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A construction for non-rank facets of stable set polytopes of webs

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A construction for non-rank facets of stable set polytopes of webs

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Abstract

Graphs with circular symmetry, called webs, are relevant w.r.t. describing the stable set polytopes of two larger graph classes, quasi-line graphs [9, 14] and claw-free graphs [8, 9]. Providing a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem [10]. However, even the problem of finding all facets of stable set polytopes of webs is open. So far, it is only known that stable set polytopes of webs with clique number ≤ 3 have rank facets only [6, 18] while there are examples with clique number ≥ 4 having non-rank facets [11, 12, 14, 16].

In this paper, we provide a construction for non-rank facets of stable set polytopes of webs. This construction is the main tool to obtain in a companion paper [17], for several fixed values of ω including all odd values at least 5, that there are only finitely many webs with clique number ω whose stable set polytopes admit rank facets only.

Keywords: web, rank-perfect graph, stable set polytope, (non-)rank facet

1 Introduction

Graphs with circular symmetry of their maximum cliques and stable sets are called *webs*: a web W_n^k is a graph with vertices $1, \dots, n$ where ij is an edge if i and j differ by at most $k \pmod n$ and $i \neq j$. The webs W_9^k on nine vertices are depicted in Figure 1. Notice that webs are also called circulant graphs C_n^k in [4] and that similar graphs $W(n, k)$ were introduced in [18].

Webs and line graphs belong to the classes of quasi-line graphs and claw-free graphs and are relevant w.r.t. describing the stable set polytopes of those larger graph classes [8, 9, 14]. The *stable set polytope* $\text{STAB}(G)$ of G is defined as the convex hull of the incidence vectors of all stable sets of the graph G . In order

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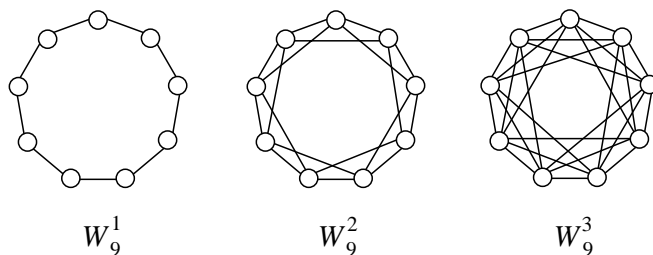


Figure 1

to describe $\text{STAB}(G)$ by means of facet-defining inequalities, the “trivial” facets $x_i \geq 0$ for all vertices i of G and the clique constraints

$$\sum_{i \in Q} x_i \leq 1$$

for all cliques $Q \subseteq G$ are necessary. These two types of facets suffice to describe $\text{STAB}(G)$ for perfect graphs G only [3]. A natural way to generalize clique constraints is to investigate *rank constraints*, that are 0/1-constraints of the form

$$\sum_{i \in G'} x_i \leq \alpha(G')$$

associated with arbitrary induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the cardinality of a maximum stable set in G' (note $\alpha(G') = 1$ holds iff G' is a clique). A graph is *rank-perfect* if all non-trivial facets of its stable set polytope are rank constraints. The class of rank-perfect graphs contains all perfect graphs [3], odd holes and odd antiholes [15], line graphs [7], and the complements of webs [20].

A characterization of the *rank* facets in stable set polytopes of claw-free graphs was given by Galluccio & Sassano [8]. They showed that all rank facets can be constructed by means of standard operations from rank constraints associated with cliques, partitionable webs, or line graphs of 2-connected, critical hypomatchable graphs. However, we are still far from having a complete description for the stable set polytopes of webs and, therefore, of quasi-line and claw-free graphs, too. Finding a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem (Grötschel, Lovász, and Schrijver [10]). Claw-free graphs are not rank-perfect: Giles & Trotter [9], Oriolo [14], and Lieblich et al. [12] found non-rank facets which occur even in the stable set polytopes of quasi-line graphs. These non-rank facets rely on combinations of joined webs.

Several further authors studied the stable set polytopes of webs. The webs W_n^1 with clique number 2 are either perfect or odd holes and, therefore, rank-perfect due to [3, 15]. (Notice that the clique number, i.e. the size of a maximum clique, of a web W_n^k is $k + 1$.) Dahl [6] showed that the webs W_n^2 with clique number 3 are rank-perfect as well. On the other hand, Kind [11] found (by means of the PORTA

software¹) examples of webs with clique number > 4 which are *not* rank-perfect, e.g., $W_{31}^4, W_{25}^5, W_{29}^6, W_{33}^7, W_{28}^8, W_{31}^9$. Oriolo [14], Liebling et al. [12], and Pêcher & Wagler [16] presented further examples of such webs.

The main contribution of this paper (Theorem 1) is a construction that enables us to obtain, from certain non-rank-perfect webs W_n^k , an infinite sequence of non-rank-perfect webs $W_{n+(k+1)}^k, W_{n+2(k+1)}^k, W_{n+3(k+1)}^k, \dots$ with the same clique number. To be more precise, we introduce the notion of *proper weak non-rank facets*. A facet $\mathbf{a}^T \mathbf{x} \leq c\alpha(G')$ of $\text{STAB}(G)$ is a *weak rank facet* w.r.t. $G' \subseteq G$, if $a_i = c$ for every vertex i of G' and if G' is rank facet-producing (i.e. $\sum_{i \in V(G')} x_i \leq \alpha(G')$ defines a facet of $\text{STAB}(G')$). A weak rank facet is *proper* if G' is not a clique and *non-rank* if it cannot be scaled to have 0/1-coefficients only (i.e. it is not a rank constraint).

Theorem 1 *If $\text{STAB}(W_n^k)$ has a proper weak non-rank facet then $\text{STAB}(W_{n+k+1}^k)$ has a proper weak non-rank facet.*

Therefore, if W_n^k has a proper weak non-rank facet then all webs $W_{n+\lambda(k+1)}^k$ ($\lambda \geq 0$) are not rank-perfect, too. Hence Theorem 1 implies the following corollary:

Corollary 2 *If there are $k + 1$ webs $W_{n_0}^k, \dots, W_{n_k}^k$ such that*

- $\text{STAB}(W_{n_i}^k)$ has a proper weak non-rank facet
- $n_i = i \pmod{k + 1}$

then all webs W_n^k with $n \geq \max\{n_0, \dots, n_k\} - k$ are not rank-perfect.

That means in particular: if we are able to provide such a set of $k + 1$ webs for a certain value of k , then there exist only finitely many rank-perfect webs W_n^k . For $k = 3$, this follows from [16] where an infinite sequence of not rank-perfect webs with clique number 4 is presented, namely $W_{33}^3, W_{42}^3, W_{51}^3, W_{60}^3, \dots$. Hence, by Corollary 2, all webs W_n^3 with $n > 56$ are not rank-perfect and there exist only finitely many rank-perfect webs W_n^3 . Similar results for all even values $k \geq 4$ are given in the companion paper [17]. Applying Corollary 2 implies:

Theorem 3 [17] *For each odd $\omega \geq 5$, there are only finitely many rank-perfect webs with clique number ω , hence, almost all of them are not rank-perfect.*

The paper is organized as follows: Section 2 is devoted to definitions and some general results which are frequently used in the sequel. The proof of Theorem 1 is given in Section 3 and we briefly discuss open problems in Section 4.

¹By PORTA it is possible to generate all facets of the convex hull of a given set of integer points, see <http://www.zib.de>

2 Definitions and general results

Let $G = (V, E)$ be a graph. If V' is any subset of the vertex set V , we denote by $G[V']$ the subgraph of G induced by V' .

Recall that a web W_n^k is a graph with vertices $1, \dots, n$ where ij is an edge if i and j differ by at most $k \pmod n$ and $i \neq j$. Webs are natural generalizations of *odd holes* and *odd antiholes*, that are chordless odd cycles of length ≥ 5 and their complements. Perfect graphs are precisely the graphs without odd holes and odd antiholes as induced subgraphs [2].

The clique number of a web W_n^k is $k + 1$ and the stability number is $\lfloor \frac{n}{k+1} \rfloor$. Unless stated otherwise, arithmetics are always performed modulo the number of vertices of the web involved in the computation. Let $1 \leq a, b \leq n$ be two vertices of a web W_n^k . We denote by $[a, b]$ the set of vertices $\{a, a + 1, a + 2, \dots, b\}$, and by Q_a the maximum clique $[a, a + k]$.

The following lemma, similar to Corollary 3.2 in [18], is frequently used in the sequel. For any finite set X , we denote its cardinality by $|X|$.

Lemma 4 *Consider a web W with clique number ω and a set V' of vertices of W . Then V' induces a subweb W' of W with clique number ω'*

- (i) *if $|Q_i \cap V'| = \omega'$ for all $i \in V'$;*
- (ii) *only if $|Q_i \cap V'| = \omega'$ for all $i \in V'$ and $|Q_i \cap V'| \geq \omega' - 1$ for all $i \notin V'$.*

Proof. The If-part is an immediate consequence of the proof of Theorem 3.1 in [18]. For the Only if-part, consider first $i \in V'$. Obviously $|Q_i \cap V'| \leq \omega'$ as $Q_i \cap V'$ is a clique of W' . If $j \notin Q_{i-\omega-1} \cup Q_i$ then j is not a neighbor of i in W' . Therefore the $2\omega' - 2$ neighbors of i in W' are exactly the set $((Q_{i-\omega-1} \cup Q_i) \cap V') - \{i\}$. Thus, $|Q_i \cap V'| = \omega'$. Now, consider $i \notin V'$. Let i' be the element of V' such that $[i' + 1, i] \cap V' = \emptyset$. Since $Q_{i'} \cap V' \subseteq \{i'\} \cup Q_i$, we have $|Q_i \cap V'| \geq \omega' - 1$. \square

Webs and line graphs belong to the classes of *quasi-line graphs* (the neighborhood of any vertex can be partitioned into two cliques) and *claw-free graphs* (the neighborhood of any vertex does not contain a stable set of size 3). The *line graph* $L(H)$ of a graph H is obtained by taking the edges of H as vertices of $L(H)$ and connecting two vertices in $L(H)$ iff the corresponding edges of H are incident. Webs and line graphs are relevant w.r.t. describing the stable set polytopes of those larger graph classes [8, 9, 14].

Recall that the stable set polytope $\text{STAB}(G)$ is the convex hull of the incidence vectors χ^S of all stable sets S of G . We denote by \mathbf{a}^T the transposed row vector of any column vector \mathbf{a} . An inequality $\mathbf{a}^T \mathbf{x} \leq b$ is said to be *valid* for $\text{STAB}(G)$, if $\mathbf{a}^T \chi^S \leq b$ holds for all stable sets S of G . A *root* of a valid inequality $\mathbf{a}^T \mathbf{x} \leq b$ is a stable set S such that $\mathbf{a}^T \chi^S = b$. A valid inequality $\mathbf{a}^T \mathbf{x} \leq b$ for $\text{STAB}(G)$ is a *facet* if and only if it has $|V(G)|$ roots with affinely independent incidence vectors (note that they have to be *linearly* independent if $b > 0$).

Let $G = (V, E)$ be a graph, F be a family of (at least three inclusion-wise) maximal cliques of G , $p \leq |F|$ be an integer, and define two sets as follows:

$$\begin{aligned} I(F, p) &= \{i \in V : |\{Q \in F : i \in Q\}| \geq p\} \\ O(F, p) &= \{i \in V : |\{Q \in F : i \in Q\}| = p - 1\} \end{aligned}$$

Oriolo [14] showed that the *clique family inequality*

$$(p - r) \sum_{i \in I(F, p)} x_i + (p - r - 1) \sum_{i \in O(F, p)} x_i \leq (p - r) \lfloor \frac{|F|}{p} \rfloor \quad (1)$$

is valid for the stable set polytope of *every* graph G where $r = |F| \bmod p$. A conjecture due to Ben Rebea says that the stable set polytopes of quasi-line graphs have clique family inequalities as only non-trivial facets, see [14].

All matrices in this paper have rational coefficients (in fact integer coefficients). If M is any square matrix, then $|M|$ stands for the determinant of M .

2.1 Rank-minimal facets of webs

Following Galluccio & Sassano [8], an inequality $\sum_{i \in V} x_i \leq \alpha(G)$ associated with a graph G with vertex set V and the graph G itself are called *rank-minimal* if and only if G is a clique or satisfies

1. $\sum_{i \in V} x_i \leq \alpha(G)$ defines a facet of $\text{STAB}(G)$, i.e. G is rank facet-producing;
2. for each $V' \subset V$, the inequality $\sum_{i \in V'} x_i \leq \alpha(G)$ does not define a facet of $\text{STAB}(G[V'])$.

All rank-minimal claw-free graphs were described in [8]. In order to state the theorem, we need the following notations.

A graph G is said to be *partitionable* if there exist two integers p and q such that G has $pq + 1$ vertices and for every vertex v of G , the induced subgraph $G \setminus \{v\}$ admits a partition into p cliques of cardinality q as well as a partition into q stable sets of cardinality p . The webs $W_{\alpha\omega+1}^{\omega-1}$ with $\alpha, \omega > 1$ are examples of partitionable graphs, including all odd holes $W_{2\alpha+1}^1$ and all odd antiholes $W_{2\omega+1}^{\omega-1}$.

A graph H is called *hypomatchable* if it does not admit a perfect matching but $H - v$ does for all vertices $v \in V(H)$ (a matching is perfect if it meets all vertices of the graph). A hypomatchable graph H is called *critical* if $H - e$ is not hypomatchable anymore for all edges $e \in E(H)$.

Theorem 5 [8] *Every rank-minimal claw-free graph is*

- a clique,
- a partitionable web, or
- the line graph of a 2-connected, critical hypomatchable graph.

We are interested in the question which rank-minimal graphs may occur as induced subgraphs of webs (recall: every web is in particular claw-free). It turns out that we essentially can exclude the third alternative of Theorem 5 due to the next lemma:

Lemma 6 *Let H be a 2-connected, critical hypomatchable graph. If its line graph $L(H)$ is an induced subgraph of a web, then $L(H)$ is a triangle or an odd hole.*

Proof. Consider a 2-connected, critical hypomatchable graph H . Since H is 2-connected, H has at least 3 vertices. Since H is critical hypomatchable, H must not admit parallel edges, i.e., H is simple.

If $|H| = 3$, then H as well as $L(H)$ is a triangle. Hence assume $|H| \geq 5$ in the sequel (note: every hypomatchable graph has an odd number of vertices). We show that H as well as $L(H)$ is an odd hole if $L(H)$ is an induced subgraph of a web.

Due to Lov'asz [13], a graph H is hypomatchable if and only if there is a sequence $H_0, H_1, \dots, H_k = H$ of graphs such that H_0 is a chordless odd cycle and for $1 \leq i \leq k$, H_i is obtained from H_{i-1} by adding a chordless odd path E_i that joins two (not necessarily distinct) vertices of H_{i-1} and has all internal vertices outside H_{i-1} . The odd paths $E_i = H_i - H_{i-1}$ are called *ears* for $1 \leq i \leq k$ and the sequence $H_0, H_1, \dots, H_k = H$ an *ear decomposition* of H .

If a hypomatchable graph H is 2-connected and has at least 5 vertices, then H admits an ear decomposition $H_0, H_1, \dots, H_k = H$ s.t. every H_i is 2-connected for $0 \leq i \leq k$ by Cornu'ejols & Pulleyblank [5] and H_0 is an odd hole (i.e. $|H_0| \geq 5$) by [19]. Moreover, in [19] is shown that we can always reorder the ears E_1, \dots, E_k of a given decomposition s.t. the decomposition starts with all ears of length ≥ 3 and ends up with all ears of length one. Thus, every 2-connected hypomatchable graph H with $|V(H)| \geq 5$ has a *proper ear decomposition* $H_0, H_1, \dots, H_k = H$ where H_0 has length ≥ 5 , each H_i is 2-connected, and, if $k > 0$, there is an index j s.t. E_1, \dots, E_j have length ≥ 3 and E_{j+1}, \dots, E_k have length one.

Consider a 2-connected hypomatchable graph H with $|V(H)| \geq 5$ and a proper ear decomposition $H_0, H_1, \dots, H_k = H$ of H . We show in the next two claims: the decomposition of H has neither ears of length 1 nor of length ≥ 3 if H is critical and $L(H)$ is an induced subgraph of a web.

Claim 1. *If $H_0, H_1, \dots, H_k = H$ contains an ear of length 1, then H is not critical hypomatchable.*

In that case, the last ear E_k of the proper ear decomposition $H_0, H_1, \dots, H_k = H$ of H is a single edge. Removing the edge E_k from $H_k = H$ yields the hypomatchable graph H_{k-1} with the same vertex set. Thus, H is not critical hypomatchable. \diamond

Claim 2. *If $H_0, H_1, \dots, H_k = H$ contains an ear of length ≥ 3 , then H is not critical hypomatchable or $L(H)$ is not an induced subgraph of a web.*

In that case, the first ear E_1 of the proper ear decomposition $H_0, H_1, \dots, H_k = H$ of H is a path of length ≥ 3 . If the endvertices u_1 and v_1 of E_1 are adjacent in H_0 (see Fig. 2(a)), then H admits a proper ear decomposition $H'_0, H'_1, \dots, H'_k = H$

with $H'_0 = H_0 \cup E_1 - \{u_1v_1\}$ and $E_2, \dots, E_k, \{u_1v_1\}$ as ear sequence (i.e. $H'_i = H'_{i-1} \cup E_{i+1}$ for $1 \leq i \leq k-1$ and $H'_k = H'_{k-1} \cup \{u_1v_1\}$). Thus, H admits an ear of length 1 and is not critical hypomatchable by Claim 1.

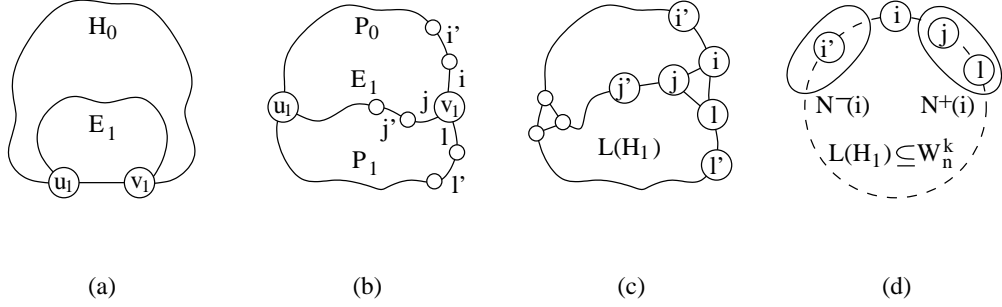


Figure 2

If the endvertices u_1 and v_1 of E_1 are non-adjacent in H_0 (see Fig. 2(b)), then there are 3 open-disjoint paths P_0, P_1, E_1 between u_1 and v_1 in H_1 : P_0 with even length ≥ 2 and P_1, E_1 with odd length ≥ 3 . Consider in H_1 the edges i, i', j, j', l, l' as shown in Fig. 2(b). Then the edges i', j', l' are pairwise disjoint (note: u_1 may be an endvertex of i' but neither of j' nor of l' because of the parity of the paths).

Assume $L(H_1)$ is an induced subgraph of a web W_n^k . We have to find a respective order of the vertices i, i', j, j', l, l' in W_n^k (recall that the line operator transforms edges of H into vertices of $L(H)$, see Fig. 2(c)). Moreover, recall that the neighborhood of every vertex x , denoted by $N(x)$, of a web W_n^k splits into two cliques $N^-(x) = \{x-k, \dots, x-1\}$ and $N^+(x) = \{x+1, \dots, x+k\}$ (where all indices are taken modulo n).

Consider $N(i)$ in W_n^k : we have $i', j, l \in N(i)$ where jl is an edge but neither $i'j$ nor $i'l$ (see Fig. 2(c)). W.l.o.g. let $i' \in N^-(i)$. Then $j, l \in N^+(i)$ follows since both $N^-(i)$ and $N^+(i)$ are cliques. Furthermore, let $j < l$ (the case $l < j$ goes analogously due to $ij, il \in E$ but $ij', il' \notin E$), i.e., assume $i+1 \leq j < l \leq i+k$ (see Fig. 2(d)).

Now, consider the vertex j' . We have $j' \in N(j)$ but $j' \notin N(i)$ (see Fig. 2(c)). This implies $j' \in N^+(j)$ (since $N^-(j) \subseteq N(i)$ by $j \in N^+(i)$), i.e., we obtain $j' \in \{j+1, \dots, j+k\}$. But $i+1 \leq j < l \leq i+k$ implies $N^+(j) \subseteq N(l)$, hence $j' \in N(l)$ in contradiction to j' and l non-adjacent (see Fig. 2(c)). Thus, $L(H_1)$ cannot be an induced subgraph of a web W_n^k .

We conclude: if E_1 connects two adjacent vertices of H_0 , then H is not critical, if E_1 connects two non-adjacent vertices of H_0 , then $L(H)$ is not an induced subgraph of a web. \diamond

Hence, we have obtained that for every 2-connected, critical hypomatchable graph H holds the following. If H has 3 vertices, then H and its line graph $L(H)$ are triangles. Otherwise, H admits a proper ear decomposition $H_0, H_1, \dots, H_k = H$

with and index j s.t. E_1, \dots, E_j have length ≥ 3 and E_{j+1}, \dots, E_k have length one. By Claim 1, there is no ear of length 1 (i.e. $j = k$). If the line graph of H is an induced subgraph of a web, then there is no ear of length ≥ 3 by Claim 1 and Claim 2 (i.e. $j = 0$). In conclusion, we obtain $k = 0$, thus H consists in the odd hole H_0 of length ≥ 5 only and $L(H)$ is an odd hole, too. \square

Remark. Claim 1 of Lemma 6 shows: if the last ear E_k of a proper ear decomposition $H_0, H_1, \dots, H_k = H$ of H has length one, then H is not critical hypomatchable. $L(H)$ is not rank-minimal by Theorem 5 in particular. The reason is the following: the graph H_{k-1} obtained by removing the edge E_k from H is 2-connected and hypomatchable, hence $L(H_{k-1})$ is rank facet-producing by Edmonds & Pulleyblank [7]. Furthermore, $\frac{|H|-1}{2} = \alpha(L(H)) = \alpha(L(H_{k-1}))$ holds by $V(H) = V(H_{k-1})$, hence $L(H)$ cannot be rank-minimal.

Since odd holes are partitionable webs, Theorem 5 and Lemma 6 imply the following corollary:

Corollary 7 *Every rank-minimal induced subgraph of a web is a clique or a partitionable web.*

2.2 Weak rank facets of webs

Recall that a facet $\mathbf{a}^T \mathbf{x} \leq c\alpha(G')$ of $\text{STAB}(G)$ is a *weak rank facet* w.r.t. $G' \subseteq G$, if $a_i = c$ for every vertex i of G' and if G' is rank facet-producing.

Lemma 8 *Let $\mathbf{a}^T \mathbf{x} \leq c\alpha(G[V'])$ be a weak rank facet of the stable set polytope of a web G . Then $c = \max\{a_i \mid i \in V(G)\}$.*

Proof. Let $\alpha' = \alpha(G[V'])$. By Corollary 7, $G[V']$ contains a rank-minimal subgraph W with $\alpha(W) = \alpha'$, which is a clique or a partitionable web. If W is a clique then $\alpha' = 1$ and it follows that $a_i \leq c$ for every vertex i , due to the stable set $\{i\}$. Hence $c = \max\{a_i \mid i \in V(G)\}$.

If W is a partitionable web then let ω' be the clique number of W . We say that two vertices a and b of W are consecutive if $[a, b] \cap W = \{a, b\}$. Obviously, there is a labeling $\{w_1, \dots, w_{|W|}\}$ of the vertices of W such that for every $1 \leq i \leq |W|$, w_i and w_{i+1} are consecutive (with arithmetics performed modulo $|W|$).

For every $1 \leq i \leq |W|$, let $S_i = \{w_{i+\omega'+1}, w_{i+2\omega'+1}, \dots, w_{i+(\alpha'-1)\omega'+1}\}$ (with indices taken modulo $|W|$). Notice that S_i is a stable set of G (due to the labeling, if w_a and w_b are adjacent and $a \leq b$ then w_a, w_{a+1}, \dots, w_b is a clique of W). Since $|W| = \alpha'\omega' + 1$, we have that $w_i \notin N_G(S_i)$ and $w_{i+1} \notin N_G(S_i)$. It follows that for every u in $[w_i, w_{i+1}]$, the set $S'_i := S_i \cup \{u\}$ is a stable set of G . Since $\mathbf{a}^T \chi^{S'_i} \leq c\alpha'$, we get $c(\alpha'-1)\omega' + a_u \leq c\alpha'$. Thus $a_u \leq c$. Therefore, spanning all consecutive pairs of W , we obtain $c = \max\{a_i \mid i \in V(G)\}$ as required. \square

2.3 A general characterization of facets

The next lemma provides a characterization when a valid inequality $\mathbf{a}^T \mathbf{x} \leq b$ is a facet of the stable set polytope of a general graph G . For that we need the following notions. A pair i, j of vertices is *a-critical* in G if there are two roots S_1 and S_2 of $\mathbf{a}^T \mathbf{x} \leq b$ such that $\{i\} = S_1 \setminus S_2$ and $\{j\} = S_2 \setminus S_1$. A subset V' of $V(G)$ is *a-connected* if the graph with vertex set V' and edge set $\{ij \mid i, j \in V', ij \text{ a-critical in } G\}$ is connected.

Lemma 9 *Let $\mathbf{a}^T \mathbf{x} \leq b$ be a valid inequality for $\text{STAB}(G)$ with $b \neq 0$. Consider a partition V_1, \dots, V_p of $V(G)$ s.t. V_i is a-connected for every $1 \leq i \leq p$. The inequality $\mathbf{a}^T \mathbf{x} \leq b$ is facet-defining if and only if there are p roots S_1, \dots, S_p with*

$$\begin{vmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_p \cap V_1| & \cdots & |S_p \cap V_p| \end{vmatrix} \neq 0$$

Proof. In order to prove the If-part, let $\mathbf{a}'^T \mathbf{x} \leq b'$ be a facet containing the face induced by the inequality $\mathbf{a}^T \mathbf{x} \leq b$.

For every $1 \leq i \leq p$, the set V_i is a-connected and so there exist λ_i such that $a_j = \lambda_i$ for all $j \in V_i$. Since for every stable set S , $\mathbf{a}^T \chi^S = b$ implies that $\mathbf{a}'^T \chi^S = b'$, V_i is \mathbf{a}' -connected. Therefore there exist λ'_i such that $a'_j = \lambda'_i$ for all $j \in V_i$. Hence we have for every $1 \leq i \leq p$,

$$\begin{aligned} \lambda_1 |S_i \cap V_1| + \dots + \lambda_p |S_i \cap V_p| &= b \\ \lambda'_1 |S_i \cap V_1| + \dots + \lambda'_p |S_i \cap V_p| &= b' \end{aligned}$$

Since

$$\begin{vmatrix} |S_1 \cap V_1| & \cdots & |S_1 \cap V_p| \\ \vdots & & \vdots \\ |S_p \cap V_1| & \cdots & |S_p \cap V_p| \end{vmatrix} \neq 0$$

holds we get $\lambda'_i = \frac{b'}{b} \lambda_i$ for every $1 \leq i \leq p$. Thus $\mathbf{a}^T \mathbf{x} \leq b$ is facet-defining.

Now let us turn to the Only if-part. Since \emptyset is not a root of the facet $\mathbf{a}^T \mathbf{x} \leq b$, there exist n roots S_1, \dots, S_n whose incidence vectors are linearly independent.

Let M be the matrix with the incidence vectors of S_1, \dots, S_n as rows. Let v_i be an element of V_i for $1 \leq i \leq p$. We add to the v_1 -th column of M the other columns related to the other elements of V_1 ; we add to the v_2 -th column of M the other columns related to the other elements of V_2 etc. This yields

$$\begin{vmatrix} \cdot & |S_1 \cap V_1| & \cdot & |S_1 \cap V_p| \\ \cdot & \vdots & \cdot & \vdots \\ \cdot & |S_n \cap V_1| & \cdot & |S_n \cap V_p| \end{vmatrix} \neq 0$$

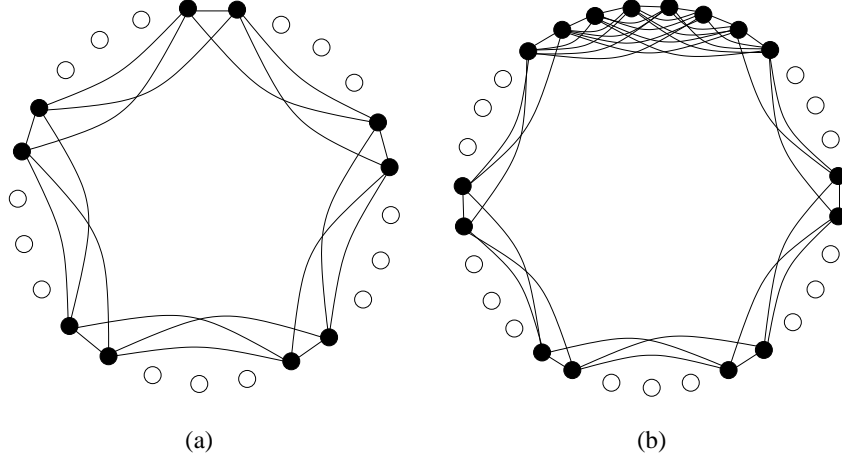


Figure 3

web W_n^k and by G^2 the web W_{n+k+1}^k . Let $\omega = k + 1$ be the clique number of both G^1 and G^2 and, for every $1 \leq i \leq n$ (resp. $1 \leq i \leq n + \omega$), let $Q_i^1 = [i, i + k]$ (resp. $Q_i^2 = [i, i + k]$) be the maximum clique of G^1 (resp. G^2) starting in i .

Since $\mathbf{a}^T \mathbf{x} \leq c\alpha_1$ is a proper weak rank facet of $\text{STAB}(G^1)$, there exists a subset V_1 of vertices of G^1 such that $\alpha_1 = \alpha(G^1[V_1])$ and $G^1[V_1]$ is rank facet-producing. Moreover, $G^1[V_1]$ has a partitionable web with vertex set W_1 , stability number α_1 , and clique number $\omega_1 \geq 2$ as induced subgraph by Corollary 7.

Notice that Q_{n-k}^1 is the maximum clique $\{n-k, \dots, n\}$ of G^1 . Let w_1, \dots, w_h be the elements in increasing order of W_1 in Q_{n-k}^1 . We have $h = \omega_1$ or $\omega_1 - 1$, by Lemma 4. For every $1 \leq i \leq h$, let q_i be the element $w_i + \omega$ of Q_{n+1}^2 and define

$$W_2 = \begin{cases} W_1 \cup \{q_1, \dots, q_{\omega_1}\} & \text{if } h = \omega_1 \\ W_1 \cup \{n+1\} \cup \{q_1, \dots, q_{\omega_1-1}\} & \text{if } h = \omega_1 - 1 \end{cases}$$

Let $V_2 = V_1 \cup Q_{n+1}^2 = V_1 \cup \{n+1, \dots, n+k+1\}$. Let \mathbf{v} be the $(n+\omega)$ -column vector $(a_1, \dots, a_n, c, \dots, c)$ and \mathbf{y} be the $(n+\omega)$ -column vector $\{a_1, \dots, a_n, 0, \dots, 0\}$.

Claim 1 *Inequality (2) is valid for $\text{STAB}(W_{n+k+1}^k)$.*

Let S be any stable set of G^2 . Let l be the vertex of S such that $[l+1, n] \cap S = \emptyset$ and let t be the vertex of S such that $[n+1, t-1] \cap S = \emptyset$. Notice that $S \setminus \{t\}$ is a stable set of G^1 . Hence we have $\mathbf{v}^T \chi^S = (\mathbf{y} + c\chi^{Q_{n+1}^2})^T \chi^S \leq c\alpha_1 + x_t \leq c(\alpha_1 + 1)$ as $x_t \leq c$ if $t \notin Q_{n+1}^2$ by Lemma 8, and $x_t = c$ if $t \in Q_{n+1}^2$. \diamond

Claim 2 *The set of vertices W_2 induces a partitionable web with stability number $\alpha_1 + 1$ and clique number ω_1 .*

Let $1 \leq v_1 \leq v_2 \leq \dots \leq v_{n'} \leq n$ be the vertices of W_1 in increasing order. We discuss the two cases $h = \omega_1$ and $h = \omega_1 - 1$.

If $h = \omega_1$ then let v be any vertex of W_2 . If v is a vertex q_i of $\{q_1, \dots, q_{\omega_1}\}$ then the set of vertices Q_v^2 meets W_2 exactly in the ω_1 vertices $\{q_i, \dots, q_{\omega_1}\} \cup \{v_1, \dots, v_{i-1}\}$, since W_1 induces a web of G^1 with clique number ω_1 by Lemma 4 (see Fig. 4). If v is a vertex w_i of $\{w_1, \dots, w_{\omega_1}\}$ then the set of vertices Q_v^2 meets W_2 precisely in the ω_1 vertices $\{w_i, \dots, w_{\omega_1}\} \cup \{q_1, \dots, q_{i-1}\}$. If v is a vertex of $W_1 \setminus \{w_1, \dots, w_{\omega_1}\}$, we obviously have $|Q_v \cap W_2| = \omega_1$ since W_1 induces a web of G^1 with clique number ω_1 due to Lemma 4.

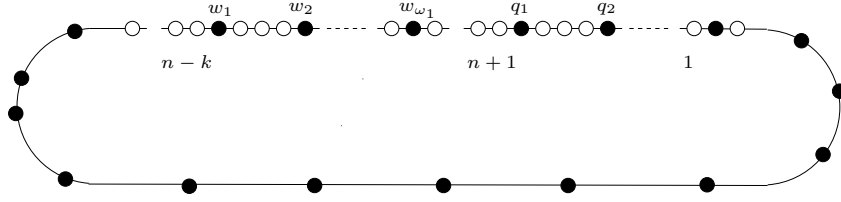


Figure 4: construction of W_2 , case $h = \omega_1$ (vertices of W_2 are drawn in black)

If $h = \omega_1 - 1$ then notice that $w_1 \neq n - k$ (otherwise Lemma 4 would imply $h = \omega_1$). Hence $n + 1 \notin \{q_1, \dots, q_h\}$. Let v be any vertex of W_2 . If v is a vertex q_i of $\{q_1, \dots, q_h\}$ then the set of vertices Q_v^2 meets W_2 exactly in the ω_1 vertices $\{q_i, \dots, q_h\} \cup \{v_1, v_2, \dots, v_{i-1}, v_i\}$, since W_1 induces a web of G^1 with clique number ω_1 by Lemma 4 (see Fig. 5). If v is a vertex w_i of $\{w_1, \dots, w_h\}$ then the set of vertices Q_v meets W_2 precisely in the ω_1 vertices $\{w_i, \dots, w_h, n + 1\} \cup \{q_1, \dots, q_{i-1}\}$, as $w_h < n + 1 < q_1$. If $v = n + 1$ then the set of vertices Q_v meets W_2 exactly in the ω_1 vertices $\{n + 1, q_1, \dots, q_h\}$. If v is a vertex of $W_1 \setminus \{w_1, \dots, w_h\}$, we obviously have $|Q_v \cap W_2| = \omega_1$ since W_1 induces a web of G^1 with clique number ω_1 due to Lemma 4.

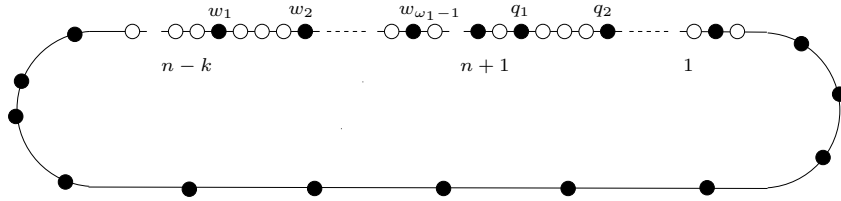


Figure 5: construction of W_2 , case $h = \omega_1 - 1$ (vertices of W_2 are drawn in black)

Hence in both cases, W_2 induces a web with clique number ω_1 (Lemma 4), with $|W| + \omega_1 = (\alpha_1 + 1)\omega_1 + 1$ vertices. Thus W_2 induces a partitionable web with stability number $\alpha_1 + 1$. \diamond

Claim 3 *The vertex set $V_2 = W_2 \cup Q_{n+1}^2$ is \mathbf{v} -connected.*

We first show that W_2 is \mathbf{v} -connected. Since $\mathbf{a}^T \mathbf{x} \leq c\alpha_1$ is a weak rank facet of $\text{STAB}(G^1)$, we have by definition $a_i = c$ for every $i \in W_1$. Hence for every

$i \in W_2$ follows $v_i = c$. Since W_2 is a partitionable web of stability number $\alpha_1 + 1$ by Claim 2, this implies that W_2 is \mathbf{v} -connected.

Let $w_1 < w_2 < \dots < w_{\omega_1}$ be the elements of W_2 in Q_{n+1}^2 (by definition of W_2 there are exactly ω_1 of them). Let S be a maximum stable set of W_2 disjoint from Q_1^2 (S exists because $W_2 \cap Q_1^2$ is a subset of a maximum clique of W_2 , and for every maximum clique Q of a partitionable graph, there exists a unique maximum stable set avoiding Q by [1]). Let s be the element of S with maximal index. Then for every $w_{\omega_1} \leq q \leq n + \omega$, the set $(S \setminus \{s\}) \cup \{q\}$ is obviously a root of inequality (2). Hence $W^2 \cup [w_{\omega_1}, n + \omega]$ is \mathbf{v} -connected. Likewise, the set $W^2 \cup [n + 1, w_1]$ is \mathbf{v} -connected.

For every $1 \leq i < \omega_1$, there exists a maximum stable set of W_2 disjoint from $Q_{w_{i+1}}^2$. Let s be the element of S with maximal index which is less or equal than w_i . Then for every $w_i \leq q \leq w_{i+1}$, the set $(S \setminus \{s\}) \cup \{q\}$ is a root of inequality (2). Hence $W^2 \cup [w_i, w_{i+1}]$ is \mathbf{v} -connected and V_2 is \mathbf{v} -connected as well. \diamond

Let $p = n - |W_1|$ and $\{1, \dots, n\} \setminus W_1 = \{y_1, \dots, y_p\}$. Due to Lemma 9, there are p roots S_1, \dots, S_p of $\mathbf{a}^T \mathbf{x} \leq c\alpha_1$ such that the incidence vectors of their restriction to $\{1, \dots, n\} \setminus W_1 = (\{1, \dots, n\} \cup Q_n) \setminus V_2$ are linearly independent, that is

$$\begin{vmatrix} |S_1 \cap \{y_1\}| & \cdots & |S_1 \cap \{y_p\}| \\ \vdots & & \vdots \\ |S_{k'} \cap \{y_1\}| & \cdots & |S_{k'} \cap \{y_p\}| \end{vmatrix} \neq 0$$

Claim 4 For every $1 \leq i \leq p$, there exists a vertex q_i of G^2 such that $S'_i = S_i \cup \{q_i\}$ is a root of inequality (2).

For every $1 \leq i \leq p$, let l_i (resp. t_i) be the element of S_i with minimal (resp. maximal) index. Let $q_i = t_i + \omega$. Obviously, q_i is not a neighbor of t_i in G^2 . If q_i is a neighbor of l_i in G^2 then $q_i + \omega - 1 - (n + \omega) \geq l_i$. Thus $t_i + \omega - 1 - n \geq l_i$, which implies that t_i is a neighbor of l_i in G^1 : a contradiction.

Hence $S'_i = S_i \cup \{q_i\}$ is a stable set of G^2 . Since q_i is a vertex of the maximum clique Q_n , it follows that S'_i is a root of inequality (2), as required. \diamond

Since $G^2[W_2]$ has stability number $\alpha_1 + 1$ (Claim 2), there is a stable set S'_0 of $G^2[V_2]$ which is a root of inequality (2).

For every $0 \leq i \leq p$ and $1 \leq j \leq p$, let $\delta_{i,j} = 1$ if $y_j \in S'_i$, 0 otherwise. By Claim 1 and 4, inequality (2) is a valid inequality with $p + 1$ \mathbf{v} -critical components $V_2, \{y_1\}, \dots, \{y_p\}$, and $p + 1$ roots S'_0, S'_1, \dots, S'_p such that

$$\begin{vmatrix} |S'_0 \cap V_2| & \delta_{0,1} & \cdots & \delta_{0,p} \\ |S'_1 \cap V_2| & \delta_{1,1} & \cdots & \delta_{1,p} \\ \vdots & \vdots & & \vdots \\ |S'_p \cap V_2| & \delta_{p,1} & \cdots & \delta_{p,p} \end{vmatrix} = \begin{vmatrix} \alpha_1 + 1 & 0 & \cdots & 0 \\ |S'_1 \cap V_2| & |S_1 \cap \{y_1\}| & \cdots & |S_1 \cap \{y_p\}| \\ \vdots & \vdots & & \vdots \\ |S'_p \cap V_2| & |S_p \cap \{y_1\}| & \cdots & |S_p \cap \{y_p\}| \end{vmatrix} \neq 0$$

Lemma 9 implies that inequality (2) defines a facet of $\text{STAB}(G^2)$. To finish the proof, it remains to show that it is a proper weak rank facet.

Claim 5 *The set V_2 is rank facet-producing and $\alpha(G^2[V_2]) = \alpha_1 + 1$.*

We have $\alpha(G^2[V_2]) \leq \alpha(G^2[V_1]) + \alpha(Q_n) \leq \alpha_1 + 1$ which further implies $\alpha(G^2[V_2]) = \alpha(G^2[W_2])$. Let v be any vertex of $V_2 \setminus W_2$. By the definition of V_2 , v is an element of Q_{n+1}^2 . Therefore $|N(v) \cap W_2| \geq \omega_1$ as $|W_2 \cap Q_{n+1}^2| = \omega_1$, by the definition of W_2 . Let δ be the element of W_2 with maximal index.

If $v < \delta$ then $(\delta - \omega) \in N(v)$. As $\delta - \omega$ is an element of W_2 by the definition of W_2 , we get $|N(v) \cap W_2| \geq \omega_1 + 1$. If $v \geq \delta$ then v has at least one neighbor in $Q_v^2 \cap W_2$, as $|Q_v^2 \cap W_2| \geq \omega_1 - 1 \geq 1$ (Lemma 4). Hence $|N(v) \cap W_2| \geq \omega_1 + 1$.

Thus, in both cases, $|N(v) \cap W_2| \geq \omega_1 + 1$. Hence $\alpha(N(x) \cap W_2) = 2$ and therefore, $G^2[V_2]$ is rank facet-producing by Galluccio & Sassano [8] (recall that W^2 is a partitionable web by Claim 2 and is, therefore, rank-minimal). \square

An immediate consequence of Theorem 11 is the main result: if $\text{STAB}(W_n^k)$ has a proper weak non-rank facet then $\text{STAB}(W_{n+k+1}^k)$ has a proper weak non-rank facet (Theorem 1).

4 Concluding remarks and open problems

The presented construction for non-rank facets of stable set polytopes of webs shows that we obtain, from every single proper weak non-rank facet in $\text{STAB}(W_n^k)$, an infinite sequence $\text{STAB}(W_n^k)$, $\text{STAB}(W_{n+(k+1)}^k)$, $\text{STAB}(W_{n+2(k+1)}^k)$, ... of not rank-perfect webs (Theorem 1).

If there is a set of webs $W_{n_0}^k, \dots, W_{n_k}^k$ such that $\text{STAB}(W_{n_i}^k)$ has a proper weak non-rank facet and $n_i = i \pmod{k+1}$ then applying this construction implies that there exist only finitely many rank-perfect webs with clique number $k+1$ (Corollary 2). Such sets of non-rank-perfect webs are presented in [16, 17] for $k=3$ and all even values $k \geq 4$; the case of all odd values $k \geq 5$ is open. We conjecture that such sets exist for the remaining cases as well:

Conjecture 12 *For every $\omega \geq 4$, there are only finitely many rank-perfect webs with clique number ω .*

According to Ben Rebea's Conjecture [14], the stable set polytopes of quasi-line graphs (and therefore of webs) have clique family inequalities as only non-trivial facets. This would particularly mean that all facets admit at most two non-zero coefficients. Notice that our construction of non-rank facets does not increase the number of non-zero coefficients. In particular, the non-rank facets presented in [16, 17] have coefficients equal to 2 and 1 only. On the other hand, Liebling et al. [12] found an infinite sequence of not rank-perfect webs where the non-rank facets admit coefficients a and $a+1$ for every $a \geq 1$. Hence we are still far from having a complete description of the stable set polytopes of webs.

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