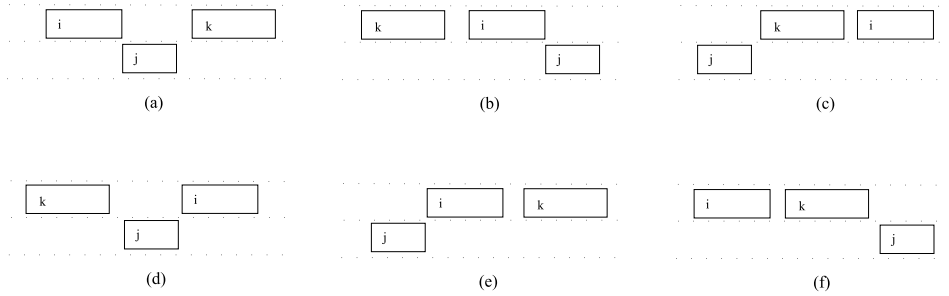


Figure 2: Possible cases for y .

Let us now analyze the demand constraints (1) for $P(G, d, s)$ (recall that these constraints are replaced by equalities in the definition of $R(G, d, s)$). If there is a node $i \notin F_s(G)$, i.e., if every point in $P(G, d, s)$ satisfies $l_i + d_i \leq r_i$

at equality, then $P(G, d, s) \subseteq \{y : y_{l_i} + d_i = y_{r_i}\}$ holds. On the other hand, if $i \in F_s(G)$, i.e., if there exists a feasible solution $z \in P(G, d, s)$ with $z_{l_i} + d_i < z_{r_i}$, then the latter inequality is a proper face of $P(G, d, s)$ and defines a facet.

Theorem 6 *If $i \in F_s(G)$, then the demand constraint $l_i + d_i \leq r_i$ defines a facet of $P(G, d, s)$.*

Proof. Call $\dim(P(G, d, s)) = k$, and let $y^0, \dots, y^k \in P(G, d, s)$ be $k + 1$ affinely independent points in P ($y^j \in \mathbf{R}^{2n+m}$). For $i = 0, \dots, k$, consider the vector \bar{y}^i obtained from y^i by replacing its r_i -entry by $y_{l_i}^i + d_i$. Note that this shrinks the interval $I(i)$ to its minimum length d_i in every y^i , leaving the remaining intervals unchanged, which does not introduce any infeasibility. These new points lie in the face F of $P(G, d, s)$ defined by $l_i + d_i \leq r_i$. Moreover, since $\dim\{y^0, \dots, y^k\} = k$ then $\dim\{\bar{y}^0, \dots, \bar{y}^k\} \geq k - 1$. But there is a point $z \in P(G, d, s)$ which does not satisfy the demand constraint $l_i + d_i \leq r_i$ at equality, and thus $\dim\{\bar{y}^0, \dots, \bar{y}^k\} = k - 1$, implying that this inequality defines a facet of $P(G, d, s)$. \square

The bound constraints (2) and the antiparallelity constraints (3)-(6) do not define facets in general. For example, suppose $P(G, d, s)$ is full-dimensional. All the points $z \in P(G, d, s)$ which satisfy $l_i \geq 0$ at equality must have $z_{x_{ij}} = 1$ for all $j \in N(i) = \{j \in V : ij \in E\}$. Thus, the face $\{z \in P(G, d, s) : z_{l_i} = 0\}$ is strictly contained in the facet $\{z \in P(G, d, s) : z_{x_{ij}} = 1\}$.

Furthermore, it is not difficult to verify that all the points $z \in P(G, d, s)$ satisfying the antiparallelity constraint (3) at equality must have $z_{x_{ik}} = z_{x_{jk}}$ for every $k \in N(i) \cap N(j)$. Thus the face defined by (3) cannot have the required dimension. The same is true for the constraints (4)-(6), showing that these inequalities do not define facets of $P(G, d, s)$ if $N(i) \cap N(j) \neq \emptyset$.

However, we present a strengthening of these constraints so that the resulting inequalities define facets of $P(G, d, s)$ and $R(G, d, s)$. To this end, we shall consider the following structure in $N(i) \cap N(j)$:

Definition 8 (Clique-star) *We say that (K, S) is a clique-star if K and S are disjoint node sets such that K induces a clique and every node $k \in S$ satisfies $d_k \leq \sum_{i \in K \setminus N(k)} d_i$.*

Definition 9 (Spanning clique-star) *Let $H = (V', E')$ be an induced subgraph of G . If V' admits a partition $V' = K \cup S$ s.t. (K, S) is a clique-star, then (K, S) is called a spanning clique-star of H .*

Proposition 1 *There exists a spanning clique-star of G , and it can be found in polynomial time.*

Proof. Let i_1, i_2, \dots, i_n be an ordering of the nodes in V such that $d_{i_k} \geq d_{i_{k+1}}$. Consider every node in this sequence and construct K and S iteratively as follows. At step k , we must decide whether i_k has to be inserted into K or into S . If there is some $i_t \in K$ with $i_k i_t \notin E$, then insert i_k into S . Otherwise, insert i_k into K . Note that in both cases (K, S) remains a clique-star due to the ordering of the nodes, so upon termination of the algorithm (K, S) is a spanning clique-star of G . This procedure is clearly polynomial. \square

Definition 10 (Clique-star inequalities) Let $ij \in E$ be an edge of G , and let (K, S) be a spanning clique-star of $N(i) \cap N(j)$. We call

$$r_i + \sum_{k \in K} d_k(x_{ik} - x_{jk}) \leq l_j + (s - d(K))x_{ji} \quad (10)$$

the clique-star inequality associated with (K, S) , where $d(K) = \sum_{k \in K} d_k$.

Lemma 2 The clique-star inequalities (10) are valid for $P(G, d, s)$.

Proof. Let $y \in P(G, d, s)$ be any integer feasible solution, and consider two cases:

Case 1: $y_{x_{ij}} = 1$. The interval $I(i)$ is located to the left of $I(j)$. Let $M \subseteq K$ be the set of nodes k such that the interval $I(k)$ is between the intervals $I(i)$ and $I(j)$, i.e., $M = \{k \in K : y_{x_{ik}} = 1 \text{ and } y_{x_{jk}} = 0\}$. Since $K \cup \{i, j\}$ is a clique, the corresponding intervals cannot overlap, and thus $y_{l_j} - y_{r_i} \geq d(M)$ implies that every $y \in P(G, d, s)$ satisfies (10).

Case 2: $y_{x_{ij}} = 0$. In this case, the interval $I(j)$ is before $I(i)$. Partition $K = L \cup M \cup R$ as follows:

$$\begin{aligned} L &= \{k \in K : y_{x_{jk}} = 0\} \\ M &= \{k \in K : y_{x_{jk}} = 1 \text{ and } y_{x_{ik}} = 0\} \\ R &= \{k \in K : y_{x_{ik}} = 1\} \end{aligned}$$

Note that $d(L) \leq y_{l_j}$ and $y_{r_i} \leq s - d(R)$. Moreover, $\sum_{k \in K} d_k(y_{x_{ik}} - y_{x_{jk}}) = -d(M)$. These observations imply:

$$\begin{aligned} y_{r_i} - y_{l_j} + \sum_{k \in K} d_k(y_{x_{ik}} - y_{x_{jk}}) &\leq s - d(L) - d(M) - d(R) \\ &= s - d(K). \end{aligned}$$

Thus, (10) is satisfied by y . \square

Let s_G be the least integer s such that $P(G, d, s)$ is nonempty, i.e., $P(G, d, s_G) \neq \emptyset$ and $P(G, d, s_G - 1) = \emptyset$. Furthermore, let $\Delta = \max_{i \in V} d_i$ denote the maximum demand.

Theorem 7 *If $g = 0$ and $s \geq s_G + 4\Delta$, then the clique-star inequalities (10) define facets of $P(G, d, s)$.*

Proof. By Lemma 2, clique star inequalities are valid for $P(G, d, s)$. We now prove that these inequalities define facets of this polytope. Let F be the face of $P(G, s)$ defined by inequality (10), and suppose that $\lambda y = \lambda_0$ for every $y \in F$. We will show that (λ, λ_0) is in fact a multiple of (10), thus proving that this inequality is facet-inducing, i.e., that F is not contained in any other facet. To this end, we prove the following sequence of claims:

Claim 1: $\lambda_{l_k} = 0$ for $k \neq j$. Let $k \neq j$ and let $y \in F$ be an integer solution with $y_{r_k} - y_{l_k} > d_k$ (which exists because $s > s_G$). Define y' to be the solution obtained from y by just setting $y'_{l_k} = y_{l_k} + 1$. Note that this new solution is feasible. Both points lie in F , implying $\lambda y = \lambda y' = \lambda_0$. Moreover, they only differ in their l_k -coordinates, and thus

$$\lambda_{l_k} y_{l_k} = \lambda_{l_k} y'_{l_k} = \lambda_{l_k} (y_{l_k} + 1).$$

Thus $\lambda_{l_k} = 0$, proving the claim. \diamond

Claim 2: $\lambda_{r_k} = 0$ for $k \neq i$. A similar construction, with points $y, y' \in F$ such that $y_{r_k} - y_{l_k} > d_k$ and $y'_{r_k} = y_{r_k} - 1$ shows that $\lambda_{r_k} = 0$ for $k \neq i$. \diamond

Claim 3: $\lambda_{x_{kl}} = 0$ if both k, l differ from i, j . Let $y \in F$ be a feasible solution with $y_{l_k} = 0$, $y_{l_l} = d_k$, and all the remaining intervals to the right of $I(k)$ (this construction is possible since $s \geq s_G + 4\Delta$, see Figure 3). Let y' be a new feasible solution obtained from y by switching intervals $I(k)$ and $I(l)$. Both solutions are in F , and thus $\lambda y = \lambda y'$. These two feasible solutions only differ in their l_k -, l_l - and x_{kl} -coordinates. Moreover, we know from the previous claims that $\lambda_{l_k} = \lambda_{r_k} = \lambda_{l_l} = \lambda_{r_l} = 0$, implying $\lambda_{x_{kl}} = 0$. \diamond

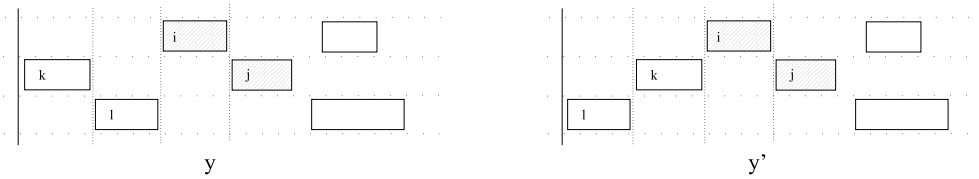


Figure 3: Feasible points for Claim 3.

Claim 4: $\lambda_{r_i} = -\lambda_{l_j}$. Let $y \in F$ be a feasible solution with $y_{r_i} = y_{l_j}$, such that it is possible to move the intervals $I(i)$ and $I(j)$ one unit to the right (this is possible since $s > s_G$). Let y' be the solution obtained after this shifting. Since both solutions are in F and we know that $\lambda_{l_i} = \lambda_{r_j} = 0$, then

$$\lambda_{r_i} y_{r_i} + \lambda_{l_j} y_{l_j} = \lambda_{r_i} (y_{r_i} + 1) + \lambda_{l_j} (y_{l_j} + 1).$$

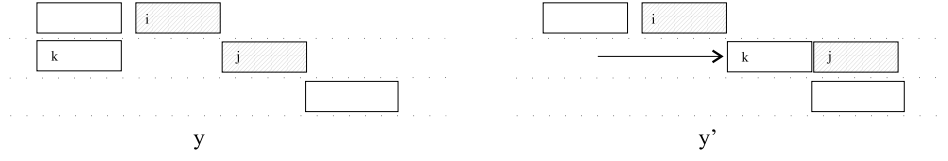


Figure 4: Feasible points for Claim 5.

This implies that $\lambda_{r_i} + \lambda_{l_j} = 0$, thus justifying the claim. \diamond

Claim 5: $\lambda_{x_{ik}} = d_k \lambda_{r_i}$ for $k \in N(i) \cap N(j)$. Let y be an integer point in F with $y_{r_i} = y_{l_j}$, and let y' be a feasible solution with only intervals k and j changed, in such a way that $y'_{l_k} = y_{r_i}$ and $y'_{l_j} = y'_{r_k} = y'_{l_k} + d_k$ (see Figure 4). This construction is possible since $s > s_G + d_i + d_j + d_k$. Both solutions lie in F , so $\lambda y = \lambda y' = \lambda_0$, and thus

$$\lambda_{l_k} y_{l_k} + \lambda_{l_j} y_{l_j} = \lambda_{l_k} y'_{l_k} + \lambda_{l_j} y'_{l_j} + \lambda_{x_{ik}}.$$

But $\lambda_{l_k} = 0$ and $y'_{l_j} = y_{r_i} + d_k$ imply $\lambda_{x_{ik}} = d_k \lambda_{r_i}$, proving the claim. \diamond

Claim 6: $\lambda_{x_{jk}} = -d_k \lambda_{r_i}$ for $k \in N(i) \cap N(j)$. A similar construction proves this claim, by considering the solutions presented in Figure 5. \diamond

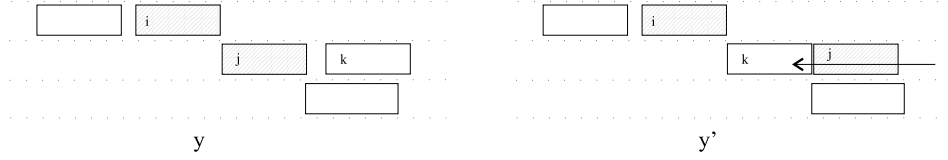


Figure 5: Feasible points for Claim 6.

Claim 7: $\lambda_{x_{ji}} = (s - d(K))$. Let $y \in F$ be any integer solution with $y_{r_i} = y_{l_j}$, and let y' be a solution with $y'_{l_i} = s - d_i$ and $y'_{l_j} = 0$ (and thus $y'_{x_{ji}} = 1$), as in Figure 6. Note that $y_{x_{ik}} - y_{x_{jk}} = 0$, $y'_{x_{ik}} = 0$, and $y'_{x_{jk}} = 1$ for $k \in N(i) \cap N(j)$. This implies that y' satisfies (10) at equality, and, therefore, $y' \in F$. Moreover, we have that

$$\begin{aligned} \lambda y &= \lambda_{r_i} y_{r_i} + \lambda_{l_j} y_{l_j} + \sum_{k \in K} (\lambda_{x_{ik}} y_{x_{ik}} - \lambda_{x_{jk}} y_{x_{jk}}) \\ &= \lambda_{r_i} (y_{r_i} - y_{r_i}) + \sum_{k \in K} \lambda_{x_{ik}} (y_{x_{ik}} - y_{x_{jk}}) = 0 \\ \lambda y' &= \lambda_{r_i} y'_{r_i} + \lambda_{l_j} y'_{l_j} + \sum_{k \in K} \lambda_{x_{jk}} y'_{x_{jk}} + \lambda_{x_{ji}} y'_{x_{ji}} \\ &= \lambda_{r_i} s + \sum_{k \in K} \lambda_{x_{jk}} + \lambda_{x_{ji}} \end{aligned}$$

$$= s + \sum_{k \in K} d_k + \lambda_{x_{ji}}$$

Thus, $\lambda_{x_{ji}} = s - d(K)$, proving the claim. \diamond

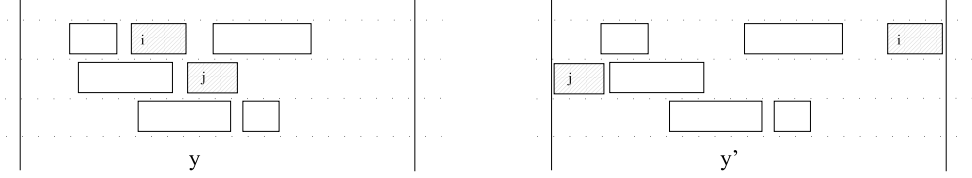


Figure 6: Feasible points for Claim 7.

This way, we have that

$$\lambda y = y_{r_i} - y_{l_j} + \sum_{k \in K} d_k (y_{x_{ik}} - y_{x_{jk}}) - (s - d(K)) y_{x_{ji}}.$$

Then λ is a multiple of the LHS of inequality (10), which implies that $\lambda_0 = 0$. Thus, the face F defined by (10) cannot be contained in any other facet of $P(G, d, s)$ and defines, therefore, itself a facet of the (full-dimensional) polytope $P(G, d, s)$. \square

Corollary 2 *Let $ij \in E$ be an edge of G such that $N(i) \cap N(j) = \emptyset$. If $g = 0$ and $s \geq s_G + 4\Delta$, then the antiparallelity constraints (3)-(6) define facets of $P(G, d, s)$.*

Corollary 3 *Let $ij \in E$. The clique-star inequality*

$$l_i + d_i + \sum_{k \in K} d_k (x_{ik} - x_{jk}) \leq l_j + (s - d(K)) x_{ji} \quad (11)$$

is valid for $R(G, d, s)$ and defines a facet of this polytope if $g = 0$ and $s \geq s_G + 4\Delta$.

Clique-star inequalities are not always facet-defining. To construct a counterexample, let $V = \{1, \dots, 5\}$ and $E = \{12, 13, 15, 23, 24, 45\}$, with $d_2 = d_3 = d_4 = 1$ and $d_1 = d_5 = 2$. Set $s = 4$, $g = 0$, $i = 4$ and $j = 2$. This instance has $\dim(P(G, d, s)) = 4$. All the feasible solutions satisfying (10) at equality are the 4 points z_a, z_b, z_c, z_d presented in Figure 7. It is not difficult to verify that $z_d = z_a - z_b + z_c$. Thus, z_d is an affine combination of the other three solutions and the dimension of the face of $P(G, d, s)$ defined by (10) is, therefore, at most 2. This shows that this clique-star inequality is not a facet of $P(G, d, s)$.

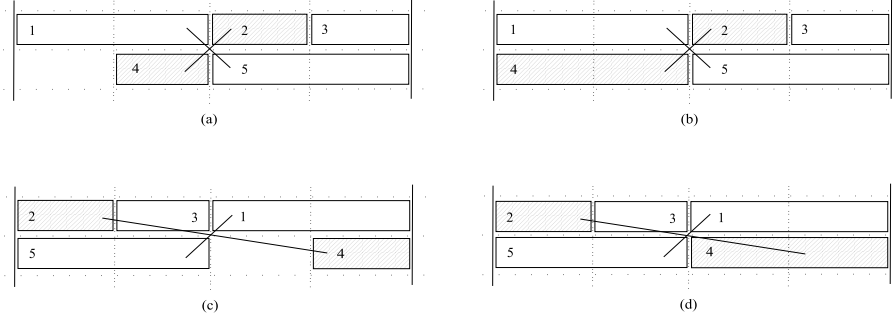


Figure 7: The only four points which satisfy (10) at equality.

Proposition 2 *The symmetric inequality of a clique-star inequality for the polytope $R(G, d, s)$ is again a clique-star inequality for the polytope.*

Proof. Let $a^T y \leq b$ be the clique-star inequality (11) associated with (K, S) . Recall that the symmetric inequality of $a^T y \leq b$ is $2a^T p - b \leq a^T y$, where $p = \frac{1}{2}(s\mathbf{1} - d, \mathbf{1})$ is the symmetry point of $R(G, d, s)$. We have that

$$\begin{aligned} 2a^T p - b &= 2 \left(\frac{(s - d_i)}{2} - \frac{(s - d_j)}{2} + \sum_{k \in K} \left(\frac{d_k}{2} - \frac{d_k}{2} \right) + \frac{d(K) - s}{2} \right) + d_i \\ &= d_j + d(K) - s \end{aligned}$$

This implies that $2a^T p - b \leq a^T y$ is the inequality:

$$d_j + d(K) - s \leq l_i - l_j + \sum_{k \in K} d_k(x_{ik} - x_{jk}) - (s - d(K))x_{ji},$$

which can be rewritten as

$$l_j + d_j + \sum_{k \in K} d_k(x_{jk} - x_{ik}) \leq l_i + (s - d(K))(1 - x_{ji}). \quad (12)$$

With the notation $x_{ij} = 1 - x_{ji}$, we have that (12) is again a clique-star inequality. \square

Clique-star inequalities describe the interaction between two adjacent intervals of feasible solutions. The same ideas can be applied to the interaction between a single node i and the border of the frequency spectrum. In this case, we must consider a spanning clique-star of $N(i)$. The following theorem can be proved in a similar way as Theorem 7.

Theorem 8 *Let $i \in V$ and let (K, S) be a spanning clique-star of $N(i)$. Then, the following inequalities are valid for $P(G, d, s)$:*

$$l_i \geq \sum_{k \in K} d_k x_{ki} \quad (13)$$

$$r_i \leq s - \sum_{k \in K} d_k x_{ik} \quad (14)$$

Moreover, if $g = 0$ and $s > s_G + 3\Delta$, then these inequalities define facets of $P(G, d, s)$.

5 Concluding remarks

In this article, we started a polyhedral study of chromatic scheduling polytopes, which arise in connection with bandwidth allocation problems. We studied the dimension and the special symmetry of these polytopes. We also identified the facet-defining constraints from the integer programming model, and presented some new facets of both polytopes.

There are many open questions concerning the dimension of these polytopes, and so it would be interesting to continue the investigations in this direction. Also, the relationship between the symmetry and the facet-defining property of valid inequalities deserves more research.

Further, more valid inequalities must be found and analyzed, since this would lead to substantial improvements for solving this kind of bandwidth allocation problems in practice with the help of cutting plane methods.

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