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## Medians of Binary Relations: Computational Complexity

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This paper was written while the author visited the Konrad-Zuse-Zentrum für Informationstechnik Berlin

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Preprint 92-4 (Januar 1992)

Herausgegeben vom  
Konrad-Zuse-Zentrum für Informationstechnik Berlin  
Heilbronner Str. 10  
1000 Berlin 31  
Verantwortlich: Dr. Klaus André  
Umschlagsatz und Druck: Rabe KG Buch-und Offsetdruck Berlin

ISSN 0933-7911

# MEDIANS OF BINARY RELATIONS: COMPUTATIONAL COMPLEXITY

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## Abstract

Let  $\mathcal{R}$  be the set of all binary relations on a finite set  $N$  and  $d$  be the symmetric difference distance defined on  $\mathcal{R}$ . For a given profile  $\Pi = (R_1, \dots, R_m) \in \mathcal{R}^m$ , a relation  $R^* \in \mathcal{R}$  that minimizes the function  $\sum_{k=1}^m d(R_k, R)$  is called a **median** relation of  $\Pi$ . A number of problems occurring in the social sciences, in qualitative data analysis and in multicriteria decision making can be modelled as problems of finding medians of a profile of binary relations. In these contexts the profile  $\Pi$  represents collected data (preferences, similarities, games) and the objective is that of finding a median relation of  $\Pi$  with some special feature (representing e. g., consensus of preferences, clustering of similar objects, ranking of teams, etc.). In this paper we analyse the computational complexity of all such problems in which the median is required to satisfy one or more of the properties: reflexivity, symmetry, antisymmetry, transitivity and completeness. We prove that whenever transitivity is required (except when symmetry and completeness are also simultaneously required) then the corresponding median problem is  $\mathcal{NP}$ -hard. In some cases we prove that they remain  $\mathcal{NP}$ -hard when the profile  $\Pi$  has a fixed number of binary relations.

## 1. Introduction

In the social choice theory a classical problem that has been largely investigated and whose origin traces back to the late eighteenth century is the problem of aggregating individual preferences (linear orders) into a social preference (a linear order). The notion of consensus of preferences plays an important role in the social sciences, a reason why many efforts have been made to find realistic models to express it (cf. Leclerc [1988a], Day [1988]).

The first mathematical approaches on problems of aggregation of preferences are credited to Borda in 1770 and Condorcet in 1785, both concerned with the design of election procedures. In a recent paper, Young [1988] discusses the model proposed by these two major figures of that time, gives some historical accounts and explains the Condorcet's theory of voting (see also Young & Levenglick [1978]).

The notion of median relation — a relation minimizing a “remoteness” function defined in terms of the symmetric difference distance — was introduced by Kemeny [1959], who investigated a method to aggregate individual preferences into a collective preference.

His method, although being of metric nature, is in fact equivalent to the CONDORCET'S MAJORITY RULE, according to which the winning collective preference should be the one supported by the largest number of votes (Young [1988], Barbut [1967], Fishburn [1977], Michaud [1987]).

In cluster analysis a similar approach was proposed by Régnier [1965], then Mirkin [1974], for solving the problem of aggregating equivalence relations into an equivalence relation (see also Zahn [1964]).

The fact that the symmetric difference distance has been used in problems occurring in many different contexts is not a pure coincidence. Axiomatics supporting its use has been investigated in several cases, cf. Kemeny [1959], Monjardet [1978], Barthélemy [1979] and Barthélemy & Monjardet [1981]. However, the median approach, as any consensus procedure, has some defects as pointed out by Fishburn [1977], Leclerc [1988a] and Barthélemy & Monjardet [1988]. This last reference gives also an overview of the developments on the algorithmic approaches and extensions of the notion of median in other structures. The results concerning its algebraic definition that generalizes to any distributive lattice (cf. Barbut [1961], Monjardet [1980]), as well as more recent results on median semilattices, resp. (semi)modular (semi)lattices can be found in Monjardet [1987, 1988], resp. Barthélemy [1981] and Leclerc [1988b]. For a unified treatment on this subject the reader should refer to Barthélemy, Flament & Monjardet [1982]; Barthélemy, Leclerc & Monjardet [1986]; Barthélemy & Monjardet [1988] and Barthélemy [1988].

In this paper we analyse the computational complexity of a class of problems of finding medians with certain properties. This class includes those classical problems such as aggregation of preferences and clustering.

The material is organized as follows. In Section 2 we give the definitions and notations to be used and present the problems to be investigated. In Section 3 the main results on the computational complexity of these problems are presented, and in Section 4 we discuss special cases concerning restricted domains.

## 2. Definition and Notations

Let  $N$  be a finite set with  $n$  objects (e. g. alternatives, candidates, teams, states, etc.) and let  $\mathcal{R}$  denote the set of all binary relations on  $N$ . Consider  $\mathcal{R}$  endowed with a metric  $d$ , the symmetric difference distance, defined as

$$d(R, S) := |R \Delta S| := |R \cup S| - |R \cap S| \quad \text{for all } R, S \in \mathcal{R}.$$

A **profile** of relations in  $\mathcal{R}$ , or a profile in  $\mathcal{R}^m$ , is an  $m$ -tuple  $\Pi = (R_1, \dots, R_m)$  where  $R_k \in \mathcal{R}$  for  $k = 1, \dots, m$ . Given a profile  $\Pi = (R_1, \dots, R_m)$  in  $\mathcal{R}^m$  a relation  $R^* \in \mathcal{R}$  that

minimizes the function

$$D(\Pi, R) := \sum_{k=1}^m d(R_k, R)$$

is called a **median relation** of  $\Pi$ .

In this general form the problem of finding a median of a given profile is trivial and not interesting. However, if we require the median to satisfy certain properties the resulting problem becomes interesting and has nice applications. So, according to the desired properties of  $R^*$  we obtain different problems, and here we consider all those arising when the properties are chosen from the set

$$P := \{ \text{Reflexive, Symmetric, Antisymmetric, Transitive, Total} \}.$$

Let us recall some definitions. A relation  $R \in \mathcal{R}$  is **reflexive** (REF) if  $(i, i) \in R$  for all  $i \in N$ ;  $R$  is **symmetric** (SYM) if  $(i, j) \in R$  implies  $(j, i) \in R$  for all  $i, j \in N$ ;  $R$  is **antisymmetric** (ASY) if  $(i, j) \in R$  and  $(j, i) \in R$  imply  $i = j$  for all  $i, j \in N$ ;  $R$  is **transitive** (TRA) if  $(i, j) \in R$  and  $(j, k) \in R$  imply  $(i, k) \in R$  for all  $i, j, k \in N$ ;  $R$  is **total** (TOT) if  $(i, j) \in R$  or  $(j, i) \in R$  for all  $i, j \in N$ .

To simplify notation we use the abbreviated form of the name

of the property (given in parentheses) to denote also the set of all relations having this property. Thus, for example, TRA denotes the set of all transitive relations in  $\mathcal{R}$ . Some relations having more than one of the properties in  $P$  are known by special names, not always standard in the literature. Here we adopt the following notation and terminology:

$\mathcal{C}$  denotes the set of all **complete preorders**, i. e.  $\mathcal{C} = \text{TRA} \cap \text{TOT}$ .

$\mathcal{T}$  denotes the set of all **tournaments**, i. e.  $\mathcal{T} = \text{ASY} \cap \text{TOT}$ .

$\mathcal{L}$  denotes the set of all **linear orders**, i. e.  $\mathcal{L} = \text{ASY} \cap \text{TRA} \cap \text{TOT}$ .

$\mathcal{O}$  denotes the set of all **partial orders**, i. e.  $\mathcal{O} = \text{ASY} \cap \text{TRA}$ .

$\mathcal{E}$  denotes the set of all **equivalence relations**, i. e.  $\mathcal{E} = \text{REF} \cap \text{SYM} \cap \text{TRA}$ .

For a subset  $\mathcal{M} \in \mathcal{R}$  the **median problem** relative to  $\mathcal{M}$ , denoted by  $MP(\mathcal{R}, \mathcal{M})$ , is defined as follows.

**Instance:** Profile  $\Pi = (R_1, \dots, R_m)$  of  $m$  relations in  $\mathcal{R}$ .

**Objective:** Find a relation  $R^* \in \mathcal{M}$  such that  $D(\Pi, R^*) = \min_{R \in \mathcal{M}} D(\Pi, R)$ .

A variant of this problem, that differs only by the fact that the size of the profile is *fixed*, is denoted by  $MP(\mathcal{R}, \mathcal{M}, m)$ . So, in this problem we have

**Instance:** Profile  $\Pi = (R_1, \dots, R_m)$  in  $\mathcal{R}^m$  where  $m$  is fixed.

We expect the reader to be familiar with the basic concepts of graph theory and complexity theory. If not, the definitions not given here can be found in Bondy & Murty [1976], resp. Garey & Johnson [1979].

A graph  $G$  with node set  $V$  and edge set  $E$  is denoted by  $G = [V, E]$ . A digraph (or directed graph)  $D$  with node set  $N$  and arc set  $A$  is denoted by  $D = (N, A)$ . A graph  $G = [V, E]$ , resp. digraph  $D = (N, A)$ , is called **complete** if  $E = \{\{u, v\} : u, v \in V, u \neq v\}$ , resp.  $A = \{(u, v) : u, v \in N, u \neq v\}$ . If  $D = (N, A)$  is a digraph with  $A = N \times N$  then  $D$  is called  **$l$ -complete** (i. e. complete with all loops). For a digraph  $D = (N, A)$ , we call the arcs in  $(N \times N) \setminus A$  **missing arcs** (analogously, missing edges in case of a graph). A digraph is called **acyclic** if it does not contain any directed cycle.

A **clique** of a graph is a complete subgraph of  $G$ . It needs not be maximal, as is sometimes assumed in the literature. A set of edges  $A$  in a graph  $G = [V, E]$  is called a **clique partitioning** of  $G$  if there is a partition  $V_1, \dots, V_k$  of  $V$  such that the subgraph induced by each  $V_i$ ,  $1 \leq i \leq k$ , is a clique in  $G$  and  $A$  is the union of all edges in  $G$  with both endnodes in the same set of the partition. In this case, if for  $1 \leq i \leq k$  the clique induced by  $V_i$  is denoted by  $Q_i$ , then we say that  $\mathcal{C}(A) := \{Q_1, \dots, Q_k\}$  is the **clique set defined by  $A$** .

### 3. Computational Complexity

We assume here that an instance of the median problems  $MP(\mathcal{R}, \mathcal{M})$  or  $MP(\mathcal{R}, \mathcal{M}, m)$  consisting of a profile  $\Pi = (R_1, \dots, R_m)$  is given by an  $(m, n^2)$ -matrix  $A = (a_{ij})$  whose rows are the characteristic vectors of the relations (indexed by the pairs  $(i, j) \in N \times N$ ). Thus,  $a_{k,e} = 1$  (resp.  $a_{k,e} = 0$ ) if  $e \in R_k$  (resp.  $e \notin R_k$ ) and clearly the size of such an instance is  $\mathcal{O}(n^2 m)$ .

It is well-known that the median problems we are considering can be formulated as 0/1 linear programs or optimization problems on weighted digraphs. In fact, it is easy to prove that

$$D(\Pi, R) = \sum_{(i,j)} w_{ij} r_{ij} + \sum_{(i,j)} \alpha_{ij},$$

where

$$(3.1) \quad \alpha_{ij} := |\{k : (i, j) \in R_k\}|,$$

$$(3.2) \quad w_{ij} := m - 2\alpha_{ij} \quad \text{and}$$

$$r = (r_{ij}) \text{ is the characteristic vector of } R.$$

Thus, each given instance of  $MP(\mathcal{R}, \mathcal{M})$  can be formulated as the 0/1 linear program:

$$(3.3) \quad \text{minimize } \sum_{(i,j)} w_{ij} r_{ij}$$

subject to:  $r = (r_{ij})$  is the characteristic vector of some relation  $R \in \mathcal{M}$ .

If the coefficients  $w_{ij}$  are interpreted as being weights associated with the arcs  $(i, j)$  of an  $l$ -complete digraph  $D_n$  on the node set  $N$ , then the problem becomes that of finding certain minimum weighted subdigraphs in  $D_n$ . For example, if  $\mathcal{M} = \mathcal{L}$  the corresponding digraph problem is a special case of the weighted feedback arc set problem or linear ordering problem, and if  $\mathcal{M} \in \mathcal{E}$  we obtain the so-called clique partitioning problem (see Reinelt [1985], Grötschel, Jünger & Reinelt [1985], Barthélemy, Guenoche & Hudry [1988], resp. Wakabayashi [1986] and Grötschel & Wakabayashi [1988]).

From the above reduction one obtains immediately the following result (excluding some trivial non-interesting cases):

**(3.4) Proposition.** *If  $\mathcal{M} \in \{SYM, ASY, TOT, ASY \cap TOT\}$  then the problem  $MP(\mathcal{R}, \mathcal{M})$  is polynomially solvable.*

□

We can also make use of the given reduction, in a more specialized way, to show that  $MP(\mathcal{R}, \mathcal{M})$  is  $\mathcal{NP}$ -hard for other subsets  $\mathcal{M}$ . Namely, we first note that the obtained digraph optimization problems are special in the sense that all of its weights  $w_{ij}$  are integers having the same parity (cf. (3.1) and (3.2)). Furthermore, we observe that whenever we have such an  $l$ -complete weighted digraph  $D_n = (N, A_n)$  with  $m := \max_{e \in A_n} |w_e|$  we can construct a profile  $\Pi = (R_1, \dots, R_m)$  in  $\mathcal{R}^m$  such that for each  $(i, j) \in N \times N$  we have  $w_{ij} = m - 2\alpha_{ij}$  with  $\alpha_{ij}$  as in (3.1). In other words, these special digraph optimization problems are also polynomially reducible to  $MP(\mathcal{R}, \mathcal{M})$  or  $MP(\mathcal{R}, \mathcal{M}, m)$ .

In order to state more formally the results we introduce first some notations. For any set  $\mathcal{M} \subset \mathcal{R}$  we denote by  $DOP(\mathcal{M}, m)$  the following **Digraph Optimization Problem** relative to  $\mathcal{M}$ .

**Instance:**  $l$ -complete digraph  $D_n = (N, A_n)$ ; weights  $w_e \in Z$  for each  $e \in A_n$ , all having the same parity and with  $\max_e |w_e| = m$ .

**Objective:** Find an arc set  $A^* \subset A_n$  such that  $A^* \in \mathcal{M}$  and  $w(A^*) := \sum_{e \in A^*} w_e$  is minimum.

We denote by  $BDOP(\mathcal{M}, \alpha)$  a variant of  $DOP(\mathcal{M}, m)$  in which the objective is the same and the instance is as follows.

**Instance:**  $l$ -complete digraph  $D_n(N, A_n)$ ; weights  $w_e \in Z$  for each  $e \in A_n$ ,  $|w_e|$  even; (fixed) integer  $\alpha$  such that  $|w_e| \leq n^\alpha$ .

The reason to introduce these problems is justified by the following result.

**(3.5) Theorem.** *Let  $\mathcal{M} \subset \mathcal{R}$  and  $m, \alpha$  be positive integers.*

(a) *If  $DOP(\mathcal{M}, m)$  is  $\mathcal{NP}$ -hard, then  $MP(\mathcal{R}, \mathcal{M}, m)$  is  $\mathcal{NP}$ -hard.*

(b) *If  $BDOP(\mathcal{M}, \alpha)$  is  $\mathcal{NP}$ -hard, then  $MP(\mathcal{R}, \mathcal{M})$  is  $\mathcal{NP}$ -hard.*

**Proof. (a):** Let  $D_n = (N, A_n)$ ,  $w$  and  $m$  be given as an instance  $\mathcal{I}$  of  $DOP(\mathcal{M}, m)$ . The corresponding instance  $\mathcal{I}'$  of  $MP(\mathcal{R}, \mathcal{M}, m)$  is constructed as follows. For each pair  $(i, j) \in A_n$  we determine the number  $z_{ij} := (m - w_{ij})/2$  and set

$$R_k := \{(i, j) \in N \times N : z_{ij} \geq k\}, \text{ for } k = 1, \dots, m,$$

obtaining this way the profile  $\Pi = (R_1, \dots, R_m)$ .

Since  $m$  is a constant, the construction of the profile  $\Pi$  can be done in  $\mathcal{O}(n^2)$  and therefore in time polynomial in the size of  $\mathcal{I}$ . The proof that an optimum solution of the instance  $\mathcal{I}'$  gives an optimum solution of  $\mathcal{I}$  is straightforward and will be omitted.

**(b):** Given an instance  $\mathcal{I} = (D_n, w, \alpha)$  of  $BDOP(\mathcal{M}, \alpha)$  we determine first the value  $q := \max_e |w_e|$ . Since  $q \leq n^\alpha$  we can construct a profile  $\Pi = (R_1, \dots, R_q)$  as in the previous case, here as well in time polynomial in the size of  $\mathcal{I}$ . The desired conclusion follows analogous to Case (a). □

The **decision versions** of  $DOP(\mathcal{M}, m)$  and  $BDOP(\mathcal{M}, \alpha)$  will be denoted by  $DDP(\mathcal{M}, m)$  and  $BDDP(\mathcal{M}, \alpha)$ , respectively. For technical reasons it will be convenient to work with a slight variant of these problems — denoted by  $DDP^*(\mathcal{M}, m)$  and  $BDDP^*(\mathcal{M}, \alpha)$  — where the instances consist of **loopless** complete digraphs. Furthermore, we consider two other properties,  $TRA^*$  and  $TOT^*$ , defined as follows. A relation  $R \in TRA^*$  iff  $(i, j) \in R$  and  $(j, k) \in R$  imply  $(i, k) \in R$  for all  $i, j, k \in N$ ,  $i \neq j \neq k \neq i$ ; and  $R \in TOT^*$  iff  $(i, j) \in R$  or  $(j, i) \in R$  for all  $i, j \in N$ ,  $i \neq j$ .

The next lemma shows that if we can prove certain  $\mathcal{NP}$ -completeness results for  $DDP^*$  or  $BDDP^*$  then we can derive analogous results for  $DDP$  or  $BDDP$ .

**(3.6) Lemma.** *Let  $\mathcal{M}^* \subset TRA^*$  and*

$$\mathcal{M} \in \{\mathcal{M}^*, \mathcal{M}^* \cap REF, \mathcal{M}^* \cap TRA, \mathcal{M}^* \cap TRA \cap REF\}.$$

*(a) If  $DDP^*(\mathcal{M}^*, m)$  is  $\mathcal{NP}$ -complete then  $DDP(\mathcal{M}, m)$  is  $\mathcal{NP}$ -complete.*

*(b) If  $BDDP^*(\mathcal{M}^*, \alpha)$  is  $\mathcal{NP}$ -complete then  $BDDP(\mathcal{M}, \alpha)$  is  $\mathcal{NP}$ -complete.*

**Proof. (a):** Let  $D_n = (N, A_n)$ ,  $w$ ,  $m$  and  $k$  be an arbitrary instance of  $DDP^*(\mathcal{M}^*, m)$ . The corresponding instance of  $DDP(\mathcal{M}, m)$ , defined by  $D'_n$ ,  $w'$ ,  $m$ ,  $k'$  is constructed as follows:  $D'_n = (N, A'_n)$  is the  $l$ -complete digraph obtained from  $D_n$  by adding to it all the missing loops, the weights  $w'_e$  are defined as:

$$w'_e := \begin{cases} w_e & \text{if } e \in A_n, \\ 0 & \text{if } e \notin A_n \text{ and } m \text{ is even,} \\ -1 & \text{if } e \notin A_n \text{ and } m \text{ is odd,} \end{cases}$$



and

$$k' := \begin{cases} k & \text{if } m \text{ is even,} \\ k - n & \text{if } m \text{ is odd.} \end{cases}$$

We claim that  $D_n$  has an arc set  $B$  such that  $B \in \mathcal{M}^*$  and  $w(B) \leq k$  iff  $D'_n$  has an arc set  $B'$  with  $B' \in \mathcal{M}$  and  $w'(B') \leq k'$ .

In fact, given  $B \subset A_n$  take  $B' := B \cup \{(i, i) : i \in N\}$ ; and conversely, given  $B' \subset A'_n$  take  $B := B' \setminus \{(i, i) : i \in N\}$ . This proves the claim and establishes the  $\mathcal{NP}$ -completeness of  $DDP(\mathcal{M}, m)$ .

(b): The proof will be omitted as it is analogous to case (a). The only difference is that we have to determine first the value  $\max_e |w_e|$ . □

For the proof of the next theorem we need the fact that the following problem is  $\mathcal{NP}$ -complete (cf. Karp [1972]):

### Acyclic Subdigraph Problem (ASP)

Instance: Digraph  $D = (N, A)$  without loops; positive integer  $k \leq |N|$ .

Question: Is there a subset  $B \subseteq A$  with  $|B| \geq k$  such that  $H = (N, B)$  is acyclic?

The next lemma (easy to be proved by induction) will be useful in theorem (3.8).

**(3.7) Lemma.** *If  $H = (N, B)$  is an acyclic digraph then there exists a graph  $H' = (N, B')$  containing  $H$ , such that  $B' \in ASY \cap TRA \cap TOT^*$ .* □

In the subsequent  $\mathcal{NP}$ -completeness proofs we shall omit the straightforward verification that the considered problems are in the class  $\mathcal{NP}$ .

**(3.8) Theorem.** *Let*

$$\mathcal{M}' = ASY \cap TRA^* \text{ and } \mathcal{M}'' = ASY \cap TRA^* \cap TOT^*.$$

(a) For every  $m \geq 2$ ,  $DDP^*(\mathcal{M}', m)$  is  $\mathcal{NP}$ -complete.

(b) For every  $m \geq 1$ ,  $DDP^*(\mathcal{M}'', m)$  is  $\mathcal{NP}$ -complete.

**Proof.** [Transformation from the Acyclic Subdigraph Problem (ASP)]

(i) Assume first that  $m \geq 2$  and let  $\mathcal{M} \in \{\mathcal{M}', \mathcal{M}''\}$ .

Suppose that  $D = (N, A)$  and  $k$  are given as an instance of ASP. Then the corresponding instance of  $DDP^*(\mathcal{M}, m)$ , defined by  $D_n$ ,  $w$ ,  $m$  and  $k'$ , is obtained as follows:

$D_n = (N, A_n)$  is the complete digraph obtained from  $D$  by adding to it all the missing arcs which are not loops; the weights  $w_e$  for  $e \in A_n$  are defined as

$$w_e := \begin{cases} -m & \text{if } e \in A, \\ -(m-2) & \text{otherwise;} \end{cases}$$

and

$$k' := -2k - \binom{n}{2}(m-2).$$

We shall prove that  $D$  has an acyclic subdigraph  $H = (N, B)$  with  $|B| \geq k$  iff  $D_n$  has a subdigraph  $H' = (N, B')$  with  $B' \in \mathcal{M}$  and  $w(B') \leq k'$ .

a) Let  $H = (N, B)$  be an acyclic subdigraph in  $D$  with  $|B| \geq k$ . Since  $H$  is also a subdigraph of  $D_n$ , then by Lemma (3.7) there exists in  $D_n$  a subdigraph  $H' = (N, B')$  containing  $H$  such that  $B' \in \mathcal{M}''$ . Moreover,

$$\begin{aligned} w(B') &= w(B) + w(B' \setminus B) \\ &\leq |B|(-m) - \left( \binom{n}{2} - |B| \right) (m-2) \leq k'. \end{aligned}$$

b) Let  $H' = (N, B')$  be a subdigraph in  $D_n$  such that  $B' \in \mathcal{M}$  and  $w(B') \leq k'$ . Since  $H'$  is acyclic, by Lemma (3.7) there exists in  $D_n$  a subdigraph  $H'' = (N, B'')$  containing  $H'$  with  $B'' \in \mathcal{M}''$ . Note that  $B''$  has at least  $k$  arcs with weight  $-m$ . Otherwise, if  $B''$  has  $l$  arcs with weight  $-m$ ,  $l \leq k-1$ , then

$$w(B') \geq w(B'') = l(-m) - \left( \binom{n}{2} - l \right) (m-2) > k'.$$

Thus, if we take  $B := \{e \in B'' : w_e = -m\}$ , clearly  $H = (N, B)$  is an acyclic subdigraph of  $D$  with  $|B| \geq k$ .

(ii) If  $m = 1$  then the above proof also holds for  $\mathcal{M} = \mathcal{M}''$ .

Since ASP is  $\mathcal{NP}$ -complete and the given transformation is polynomial, the result follows. □

We want to prove in the sequel that  $\text{DDP}^*(\mathcal{M}, m)$  for  $\mathcal{M} = \text{SYM} \cap \text{TRA}^*$  is  $\mathcal{NP}$ -complete. For that, we introduce an  $\mathcal{NP}$ -complete problem called **Restricted Clique Partitioning Problem**, denoted by RCPP, defined as follows:

Instance: Complete graph  $K_n = [V, E]$ , weights  $w_e \in \{-1, 0, 1\}$  for each  $e \in E$ , integer  $k$ .

Question: Is there a clique partitioning  $A \subset E$  such that  $w(A) \leq k$ ? (That is, is there a partition of the node set  $V_n$  such that the sum of the weights of all edges with both endnodes in the same set of the partition is less or equal to  $k$ ?)

As the  $\mathcal{NP}$ -completeness proof of RCPP is long, in order not to break the continuity, we shall postpone it to the end of this section (see Theorem 3.14).

**(3.9) Theorem.** *Let*

$$\mathcal{M} = \text{SYM} \cap \text{TRA}^*.$$

*Then  $\text{DDP}^*(\mathcal{M}, m)$  is  $\mathcal{NP}$ -complete for every  $m \geq 1$ .*

**Proof.** [Transformation from RCPP]

Note that it suffices to prove for  $m = 1$ . Let  $K_n = [V_n, \mathcal{E}_n]$ ,  $w$  and  $k$  be an arbitrary instance of RCPP and assume that  $V_n = \{1, 2, \dots, n\}$ . The corresponding instance of  $\text{DDP}^*(\mathcal{M}, 1)$  defined by  $D_n$ ,  $w'$  and  $k'$ , is constructed as follows:  $D_n = (N, A_n)$  is a complete digraph with node set  $N = V_n$ , the weights  $w'_e$  for  $e \in A_n$  are defined as

$$w'_{ij} := \begin{cases} 1 & \text{if } (w_{ij} = 1) \text{ or } (w_{ij} = 0 \text{ and } i < j), \\ -1 & \text{if } (w_{ij} = -1) \text{ or } (w_{ij} = 0 \text{ and } i > j); \end{cases}$$

and  $k' := k$ .

It is immediate that, if  $K_n = [V_n, \mathcal{E}_n]$  has a clique partitioning  $A$  with  $w(A) \leq k$ , then  $B := \{ij, ji : \{i, j\} \in A\}$ , is an arc set in  $D_n$  such that  $B \in \mathcal{M}$  and  $w'(B) = w(A)$ . Conversely, if  $D_n$  has an arc set  $B \in \mathcal{M}$  with  $w'(B) \leq k'$ , then it is easy to see that the set  $A := \{\{i, j\} : ij \in B\}$  is a clique partitioning of  $K_n$  with  $w(A) = w'(B)$ . Since RCPP is  $\mathcal{NP}$ -complete (cf. Theorem (3.14)), and the given transformation is polynomial, the result follows. □

The following theorem combines Lemma (3.6) and the Theorems (3.5), (3.8) and (3.9).

**(3.10) Theorem.** *Let  $\mathcal{M}'$  and  $\mathcal{M}''$  be such that*

$$\mathcal{M}' \in \left\{ \begin{array}{ll} \text{ASY} \cap \text{TOT}^* \cap \text{TRA}^*, & \text{ASY} \cap \text{TOT}^* \cap \text{TRA}, \\ \text{SYM} \cap \text{TRA}^*, & \text{SYM} \cap \text{TRA} \end{array} \right\},$$

$$\mathcal{M}'' \in \{\text{ASY} \cap \text{TRA}^*, \text{ASY} \cap \text{TRA}\}.$$

*Then the median problem  $\text{MP}(\mathcal{R}, \mathcal{M}, m)$  is  $\mathcal{NP}$ -hard*

- i) for every  $m \geq 1$  and  $\mathcal{M} \in \{\mathcal{M}', \mathcal{M}' \cap \text{REF}\}$ ;*
- ii) for every  $m \geq 2$  and  $\mathcal{M} \in \{\mathcal{M}'', \mathcal{M}'' \cap \text{REF}\}$ .*

□

Two more cases need to be analysed. Namely, when  $\mathcal{M} \in \{\text{TRA}^*, \text{TRA}^* \cap \text{TOT}^*\}$ . For these cases, we shall consider the decision problem  $\text{BDDP}^*$ .

(3.11) Theorem. Let

$$\mathcal{M} = \text{TRA}^* \cap \text{TOT}^*.$$

Then the problem  $\text{BDDP}^*(\mathcal{M}, 4)$  is  $\mathcal{NP}$ -complete.

**Proof.** By Theorem (3.8), the problem  $Q := \text{DDP}^*(\mathcal{M}^*, 2)$  with  $\mathcal{M}^* = \text{ASY} \cap \text{TRA}^* \cap \text{TOT}^*$  is  $\mathcal{NP}$ -complete. We want to prove that  $Q$  is polynomially transformable to  $\tilde{Q} := \text{BDDP}^*(\mathcal{M}, 4)$ . Let  $D_n = (N, A_n)$ ,  $w$  and  $k$  be given as an instance of  $Q$ . Note that, we may assume that  $k < n^2$ , otherwise  $Q$  is trivially solvable. Suppose  $N = \{1, 2, \dots, n\}$ ,  $n \geq 2$ . The corresponding instance of  $\tilde{Q}$  defined by  $\tilde{D}_p$ ,  $\tilde{w}$  and  $\tilde{k}$  is constructed as follows:  $\tilde{D}_p = (\tilde{N}, \tilde{A})$  is the complete digraph of order  $p = 2n$  with node set  $\tilde{N} := \{i_1, i_2 : i \in N\}$ . To define the weights  $\tilde{w}_e$  for  $e \in \tilde{A}$  we let

$$R := \bigcup_{1 \leq i < j \leq n} R_{ij} \text{ where } R_{ij} := \{(i_1, j_1), (j_1, i_2)\}$$

and set

$$\tilde{w}_e = \begin{cases} 0 & \text{if } e = (i_2, i_1), i \in N \\ L & \text{if } e = (i_1, i_2), i \in N \\ w_{ij} & \text{if } e = (i_1, j_1), e \in R \\ w_{ji} & \text{if } e = (j_1, i_2), e \in R \\ M & \text{otherwise,} \end{cases}$$

where

$$M := 4n^2 \text{ and } L := 2n^4.$$

Observe that  $|\tilde{w}_e|$  is even and  $|\tilde{w}_e| \leq p^4$  for every  $e \in \tilde{A}$ .

The parameter  $\tilde{k}$  is defined as

$$\tilde{k} := k + CM, \text{ where } C := \binom{2n}{2} - n - \binom{n}{2} = 3 \binom{n}{2}.$$

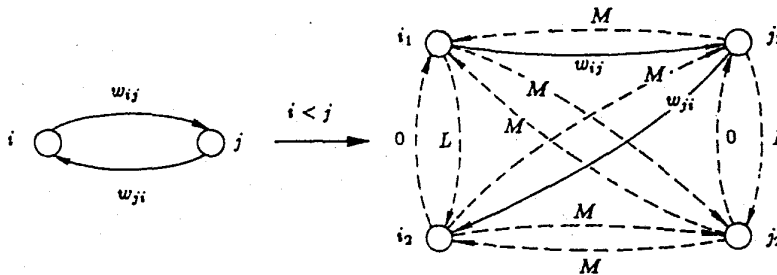


Figure 1

We shall prove that  $D_n = (N, A_n)$  has a subdigraph  $H = (N, B)$  with  $B \in \mathcal{M}^*$  and  $w(B) \leq k$  iff  $\tilde{D}_p = (\tilde{N}, \tilde{A})$  has a subdigraph  $\tilde{H} = (\tilde{N}, \tilde{B})$  with  $\tilde{B} \in \mathcal{M}$  and  $\tilde{w}(\tilde{B}) \leq \tilde{k}$ .

It is clear that  $M, L$  and  $\tilde{k}$  were chosen conveniently so that the above claim can be shown to hold. Before we give the proof, let us explain the idea behind the choice of the values for  $M, L$  and  $\tilde{k}$ .

Note that for each pair  $i, j, 1 \leq i < j \leq n$ , the arcs  $(i, j)$  and  $(j, i)$  in  $D_n$  correspond to the arcs  $(i_1, j_1)$  and  $(j_1, i_2)$  in  $\tilde{D}_p$ , respectively, and that the assigned weights agree correspondingly. See Figure 1. Given a subdigraph  $\tilde{H} = (\tilde{N}, \tilde{B})$  in  $\tilde{D}_p$  with  $\tilde{B} \in \mathcal{M}$  and  $\tilde{w}(\tilde{B}) \leq \tilde{k}$ , we want to construct a subdigraph  $H = (N, B)$  in  $D_n$  with  $B \in \mathcal{M}^*$  and  $w(B) \leq k$ . So we want  $\tilde{H}$  to have exactly one of the arcs  $(i_1, j_1), (j_1, i_2)$  for each pair  $i, j, 1 \leq i < j \leq n$  (so that the corresponding arcs in  $D_n$  can be set into  $B$ ). Thus we choose  $L$  conveniently (according to  $\tilde{k}$ ) so that both of  $(i_1, j_1)$  and  $(j_1, i_2)$  cannot be in any transitive subdigraph  $\tilde{H}$  with  $\tilde{w}(\tilde{B}) \leq \tilde{k}$ . This can be accomplished by choosing  $L$  so that whenever both of these arcs are chosen to be in a transitive subdigraph  $\tilde{H} = (\tilde{N}, \tilde{B})$ , then the choice of  $(i_1, i_2)$  forced by the transitivity gives that  $\tilde{w}(\tilde{B}) > \tilde{k}$ . The values for  $\tilde{k}$  and  $M$  are so chosen that  $\tilde{H}$  must be a subdigraph consisting of :

- i) all arcs with weight 0;
- ii) exactly one of the arcs  $(i_1, j_1), (j_1, i_2)$  for each pair  $i, j, 1 \leq i < j \leq n$ ;
- iii) exactly  $C := \binom{2n}{2} - n - \binom{n}{2}$  arcs with weight  $M$ .

a) Given a subdigraph  $H = (N, B)$  in  $D_n$  with  $B \in \mathcal{M}^*$  and  $w(B) \leq k$ , construct  $\tilde{H} = (\tilde{N}, \tilde{B})$  by setting:

$$\tilde{B} := \tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3,$$

where

$$\begin{aligned} \tilde{B}_1 &:= \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2) : 1 \leq i < j \leq n \text{ and } (i, j) \in B\}, \\ \tilde{B}_2 &:= \{(j_1, i_1), (j_1, i_2), (j_2, i_1), (j_2, i_2) : 1 \leq i < j \leq n \text{ and } (j, i) \in B\}, \\ \tilde{B}_3 &:= \{(i_2, i_1) : i \in N\}. \end{aligned}$$

Notice that  $|\tilde{B}| = \binom{2n}{2}$  and  $\tilde{B} \in \text{TOT}^*$ .

For each pair  $e = (i, j), 1 \leq i, j \leq n, i \neq j$ , let  $S_e$  be the following *basic* subdigraph:

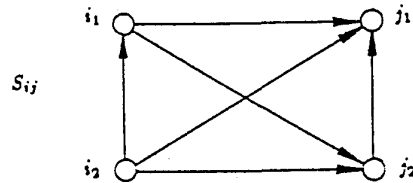


Figure 2.

Clearly,  $\tilde{H} = (\tilde{N}, \tilde{B})$  is the union of all basic subdigraphs  $S_e$  each corresponding to an arc  $e \in B$ . By inspection, it is easy to see that these subdigraphs  $S_e$  are transitive. Thus

it remains to be proved that if  $e := (i_r, j_s)$  and  $f := (j_s, l_t)$ , with  $r, s, t \in \{1, 2\}$ , are arcs of  $\tilde{B}$  not in the same basic subdigraph, then  $g := (i_r, l_t) \in \tilde{B}$ . Notice that when  $i = j$  or  $j = l$  then  $e$  and  $f$  are in a same basic subdigraph. Furthermore,  $\tilde{B}$  has no arcs such as  $e$  and  $f$  where  $i = l$ . Therefore we may assume that  $i, j, l$  are pairwise distinct.

Since  $e := (i_r, j_s) \in \tilde{B}$ , then

$$\text{if } \begin{cases} i < j \text{ then } e \in \tilde{B}_1 \text{ and } (i, j) \in B, \\ i > j \text{ then } e \in \tilde{B}_2 \text{ and } (i, j) \in B. \end{cases}$$

Similarly,  $f := (j_s, l_t) \in \tilde{B}$  implies that

$$\text{if } \begin{cases} j < l \text{ then } f \in \tilde{B}_1 \text{ and } (j, l) \in B, \\ j > l \text{ then } f \in \tilde{B}_2 \text{ and } (j, l) \in B. \end{cases}$$

Thus,  $(i, j) \in B$  and  $(j, l) \in B$ . Since  $B \in \text{TRA}^*$ ,  $(i, l) \in B$ . If  $i < l$  then  $g \in \tilde{B}_1$ , otherwise  $g \in \tilde{B}_2$ . Hence,  $g \in \tilde{B}$ . This completes the proof that  $\tilde{B} \in \text{TRA}^*$ .

Now let us prove that  $\tilde{w}(\tilde{B}) \leq \tilde{k}$ . Notice that  $\tilde{B}_1, \tilde{B}_2$  and  $\tilde{B}_3$  are pairwise disjoint,  $R \cap \tilde{B}_3 = \emptyset$  and  $|R \cap \tilde{B}| = |R \cap (\tilde{B}_1 \cup \tilde{B}_2)| = \frac{|R|}{2} = \binom{n}{2}$ .

Thus,

$$\begin{aligned} \tilde{w}(\tilde{B}) &= \tilde{w}(\tilde{B} \cap R) + \tilde{w}((\tilde{B}_1 \cup \tilde{B}_2) \setminus R) + \tilde{w}(\tilde{B}_3) \\ &= w(B) + (|\tilde{B}| - |\tilde{B}_3| - |\tilde{B} \cap R|)M \\ &= w(B) + \left( \binom{2n}{2} - n - \binom{n}{2} \right) M \leq k + CM = \tilde{k}. \end{aligned}$$

**b)** Let  $\tilde{H} = (\tilde{N}, \tilde{B})$  be a subdigraph of  $\tilde{D}_p$  with  $\tilde{B} \in \mathcal{M}$  and  $\tilde{w}(\tilde{B}) \leq \tilde{k}$ . Then the following holds:

(b<sub>1</sub>)  $\tilde{B}$  does not contain an arc  $e$  with  $\tilde{w}_e = L$ .

Suppose  $\tilde{B}$  contains such an arc  $e$ . Then

$$\tilde{w}(\tilde{B}) \geq L - \sum_{e \in A_n} |w_e| + 2 \binom{n}{2} M.$$

Since  $\sum_{e \in A_n} |w_e| \leq 2n^2$ , it follows that  $\tilde{w}(\tilde{B}) \geq 6n^4 - 4n^3 - 2n^2$ . On the other hand,  $\tilde{k} = k + CM < n^2 + \left(\frac{3n(n-1)}{2}\right)4n^2 = 6n^4 - 6n^3 + n^2$  and therefore,  $\tilde{w}(\tilde{B}) > \tilde{k}$ , a contradiction.

(b<sub>2</sub>)  $\tilde{B}$  contains all arcs  $e$  with  $\tilde{w}_e = 0$ .

This follows immediately from (b<sub>1</sub>) and the fact that  $\tilde{B} \in \text{TOT}^*$ .

(b<sub>3</sub>) For every pair  $(i, j)$ ,  $1 \leq i < j \leq n$ ,  $|\tilde{B} \cap R_{ij}| \leq 1$ .

Suppose there is a pair  $(i, j)$  such that  $\tilde{B}$  contains both of  $(i_1, j_1)$  and  $(j_1, i_2)$ . Since  $\tilde{B} \in \text{TRA}^*$ , this implies that  $(i_1, i_2) \in \tilde{B}$ ; but as  $\tilde{w}(i_1, i_2) = L$ , this contradicts (b<sub>1</sub>).

(b<sub>4</sub>)  $\tilde{B}$  contains exactly  $C$  arcs with weight  $M$ .

Suppose  $\tilde{B}$  has more than  $C$  arcs with weight  $M$ . Thus

$$\tilde{w}(\tilde{B}) \geq (C+1)M - \sum_{e \in A_n} |w_e| > \tilde{k},$$

a contradiction. So,  $\tilde{B}$  can have at most  $C$  arcs with weight  $M$ . On the other hand, since  $|\tilde{B}| \geq \binom{2n}{2}$  and  $\tilde{B}$  contains  $n$  arcs with weight 0 (by (b<sub>2</sub>)), at most  $\binom{n}{2}$  arcs of  $R$  (by (b<sub>3</sub>)) and no arcs with weight  $L$  (by (b<sub>1</sub>)), then  $\tilde{B}$  must contain at least  $\binom{2n}{2} - n - \binom{n}{2} =: C$  arcs with weight  $M$ . Thus  $\tilde{B}$  contains exactly  $C$  arcs with weight  $M$ .

(b<sub>5</sub>) For every pair  $(i, j)$ ,  $1 \leq i < j \leq n$ ,  $|\tilde{B} \cap R_{ij}| = 1$ .

Since  $\binom{2n}{2} \leq |\tilde{B}| = n + C + |\tilde{B} \cap R|$ , it follows that  $|\tilde{B} \cap R| \geq \binom{2n}{2} - n - C = \binom{n}{2}$ . If for some pair  $(i, j)$ ,  $1 \leq i < j \leq n$ ,  $|\tilde{B} \cap R_{ij}| < 1$  then by (b<sub>3</sub>)  $|\tilde{B} \cap R| < \binom{n}{2}$ , a contradiction. Thus, the statement is proved.

(b<sub>6</sub>)  $\tilde{B}$  has no double arcs.

Immediate from (b<sub>1</sub>), (b<sub>4</sub>) and (b<sub>5</sub>).

(b<sub>7</sub>)  $\tilde{w}(\tilde{B} \cap R) \leq k$ .

Clearly,  $\tilde{w}(\tilde{B}) = \tilde{w}(\tilde{B} \cap R) + CM$ . Thus,  $\tilde{w}(\tilde{B} \cap R) \leq \tilde{k} - CM = k$ . But  $(i_1, j_2) \in \tilde{B} \cap R$  and  $(j_2, j_1) \in \tilde{B}$  imply  $(i_1, j_1) \in \tilde{B}$ . Thus,  $(i_1, j_1) \in \tilde{B}$ . Analogously, analysing the cases  $j < l$  and  $j > l$  we conclude that  $(j_1, l_1) \in \tilde{B}$ .

Since  $\tilde{B} \in \text{TRA}^*$ , then  $(i_1, j_1) \in \tilde{B}$  and  $(j_1, l_1) \in \tilde{B}$  imply that  $(i_1, l_1) \in \tilde{B}$ . Thus, if  $i < l$  then  $(i_1, l_1) \in \tilde{B} \cap R$ , and therefore  $(i, l) \in B$ . Suppose  $i > l$ . By (b<sub>6</sub>)  $(i_1, l_1) \in \tilde{B}$  implies  $(l_1, i_1) \notin \tilde{B}$ . By (b<sub>5</sub>), if  $l < i$  and  $(l_1, i_1) \notin \tilde{B} \cap R$  then  $(i_1, l_2) \in \tilde{B} \cap R$ . But then,  $(i, l) \in B$  and therefore  $B \in \text{TRA}^*$ .

Since the given transformation is polynomial, it follows that  $\tilde{Q}$  is  $\mathcal{NP}$ -complete.  $\square$

A construction similar to the one presented in the proof of Theorem (3.11) leads to the following result.

**(3.12) Theorem.** *Let*

$$\mathcal{M} = \text{TRA}^*.$$

*Then the problem  $BDDP^*(\mathcal{M}, 6)$  is  $\mathcal{NP}$ -complete.*

**Proof.** Let  $Q := \text{DDP}^*(\mathcal{M}, 2)$  with  $\mathcal{M}^* = \text{ASY} \cap \text{TRA}^* \cap \text{TOT}^*$  be the  $\mathcal{NP}$ -complete problem considered in Theorem (3.8). Our aim is to prove that  $Q$  is polynomially transformable to  $\tilde{Q} := \text{BDDP}^*(\mathcal{M}, 6)$ . For that, let us assume that  $D_n = (N, A_n)$ ,  $w$  and  $k$ ,  $k < n^2$ , are given as an instance of  $Q$ , and let us construct the corresponding instance of  $\tilde{Q}$ .

Let  $\tilde{D}_p = (\tilde{N}, \tilde{A})$  be the complete digraph of order  $p = 2n$  with node set  $\tilde{N} := \{i_1, i_2 : i \in N\}$ .

To define the weights  $\tilde{w}_e$ , set

$$\begin{aligned} R &:= \bigcup_{1 \leq i < j \leq n} R_{ij}, & \text{where } R_{ij} &:= \{(i_1, j_1), (j_1, i_2)\}, \\ \bar{R} &:= \bigcup_{1 \leq i < j \leq n} \bar{R}_{ij}, & \text{where } \bar{R}_{ij} &:= \{(j_1, i_1), (i_2, j_1)\}. \end{aligned}$$

Let  $M$  be the *smallest even* integer such that

$$M > k + 2n^2,$$

and set

$$\begin{aligned} M^* &:= \left( \binom{n}{2} + 1 \right) M, \\ L &:= M + \binom{n}{2} (M^* + M). \end{aligned}$$

Now define  $\tilde{w}_e$  for each  $e \in \tilde{A}$ , as follows :

$$\tilde{w}_e = \begin{cases} -M^* & \text{if } e = (i_2, i_1), i \in N \\ L & \text{if } e = (i_1, i_2), i \in N \\ w_{ij} - M^* & \text{if } e = (i_1, j_1), e \in R \\ w_{ji} - M^* & \text{if } e = (j_1, i_2), e \in R \\ M & \text{if } e \in \bar{R} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that  $|\tilde{w}_e|$  is even and  $|\tilde{w}_e| \leq p^6$  for every  $e \in \tilde{A}$ .



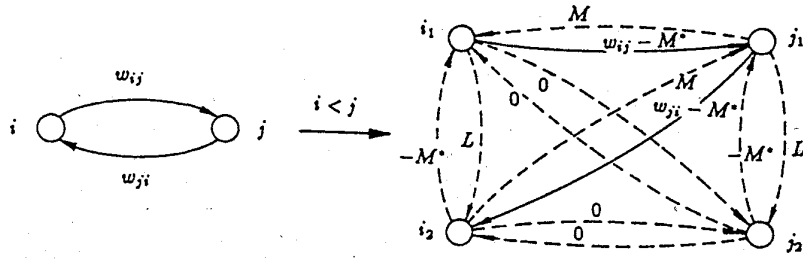


Figure 3.

Let

$$\tilde{k} := k + \binom{n}{2}M - \left(\binom{n}{2} + n\right)M^*.$$

We claim that  $D_n = (N, A_n)$  has a subdigraph  $H = (N, B)$  with  $B \in \mathcal{M}^*$  and  $w(B) \leq k$  iff  $\tilde{D}_p = (\tilde{N}, \tilde{A})$  has a subdigraph  $\tilde{H} = (\tilde{N}, \tilde{B})$  with  $\tilde{B} \in TRA^*$  and  $\tilde{w}(\tilde{B}) \leq \tilde{k}$ .

a) Given  $H = (N, B)$  in  $D_n$  with  $B \in \mathcal{M}^*$  and  $w(B) \leq k$ , let  $\tilde{H} = (\tilde{N}, \tilde{B})$  be the subdigraph of  $\tilde{D}_p$  defined by :

$$\tilde{B} := \tilde{B}_1 \cup \tilde{B}_2 \cup \tilde{B}_3,$$

where

$$\tilde{B}_1 := \{(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2) : 1 \leq i < j \leq n \text{ and } (i, j) \in B\},$$

$$\tilde{B}_2 := \{(j_1, i_1), (j_1, i_2), (j_2, i_1), (j_2, i_2) : 1 \leq i < j \leq n \text{ and } (j, i) \in B\},$$

$$\tilde{B}_3 := \{(i_2, i_1) : 1 \leq i \leq n\}.$$

Then

$$\begin{aligned} \tilde{w}(\tilde{B}) &= \tilde{w}(\tilde{B} \cap R) + \tilde{w}(\tilde{B} \setminus R) \\ &= \tilde{w}(\tilde{B} \cap R) + \tilde{w}(\tilde{B}_1 \cup \tilde{B}_2 \setminus R) + \tilde{w}(\tilde{B}_3) \\ &= w(B) - \binom{n}{2}M^* + \binom{n}{2}M - nM^* \\ &= w(B) - \left(\binom{n}{2} + n\right)M^* + \binom{n}{2}M \\ &\leq k - \left(\binom{n}{2} + n\right)M^* + \binom{n}{2}M \leq \tilde{k}. \end{aligned}$$

Using the fact that  $B \in \mathcal{M}^*$  it is not difficult to prove that  $\tilde{B} \in TRA^*$ . Indeed, the proof is analogous to the one present for Theorem (3.11), and therefore it will be omitted.

b) Let  $\tilde{H} = (\tilde{N}, \tilde{B})$  be a subdigraph of  $\tilde{D}_p$  with  $\tilde{B} \in TRA^*$  and  $\tilde{w}(\tilde{B}) \leq \tilde{k}$ . Based on  $\tilde{H}$  we want to construct a transitive tournament  $H = (N, B)$  in  $D_n$  with  $w(B) \leq k$ . For that, we first observe that  $\tilde{H}$  has the following properties :

(b<sub>1</sub>)  $\tilde{B}$  does not contain an arc  $e$  with  $\tilde{w}_e = L$ .

Suppose  $\tilde{B}$  contains such an arc  $e$ . Then

$$\begin{aligned}\tilde{w}(\tilde{B}) &\geq L - nM^* - \sum_{e \in A} |w_e| - n(n-1)M^* \\ &\geq M + \binom{n}{2}(M^* + M) - nM^* - 2n^2 - n(n-1)M^* \\ &> k - \binom{n}{2}M^* - nM^* + \binom{n}{2}M = \tilde{k},\end{aligned}$$

a contradiction.

(b<sub>2</sub>) For every pair  $(i, j)$ ,  $1 \leq i < j \leq n$ ,  $|\tilde{B} \cap R_{ij}| = 1$ .

Suppose there is a pair  $(i, j)$  such that  $|\tilde{B} \cap R_{ij}| = 2$ . In this case, since  $\tilde{B} \in TRA^*$ , it follows that  $(i_1, i_2) \in \tilde{B}$ , contradicting (b<sub>1</sub>). Thus,  $|\tilde{B} \cap R_{ij}| \leq 1$  for every pair  $(i, j)$ ,  $1 \leq i < j \leq n$ . Now suppose there is a pair  $(i, j)$  such that  $|\tilde{B} \cap R_{ij}| = 0$ . Then

$$\tilde{w}(\tilde{B}) \geq -\left(\binom{n}{2} - 1\right)M^* - nM^* - \sum_{e \in A_n} |w_e|.$$

Using the fact that  $\sum_{e \in A_n} |w_e| \leq 2n^2$  and making some substitutions we get  $\tilde{w}(\tilde{B}) > \tilde{k}$ . Since this contradicts our assumption, we conclude that (b<sub>2</sub>) holds.

(b<sub>3</sub>) For every  $i$ ,  $1 \leq i \leq n$ ,  $(i_2, i_1) \in \tilde{B}$ .

Suppose for some  $i$ ,  $1 \leq i \leq n$ ,  $(i_2, i_1) \notin \tilde{B}$ . Then

$$\begin{aligned}\tilde{w}(\tilde{B}) &\geq -(n-1)M^* - \sum_{e \in A_n} |w_e| - \binom{n}{2}M^* \\ &\geq \left(\binom{n}{2} + 1\right)M - nM^* - 2n^2 - \binom{n}{2}M^* \\ &> \binom{n}{2}M + k - \left(\binom{n}{2} + n\right)M^* = \tilde{k},\end{aligned}$$

a contradiction.

(b<sub>4</sub>) For every pair  $(i, j)$ ,  $1 \leq i < j \leq n$ ,  $|\tilde{B} \cap \bar{R}_{ij}| = 1$ .

By (b<sub>2</sub>), for every pair  $(i, j)$ ,  $1 \leq i < j \leq n$ , exactly one of the arcs  $(i_1, j_1)$  or  $(j_1, i_2)$  is in  $\tilde{B}$ . If  $(i_1, j_1) \in \tilde{B}$ , since  $(i_2, i_1) \in \tilde{B}$ , it follows that  $(i_2, j_1) \in \tilde{B}$ . Analogously, if  $(j_1, i_2) \in \tilde{B}$  then  $(j_1, i_1) \in \tilde{B}$ . Thus,  $|\tilde{B} \cap \bar{R}_{ij}| \geq 1$ . Now suppose there is a pair  $(i, j)$ ,

$1 \leq i < j \leq n$ , such that  $|\tilde{B} \cap \bar{R}_{i,j}| > 1$ . This implies that  $\tilde{B}$  has more than  $\binom{n}{2}$  arcs with weight  $M$  and therefore

$$\begin{aligned} \tilde{w}(\tilde{B}) &\geq \left( \binom{n}{2} + 1 \right) M - \sum_{e \in A_n} |w_e| - \binom{n}{2} M^* - nM^* \\ &> \binom{n}{2} M + k - \left( \binom{n}{2} + n \right) M^* = \tilde{k}, \end{aligned}$$

a contradiction. So, we have proved that (b<sub>4</sub>) holds.

(b<sub>5</sub>)  $\tilde{w}(\tilde{B} \cap R) \leq k - \binom{n}{2} M^*$ .

This follows from the fact that  $\tilde{w}(\tilde{B} \cap R) = \tilde{w}(\tilde{B}) + nM^* - \binom{n}{2} M$  and  $\tilde{w}(\tilde{B}) \leq \tilde{k}$ .

Now let  $H = (N, B)$  be the subdigraph of  $D_n$  with

$$\begin{aligned} B := &\{(i, j) : (i_1, j_1) \in \tilde{B} \cap R, \quad 1 \leq i < j \leq n\} \cup \\ &\{(j, i) : (j_1, i_2) \in \tilde{B} \cap R, \quad 1 \leq i < j \leq n\}. \end{aligned}$$

We claim that  $B \in \mathcal{M}^*$  and  $w(B) \leq k$ . Note that  $|B| = |\tilde{B} \cap R| = \binom{n}{2}$ . Furthermore,  $w(B) = \tilde{w}(\tilde{B} \cap R) + \binom{n}{2} M^*$ . Thus, by (b<sub>5</sub>) it follows that  $w(B) \leq k$ . The definition of  $B$  and fact (b<sub>2</sub>) yield immediately that  $B \in \text{ASY} \cap \text{TOT}^*$ . So it remains to be shown that  $B \in \text{TRA}^*$ .

Let  $(i, j)$  and  $(j, l)$  be arcs of  $B$ ,  $i \neq j \neq l \neq i$ . If  $i < j$  and  $(i, j) \in B$  then  $(i_1, j_1) \in \tilde{B} \cap R$ . If  $i > j$  and  $(i, j) \in B$  then  $(i_1, j_2) \in \tilde{B} \cap R$ . In the latter case, since  $(j_2, j_1) \in \tilde{B}$  and  $\tilde{B} \in \text{TRA}^*$ , it follows that  $(i_1, j_1) \in \tilde{B}$ . From an analogous analysis of cases  $j < l$  and  $j > l$ , we conclude that  $(j_1, l_1) \in \tilde{B}$ . Thus,  $(i_1, j_1) \in \tilde{B}$  and  $(j_1, l_1) \in \tilde{B}$ , and therefore  $(i_1, l_1) \in \tilde{B}$ . If  $i < l$ , then  $(i_1, l_1) \in \tilde{B} \cap R$ , and hence  $(i, l) \in B$ . Suppose  $l < i$ . Then  $(i_1, l_1) \in \tilde{B} \cap \bar{R}$ , and therefore by (b<sub>4</sub>) it follows that  $(l_2, i_1) \notin \tilde{B}$ . In this case,  $(l_1, i_1) \notin \tilde{B}$ ; otherwise  $(l_2, l_1) \in \tilde{B}$  and  $(l_1, i_1) \in \tilde{B}$  would imply  $(l_2, i_1) \in \tilde{B}$ , a contradiction. But if  $(l_1, i_1) \notin B$ , by (b<sub>2</sub>) we conclude that  $(i_1, l_2) \in \tilde{B}$ . Thus  $(i, l) \in B$ , and this proves that  $B \in \text{TRA}^*$ .

Clearly, the transformation of  $Q$  to  $\tilde{Q}$  is polynomial and therefore  $\tilde{Q}$  is  $\mathcal{NP}$ -complete.  $\square$

The next theorem summarizes the results implied by the last two theorems together with Lemma (3.6) and Theorem (3.5).

**(3.13) Theorem.** Let  $\mathcal{M}' \in \{\text{TRA}^* \cap \text{TOT}^*, \text{TRA} \cap \text{TOT}^*, \text{TRA}^*, \text{TRA}\}$  and  $\mathcal{M} \in \{\mathcal{M}', \mathcal{M}' \cap \text{REF}\}$ .

Then the median problem  $\text{MP}(\mathcal{R}, \mathcal{M})$  is  $\mathcal{NP}$ -hard.  $\square$

In Table 4 we give an overview of the obtained results. Each row of the table corresponds to a median problem  $MP(\mathcal{R}, \mathcal{M})$ , where  $\mathcal{M}$  is the intersection of the sets specified by the boxes marked with heavy lines. The traced lines in the boxes corresponding to the column REF indicate that this property may be included or not, without changing the complexity status of the problem. In the cases indicated with "†" we have shown that the restricted version  $MP(\mathcal{R}, \mathcal{M}, m)$  is also  $\mathcal{NP}$ -hard.

REF	SYM	ASY	TOT*	TRA*	TRA	
Any Combination						} Easy
						} $\mathcal{NP}$ -hard

nodes and completing it with all the missing edges, which are not loops. Assume that  $V' := V \cup X \cup Y$ , where  $|X| = |Y| = n$ . Assign weights  $w_e$  to each edge  $e \in E'$  by setting

$$w_e := \begin{cases} 1 & \text{if } e \in E \cup (X : Y), \\ -1 & \text{if } e \in (V : X \cup Y), \\ 0 & \text{otherwise;} \end{cases}$$

and let

$$k' := |E| - k - n^2.$$

We claim that  $G$  has a cut  $C$  with  $|C| \geq k$  iff  $G'$  has a clique partitioning  $A$  with  $w(A) \leq k'$ .

a) Assume that  $C = E(V_1 : V \setminus V_1)$  is a cut in  $G$  with  $|C| \geq k$ , and let  $V_2 := V \setminus V_1$ . Then the edge set  $A := E'(V_1 \cup X) \cup E'(V_2 \cup Y)$  is a clique partitioning of  $G'$  with

$$\begin{aligned} w(A) &= w(E'(V_1 \cup X)) + w(E'(V_2 \cup Y)) \\ &= -|(V_1 : X)| - |(V_2 : Y)| + |E \setminus C| \\ &= -n^2 + |E| - |C| \leq -n^2 + |E| - k =: k'. \end{aligned}$$

b) Assume that  $G'$  has a clique partitioning  $A$  with  $w(A) \leq k'$ . We want to prove that  $G$  has a cut  $C$  with  $|C| \geq k$ . Let us assume for the moment that the following holds :

Claim 1:  $G'$  has a clique partitioning  $A'$  with  $w(A') \leq k'$  and  $\mathcal{C}(A') = \{Q_1, Q_2\}$ , where  $Q_1$  and  $Q_2$  are such that  $VQ_1 = X \cup V_1$  and  $VQ_2 = Y \cup V_2$  for some nonempty subsets  $V_1$  and  $V_2$  of  $V$ .

Note that, in this case,  $w(A') = w(EQ_1) + w(EQ_2) = -n^2 + |E(V_1)| + |E(V_2)| = -n^2 + |E| - |E(V_1 : V_2)|$ , and since  $w(A') \leq k' = -n^2 + |E| - k$ , it follows that  $|E(V_1 : V_2)| \geq k$ , and therefore  $C := E(V_1 : V_2)$  defines the desired cut  $C$  in  $G$ .

Thus, in order to complete the proof it remains to be shown that Claim 1 holds. Before that, for notational convenience, let us give names to the different types of cliques we shall consider. According to its intersection with the sets  $V, X, Y$ , a clique  $H = [VH, EH]$  may be of one of the following types (see Figure 5) :

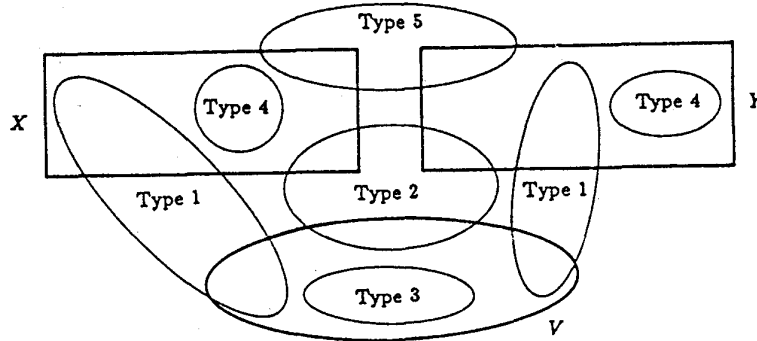


Figure 5.

Type 1 :

$(X, V)$ -intersecting (if  $VH \cap X \neq \emptyset$ ,  $VH \cap V \neq \emptyset$  and  $VH \cap Y = \emptyset$ ) or  
 $(Y, V)$ -intersecting (if  $VH \cap Y \neq \emptyset$ ,  $VH \cap V \neq \emptyset$  and  $VH \cap X = \emptyset$ ).

Type 2 :

$(X, Y, V)$ -intersecting (if  $VH \cap X \neq \emptyset$ ,  $VH \cap Y \neq \emptyset$  and  $VH \cap V \neq \emptyset$ ).

Type 3 :

$V$ -included (if  $\emptyset \neq VH \subseteq V$ ).

Type 4 :

$X$ -included (if  $\emptyset \neq VH \subseteq X$ ) or  $Y$ -included (if  $\emptyset \neq VH \subseteq Y$ ).

Type 5 :

$(X, Y)$ -intersecting (if  $VH \cap X \neq \emptyset$ ,  $VH \cap Y \neq \emptyset$  and  $VH \cap V = \emptyset$ ).

Note that according to the given definitions the desired clique partitioning  $A'$  of Claim 1 must be such that  $\mathcal{C}(A') = \{Q_1, Q_2\}$ , where  $Q_1$  and  $Q_2$  are both of Type 1.

For simplicity, we say that a clique partitioning  $A_1$  is *better* than a clique partitioning  $A_2$  if either  $w(A_1) < w(A_2)$ , or  $w(A_1) = w(A_2)$  and  $|\mathcal{C}(A_1)| < |\mathcal{C}(A_2)|$ .

Since each clique partitioning  $A$  is bijectively associated with the clique set  $\mathcal{C}(A)$ , when we refer to a clique partitioning  $B$  obtained from  $A$  by replacing some of the cliques in  $\mathcal{C}(A)$  with others, we are in fact defining how  $\mathcal{C}(B)$  is constructed and therefore defining in this way the arc set  $B$ .

Now consider the following Claim 2 to be used in the proof of Claim 1.

Claim 2: Let  $Q$  be a clique partitioning of  $G'$  and assume that  $Q_1, \dots, Q_l$ ,  $l \geq 2$ , are cliques in  $\mathcal{C}(Q)$  all of which are  $(W, V)$ -intersecting, where  $W = X$  or  $W = Y$ . Let  $\alpha := \sum_{i=1}^l |VQ_i \cap W|$  and  $\beta := \sum_{i=1}^l |VQ_i \cap V|$ . If  $\alpha \geq \beta$ , then the clique partitioning  $Q'$  obtained from  $Q$  by replacing the cliques  $Q_1, \dots, Q_l$  with the clique  $G'[VQ_1 \cup \dots \cup VQ_l]$  is such that  $w(Q') < w(Q)$ .

The proof of Claim 2 will be omitted as it can be obtained without any difficulty by induction on  $l$ . (For  $l \geq 3$  prove that there exist two cliques  $Q_i$  and  $Q_j$ ,  $1 \leq i < j \leq l$ , such that  $|(VQ_i \cup VQ_j) \cap W| \geq |(VQ_i \cup VQ_j) \cap V|$ .)

### Proof of Claim 1

Let  $\mathcal{A}$  be the set of the clique partitionings  $\tilde{A}$  of  $G$  with  $w(\tilde{A}) \leq k$  and such that  $\mathcal{C}(\tilde{A})$  contains the smallest number possible  $p$  of cliques of Type 2. Clearly,  $\mathcal{A} \neq \emptyset$  and  $p \geq 0$ . Our aim is to prove first that  $p = 0$ , and then show the existence of the desired clique partitioning  $A'$ .

Let us start by assuming that  $p \geq 1$ . Now let  $\hat{A}$  be a *best* clique partitioning in  $\mathcal{A}$ , and let  $H_1, \dots, H_p$  be the cliques of Type 2 contained in  $\mathcal{C}(\hat{A})$ .

For  $1 \leq i \leq p$  let

$$\begin{aligned} X_i &:= VH_i \cap X, & x_i &:= |X_i|, \\ Y_i &:= VH_i \cap Y, & y_i &:= |Y_i|, \\ V_i &:= VH_i \cap V, & v_i &:= |V_i|. \end{aligned}$$

Suppose  $\mathcal{C}(\hat{A})$  contains a clique  $H_i$ ,  $1 \leq i \leq p$ , such that  $v_i \leq x_i$ . Then we can split  $H_i$  into a clique of Type 1,  $G'[X_i \cup V_i]$ , and a clique of Type 4,  $G'[Y_i]$ , obtaining this way a clique partitioning  $\hat{A}$  with  $w(\hat{A}) \leq w(\hat{A})$  and with  $\mathcal{C}(\hat{A})$  containing  $p-1$  cliques of Type 2. Since this contradicts the choice of  $\hat{A}$ , we conclude that  $v_i > x_i$  for  $i = 1, \dots, p$ .

By symmetry, we also conclude that  $v_i > y_i$  for  $i = 1, \dots, p$ .

It is immediate that  $\mathcal{C}(\hat{A})$  contains no  $(X, Y)$ -intersecting cliques. Otherwise, a better clique partitioning could be obtained from  $\hat{A}$  by replacing each  $(X, Y)$ -intersecting clique with 2 cliques, one being  $X$ -included and the other  $Y$ -included.

It is also easy to see that  $\mathcal{C}(\hat{A})$  contains no  $X$ -included and no  $Y$ -included cliques. For, if  $H$  were an  $X$ -included clique in  $\mathcal{C}(\hat{A})$  then by replacing the clique  $H_1$  with the clique  $H_1 \cup H$  we could obtain a better clique partitioning in  $\mathcal{A}$  (since  $v_1 > y_1$ ). By symmetry, the same holds with respect to  $Y$ -included cliques.

Since  $\sum_{i=1}^p x_i < \sum_{i=1}^p v_i \leq n = |X|$  and  $\sum_{i=1}^p y_i < |Y|$ , then  $\mathcal{C}(\hat{A})$  must contain  $(X, V)$ -intersecting cliques, say  $\tilde{H}_1, \dots, \tilde{H}_h$ ,  $h \geq 1$ , and  $(Y, V)$ -intersecting cliques, say  $\tilde{Q}_1, \dots, \tilde{Q}_q$ ,  $q \geq 1$ .

$$\text{Let } \alpha := \bigcup_{i=1}^h (V\tilde{H}_i \cap X) \text{ and } \beta := \bigcup_{i=1}^h (V\tilde{H}_i \cap V).$$

Since  $\alpha \geq \beta$ , if  $h \geq 2$  then by Claim 2 the cliques  $\tilde{H}_1, \dots, \tilde{H}_h$  can be replaced with the clique  $\bigcup_{i=1}^h \tilde{H}_i$  yielding this way a better clique partitioning in  $\mathcal{A}$ . By symmetry, if  $q \geq 2$  then a better clique partitioning can also be obtained. Thus, we conclude that  $h = 1$  and  $q = 1$ , and for simplicity we let  $H := \tilde{H}_1$  and  $Q := \tilde{Q}_1$ .

If  $\mathcal{C}(\hat{A})$  contains  $V$ -included cliques, say  $H'_1, \dots, H'_l$ ,  $l \geq 1$ , then it is easy to see that these cliques can be combined with the clique  $H$  giving this way a better clique partitioning. It suffices to note that  $|VH \cap X| > |VH \cap V| + \bigcup_{i=1}^l |VH'_i|$ .

Thus, we conclude that  $\mathcal{C}(\hat{A}) = \{H_1, \dots, H_p\} \cup \{H, Q\}$ .

Let  $H_X := VH \cap X$ ,  $h_X := |H_X|$ ;  $H_V := VH \cap V$ ,  $h_V := |H_V|$ ;  $Q_Y := VQ \cap Y$ ,  $q_Y := |Q_Y|$ ;  $Q_V := VQ \cap V$ ,  $q_V := |Q_V|$ . Note that  $h_X > h_V$  and  $q_Y > q_V$ .

Let us now focus our attention on the cliques  $H$ ,  $Q$  and  $H_1$ .

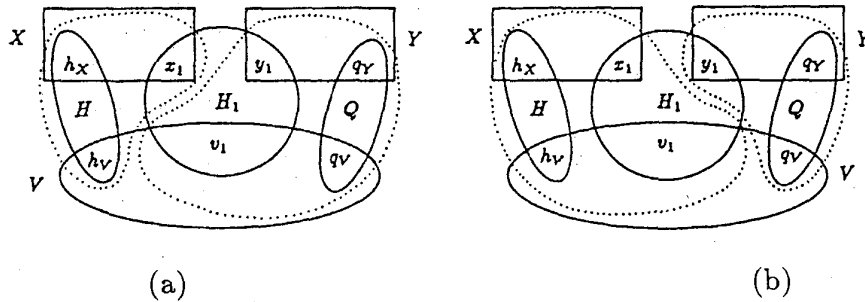


Figure 6.

Suppose  $v_1 \leq h_V + y_1$ .

In this case, let  $\tilde{A}$  be the clique partitioning obtained from  $\hat{A}$  by splitting  $C_1$  into the new cliques  $G'[VH \cup X_1]$  and  $G'[VQ \cup Y_1 \cup V_1]$ , and preserving the (old) cliques  $H_2, \dots, H_p$  (see Figure 6.a). Thus,  $\mathcal{C}(\tilde{A})$  contains  $p - 1$  cliques of Type 2 and  $w(\tilde{A}) = w(\hat{A}) - x_1 h_V - x_1 y_1 + x_1 v_1 - y_1 q_V - v_1 q_Y + |E(V_1 : Q_V)| \leq w(\hat{A}) - x_1(h_V + y_1 - v_1) - y_1 q_V - v_1(q_Y - q_V)$ .

Since  $h_V + y_1 - v_1 \geq 0$  and  $q_Y - q_V > 0$  we conclude that  $w(\tilde{A}) < w(\hat{A}) \leq k'$ , and therefore we have a contradiction to the choice of  $\hat{A}$ .

Assume now that  $v_1 > h_V + y_1$ .

In this case, let  $\tilde{A}$  be a clique partitioning obtained from  $\hat{A}$  by performing a splitting of  $C_1$ , symmetric to the previous one. That is,  $\tilde{A}$  consists of the new cliques  $G'[VH \cup X_1 \cup V_1]$  and  $G'[VQ \cup Y_1]$ , and the cliques  $H_2, \dots, H_p$  (see Figure 6.b). Thus  $\mathcal{C}(\tilde{A})$  contains  $p - 1$  cliques of Type 2 and

$$\begin{aligned} w(\tilde{A}) &\leq w(\hat{A}) - y_1 q_V - x_1 y_1 + y_1 v_1 - x_1 h_V - v_1 h_X + v_1 h_V \\ &= w(\hat{A}) - y_1 q_V + y_1(v_1 - x_1) + h_V(v_1 - x_1) - v_1 h_X \\ &= w(\hat{A}) - y_1 q_V + (y_1 + h_V)(v_1 - x_1) - v_1 h_X . \end{aligned}$$

Since  $y_1 + h_V < v_1$  and  $v_1 > x_1$ , it follows that

$$w(\tilde{A}) < w(\hat{A}) - y_1 q_V + v_1(v_1 - x_1) - v_1 h_X .$$

Now using the fact that  $h_X = n - (x_1 + \dots + x_p) = (h_V + q_V + v_1 + \dots + v_p) - (x_1 + \dots + x_p) > v_1 - x_1$ , we obtain that  $w(\tilde{A}) < w(\hat{A})$ , again a contradiction to the choice of  $\hat{A}$ .

This completes the proof that  $p = 0$ .

Now let us assume that  $A'$  is a best clique partitioning in  $\mathcal{A}$  and that  $\mathcal{C}(A')$  contains no cliques of Type 2.



It is immediate that  $\mathcal{C}(A')$  must contain at least a clique of Type 1; otherwise we would have  $w(A') \geq 0$  and therefore  $w(A') > k'$ , a contradiction. Let  $Q_1$  be a clique of Type 1 contained in  $\mathcal{C}(A')$ , and assume w.l.o.g. that  $Q_1$  is  $(X, V)$ -intersecting.

Clearly,  $\mathcal{C}(A')$  contains no  $(X, Y)$ -intersecting cliques. It is also immediate that  $\mathcal{C}(A')$  contains no  $X$ -included cliques, since they could all be combined with  $Q_1$  giving this way a better clique partitioning.

If  $\mathcal{C}(A')$  contains other  $(X, V)$ -intersecting different from  $Q_1$ , say  $H_1, \dots, H_h$ , then by Claim 2, if we set  $Q_1 := Q_1 \cup H_1 \cup \dots \cup H_h$ , then we obtain a better clique partitioning. Thus, we conclude that  $\mathcal{C}(A')$  contains a unique  $(X, V)$ -intersecting.

If  $\mathcal{C}(A')$  contains a  $V$ -included clique, say  $H$ , then (since  $n = |VQ_1 \cap X| > |VQ_1 \cap V|$ ) we can set  $Q_1 := Q_1 \cup H$  and obtain a better clique partitioning.

If  $\mathcal{C}(A')$  contains no  $(Y, V)$ -intersecting cliques, then it consists of the clique  $Q_1 = G'[X \cup V]$  and some  $Y$ -included cliques, and therefore  $w(A') = -n^2 + |E| > k'$ , a contradiction. Thus, let  $Q_2$  be a  $(Y, V)$ -intersecting clique in  $\mathcal{C}(A')$ . If  $\mathcal{C}(A')$  contains  $Y$ -included cliques and/or other  $(Y, V)$ -intersecting cliques, by performing analogous transformations to the ones we defined with respect to  $Q_1$ , we can construct a better clique partitioning. Hence, we conclude that  $Q_2$  is the unique  $(Y, V)$ -intersecting clique in  $\mathcal{C}(A')$  and therefore,  $\mathcal{C}(A') = \{Q_1, Q_2\}$  with  $Q_1$  and  $Q_2$  both of Type 1.

Thus, we have proved that Claim 1 holds, and therefore we have completed the proof of the theorem. □

#### 4. The case of restricted domains

In the preceding section we have proved the  $\mathcal{NP}$ -hardness of  $\text{MP}(\mathcal{R}, \mathcal{M})$  or  $\text{MP}(\mathcal{R}, \mathcal{M}, m)$  for certain subsets  $\mathcal{M} \subset \mathcal{R}$ . One may now ask whether the following special cases of these problems have also the same complexity: Instead of  $\mathcal{R}$  (an unrestricted domain of the relations in the profile  $\Pi$ ), we may have that the given relations are endowed with some properties from  $P$ . In this case, instead of  $\mathcal{R}$ , we have a subset  $\mathcal{M}' \subset \mathcal{R}$  and we are lead to the problems  $\text{MP}(\mathcal{M}', \mathcal{M})$  and  $\text{MP}(\mathcal{M}', \mathcal{M}, m)$  defined analogously. In other words, when we consider that the domain is  $\mathcal{R}$ , this means that we have no information about the properties of the input relations, and if specify a subset  $\mathcal{M}' \subset \mathcal{R}$  this means that the input relations are known to be in  $\mathcal{M}'$  (they belong to a restricted domain).

When  $\mathcal{M}' = \mathcal{M} = \mathcal{L}$  (the given relations are linear orders and the objective relation is also a linear order), Orlin [1981] (in an unpublished manuscript) and Bartholdi, Tovey and Trick [1988] proved that  $\text{MP}(\mathcal{L}, \mathcal{L})$  is  $\mathcal{NP}$ -hard. Note that this implies that the more general problem  $\text{MP}(\mathcal{R}, \mathcal{L})$  is  $\mathcal{NP}$ -hard, but not that  $\text{MP}(\mathcal{R}, \mathcal{L}, m)$  is  $\mathcal{NP}$ -hard — the result

we have shown. So it remains now to settle whether  $\text{MP}(\mathcal{L}, \mathcal{L}, m)$  is  $\mathcal{NP}$ -hard or not for a fixed  $m \geq 2$ .

Let us turn now to the case of equivalence relations. Krivánek and Morávek [1986] proved that  $\text{MP}(\text{SYM}, \mathcal{E}, 1)$  is  $\mathcal{NP}$ -hard. This result yields as a corollary the fact that  $\text{MP}(\mathcal{R}, \mathcal{E}, m)$  is  $\mathcal{NP}$ -hard (Theorem 3.9). Furthermore, from it one can also derive that  $\text{MP}(\mathcal{E}, \mathcal{E})$  is  $\mathcal{NP}$ -hard. The reduction given by Krivánek and Morávek is from a problem on hierarchical-tree clustering, whose  $\mathcal{NP}$ -completeness proof is very laborious. For the sake of completeness of the class of results covered in this paper we have included our weaker result. We should observe however, that in this case rather than the  $\mathcal{NP}$ -hardness of  $\text{MP}(\mathcal{R}, \mathcal{E}, m)$ , the interest lies more on the Theorem 3.14 from which the result could be derived.

There remains a number of open problems concerning the computational complexity status of  $\text{MP}(\mathcal{M}', \mathcal{M}, m)$  for some combinations of  $\mathcal{M}$  and  $\mathcal{M}'$ . Let us recall that the problems we know to be  $\mathcal{NP}$ -hard are  $\text{MP}(\mathcal{R}, \mathcal{L}, m)$ ,  $\text{MP}(\mathcal{R}, \mathcal{P}, m)$  for  $m \geq 2$ , and  $\text{MP}(\text{SYM}, \mathcal{E}, m)$ . It would certainly be interesting to establish the complexity of the problems:  $\text{MP}(\mathcal{T}, \mathcal{L}, m)$ ,  $\text{MP}(\mathcal{L}, \mathcal{L}, m)$  and  $\text{MP}(\mathcal{E}, \mathcal{E}, m)$ .

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