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Polynomial Inequalities Representing Polyhedra*

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POLYNOMIAL INEQUALITIES REPRESENTING POLYHEDRA⋆

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Abstract. Our main result is that every n-dimensional polytope can be described by at most $(2n-1)$ polynomial inequalities and, moreover, these polynomials can explicitly be constructed. For an n-dimensional pointed polyhedral cone we prove the bound $2n-2$ and for arbitrary polyhedra we get a constructible representation by $2n$ polynomial inequalities.

1. Introduction

By a striking result of Bröcker and Scheiderer (see [Sch89], [Brö91], [BCR98] and [Mah89]), every basic closed semi-algebraic set of the form

$$\mathcal{S} = \{x \in \mathbb{R}^n : \mathfrak{f}_1(x) \geq 0, \ldots, \mathfrak{f}_l(x) \geq 0\},$$

where $\mathfrak{f}_i \in \mathbb{R}[x]$, $1 \leq i \leq l$, are polynomials, can be represented by at most $n(n+1)/2$ polynomials, i.e., there exist polynomials $\mathfrak{p}_1, \ldots, \mathfrak{p}_{n(n+1)/2} \in \mathbb{R}[x]$ such that

$$\mathcal{S} = \left\{x \in \mathbb{R}^n : \mathfrak{p}_1(x) \geq 0, \ldots, \mathfrak{p}_{n(n+1)/2}(x) \geq 0\right\}.$$

Moreover, in the case of basic open semi-algebraic sets, i.e., $\geq$ is replaced by strict inequality, one can even bound the maximal number of polynomials needed by the dimension n instead of $n(n+1)/2$. Rephrasing the results in terms of semi-algebraic geometry, the *stability index* of every basic closed or open semi-algebraic set is $n(n+1)/2$ or n, respectively. Both bounds are best possible.

No explicit constructions, however, of such systems of polynomials are known nor whether the upper bound $n(n+1)/2$ can be improved for semi-algebraic sets having additional structure such as convexity. Even in the very special case of n-dimensional polyhedra almost nothing was known. In [Brö91, Example 2.10] or in [ABR96, Example 4.7] a description of a regular convex m-gon in the plane by two polynomials is given. This result was generalised to arbitrary convex polygons and three polynomial inequalities by vom Hofe [vH92]. Bernig [Ber98] proved that, for $n = 2$, every convex polygon can even be represented by two polynomial inequalities. In [GH03] a construction of $O(n^n)$ polynomial inequalities representing an n-dimensional simple polytope is given. Based on ideas from [Bos03], here we give, in particular, an explicit construction of $(2n-1)$ polynomials describing an arbitrary n-dimensional polytope. Hence the general upper bound of $n(n+1)/2$ polynomials can be improved (at

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least) for n-dimensional polytopes, and we conjecture that the dimension itself is the right value for this special case.

In order to state our results we fix some notation. A polyhedron $P \subset \mathbb{R}^n$ is the intersection of finitely many closed halfspaces, i.e., we can write it as

$$P = \{x \in \mathbb{R}^n : a_i \cdot x \leq b_i,\, 1 \leq i \leq m\}\,,$$

for some $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$. Here $a \cdot x$ denotes the standard inner product on $\mathbb{R}^n$. If P is bounded then it is called a polytope. A pointed polyhedral cone $C \subset \mathbb{R}^n$ with apex at the origin is the intersection of finitely many closed halfspaces of the type

$$C = \left\{x \in \mathbb{R}^d : a_i \cdot x \leq 0,\, 1 \leq i \leq m\right\},$$

$a_i \in \mathbb{R}^n$. For polynomials $\mathfrak{p}_i \in \mathbb{R}[x]$, $1 \leq i \leq l$, we denote by

$$\mathcal{P}(\mathfrak{p}_1, \dots, \mathfrak{p}_l) := \{x \in \mathbb{R}^n : \mathfrak{p}_1(x) \geq 0, \dots, \mathfrak{p}_l(x) \geq 0\}$$

the associated basic closed semi-algebraic set generated by the polynomials.

Theorem 1.1. *Let $C \subset \mathbb{R}^n$ be an n-dimensional pointed polyhedral cone. Then we can construct $(2n-2)$ polynomials $\mathfrak{p}_i \in \mathbb{R}[x]$, $1 \leq i \leq 2n-2$, such that $C = \mathcal{P}(\mathfrak{p}_1, \dots, \mathfrak{p}_{2n-2})$.*

The case of polytopes can be derived as a consequence of the construction behind Theorem 1.1 and here we get

Theorem 1.2. *Let $P \subset \mathbb{R}^n$ be an n-dimensional polytope. Then we can construct $(2n-1)$ polynomials $\mathfrak{p}_i \in \mathbb{R}[x]$, $1 \leq i \leq 2n-1$, such that $P = \mathcal{P}(\mathfrak{p}_1, \dots, \mathfrak{p}_{2n-1})$.*

At the end of Section 3 (see Definition 3.3) we will give an explicit description of the polynomials we employ. The construction behind the proof of Theorem 1.2 or Theorem 1.1 can also be applied to the interior of a polytope or a cone which are open semi-algebraic sets. Furthermore, in [GH03, Proposition 2.5] it is shown how a representation of a polytope by polynomial inequalities can be used to get a representation of a polyhedron by polynomials. Applying this proposition to Theorem 1.2 leads to

Corollary 1.3. *Let $P \subset \mathbb{R}^n$ be an n-dimensional polyhedron. Then we can construct $2n$ polynomials $\mathfrak{p}_i \in \mathbb{R}[x]$, $1 \leq i \leq 2n$, such that $P = \mathcal{P}(\mathfrak{p}_1, \dots, \mathfrak{p}_{2n})$.*

The paper is organised as follows. In Section 2 we give, for a pointed cone C, a construction of two polynomials $\mathfrak{p}_{C,\varepsilon}, \mathfrak{p}_0$ such that C is "nicely approximated" by $\mathcal{P}(\mathfrak{p}_{C,\varepsilon}, \mathfrak{p}_0)$. Then, for a face $F = C \cap \{x \in \mathbb{R}^n : a_i \cdot x = 0, i \in I_F\}$ of C, we apply this construction to the cone $C_F = \{x \in \mathbb{R}^n : a_i \cdot x \leq 0, i \in I_F\}$, where I_F denotes the index set of active constraints of F. In that way we get an approximation of C_F by a semi-algebraic set of the type $\mathcal{P}(\mathfrak{p}_{C_F,\varepsilon}, \mathfrak{p}_F)$. In Section 3 we study the relations between the set $\mathcal{P}(\mathfrak{p}_{C_{F\cap G},\varepsilon}, \mathfrak{p}_{F\cap G})$ and $\mathcal{P}(\mathfrak{p}_{C_F,\varepsilon}, \mathfrak{p}_F)$, $\mathcal{P}(\mathfrak{p}_{C_G,\varepsilon}, \mathfrak{p}_G)$ for two different faces F and G of the same dimension. Thereby, it turns out that we may multiply all polynomials $\mathfrak{p}_{C_F,\varepsilon}$ belonging to faces of the same dimension as well as the polynomials $\mathfrak{p}_F$ in order to get a representation of a pointed polyhedral cone by polynomials. In Section 4 we give a brief outlook why we are interested in such a polynomial representation of polytopes and what might be achievable by such a representation with respect to hard combinatorial optimisation problems.

2. Approximating cones

In the following we use some standard terminology and facts from the theory of polyhedra for which we refer to the books [MS71] and [Zie95]. For the approximation of a cone by a closed semi-algebraic set consisting of two polynomials we need a lemma about the approximation of a polytope by a strictly convex polynomial which was already shown in [GH03, Lemma 2.6]. Since it is essential for the explicit construction of the polynomials we state it here. To this end, let B^n be the n-dimensional unit ball centred at the origin. The diameter of a polytope is denoted by $\mathrm{diam}(P)$, i.e., $\mathrm{diam}(P) = \max\{\|x-y\| : x,y \in P\}$, where $\|\cdot\|$ denotes the Euclidean norm.

Lemma 2.1. *Let $P = \{x \in \mathbb{R}^n : a_i \cdot x \leq b_i,\ 1 \leq i \leq m\}$ be an n-dimensional polytope. For $1 \leq i \leq m$ let*

$$\mathfrak{v}_i(x) := \frac{2a^i \cdot x - h(a_i) + h(-a_i)}{h(a_i) + h(-a_i)},$$

where $h(a) := \max\{a \cdot x : x \in P\}$ is the support function of P. Let $\varepsilon > 0$, choose an integer k such that $k > \ln(m)/(2\ln(1 + \frac{2\varepsilon}{(n+1)\mathrm{diam}(P)}))$, and set

$$\mathfrak{p}_{P,\varepsilon}(x) := \sum_{i=1}^{m} \frac{1}{m}\, [\mathfrak{v}_i(x)]^{2\,k} \qquad \textit{and} \qquad K_\varepsilon := \{x \in \mathbb{R}^n : \mathfrak{p}_{P,\varepsilon}(x) < 1\}.$$

Then we have $P \subset K_\varepsilon \subset P + \varepsilon\, B^n$.

Proof. [GH03, Lemma 2.6]. □

Now let

$$C = \{x \in \mathbb{R}^n : a_i \cdot x \leq 0,\ 1 \leq i \leq m\}\,, \tag{2.1}$$

be a pointed n-dimensional cone with $\|a_i\| = 1$, $1 \leq i \leq m$. The set of all k-dimensional faces (k-faces for short) is denoted by $\mathcal{F}_k$, $0 \leq k \leq n-1$. For a k-face F, we denote by $I_F := \{i : a_i \cdot x = 0 \text{ for all } x \in F\}$ the set of active constraints. We always assume that our representation (2.1) of C is non-redundant, hence $\{x \in C : a_i \cdot x = 0\}$ is an $(n-1)$-face (facet) of C for $1 \leq i \leq m$. For each F, let

$$u_F := \frac{\sum_{i \in I_F} a_i}{\|\sum_{i \in I_F} a_i\|} \quad \text{and} \quad \mathfrak{p}_F(x) := -u_F \cdot x. \tag{2.2}$$

u_F is an outer unit normal vector of the face F, i.e., $F = C \cap \{x \in \mathbb{R}^n : \mathfrak{p}_F(x) = 0\}$ and $C \setminus F \subset \{x \in \mathbb{R}^n : \mathfrak{p}_F(x) > 0\}$. The only vertex, i.e., 0-face, of C is the origin, and in this case, we denote the above outer unit normal vector and the polynomial by u_0 and $\mathfrak{p}_0$, respectively. In the next lemma we construct a basic closed semi-algebraic set consisting of two polynomials that gives a nice and controllable approximation of C. In what follows we will often use some constants depending on the cone or polytope. All of these constants are explicitly computable by elementary methods, but in order to keep the presentation simple we do not go into the details here.

Lemma 2.2. *For every* $\varepsilon \in (0, 1/2]$ *we can construct a polynomial* $\mathfrak{p}_{C,\varepsilon}(x)$ *such that*

i) $\{x + \varepsilon\,(u_0 \cdot x)B^n : x \in C\} \subset \mathcal{P}(\mathfrak{p}_{C,\varepsilon}, \mathfrak{p}_0) \subset \{x + \omega_C\,\varepsilon\,(u_0 \cdot x)B^n : x \in C\}$,

ii) $\{x \in \mathbb{R}^n : \mathfrak{p}_{C,\varepsilon}(x) = 0, \mathfrak{p}_0(x) = 0\} = \{0\}$,

iii) $\{x + \varepsilon\,(u_0 \cdot x)B^n : x \in C,\ \mathfrak{p}_0(x) > 0\} \subset \{x \in \mathbb{R}^n : \mathfrak{p}_{C,\varepsilon}(x) > 0\}$,

where $\omega_C \geq 1$ *is a constant depending only on* C.

Proof. Firstly, observe that for $n = 1$ there is nothing to do, because we may set $\mathfrak{p}_{C,\varepsilon}(x) := \mathfrak{p}_0(x)$ and $\omega_C = 1$, say. So let $n \geq 2$. For ease of notation we may assume that $-u_0 = e_n$, the n-th unit vector, which can be achieved by a suitable rotation. Due to this choice $C \cap \{x \in \mathbb{R}^n : x_n = 1\}$ is an $(n-1)$-dimensional polytope P, which we identify with its image under the orthogonal projection onto $\mathbb{R}^{n-1}$. Thus let $P = \{x \in \mathbb{R}^{n-1} : \widetilde{a}_i \cdot x \leq \widetilde{b}_i,\ 1 \leq i \leq m\}$, for some $\widetilde{a}_i \in \mathbb{R}^{n-1}$, $\|\widetilde{a}_i\| = 1$, $\widetilde{b}_i \in \mathbb{R}$. With this notation we may write C as the homogenisation of P, i.e., $C = \{x_n\,(x, 1)^\intercal : x \in P,\ x_n \geq 0\}$. For $\mu \geq 0$ let

$$P_\mu = \{x \in \mathbb{R}^{n-1} : \widetilde{a}_i \cdot x \leq \widetilde{b}_i + \mu,\ 1 \leq i \leq m\}.$$

Then

$$P + \mu\,B^{n-1} \subset P_\mu \subset P + \omega_P\,\mu\,B^{n-1},$$

for a certain constant $\omega_P \geq 1$ depending only on P. From Lemma 2.1 we get that, for every $\nu > 0$, we can construct a strictly convex polynomial $\mathfrak{p}_{P_{\mu,\nu}}$ such that

$$P_\mu \subset \left\{x \in \mathbb{R}^{n-1} : \mathfrak{p}_{P_{\mu,\nu}}(x) < 1\right\} \subset P_\mu + \nu\,B^{n-1}. \tag{2.3}$$

In particular, $\mathfrak{p}_{P_{\mu,\nu}}$ can be written as $\mathfrak{p}_{P_{\mu,\nu}}(x) = \sum_{i=1}^m \lambda_i[\widetilde{a}_i \cdot x - \alpha_i]^{2k}$ for certain constants $\lambda_i \in \mathbb{R}_{>0}$, $\alpha_i \in \mathbb{R}$, $k \in \mathbb{N}$, depending on P_μ and ν (cf. Lemma 2.1). For a scalar $x_n > 0$ we immediately get

$$\begin{aligned} x_n\,P_\mu &\subset \{x \in \mathbb{R}^{n-1} : \sum_{i=1}^m \lambda_i[\widetilde{a}_i \cdot x - x_n\alpha_i]^{2k} < (x_n)^{2k}\} \\ &\subset x_n\,P + x_n(\nu + \omega_P\mu)\,B^{n-1}. \end{aligned} \tag{2.4}$$

Since $\widetilde{a}_1, \ldots, \widetilde{a}_m$ are the outer normal vectors of an $(n-1)$-dimensional polytope, these inclusions hold for $x_n = 0$ as well, if we replace $<$ by $\leq$. Hence, with

$$\overline{\mathfrak{p}}_{P_{\mu,\nu}}(x) = (x_n)^{2k} - \sum_{i=1}^m \lambda_i[\widetilde{a}_i \cdot (x_1, \ldots, x_{n-1})^\intercal - x_n\alpha_i]^{2k}$$

and $\mathfrak{p}_0(x) = x_n$, for $x = (x_1, \ldots, x_n)^\intercal \in \mathbb{R}^n$, we get

$$\begin{aligned} &\text{i)} \quad \{x \in \mathbb{R}^n : \overline{\mathfrak{p}}_{P_{\mu,\nu}}(x) = 0,\ \mathfrak{p}_0(x) = 0\} = \{0\}, \\ &\text{ii)} \quad x_n\,P_\mu \subset \{x \in \mathbb{R}^n : \overline{\mathfrak{p}}_{P_{\mu,\nu}}(x) \geq 0\},\ \text{for } x_n \geq 0, \\ &\text{iii)} \quad x_n\,P_\mu \subset \{x \in \mathbb{R}^n : \overline{\mathfrak{p}}_{P_{\mu,\nu}}(x) > 0\},\ \text{for } x_n > 0. \end{aligned} \tag{2.5}$$

From (2.4) we conclude that

$$\mathcal{P}(\overline{\mathfrak{p}}_{P_{\mu,\nu}}, \mathfrak{p}_0) \subset \{x + x_n(\nu + \omega_P \mu)B^n,\, x \in C\}\,. \tag{2.6}$$

With $\gamma = \max\{(1 - a_i \cdot e_n)^{-1/2} : 1 \le i \le m\}$ and by some elementary calculations we get for $y \in \left\{x + x_n \left(\frac{\mu}{\mu+\gamma}\right) B^n,\, x \in C\right\}$ that

$$(y_1, \ldots, y_{n-1})^\intercal \in y_n P_\mu. \tag{2.7}$$

Thus we have by (2.5) ii)

$$\left\{x + x_n \left(\frac{\mu}{\mu+\gamma}\right) B^n,\, x \in C\right\} \subset \mathcal{P}(\overline{\mathfrak{p}}_{P_{\mu,\nu}}, \mathfrak{p}_0). \tag{2.8}$$

Now, for a given $\varepsilon \in (0, 1/2]$, we may choose μ and ν such that $\mu/(\mu + \gamma) = \varepsilon$ and $\nu + \omega_P \mu \le 4\gamma\omega_P\varepsilon$. With $\omega_C := 4\gamma\omega_P$ and $\mathfrak{p}_{C,\varepsilon} := \overline{\mathfrak{p}}_{P_{\mu,\nu}}$ for this special choice of parameters we get by (2.6) and (2.8) the statement i) of the lemma. Property ii) is an immediate consequences of (2.5) i) and the last statement follows from (2.7) and (2.5) iii). □

Remark 2.3.

i) *The main geometric message of Lemma 2.2 is that we can construct a cone of the type $\mathcal{P}(\mathfrak{p}_{C,\varepsilon}, \mathfrak{p}_0)$, which is not too far away from C, but at the same time we also know that $\mathcal{P}(\mathfrak{p}_{C,\varepsilon}, \mathfrak{p}_0)$ is not too close to C. This property of $\mathcal{P}(\mathfrak{p}_{C,\varepsilon}, \mathfrak{p}_0)$ plays a key role in our construction.*

ii) *As constant ω_P in the above proof we can take $R(P)/r(P)$, where $R(P)$ and $r(P)$ denote the radii of two concentric balls such that $x + r(P)\, B^n \subset P \subset x + R(P)\, B^n$.*

For a k-face F of C, let $C_F = \{x \in \mathbb{R}^n : a_i \cdot x \le 0,\, i \in I_F\}$ be the face-cone of F. C_F is an n-dimensional cone containing a k dimensional linear subspace, namely $\operatorname{lin}(F)$, the linear hull of F. The $(n-k)$-dimensional orthogonal complement $\operatorname{lin}(F)^\perp$ of $\operatorname{lin}(F)$ is given by $\operatorname{lin}\{a_i : i \in I_F\}$. If we apply the construction of Lemma 2.2 to $C_F \cap \operatorname{lin}(F)^\perp$ (in the space $\operatorname{lin}(F)^\perp$) we get a generalisation of Lemma 2.2 from the face-cone of the vertex to arbitrary k-faces of C.

Corollary 2.4. *Let F be a k-face of C with $0 \le k \le n-1$. For every $\varepsilon \in (0, 1/2]$ we can construct a polynomial $\mathfrak{p}_{C_F,\varepsilon}(x)$ such that*

i) $\{x + \varepsilon\,(u_F \cdot x)B^n : x \in C_F\} \subset \mathcal{P}(\mathfrak{p}_{C_F,\varepsilon}, \mathfrak{p}_F)$
$\subset \{x + \omega_{C_F}\,\varepsilon\,(u_F \cdot x)B^n : x \in C_F\}\,,$

ii) $\{x \in \mathbb{R}^n : \mathfrak{p}_{C_F,\varepsilon}(x) = 0, \mathfrak{p}_F(x) = 0\} = \operatorname{lin}(F),$

iii) $\{x + \varepsilon\,(u_F \cdot x)B^n : x \in C_F,\, \mathfrak{p}_F(x) > 0\} \subset \{x \in \mathbb{R}^n : \mathfrak{p}_{C_F,\varepsilon}(x) > 0\}\,,$

where $\omega_{C_F} \ge 1$ is a constant depending only on C.

We note that, for a facet F of C and $\varepsilon \in (0, 1/2]$, we just have (cf. proof of Lemma 2.2)

$$\mathfrak{p}_{C_F,\varepsilon}(x) = \mathfrak{p}_F(x) = -u_F \cdot x. \tag{2.9}$$

3. Multiplying polynomial inequalities

The main objective of our proof strategy is to multiply, for each $k \in \{0, \dots, n-1\}$, all the polynomials $\mathfrak{p}_{C_F,\varepsilon}$, $F \in \mathcal{F}_K$, and $\mathfrak{p}_F$, $F \in \mathcal{F}_k$, such that for a special choice of the parameters ε, the arising $2n$ polynomials give a complete description of the cone C. To this end, we have to study, for two k-faces F and G, the relations between $\mathcal{P}(\mathfrak{p}_{C_F,\varepsilon}, \mathfrak{p}_F)$, $\mathcal{P}(\mathfrak{p}_{C_G,\varepsilon}, \mathfrak{p}_G)$, and $\mathcal{P}(\mathfrak{p}_{C_{F\cap G},\varepsilon}, \mathfrak{p}_{F\cap G})$.

Lemma 3.1. *Let F, G be k-faces of C and let $\varepsilon_k \in (0, 1/2]$. Then we can find an $\varepsilon_{F,G} \in (0, 1/2]$ such that*

$$\begin{aligned}\big\{x + \varepsilon_{F,G}\,(u_{F\cap G}\cdot x)B^n : x \in C_{F\cap G}\big\} \subset \qquad&\\ \{x + \varepsilon_k\,(u_F \cdot x)\,B^n : x \in C_F,\ -u_F\cdot x > 0\}&\\ \cup\,\{x + \varepsilon_k\,(u_G \cdot x)\,B^n : x \in C_G,\ -u_G\cdot x > 0\}&\\ \cup\,(\mathrm{lin}(F) \cap \mathrm{lin}(G))\,.&\end{aligned}$$

Proof. Let $C_{F\cap G} = \mathrm{lin}(F \cap G) + \mathrm{cone}\{v_1, \dots, v_r\}$ for some points $v_i \in \mathrm{lin}(F \cap G)^\perp$, where cone denotes the conical hull. Since both, $\frac{1}{2}(u_F + u_G)$ and $u_{F\cap G}$, are outer normal vectors of the face $F \cap G$ we find that

$$\rho = \min\left\{\frac{\frac{1}{2}(u_F + u_G)\cdot v_i}{u_{F\cap G}\cdot v_i} : 1 \le i \le r\right\} > 0.$$

Hence, for $x \in C_{F\cap G}$, we get

$$\max\{-u_F\cdot x, -u_G\cdot x\} \ge \frac{1}{2}(-u_F - u_G)\cdot x \ge \rho(-u_{F\cap G})\cdot x. \tag{3.1}$$

If $u_{F\cap G}\cdot x = 0$ then $x \in \mathrm{lin}(F \cap G) \subset \mathrm{lin}(F) \cap \mathrm{lin}(G)$. Otherwise we have $-u_{F\cap G}\cdot x > 0$, and with $\varepsilon_{F,G} := \min\{\rho\varepsilon_k, 1/2\}$ and (3.1) we get the required inclusion. □

As a corollary we get that we can find ε_k, $0 \le k \le n-1$, such that a cone of the type $\mathcal{P}(\mathfrak{p}_{C_{F\cap G},\varepsilon_{\dim(F\cap G)}}, \mathfrak{p}_{F\cap G})$, $F, G \in \mathcal{F}_k$, is covered by the interior of $\mathcal{P}(\mathfrak{p}_{C_F,\varepsilon_k}, \mathfrak{p}_F)$, the interior of $\mathcal{P}(\mathfrak{p}_{C_G,\varepsilon_k}, \mathfrak{p}_G)$, and the linear space $\mathrm{lin}(F)\cap\mathrm{lin}(G)$.

Corollary 3.2. *We can determine positive constants $\varepsilon_k \le 1/2$, $0 \le k \le n-1$, such that for any pair of two different k-faces F and G of C, $k \in \{0, \dots, n-1\}$,*

$$\begin{aligned}&\mathcal{P}(\mathfrak{p}_{C_{F\cap G},\varepsilon_{\dim(F\cap G)}}, \mathfrak{p}_{F\cap G}) \subset\\ &\quad\{x \in \mathbb{R}^n : \mathfrak{p}_{C_F,\varepsilon_k}(x) > 0, \mathfrak{p}_F(x) > 0\}\\ &\cup\{x \in \mathbb{R}^n : \mathfrak{p}_{C_G,\varepsilon_k}(x) > 0, \mathfrak{p}_G(x) > 0\}\\ &\cup\{x \in \mathbb{R}^n : \mathfrak{p}_{C_F,\varepsilon_k}(x) = 0, \mathfrak{p}_F(x) = 0, \mathfrak{p}_{C_G,\varepsilon_k}(x) = 0, \mathfrak{p}_G(x) = 0\}\,.\end{aligned} \tag{3.2}$$

Proof. By (2.9) we may set $\varepsilon_{n-1} := 1/2$ and in view of Corollary 2.4 and Lemma 3.1 we just have to say how to calculate the numbers ε_k, $0 \le k \le n-2$. For two faces $F, G \in \mathcal{F}_k$ the proof of Lemma 3.1 (the $\varepsilon_{F,G}$ constructed there) leads to an upper bound on $\varepsilon_{\dim(F\cap G)}$ provided we know ε_k. Hence, for $k = n-2, \dots, 0$, we can calculate suitable numbers ε_k via

$$\varepsilon_k := \min_{k+1\le l\le n-1} \min_{F,G\in\mathcal{F}_l} \{\varepsilon_{F,G} : \dim(F\cap G) = k\}.$$

$\square$

Since every $(n-2)$-face H of C is given by the intersection of two uniquely determined facets F and G of C we may even set (cf. (2.9))

$$\varepsilon_{n-2} := 1/2, \quad \mathfrak{p}_{C_H,\varepsilon_{n-2}}(x) := \mathfrak{p}_H(x) = -u_H \cdot x \tag{3.3}$$

without violating the validity of Corollary 3.2.

Now we come to the definition of the polynomials, which give us a representation of an n-dimensional pointed polyhedral cone and to the proofs of Theorem 1.1 and Theorem 1.2.

Definition 3.3. *Let ε_k, $0 \le k \le n-1$, be chosen according to Corollary 3.2 and* (3.3)*. For $F \in \mathcal{F}_k$, let $\mathfrak{p}_F, \mathfrak{p}_{C_F,\varepsilon_k} \in \mathbb{R}[x]$ be given as in* (2.2)*, Lemma 2.2,* (2.9)*, and* (3.3)*. Then, for $k = 0, \dots, n-1$, let*

$$\mathfrak{P}_{k,1}(x) := \prod_{F\in\mathcal{F}_k} \mathfrak{p}_F(x) \quad \text{and} \quad \mathfrak{P}_{k,2}(x) := \prod_{F\in\mathcal{F}_k} \mathfrak{p}_{C_F,\varepsilon_k}(x).$$

Proof of Theorem 1.1. First we show that

$$C = \{x \in \mathbb{R}^n : \mathfrak{P}_{k,1}(x) \ge 0,\, \mathfrak{P}_{k,2}(x) \ge 0,\, k = 0, \dots, n-1\}.$$

The inclusion $\subset$ is obvious. So let $y \notin C$, but suppose that y satisfies all the polynomial inequalities. Since $y \notin C$ one of the facet defining inequalities has to be violated, i.e., there exists an $(n-1)$-face F with $\mathfrak{p}_F(y) < 0$. Hence we may define $p \in \{0, \dots, n-1\}$ as the minimum number (index) for which one of the factors in the polynomials $\mathfrak{P}_{p,1}(x)$ or $\mathfrak{P}_{p,2}(x)$ is violated. Since both, $\mathfrak{P}_{0,1}(x)$ and $\mathfrak{P}_{0,2}(x)$, consists only of one polynomial we have $p \in \{1, \dots, n-1\}$.

Let $F \in \mathcal{F}_p$ such that $\mathfrak{p}_F(y) < 0$ or $\mathfrak{p}_{C_F,\varepsilon_p}(y) < 0$. Since $\mathfrak{P}_{p,1}(y) \ge 0$ and $\mathfrak{P}_{p,2}(y) \ge 0$ there must exist a $G \in \mathcal{F}_p$ with $\mathfrak{p}_G(y) \le 0$ (in the case that $\mathfrak{p}_F(y) < 0$) or with $\mathfrak{p}_{C_G,\varepsilon_p}(y) \le 0$ (if $\mathfrak{p}_{C_F,\varepsilon_p}(y) < 0$). Thus we know that y is neither contained in the interior of the cone $\mathcal{P}(\mathfrak{p}_{C_F,\varepsilon_l}, \mathfrak{p}_F)$ nor in the interior of $\mathcal{P}(\mathfrak{p}_{C_G,\varepsilon_l}, \mathfrak{p}_G)$ nor in the linear space $\mathrm{lin}(F) \cap \mathrm{lin}(G)$. By the choice of $\varepsilon_{\dim(F\cap G)}$ and Corollary 3.2, however, those points y are cut off by the cone $\mathcal{P}(\mathfrak{p}_{C_{F\cap G},\varepsilon_{\dim(F\cap G)}}, \mathfrak{p}_{F\cap G})$. Thus we must have

$$y \notin \mathcal{P}(\mathfrak{p}_{C_{(F\cap G)},\varepsilon_{\dim(F\cap G)}}, \mathfrak{p}_{F\cap G})$$

contradicting the minimum property of p. Finally, we observe that by (2.9) $\mathfrak{P}_{n-1,1} = \mathfrak{P}_{n-1,2}$, by (3.3) $\mathfrak{P}_{n-2,1} = \mathfrak{P}_{n-2,2}$ and hence we only have $2n-2$ polynomials. $\square$

The key to this algebraic proof are the special geometric properties i) to iii) of the approximative sets introduced in Corollary 2.4. These relations in combination with the result of Corollary 3.2 ensure that, for each pair of faces F, G, the set $\mathcal{P}(\mathfrak{p}_{C_{F\cap G},\varepsilon_{\dim(F\cap G)}}, \mathfrak{p}_{F\cap G})$ is contained in a special way in the union of the corresponding sets constructed for F, G respectively, and this inclusion allows us to multiply those polynomials the latter are based on.

Proof of Theorem 1.2. Let $P \subset \mathbb{R}^n$ be an n-dimensional polytope and let $C \subset \mathbb{R}^{n+1}$ be the $(n+1)$-dimensional pointed polyhedral cone $C = \{x_{n+1}(x, 1)^\intercal : x \in P\}$. Theorem 1.1 shows that we construct $2n$ polynomials describing C, where, in particular, one polynomial ($\mathfrak{P}_{0,1}(x)$ in the notation of Definition 3.3) describes just a supporting hyperplane of C at the origin. Fixing the last coordinate to $x_{n+1} = 1$ in these polynomials gives a representation of P by $2n$ polynomials. The polynomial $\mathfrak{P}_{0,1}(x)$, however, is apparently redundant for the polytope. □

Remark 3.4. *We want to remark that for a polytope $P = \{x \in \mathbb{R}^n : a_i \cdot x \leq b_i, 1 \leq i \leq m\}$ with rational input data a_i, b_i all the constants involved in the construction of the polynomials $\mathfrak{p}_{C_F,\varepsilon}$ can be substituted by certain rational numbers. Moreover, these numbers can be calculated by well known methods from Linear Programming or Computational Geometry (cf.* [Bos03]*).*

4. Outlook

The usual method to attack hard combinatorial optimisation problems is the polyhedral approach. The basic idea here is a "change of the representation" of the problem, namely, to represent combinatorial objects (such as the tours of a travelling salesman, the independent sets of a matroid, or the stable sets in a graph) as the vertices of a polytope. If one can find complete or tight partial representations of polytopes of this type by linear equations and inequalities, linear programming (LP) techniques can be employed to solve the associated combinatorial optimisation problem, see [GLS93]. Even in the case where only partial inequalities of the polyhedra associated with combinatorial problems are known, LP techniques (such as cutting planes and column generation) have resulted in very successful exact or approximate solution methods. One prime example for this methodology is the travelling salesman problem, see [ABCC98] and the corresponding web page at `http://www.math.princeton.edu/tsp/`. Progress of the type may also be possible via a "polynomial-representation approach". Of course, since the degree of the polynomials in a such a polynomial representation is in general very high (see e.g. [GH03, Proposition 2.1]), and since polynomial inequalities are much harder to treat than linear inequalities, we can not expect that such an exact polynomial representation yields immediately a new method for combinatorial optimisation problems. However, if we can answer questions like how well can we construct a small number of "simple" polynomials $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ such that a given polytope (or a general closed semi-algebraic set) is well approximated by the corresponding polynomials, or how well can it be described or approximated by polynomials of total degree k, then we believe that those results lead to a new approach to combinatorial optimisation problems via non-linear methods. We do know, of course, that these indications of possible future results are mere speculation. Visions of this type, however, were the starting point of the results presented in this paper.

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