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On Non-Rank Facets of Stable Set Polytopes of Webs with Clique Number Four

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On Non-Rank Facets of Stable Set Polytopes of Webs with Clique Number Four

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Abstract

Graphs with circular symmetry, called webs, are relevant for describing the stable set polytopes of two larger graph classes, quasi-line graphs [8, 12] and claw-free graphs [7, 8]. Providing a decent linear description of the stable set polytopes of claw-free graphs is a long-standing problem [9]. However, even the problem of finding all facets of stable set polytopes of webs is open. So far, it is only known that stable set polytopes of webs with clique number ≤ 3 have rank facets only [5, 15] while there are examples with clique number > 4 having non-rank facets [10, 12, 11]. The aim of the present paper is to treat the remaining case with clique number $= 4$: we provide an infinite sequence of such webs whose stable set polytopes admit non-rank facets.

Key words: web, rank-perfect graph, stable set polytope, (non-)rank facet

1 Introduction

A natural generalization of odd holes and odd antiholes are graphs with circular symmetry of their maximum cliques and stable sets, called webs: a web W_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge iff i and j differ by at most k (modulo n) and $i \neq j$. These graphs belong to the classes of quasi-line graphs and claw-free graphs and are, besides line graphs, relevant for describing the stable set polytopes of those larger graph classes [7, 8, 12]. All facets of the stable set polytope of line graphs are known from matching theory [6]. In contrary, we are still far from having a complete description for the stable set polytopes of webs and, therefore, of quasi-line and claw-free graphs, too.

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In particular, as shown by Giles & Trotter [8], the stable set polytopes of claw-free graphs contain facets with a much more complex structure than those defining the matching polytope. Oriolo [12] discussed which of them can occur in quasi-line graphs. In particular, these non-rank facets rely on certain combinations of joined webs.

Several further authors studied the stable set polytopes of webs. Obviously, webs with clique number 2 are either perfect or odd holes (their stable set polytopes are known due to [2, 13]). Dahl [5] studied webs with clique number 3 and showed that their stable set polytopes admit rank facets only. On the other hand, Kind [10] found (by means of the PORTA software[1]) examples of webs with clique number > 4 whose stable set polytopes have *non*-rank facets. Oriolo [12] and Liebling et al. [11] presented further examples of such webs.

In [12], Oriolo asked whether the stable set polytopes of webs with clique number $= 4$ admit rank facets only. The aim of the present paper is to answer that question by providing an infinite sequence of webs with clique number $= 4$ whose stable set polytopes have *non*-rank facets.

2 Some Known Results on Stable Set Polytopes

The *stable set polytope* $\text{STAB}(G)$ of G is defined as the convex hull of the incidence vectors of all stable sets of the graph $G = (V, E)$ (a set $V' \subseteq V$ is a stable set if the nodes in V' are mutually non-adjacent). A linear inequality $a^T x \leq b$ is said to be *valid* for $\text{STAB}(G)$ if it holds for all $x \in \text{STAB}(G)$. We call a stable set S of G a *root* of $a^T x \leq b$ if its incidence vector χ^S satisfies $a^T \chi^S = b$. A valid inequality for $\text{STAB}(G)$ is a *facet* if and only if it has $|V|$ roots with affinely independent incidence vectors. (Note that the incidence vectors of the roots of $a^T x \leq b$ have to be *linearly* independent if $b > 0$.)

The aim is to find a system $Ax \leq b$ of valid inequalities s.t. $\text{STAB}(G) = \{x \in \mathbb{R}_+^{|G|} : Ax \leq b\}$ holds. Such a system is unknown for the most graphs and it is, therefore, of interest to study certain linear relaxations of $\text{STAB}(G)$ and to investigate for which graphs G these relaxations coincide with $\text{STAB}(G)$.

One relaxation of $\text{STAB}(G)$ is the *fractional stable set polytope* $\text{QSTAB}(G)$ given by all "trivial" facets, the *nonnegativity constraints*

$$x_i \geq 0 \tag{0}$$

for all nodes i of G and by the *clique constraints*

$$\sum_{i \in Q} x_i \leq 1 \tag{1}$$

[1]By PORTA it is possible to generate all facets of the convex hull of a given set of integer points, see http://www.zib.de

for all cliques $Q \subseteq G$ (a set $V' \subseteq V$ is a clique if the nodes in V' are mutually adjacent). Obviously, a clique and a stable set have at most one node in common. Therefore, QSTAB(G) contains all incidence vectors of stable sets of G and STAB$(G) \subseteq$ QSTAB(G) holds for all graphs G. The two polytopes coincide precisely for perfect graphs [2, 13].

A graph G is called *perfect* if, for each (node-induced) subgraph $G' \subseteq G$, the chromatic number $\chi(G')$ equals the clique number $\omega(G')$. That is, for all $G' \subseteq G$, as many stable sets cover all nodes of G' as a maximum clique of G' has nodes (maximum cliques resp. maximum stable sets contain a maximal number of nodes).

In particular, for all imperfect graphs G follows STAB$(G) \subset$ QSTAB(G) and, therefore, further constraints are needed to describe their stable set polytopes.

A natural way to generalize clique constraints is to investigate *rank constraints*

$$\sum_{i \in G'} x_i \ \leq \ \alpha(G') \tag{2}$$

associated with *arbitrary* (node-)induced subgraphs $G' \subseteq G$ where $\alpha(G')$ denotes the stability number of G', i.e., the cardinality of a maximum stable set in G' (note that $\alpha(G') = 1$ holds iff G' is a clique). For convenience, we often write (2) in the form $x(G') \leq \alpha(G')$.

Let RSTAB(G) denote the *rank polytope* of G given by all nonnegativity constraints (0) and all rank constraints (2). A graph G is called *rank-perfect* [16] if STAB(G) coincides with RSTAB(G).

By construction, every perfect graph is rank-perfect. Further graphs which are rank-perfect by definition are *near-perfect* [14] (resp. *t-perfect* [2], *h-perfect* [9]) graphs, where rank constraints associated with cliques and the graph itself (resp. edges and odd cycles, cliques and odd cycles) are allowed.

A result of PADBERG [13] shows that minimally imperfect graphs are near-perfect. (A graph is called *minimally imperfect* if it is imperfect but all proper induced subgraphs are perfect. Berge [1] conjectured and Chudnovsky, Robertson, Seymour & Thomas [4] proved recently that chordless odd cycles C_{2k+1} with $k \geq 2$, termed *odd holes*, and their complements $\overline{C}_{2k+1}$, called *odd antiholes*, are the only minimally imperfect graphs. The complement $\overline{G}$ has the same node set as G, but two nodes are adjacent iff they are non-adjacent in G.)

Moreover, line graphs are rank-perfect by [6], as their stable set polytopes admit as only non-trivial facets rank constraints associated with cliques and line graphs of 2-connected hypomatchable graphs. (The *line graph* $L(H)$ of a graph H is obtained by taking the edges of H as nodes of $L(H)$ and connecting two nodes in $L(H)$ iff the corresponding edges of H are incident. A graph H is called *hypomatchable* if, for all nodes v of H, the subgraph $H - v$ admits a matching meeting all nodes. Since matchings of H correspond to stable sets of $L(H)$, the description of the matching polytope due to [6] implies a description of the stable set polytope for line graphs.)

A generalization of line graphs is the class of *quasi-line graphs* where the neighborhood of any node can be partitioned into two cliques. A superclass of quasi-line graphs consists of all *claw-free graphs* where the neighborhood of any node does not contain a stable set of size 3. A characterization of the rank polytope of claw-free graphs was given by Galluccio & Sassano [7]. They showed that all rank facets can be constructed by means of standard operations from rank constraints associated with cliques, partitionable webs $W_{\alpha\omega+1}^{\omega-1}$, or line graphs of 2-connected, (edge-)critical hypomatchable graphs. However, claw-free graphs are not rank-perfect and finding all facets of the stable set polytopes of claw-free graphs is a long-standing problem [9]. Giles & Trotter [8] found, e.g., non-rank facets which occur even in the stable set polytopes of quasi-line graphs. These non-rank facets rely on combinations of joined webs.

Recall that a web W_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge if i and j differ by at most k (i.e., if $|i-j| \leq k \bmod n$) and $i \neq j$. We assume $k \geq 1$ and $n \geq 2(k+1)$ in the sequel in order to exclude the degenerated cases when W_n^k is a stable set or a clique. W_n^1 is a hole and W_{2k+1}^{k-1} an odd antihole for $k \geq 2$. All webs W_9^k on nine nodes are depicted in Figure 1. Note that webs are also called circulant graphs C_n^k [3]. Furthermore, graphs $W(n,k)$ with $n \geq 2$, $1 \leq k \leq \frac{1}{2}n$ and $W(n,k) = \overline{W}_n^{k-1}$ were introduced in [15].

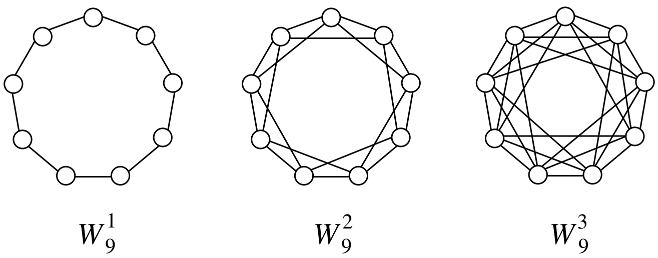

Figure 1

So far, the following is known about stable set polytopes of webs. The webs W_n^1 are holes, hence they are perfect if n is even and minimally imperfect if n is odd (recall that we suppose $n \geq 2(k+1)$). Thus, all webs with clique number 2 are particularly near-perfect and, in addition, all webs with stability number 2 and W_{11}^2 by [14, 16] (note $\omega(W_n^k) = k+1$ and $\alpha(W_n^k) = \lfloor \frac{n}{k+1} \rfloor$). Dahl [5] showed that all webs W_n^2 with clique number 3 are rank-perfect. But there are several webs with clique number > 4 known to be *not* rank-perfect [10, 12, 11], e.g., W_{31}^4, W_{25}^5, W_{29}^6, W_{33}^7, W_{28}^8, W_{31}^9.

In order to answer the question whether the webs with clique number $= 4$ are rank-perfect or not, we first analyze the structure of the known non-rank facets of webs with higher clique number in Section 3 and then investigate in Section 4 a similar construction for the webs W_n^3 that gives rise to an infinite sequence of webs with clique number $= 4$ having non-rank facets: W_{3l}^3 is not rank-perfect for every $l \geq 11$ with $2 = l \bmod 3$ (see Theorem 9).

3 Structure of Known Non-Rank Facets of Webs

A facet $a^T x \leq b$ of the stable set polytope of a web W_n^k is *reduced* if at most $\omega(W_n^k)-1=k$ consecutive coefficients a_i are maximal (a_i and a_j are consecutive iff $j = i \pm 1 \pmod n$ and a_i is maximal iff $a_i = \max\{a_j : 1 \leq j \leq n\}$).

Reduced facets play an important role in stable set polytopes of webs. For instance, Dahl's description of the stable set polytope of webs with clique number 3 was done in two steps: first, he proved that to get such a description, it is enough to characterize reduced facets (Lemma 4.2 [5]); second, he provided such a characterization (Theorem 4.3 [5]).

Furthermore, looking at the known non-rank facets of webs, we observe that the reduced ones admit a certain structure: they are clique family inequalities introduced in [12] and are associated with induced subwebs.

Let $G = (V,E)$ be a graph, $\mathcal{F}$ be a family of (at least three inclusion-wise) maximal cliques of G, $p \leq |\mathcal{F}|$ be an integer, and define two sets as follows:

$$\begin{aligned} I(\mathcal{F},p) &= \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| \geq p\} \\ O(\mathcal{F},p) &= \{i \in V : |\{Q \in \mathcal{F} : i \in Q\}| = p-1\} \end{aligned}$$

Oriolo [12] showed that the *clique family inequality*

$$(p-r)\sum_{i\in I(\mathcal{F},p)} x_i + (p-r-1)\sum_{i\in O(\mathcal{F},p)} x_i \leq (p-r)\lfloor\frac{|\mathcal{F}|}{p}\rfloor \tag{3}$$

is valid for the stable set polytope of *every* graph G where $r = |\mathcal{F}| \bmod p$ and $r > 0$.

We are interested in the clique family inequalities *associated with proper subwebs* $W_{n'}^{k'}$ of W_n^k where $\mathcal{F} = \{Q_i : i \in W_{n'}^{k'}\}$ is chosen as clique family, $p = k'+1$, and $Q_i = \{i, \ldots, i+k\}$ denotes the maximum clique of W_n^k starting in node i. In order to explore the special structure of such inequalities, we need the following result due to Trotter [15].

Lemma 1 [15] *$W_{n'}^{k'}$ is an induced subweb of W_n^k if and only if*

(i) *$n(k'+1) \geq n'(k+1)$ and $nk' \leq n'k$ holds,*

(ii) *there is a subset $V' = \{i_1, \ldots, i_{n'}\} \subseteq V(W_n^k)$ s.t. $|V' \cap Q_{i_j}| = k'+1$ for every $1 \leq j \leq n'$.*

We now prove the following.

Lemma 2 *Let $W_{n'}^{k'} \subset W_n^k$ be a proper induced subweb. The clique family inequality of STAB(W_n^k) associated with $W_{n'}^{k'}$ is*

$$(k'+1-r)\sum_{i\in I(\mathcal{F},p)} x_i + (k'-r)\sum_{i\in O(\mathcal{F},p)} x_i \leq (k'+1-r)\,\alpha(W_{n'}^{k'}) \tag{4}$$

where $r = n' \bmod (k'+1)$, $r > 0$ and $W_{n'}^{k'} \subseteq I(\mathcal{F},p)$ holds.

Proof. Let $W_{n'}^{k'}$ be a proper subweb of W_n^k and recall that the clique family inequality associated with $W_{n'}^{k'}$ is given by $(\mathcal{F}, p)$ with $\mathcal{F} = \{Q_i : i \in W_{n'}^{k'}\}$ and $p = k' + 1$. Obviously $|\mathcal{F}| = |W_{n'}^{k'}| = n'$ follows. Let $V' = \{i_1, \ldots, i_{n'}\}$ be the node set of $W_{n'}^{k'}$ in W_n^k. Lemma 1(ii) implies that $Q_{i_j} = \{i_j, \ldots, i_j + k\}$ contains the nodes $i_j, \ldots, i_{j+k'}$ from V'. Obviously, the node $i_{j+k'}$ belongs exactly to the $(k'+1)$ cliques $Q_{i_j}, \ldots, Q_{i_{j+k'}}$ from $\mathcal{F}$. Since all indices are taken modulo n, every node in $W_{n'}^{k'}$ is covered precisely $(k'+1)$ times by $\mathcal{F}$ and $p = k' + 1$ yields, therefore, $W_{n'}^{k'} \subseteq I(\mathcal{F}, p)$. Furthermore, $|\mathcal{F}| = n'$ and $p = \omega(W_{n'}^{k'})$ implies $\lfloor \frac{|\mathcal{F}|}{p} \rfloor = \alpha(W_{n'}^{k'})$. Hence the clique family inequality given by $(\mathcal{F}, p)$ is (4) which finishes the proof. $\Box$

Let's illustrate that with the help of the smallest not rank-perfect web W_{25}^5. Its non-rank facets are clique family inequalities associated with induced subwebs $W_{10}^2 \subseteq W_{25}^5$ (note that the node sets 1, 2, 6, 7, 11, 12, 16, 17, 21, 22 and 1, 3, 6, 8, 11, 13, 16, 18, 21, 23 both induce a $W_{10}^2 \subseteq W_{25}^5$, see the black nodes in Figure 2).

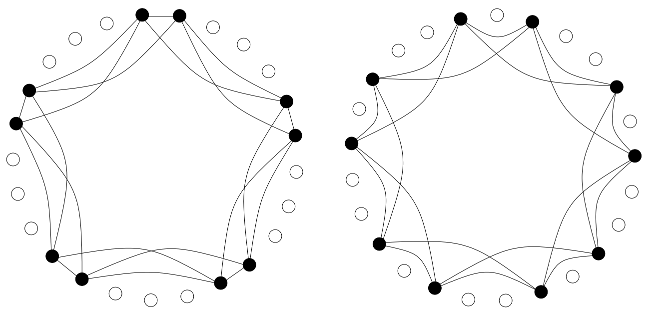

Figure 2: The induced subwebs $W_{10}^2 \subseteq W_{25}^5$

Choosing $\mathcal{F} = \{Q_i : i \in W_{10}^2\}$ yields $p = \omega(W_{10}^2) = 3$ in both cases. All remaining nodes are covered $\frac{10}{25-10}(5-2) = 2$ times, hence $O(\mathcal{F}, p) = W_{25}^5 - W_{10}^2$ follows. In particular, $O(\mathcal{F}, p)$ induces the subweb W_{15}^3 of W_{25}^5 and the corresponding clique family inequality is

$$2x(W_{10}^2) + 1x(W_{15}^3) \leq 2\alpha(W_{10}^2)$$

due to $r = |\mathcal{F}| \bmod p = 1$ and yields a non-rank facet of STAB(W_{25}^5). Notice that in the previous example, W_{25}^5 partitions into two induced subwebs W_{10}^2 and W_{15}^3. *All* known reduced non-rank facets are of this kind, i.e., they are clique family inequalities associated with an induced subweb such that the remaining part also induces a subweb.

In order to answer the question whether the webs W_n^3 are rank-perfect or not, we look, therefore, for possible partitions of W_n^3 into two disjoint subwebs and investigate the associated clique family inequalities (4). Lemma 1(i) shows that

$$\begin{array}{lll} W_{n'}^2 \subseteq W_n^3 & \text{with} & \frac{2}{3}n \leq n' \leq \frac{3}{4}n \\ W_{n''}^1 \subseteq W_n^3 & \text{with} & \frac{1}{3}n \leq n'' \leq \frac{2}{4}n \end{array}$$

are the only possible subwebs since $W_{n'}^{k'} \subset W_n^k$ implies $k' < k$ by [15] again. Hence, there are only two possibilities to partition W_n^3 into two disjoint subwebs:

$$2|n \text{ and } W_n^3 = W_{\frac{1}{2}n}^1 \cup W_{\frac{1}{2}n}^1$$
$$3|n \text{ and } W_n^3 = W_{\frac{1}{3}n}^1 \cup W_{\frac{2}{3}n}^2$$

First, consider the clique family inequalities associated with $W_{\frac{n}{2}}^1 \subset W_n^3$ resp. $W_{\frac{1}{3}n}^1 \subset W_n^3$. Then $\mathcal{F} = \{Q_i : i \in W_{\frac{n}{2}}^1\}$ resp. $\mathcal{F} = \{Q_i : i \in W_{\frac{1}{3}n}^1\}$ yields $p = 2$ by Lemma 2. Due to $r > 0$, we obtain $(p-r) = 1$ and $(p-r-1) = 0$ in this case, hence the associated clique family inequalities (4) cannot be *non*-rank constraints.

Thus, turn to the clique family inequality associated with $W_{\frac{2}{3}n}^2 \subset W_n^3$. This means in particular, that n is divisible by 3, i.e., we have $n = 3l$ (for some $l \geq 3$ by $n \geq 2(k+1)$) and $W_{3l}^3 = W_{2l}^2 \cup W_{1l}^1$. Choosing $\mathcal{F} = \{Q_i : i \in W_{2l}^2\}$ implies $p = 3$ and the associated clique family inequality (4) is a *non*-rank constraint provided $r = 1$ and $O(\mathcal{F}, p) \neq \emptyset$ holds.

In order to determine $O(\mathcal{F}, p)$, we have to study the distribution of the nodes of W_{2l}^2 in W_{3l}^3. For that we need the following notation. Let $V' \subseteq V(W_n^k)$, then a maximal set of consecutive nodes of W_n^k belonging to V' is called an *interval* of V'. Furthermore, an interval containing d nodes is called a *d-interval* (e.g., $\{i, i+1, i+2\}$ is a 3-interval of V' if $i, i+1, i+2 \in V'$ but $i-1, i+3$ not).

Lemma 3 *If $W_{2l}^2 \subseteq W_{3l}^3$, then $V(W_{2l}^2)$ and $V(W_{3l}^3 - W_{2l}^2)$ consist of 2-intervals and 1-intervals only, respectively.*

Proof. Let $V = V(W_{3l}^3)$ and $V' = V(W_{2l}^2)$.

Claim 1. $V - V'$ must consist of 1-intervals only. Assume there is a node $i \in V'$ s.t. $i+1, i+2 \in V - V'$. Then the maximum clique $Q_i \subseteq W_{3l}^3$ starting in i cannot contain 3 but at most 2 nodes of V', a contradiction to Lemma 1(ii) (recall $Q_i = \{i, i+1, i+2, i+3\}$ and $i+1, i+2 \in V - V'$). $\diamond$

Claim 2. V' cannot contain a 1-interval. Suppose there is a node $i \in V'$ s.t. $i-1, i+1 \in V - V'$. Then we have $i-2 \in V'$ by Claim 1 and the maximum clique $Q_{i-2} \subseteq W_{3l}^3$ starting in $i-2$ does not contain 3 but only the 2 nodes $i-2$ and i of V', a contradiction to Lemma 1(ii) again. $\diamond$

Claim 3. V' must consist of 2-intervals only. Claim 1 shows that the number of intervals in $V - V'$ equals l (since $V - V'$ contains l nodes). Obviously, $V - V'$ and V' have the same number of intervals. $|V'| = 2l$ and Claim 2 imply, therefore, that every of the l intervals of V' contains 2 nodes. $\square$

In particular, $W_{3l}^3 - W_{2l}^2$ contains every third node of W_{3l}^3. This implies:

Lemma 4 *Let $W^2_{2l} \subset W^3_{3l}$. Then $W^3_{3l} - W^2_{2l}$ is the hole W^1_{1l} and the clique family inequality*

$$(3-r)\,x(W^2_{2l}) + (2-r)\,x(W^1_{1l}) \leq (3-r)\,\alpha(W^2_{2l}) \tag{5}$$

of STAB(W^3_{3l}) associated with W^2_{2l} is a non-*rank constraint if $r = 2l \bmod 3 = 1$.*

Proof. Let $W^2_{2l} \subset W^3_{3l}$. Then Lemma 3 shows that $W^3_{3l} - W^2_{2l}$ is the hole W^1_{1l}. Choosing $\mathcal{F} = \{Q_i : i \in W^2_{2l}\}$ implies $p = 3$. Every node $i \in W^1_{1l}$ belongs to exactly the two cliques Q_{i-2} and Q_{i-1} in $\mathcal{F}$ since $i-2, i-1 \in W^2_{2l}$ but $i-3 \in W^1_{1l}$ follows from Lemma 3. Thus, every node in W^1_{1l} is covered exactly twice by $\mathcal{F}$ and $W^1_{1l} \subseteq O(\mathcal{F}, p)$ follows. $W^2_{2l} \subseteq I(\mathcal{F}, p)$ by Lemma 2 and $W^2_{2l} \cup W^1_{1l} = W^3_{3l}$ finishes the proof. $\Box$

We investigate in the next section in which cases the non-rank constraint (5) yields a facet of STAB(W^3_n).

4 Non-Rank Facets of STAB(W^3_n)

Throughout this section, let n be divisible by 3 (i.e., $n = 3l$ for some $l \geq 3$ by $n \geq 2(k+1)$) and $2 = l \bmod 3$ (i.e., $1 = 2l \bmod 3$). Consider a partition $W^2_{2l} \cup W^1_{1l}$ of W^3_{3l} into disjoint subwebs. The clique family inequality (5) of STAB(W^3_{3l}) associated with W^2_{2l} is the non-rank constraint

$$2x(W^2_{2l}) + 1x(W^1_{1l}) \leq 2\alpha(W^2_{2l}) \tag{$*$}$$

due to Lemma 4. The aim of this section is to prove that $(*)$ is a facet of STAB(W^3_{3l}) whenever $l \geq 11$.

For that, we have to present $3l$ roots of $(*)$ whose incidence vectors are linearly independent. (Recall that a root of $(*)$ is a stable set of W^3_{3l} satisfying $(*)$ at equality.)

It follows from [15] that a web W^k_n produces the full rank facet $x(W^k_n) \leq \alpha(W^k_n)$ iff $(k+1) \not| \, n$. Thus W^2_{2l} is facet-producing if $2 = l \bmod 3$ and the maximum stable sets of W^2_{2l} yield already $2l$ roots of $(*)$ with linearly independent incidence vectors.

Let $V = V(W^3_{3l})$ and $V' = V(W^2_{2l})$. We need a set $\mathcal{S}$ of further l roots of $(*)$ which have a non-empty intersection with $V - V'$, called *mixed roots*, and are independent, too, in order to prove that $(*)$ is a facet of STAB(W^3_{3l}).

First, we show that there is no such set $\mathcal{S}$ of l mixed roots in the two smallest cases with $l \geq 3$ and $2 = l \bmod 3$ (i.e., if $l = 5, 8$ and $n = 15, 24$) but that there exists such a set $\mathcal{S}$ for every $l \geq 11$.

Proposition 5 *The constraint $(*)$ is not a facet of* STAB(W^3_{15}) *or* STAB(W^3_{24}).

Proof. Let S be a mixed root of $(*)$ then we have

$$2|S \cap V'| + 1|S \cap (V - V')| = 2\alpha(W_{2l}^2)$$

due to the coefficients of $(*)$. This implies that we need $2x$ nodes in $S \cap (V - V')$ if $|S \cap V'| = \alpha(W_{2l}^2) - x$. In particular, $S \not\subseteq V'$ yields $|S| > \alpha(W_{2l}^2)$.

If $l = 5$, then $\alpha(W_{10}^2) = \lfloor \frac{10}{3} \rfloor = 3 = \lfloor \frac{15}{4} \rfloor = \alpha(W_{15}^3)$ implies that $(*)$ cannot have any root S with $S \not\subseteq W_{10}^2$.

In the case $l = 8$, we have $|S| > \alpha(W_{16}^2) = 5$ for every root $S \not\subseteq W_{16}^2$. This implies $|S| + |N^+(S)| = 4|S| \geq 24 = 3l$ (where $N^+(S)$ denotes the union of $N^+(i) = \{i+1, i+2, i+3\}$ for all nodes $i \in S$). Thus, S has to contain every 4th node of V. But there are only 4 stable sets in W_{24}^3 of that type, namely

$$\begin{array}{lcl} S_1 & = & \{1, 5, \ \ 9, 13, 17, 21\} \\ S_2 & = & \{2, 6, 10, 14, 18, 22\} \\ S_3 & = & \{3, 7, 11, 15, 19, 23\} \\ S_4 & = & \{4, 8, 12, 16, 20, 24\} \end{array}$$

instead of the needed $l = 8$ mixed roots. Consequently, in the two cases $l = 5$ resp. $l = 8$ there are not enough roots and $(*)$ is, therefore, neither a facet of $\mathrm{STAB}(W_{15}^3)$ nor of $\mathrm{STAB}(W_{24}^3)$. □

We now show that there exists a set $\mathcal{S}$ of l mixed roots of $(*)$ whenever $l \geq 11$. Due to $2 = l \bmod 3$, we set $l = 2 + 3l'$ and obtain $|V| = 3l = 6 + 9l'$. Thus, V can be partitioned into 2 blocks D_1, D_2 with 3 nodes each and l' blocks $B_1, \ldots, B_{l'}$ with 9 nodes each s.t. every block ends with a node in $V - V'$ (this is possible since every third node of V belongs to $V - V'$ due to Lemma 3, say $i \in V'$ if $3 \nmid i$ and $i \in V - V'$ if $3|i$). Figure 3 shows a block D_i and a block B_j (where circles represent nodes in V' and squares represent nodes in $V - V'$). For the studied mixed roots of $(*)$ we choose the black filled nodes in Figure 3:

D_i $\qquad\qquad$ B_j

○ ○ ■ | ○ ○ □ ● ○ □ ○ ● □

Figure 3

Lemma 6 *Any set S containing the 3rd node of the blocks D_1, D_2 and the 4th and 8th node of any block B_j is a root of $(*)$ with $|S \cap V'| = 2l'$, $|S \cap (V - V')| = 2$ for every ordering $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ of the blocks s.t. D_1, D_2 are not neighbored.*

Remark. Note that $D_1 = \{3i+1, 3i+2, 3i+3\}$ for some $0 \leq i < l$. Moreover, D_1 and D_2 are not necessarily neighbored only if $l' > 1$ (i.e., there is no suitable ordering if $l = 5$).

Proof. Consider a set S constructed that way. Since every block ends with a node in $V - V'$ by definition and every third node of V is in $V - V'$ by Lemma 3, we have that the last node of D_i and the 3rd, 6th, and 9th node of B_j belong to $V - V'$ while all other nodes are in V'. Thus, the two last nodes in D_1 and D_2 are the two studied nodes in $S \cap (V - V')$ and the 4th and 8th node in B_j for $1 \leq j \leq l'$ are the studied $2l'$ nodes in $S \cap V'$ (see Figure 3).

S is a stable set provided the two blocks D_1 and D_2 are not neighbored: Obviously, there is no edge between the 4th and 8th node of any block B_j. Thus, we only have to discuss what happens between two consecutive blocks. Since the first 3 nodes of every block B_j do not belong to S, there is no problem with having any block before B_j, i.e., $B_k B_j$ or $D_i B_j$. For the remaining case $B_j D_i$, notice that the last node of B_j and the first two nodes of D_i do not belong to S and there cannot be an edge between two nodes of S in that case, too.

This shows that S is a stable set satisfying $|S \cap V'| = 2l'$ and $|S \cap (V - V')| = 2$. Due to $\alpha(W^2_{2l}) = \lfloor \frac{2(2+3l')}{3} \rfloor = 2l' + 1$, the set S is finally a root of $(*)$. □

Lemma 6 implies that there are mixed roots S of $(*)$ with $|S| = 2 + 2l'$ if $l' \geq 2$. The next step is to show that there are l such roots if $l' \geq 3$ (resp. $l \geq 11$).

In the sequel, we denote by $S_{i,m}$ the stable set constructed as in Lemma 6 where i is the third node in D_1 and $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$. If there are more than $\lfloor \frac{l'}{2} \rfloor$ blocks between D_1 and D_2, there are less than $\lfloor \frac{l'}{2} \rfloor$ blocks between D_2 and D_1. Hence it suffices to consider $m \leq \lfloor \frac{l'}{2} \rfloor$.

Clearly, $S_{i,m}$ contains a second node from $V - V'$, namely, the third node $i + 9m + 3$ of block D_2. If $2|l'$ and $m = \frac{l'}{2}$, then $(i + 9m + 3) + 9m + 3 = i + 9l' + 6 = i (\mathrm{mod}\, n)$ and, therefore, $S_{i,m} = S_{i+9m+3,m}$ follows.

Remark. If $l' = 2$, then the only ordering of the blocks avoiding that D_1 and D_2 are neighbored is D_1, B_1, D_2, B_2. Hence, we find only $\frac{l}{2}$ mixed roots with $2 + 2l'$ nodes, namely, the stable sets S_1, S_2, S_3, S_4 of C^3_{24} presented above.

We are supposed to construct *distinct* mixed roots $S_{i,m}$ of $(*)$ with $2 + 2l'$ nodes, hence we choose orderings $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ with $1 \leq m < \frac{l'}{2}$ and obtain easily:

Lemma 7 *If $l' \geq 3$, then the stable sets $S_{i,m}$ for each $i \in V - V'$ obtained from any ordering $V = D_1, B_1, \ldots, B_m, D_2, B_{m+1}, \ldots, B_{l'}$ with $1 \leq m < \frac{l'}{2}$ yield $|V - V'| = l$ roots of $(*)$ with $2 + 2l'$ nodes each.*

Consequently, we can always choose a set of $3l$ roots of $(*)$ if $l' \geq 3$ resp. $l \geq 11$. If $\mathcal{S}$ is a set of l distinct mixed roots, denote by $A_{\mathcal{S}}$ the square matrix containing the incidence vectors of the $2l$ maximum stable sets of W^2_{2l} and the l mixed roots in $\mathcal{S}$. $A_{\mathcal{S}}$ can be arranged s.t. the first $2l$ and the last l columns correspond to the nodes in W^2_{2l} and W^1_{1l}, respectively, and the first $2l$ rows contain the incidence vectors of the maximum stable sets of W^2_{2l} where the last rows contain the incidence vectors of the l mixed roots in $\mathcal{S}$. (Note that the nodes corresponding to the last l columns of $A_{\mathcal{S}}$ are $3, 6, \ldots, 3l$.) Then $A_{\mathcal{S}}$ has the block structure

$$A_{\mathcal{S}} = \left(\begin{array}{c|c} A_{11} & 0 \\ \hline A_{21} & A_{22} \end{array} \right)$$

where the $2l \times 2l$-matrix A_{11} is invertible since W_{2l}^2 is facet-producing by [15] (in the considered case with $1 = 2l \bmod 3$ resp. $2 = l \bmod 3$).

It is left to find a set $\mathcal{S}$ of l distinct mixed roots s.t. A_{22} is an invertible $l \times l$-matrix (then $A_{\mathcal{S}}$ is invertible due to its block structure).

Lemma 8 *For every $l \geq 11$, there is a set $\mathcal{S}$ of l mixed roots of $(*)$ containing 2 nodes from $V - V'$ s.t. the $l \times l$-submatrix A_{22} of $A_{\mathcal{S}}$ is invertible.*

Proof. Every root $S_{i,m}$ of $(*)$ corresponds to a row in $(A_{21}|A_{22})$ of $A_{\mathcal{S}}$ having precisely two 1-entries in the columns belonging to A_{22} (by $|S_{i,m} \cap (V - V')| = 2$ for all $i \in V - V'$). Lemma 7 ensures that no such roots coincide if $1 \leq m < \frac{l'}{2}$ for all $i \in V - V'$.

The idea of finding cases when A_{22} is invertible goes as follows: Let $S_{3j,1}$ for $1 \leq j \leq l - 4$ be the first $l - 4$ roots in $\mathcal{S}$ with $S_{3j,1} \cap (V - V') = \{3j, 3(j + 4)\}$. Choose as the remaining 4 roots in $\mathcal{S}$ the stable sets $S_{3j,2}$ for $l - 10 \leq j \leq l - 7$ with $S_{3j,2} \cap (V - V') = \{3j, 3(j + 7)\}$. Then take their incidence vectors $\chi^{S_{3j,1}}$ for $1 \leq j \leq l - 4$ as the first $l - 4$ rows and $\chi^{S_{3j,2}}$ for $l - 10 \leq j \leq l - 7$ as the last 4 rows of $(A_{21}|A_{22})$. By construction, A_{22} is the following $l \times l$-matrix (1-entries are shown only):

	1	$\ldots$	5	$\ldots$	l–11	l–10	l–9	l–8	l–7	l–6	l–5	l–4	l–3	l–2	l–1	l
1	1		1													
$\vdots$		$\ddots$			$\ddots$											
$\vdots$				$\ddots$			$\ddots$									
l–11					1				1							
l–10						1				1						
l–9							1				1					
l–8								1				1				
l–7									1				1			
l–6										1				1		
l–5											1				1	
l–4												1				1
l–3						1							1			
l–2							1							1		
l–1								1							1	
l									1							1

A_{22} has only 1-entries on the main diagonal (coming from the first nodes in $V - V'$ of $S_{3j,1}$ for $1 \leq j \leq l - 4$ and from the second nodes in $V - V'$ of $S_{3j,2}$ for

$l-10 \leq j \leq l-7$). The only non-zero entries of A_{22} below the main diagonal come from the first nodes in $V - V'$ of $S_{3j,2}$ for $l-10 \leq j \leq l-7$. Hence, A_{22} has the form

$$A_{22} = \left(\begin{array}{c|c} A'_{22} & \\ \hline 0 & A''_{22} \end{array} \right)$$

where both A'_{22} and A''_{22} are invertible due to the following reasons:

A'_{22} is an $(l-11) \times (l-11)$-matrix having 1-entries on the main diagonal and 0-entries below the main diagonal by construction. Hence A'_{22} is clearly invertible.

A''_{22} is an 11×11-matrix which has obviously the circular 1's property. In other words, A''_{22} is equivalent to the matrix $A(\overline{C}_{11})$ containing the incidence vectors of the maximum stable sets of the odd antihole $\overline{C}_{11}$ as rows. Since $A(\overline{C}_{11})$ is invertible due to Padberg [13], the matrix A''_{22} is invertible, too. (Note that $l = 11$ implies $A_{22} = A''_{22}$.)

This completes the proof that A_{22} is invertible for every $l \geq 11$ if we choose the set $\mathcal{S}$ of l roots of $(*)$ as constructed above. □

Remark. Note that there are cases where it is possible to choose the l mixed roots of one type $S_{3j,k}$ only. E.g., the l roots $S_{3j,1}$ for $1 \leq j \leq l$ yield an invertible $l \times l$-matrix A_{22} whenever l is odd (then A_{22} has the circular 1's property and corresponds to the matrix containing the incidence vectors of the maximum stable sets of the odd antihole $\overline{C}_l$ as rows). Moreover, if $2|l$ but $4 \not| \, l$ resp. $4|l$ but $8 \not| \, l$, the roots $S_{3j,1}$ for $1 \leq j \leq l$ yield a matrix A_{22} which can be partitioned into 2 resp. 4 invertible blocks with the circular 1's property. However, there are cases left where such a partition is not possible when using mixed roots of the same type only (e.g. the case $n = 96$ and all further cases with $8|l$). This let us use mixed roots of different types for the construction.

Finally, we have shown that, for every $l \geq 11$, there are $3l$ roots of $(*)$ whose incidence vectors are linearly independent: The maximum stable sets of W^2_{2l} yield the first $2l$ independent roots (since W^2_{2l} is facet-producing in the considered case with $2 = l \bmod 3$ by Trotter [15]). Lemma 8 shows that there are further l mixed roots which are independent, too. This implies:

Theorem 9 *For any $W^2_{2l} \subseteq W^3_{3l}$ where $2 = l \bmod 3$ and $l \geq 11$, the clique family inequality*

$$2x(W^2_{2l}) + 1x(W^1_{1l}) \leq 2\alpha(W^2_{2l})$$

associated with W^2_{2l} is a non-rank facet of STAB(W^3_{3l}).

This gives us an infinite sequence of not rank-perfect webs W^3_{3l} with clique number 4, namely W^3_{33}, W^3_{42}, W^3_{51}, W^3_{60}, ... and answers the question whether the webs W^3_n with clique number 4 are rank-perfect negatively.

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