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## Calculation of clinch and elimination numbers for sports leagues with multiple tiebreaking criteria

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## Abstract

The clinch (elimination) number is a minimal number of future wins (losses) needed to clinch (to be eliminated from) a specified place in a sports league. Several optimization models and computational results are shown in this paper for calculating clinch and elimination numbers in the presence of predefined multiple tiebreaking criteria. The main subject of this paper is to provide a general algorithmic framework based on integer programming with utilizing possibly multilayered upper and lower bounds.

## 1 Introduction

At a very early stage of a season of some sports league, if a team wins all of its remaining games, then the team will secure the pennant race; and conversely, if a team loses all of its remaining games, then the team will be in the cellar. At any moment during the season, unless a team has a chance to be eliminated from some specified situation (such as league championship or playoff berth) even when the team wins all remaining games, there exists a minimal number of future wins needed for the team to achieve the situation; and conversely, unless a team has a chance to achieve the specified situation even when the team loses all remaining games, there exists a minimal number of future losses needed to be eliminated from the situation. These numbers of future wins and losses are respectively called the *clinch* and *elimination* numbers for the team at the moment. The purpose of this paper is to provide a general algorithmic framework of calculating clinch and elimination numbers that is valid for various situations in various sports leagues.

Computing cost varies from league to league. The size of a league, namely the number of teams involved in the league, obviously affects the cost, but there are two structural factors that significantly affect the cost. One such factor is the treatment of ties (draws). If ties are not allowed (as in Major League Baseball (MLB)), if a tie is converted to a fixed score (e.g., a loss, or a pair of a half win and a half loss), or similarly if some winning point system is used (as in most football leagues where three/one/zero points are awarded for a win/tie/loss), then everything will be done in a linear world (as far as the second factor explained below is not concerned). On the other hand,

if ties are allowed and especially if the winning percentage (WP) defined as the number of wins divided by the total number of wins and losses is used for team standings (as in Japan’s Nippon Professional Baseball (NPB) leagues and Taiwan’s Chinese Professional Baseball League (CPBL)), then nonlinearity resulting from the winning percentage directly influences the computation. Note that a tie is then worth a pair of WP wins and  $(1 - \text{WP})$  losses and the value of a tie becomes higher as the WP goes up.

The second influential factor to the computing cost is the presence of tiebreaking criteria for season standings, which plays a more crucial role from a computational point of view. Some sports leagues permit joint champions, and in some leagues one-game, a-series-of-games or round-robin tiebreakers are additionally played among the tied competitors, but some leagues provide tiebreaking criteria, which include head-to-head (considering only results of games among the tied), intra-district (in case of multiple districts in the league), scoring differential (the difference between points scored and those conceded) and so on. In MLB, tiebreakers are played among teams tied for a division championship or a league’s second wildcard berth, and tiebreaking rules are currently used only for home-ground advantage and postseason seedings. In the two leagues of NPB, the following four criteria are used in this order: ① WP, ② head-to-head WP, ③ intra-league WP, ④ previous season’s ranking (for the Pacific league); ① WP, ② number of wins, ③ head-to-head WP, ④ previous season’s ranking (for the Central league). When a variety of tiebreaking criteria must be taken into consideration in calculating clinch and eliminations numbers, the resulting model can be logically complicated and computationally expensive to solve.

Mathematical research on clinch and elimination started in Schwartz [18] and Hoffmann–Rivlin [9]. Earlier proposed solutions since then were mostly based on network flow [13, 19], and the complexity of this kind of problems were thoroughly investigated in [2, 11, 8, 12, 6]. On the other hand, another approach based on integer programming became tractable thanks to the advancement on nonlinear MIP technology [15, 1, 14, 7, 10]. Incorporating tiebreaking criteria was discussed in Ribeiro–Urrutia [14] for Brazilian national football league (Brasileirão), in Cheng–Steffy [7] and Russell–van Beek [16, 17] for North American national hockey league (NHL). There is no wildcard option in the former, and the subject of the latter papers is to detect clinching and elimination of each team at a date during the season.

In this paper, we will pick up a Japanese professional men’s basketball league named B.LEAGUE as a target sports league, and show our generic framework for calculating clinch and elimination numbers in the presence of multiple tiebreaking criteria. The B.LEAGUE [3] began in September 2016 and is operated by the Japan Professional Basketball League. The league consists of three divisions; the top two divisions have 18 teams (called clubs) each with a system of promotion and relegation between the first and second divisions. Each of these two divisions is further divided into three districts (East, Middle and West districts; also called conferences), each of which consists of 6 teams. The third division is currently made up of 7 professional and 3 corporate teams. We confine ourselves in this paper to the top two divisions respectively called B1 and B2.

In both divisions, each team plays a 60-game schedule, which consisted of 36 games against teams within their own district (8 games against three teams and 6 games against the remaining two teams) and 24 games against teams in the other two districts (2 games against each team) in the 2016–2017 season. The game schedule is subject to change—in the 2017–2018 season, it consisted of 30 intra-district games (6 games against each) and 30 inter-district games (4 games against three teams and 2 games against the remaining nine teams). For division B1, the top two teams of each district will qualify for the championship tournament along with the two teams that finish with the best records but do not finish in the top two of their district. The quarter-final and semi-final

rounds of the championship tournament will be played at the home court of the team that finished with a better record during the season. For division B2, the winner of each district plus a team with the best record from the remaining 15 teams will qualify for the B2 playoff tournament. The semi-finals of the B2 playoff will be played at the home venue of a higher-ranked team.

The four B1 teams with the worst regular-season records will contest a tournament to avoid relegation to B2. The first round will be played at the home venue of a better ranked team. The two losing teams from this first round will be automatically relegated to B2 and replaced by the winner and runner-up of the B2 playoff tournament. The two B1 teams that win the first round of the relegation tournament will meet in a single match at a neutral venue, with the winner remaining in B1. The loser of this match will contest a final relegation match at a neutral venue against the B2's third placed team. The B2 team with the worst regular-season record has to play a single match with the top B3 team at a neutral venue. This promotion and relegation system is subject to change in circumstances where any of the lower division teams that qualify for promotion to the upper division does not hold a full upper-division license with the league.

Ties are rare in basketball due to its high-scoring nature, but if the score is tied at the end of regulation, the game goes into as many extra periods or overtimes as necessary to determine a winner. At the end of the season, the top two divisions of B.LEAGUE follow the tiebreaking criteria

- ① WP (equivalent to the number of wins),
- ② WP among the tied teams (equivalent to the average number of wins per game),
- ③ Scoring differential among the tied teams,
- ④ Scoring average per game among the tied teams,
- ⑤ Scoring differential during the season,
- ⑥ Scoring average during the season (equivalent to the total score),
- ⑦ Drawing (if determined necessary by the board of directors)

in this order for the entire season's rankings [4, 5]. The league uses the term WP and average scores because the number of games allocated initially to each pair of teams is not even.

The situations for which clinch and elimination numbers are calculable in the top two divisions of this league then include

- Intra-district championship (B1/B2) [a],
- Championship tournament qualification (B1) [a, c],
- Avoiding the relegation tournament (B1) [b],
- B2 playoff tournament qualification [a, c],
- Home-ground advantage for the championship tournament (B1) [a, c],
- Home-ground advantage for the relegation tournament (B1) [b],
- Home-ground advantage for the B2 playoff tournament [a, c],

- Avoiding the relegation match (B2) [b].

Calculating clinch and elimination numbers for these situations are essentially the combinations of the following three types of calculation:

- a. Clinch and elimination for intra-district ranking,
- b. Clinch and elimination for intra-division ranking,
- c. Clinch and elimination for wildcard berths.

Alphabets inside the brackets after the eight situations mentioned above indicate which type of calculation is necessary for that particular situation.

This paper is composed of eight sections. After introducing the set of possible future scenarios in section 2, we start with the calculation of intra-district clinch and elimination numbers in section 3. Intra-division clinch and elimination follow in section 4. Wildcard-related clinch and elimination need special treatments, which are given in section 5. Upper and lower bounds of each optimization models as well as some related topics are discussed in section 6. Several numerical experiments are presented in section 7. Concluding remarks are given in section 8.

## 2 Scenario set

Our notation basically follows Adler et al. [1]. Let  $L$  be the set of teams in a division (or a league in general) and consist of, say, three disjoint subsets  $D_l$ ,  $l = 1, 2, 3$ , each of which respectively corresponds to the set of teams in a district; then  $L = \bigcup_{l=1}^3 D_l$ . Let  $n_l$  be the number of teams in each district; namely  $n_l := |D_l|$ . Suppose we are given the current win-loss records of all teams and the remaining schedule of games in  $L$ . Let  $w_{ij}$  be the current number of wins of team  $i \in L$  against team  $j \in L$ , and let  $g_{ij}$  be the number of remaining games between teams  $i$  and  $j$ . The current number of losses of team  $i$  against team  $j$  is then given by  $w_{ji}$ .  $w = (w_{ij})$  and  $g = (g_{ij})$  can be respectively seen as nonsymmetric and symmetric square matrices of order  $n := \sum_{l=1}^3 n_l$  with zero diagonals. Assume each team in  $L$  plays  $M$  games in a season. In current B.LEAGUE,  $n_l = 6$  for all  $l$  and  $M = 60$ .

Let  $x_{ij}$  be the number of future wins of team  $i \in L$  against team  $j \in L$ . If there is no ties in each game, any matrix  $x = (x_{ij}) \in \mathbb{Z}^{n \times n}$  satisfying the conditions

$$\begin{cases} x_{ij} + x_{ji} = g_{ij} & (\forall i, j \in L, i < j) \\ x_{ii} = 0 & (\forall i \in L) \\ x_{ij} \geq 0 & (\forall i, j \in L) \\ x_{ij} \in \mathbb{Z} & (\forall i, j \in L) \end{cases} \quad (1)$$

represents a possible future scenario, where  $\mathbb{Z}$  is the set of integers. Given  $w$  and  $g$ , let  $X$  be the set of scenarios satisfying (1). If ties are allowed in the league, the first equality should be replaced with

$$x_{ij} + x_{ji} \leq g_{ij} \quad (\forall i, j \in L, i < j).$$

Based on this set of possible future scenarios, we construct several optimization models in order to calculate clinch and elimination numbers for the eight situations listed in the previous section.

### 3 Clinch and elimination for intra-district ranking

Let us start with the calculation of intra-district  $k$ -th place clinch and elimination numbers, where  $k = 1, 2, \dots, n_l - 1$ . The cases of  $k = 1$  and  $k = 2$  will be used later.

#### 3.1 Clinching $k$ -th place

Clinching  $k$ -th place means that there is no chance of finishing in  $(k + 1)$ -th place or worse even if the team loses all of its remaining games. Since it is not appropriate to take the third criterion on the scoring differential into consideration, we hereafter define the  $k$ -th place clinch number as a minimal number of future wins needed for a team to clinch the  $k$ -th place without considering the scoring differentials. With this definition, we consider the following optimization problem for finding a maximal number of future wins of team  $a$  under the condition that the team finishes in  $(k + 1)$ -th place or worse. Let team  $a$  belong to  $D_{l'}$ , where  $l' = 1, 2$  or  $3$ .

$$\begin{aligned}
& \max_{\substack{x \in X \\ \alpha, \beta, \lambda \in \{0, 1\}^{n_{l'}}}} \sum_{j \in L} x_{aj} \\
& \text{subject to} \quad \boxed{\sum_{j \in L} (w_{aj} + x_{aj}) \leq \sum_{j \in L} (w_{ij} + x_{ij}) + (M + 1) \alpha_i - 1 \quad (\forall i \in D_{l'})} \\
& \quad \boxed{\begin{aligned} & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i \quad (\forall i \in D_{l'}) \\ & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \geq \lambda_i \quad (\forall i \in D_{l'}) \\ & \frac{\sum_{j \in D_{l'}} (w_{aj} + x_{aj})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j)} \leq \frac{\sum_{j \in D_{l'}} (w_{ij} + x_{ij})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j)} + \beta_i \\ & \quad (\forall i \in D_{l'}) \\ & \lambda_i \leq \beta_i \quad (\forall i \in D_{l'}), \quad \beta_a = 1 \end{aligned}} \\
& \quad \sum_{i \in D_{l'}} \alpha_i + \sum_{i \in D_{l'}} \beta_i = 2n_{l'} - k
\end{aligned}$$

The constraints of this maximization problem indicate that there exist at least  $k$  teams in the same district that are either superior to team  $a$  in the number of wins (see the first block) or not inferior to team  $a$  in the average number of wins per game among the teams tied in the number of wins (see the second block). The binary variables  $\alpha_i$ ,  $\beta_i$  and  $\lambda_i$  are assigned to all teams in  $D_{l'}$ , and, for any feasible solution, the following relations hold:

$$\begin{aligned}
& \alpha_a = 1, \quad \lambda_a = 0, \\
& \alpha_i + \beta_i \geq 1 \quad (\forall i \in D_{l'}).
\end{aligned}$$

Note that if either  $\alpha_i$  or  $\beta_i$  is zero, then the other is necessarily one, from which the last inequality follows. Team  $i$  is tied to team  $a$  if and only if  $\lambda_i = 0$ , and the teams with  $\beta_i = 0$  are chosen only from these tied teams. Exactly  $k$  teams in the same district are thus chosen as either  $\alpha_i = 0$  or  $\beta_i = 0$ .

With removing redundancy of the constraints, we can consider the following equivalent problem instead.

**Problem  $(C_{l'}^k)$ :**

$$\begin{aligned}
& \max_{\substack{x \in X \\ \alpha, \beta, \lambda, \lambda_1, \lambda_2 \in \{0,1\}^{n_{l'}}}} \sum_{j \in L} x_{aj} \\
\text{subject to } & \left[ \begin{aligned} & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i \quad (\forall i \in D_{l'}) \\ & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M+1) \lambda_{1i} - 1 \quad (\forall i \in D_{l'}) \\ & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M+1) \lambda_{2i} + 1 \quad (\forall i \in D_{l'}) \\ & \lambda_i + \lambda_{1i} + \lambda_{2i} = 2 \quad (\forall i \in D_{l'}) \end{aligned} \right] \\
& \left[ \lambda_{1i} \leq \alpha_i \quad (\forall i \in D_{l'}) \right] \\
& \left[ \begin{aligned} & \frac{\sum_{j \in D_{l'}} (w_{aj} + x_{aj})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j)} \leq \frac{\sum_{j \in D_{l'}} (w_{ij} + x_{ij})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j)} + \beta_i \\ & \quad (\forall i \in D_{l'}) \\ & \lambda_i \leq \beta_i \quad (\forall i \in D_{l'}), \quad \beta_a = 1 \end{aligned} \right] \\
& \sum_{i \in D_{l'}} \alpha_i + \sum_{i \in D_{l'}} \beta_i = 2n_{l'} - k
\end{aligned}$$

where additional binary variables  $\lambda_{1i}$  and  $\lambda_{2i}$  are assigned to all teams in  $D_{l'}$  for team classification. All teams in  $D_{l'}$  are now classified into three categories: tied ( $\lambda_i = 0$ ), superior ( $\lambda_{1i} = 0$ ), or inferior ( $\lambda_{2i} = 0$ ) to team  $a$  in the number of wins (see the dashed block). The teams with  $\alpha_i = 0$  are chosen from the superior teams with  $\lambda_{1i} = 0$  (see the first solid block), and the teams with  $\beta_i = 0$  are chosen from the tied teams with  $\lambda_i = 0$  (see the second solid block).

After solving this maximization problem, we have the intra-district  $k$ -th place clinch number  $\#C_{l'}^k$  of team  $a$  as in Figure 1, where  $\bar{z}_{l'}^k$  denotes the optimal objective function value of problem  $(C_{l'}^k)$ .

### 3.2 Being eliminated from $k$ -th place

The elimination number of losses can be calculated in a similar but dual manner. Being eliminated from  $k$ -th place means that there is no chance of finishing in  $k$ -th place or better even if the team



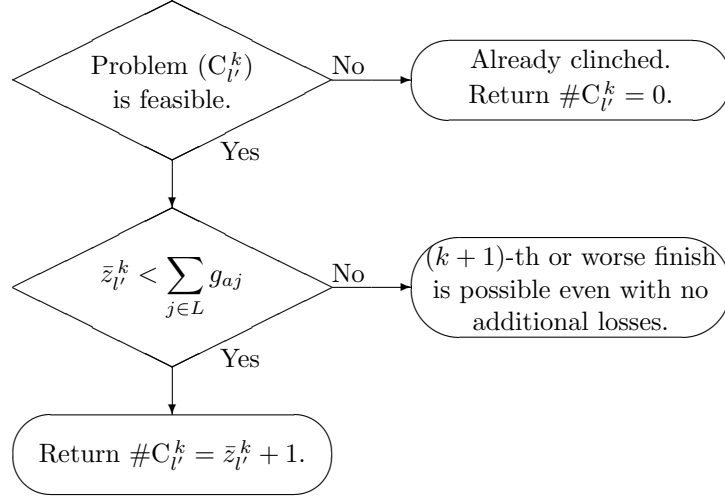


Figure 1: Finding the intra-district  $k$ -th place clinch number  $\#C_{l'}^k$

wins all of its remaining games. We define the  $k$ -th place elimination number as a minimal number of future losses needed for a team to be eliminated from the  $k$ -th place without considering the score differentials. We thus consider an optimization problem for finding a maximal number of future losses of team  $a \in D_{l'}$  under the condition that the team finishes in  $k$ -th place or better. The problem is written as

$$\begin{aligned}
 & \max_{\substack{\alpha, \beta, \lambda \in \{0,1\}^{n_{l'}} \\ x \in X}} \sum_{j \in L} x_{ja} \\
 & \text{subject to} \quad \boxed{\sum_{j \in L} (w_{aj} + x_{aj}) \geq \sum_{j \in L} (w_{ij} + x_{ij}) - (M+1)\alpha_i + 1 \quad (\forall i \in D_{l'})} \\
 & \quad \boxed{\begin{aligned}
 & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M\lambda_i \quad (\forall i \in D_{l'}) \\
 & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \geq \lambda_i \quad (\forall i \in D_{l'}) \\
 & \frac{\sum_{j \in D_{l'}} (w_{aj} + x_{aj})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j)} \geq \frac{\sum_{j \in D_{l'}} (w_{ij} + x_{ij})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j)} - \beta_i \\
 & \lambda_i \leq \beta_i \quad (\forall i \in D_{l'}), \quad \beta_a = 1
 \end{aligned}}
 \end{aligned}$$

$$\sum_{i \in D_{l'}} \alpha_i + \sum_{i \in D_{l'}} \beta_i = n_{l'} + k$$

and hence as

**Problem (E<sub>l'</sub><sup>k</sup>):**

$$\begin{aligned} & \max_{\substack{x \in X \\ \alpha, \beta, \lambda, \lambda_1, \lambda_2 \in \{0,1\}^{n_{l'}}}} \sum_{j \in L} x_{ja} \\ \text{subject to} & \left[ \begin{aligned} & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i \quad (\forall i \in D_{l'}) \\ & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M+1) \lambda_{1i} - 1 \quad (\forall i \in D_{l'}) \\ & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M+1) \lambda_{2i} + 1 \quad (\forall i \in D_{l'}) \\ & \lambda_i + \lambda_{1i} + \lambda_{2i} = 2 \quad (\forall i \in D_{l'}) \end{aligned} \right] \\ & \boxed{\lambda_{2i} \leq \alpha_i \quad (\forall i \in D_{l'})} \\ & \boxed{\begin{aligned} & \frac{\sum_{j \in D_{l'}} (w_{aj} + x_{aj})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j)} \geq \frac{\sum_{j \in D_{l'}} (w_{ij} + x_{ij})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j)} - \beta_i \\ & \lambda_i \leq \beta_i \quad (\forall i \in D_{l'}), \quad \beta_a = 1 \end{aligned}} \\ & \sum_{i \in D_{l'}} \alpha_i + \sum_{i \in D_{l'}} \beta_i = n_{l'} + k \end{aligned}$$

where it holds that

$$\begin{aligned} \alpha_a &= 1, \quad \lambda_a = 0, \\ \alpha_i + \beta_i &\geq 1 \quad (\forall i \in D_{l'}) \end{aligned}$$

for any feasible solution. The binary variables are the same as in the previous subsection. The dashed block for team classification is also the same. The teams with  $\alpha_i = 0$  are now chosen from the inferior teams with  $\lambda_{2i} = 0$  (see the first solid block), and the inequality of the second criterion is reversed (see the second solid block). Exactly  $(n_{l'} - k)$  teams in the same district are then chosen either as  $\alpha_i = 0$  or as  $\beta_i = 0$ .

After solving this maximization problem, the  $k$ -th place elimination number  $\#E_{l'}^k$  of team  $a$  is given as in Figure 2, where  $\bar{y}_{l'}^k$  denotes the optimal objective function value of (E<sub>l'</sub><sup>k</sup>).

It should be noted that finding a maximal number of future losses is equal to finding a minimal number of future wins since

$$\sum_{j \in L} x_{aj} + \sum_{j \in L} x_{ja} = \sum_{j \in L} g_{aj}$$

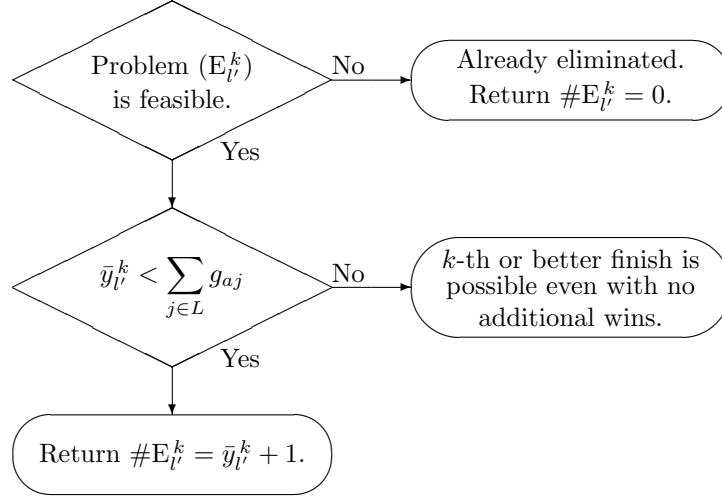


Figure 2: Finding the intra-district  $k$ -th place elimination number  $\#E_{l'}^k$

follows from the first equality in (1) and this righthand side is fixed. An elimination number of losses can thus be expressed equivalently as an elimination number of wins, which indicates a maximal number of future wins for being eliminated.

## 4 Clinch and elimination for intra-division ranking

This section deals with the  $k$ -th place clinch and elimination numbers in each division, where  $k = 1, 2, \dots, n - 1$ . The three cases of  $k = n - 4, n - 2, n - 1$  are important in B.LEAGUE. The logical structure of intra-division clinch and elimination is essentially the same as that of intra-district clinch and elimination.

### 4.1 Clinching $k$ -th place

The  $k$ -th place intra-division clinch number for team  $a$  can be calculated by solving

**Problem (C<sup>k</sup>):**

$$\max_{\substack{x \in X \\ \gamma^l, \delta^l, \lambda^l, \lambda_1^l, \lambda_2^l \in \{0,1\}^{n_l}, l \in \{1,2,3\}}} \sum_{j \in L} x_{aj}$$

$$\begin{aligned}
& \text{subject to} \quad \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
& \quad \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M + 1) \lambda_{1i}^l - 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
& \quad \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M + 1) \lambda_{2i}^l + 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
& \quad \lambda_i^l + \lambda_{1i}^l + \lambda_{2i}^l = 2 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
& \quad \lambda_{1i}^l \leq \gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
& \quad \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + x_{aj})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^l)} \leq \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + x_{ij})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^l)} + \delta_i^l \\
& \quad \lambda_i^l \leq \delta_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}), \quad \delta_a^{l'} = 1 \\
& \quad \sum_{l=1}^3 \sum_{i \in D_l} \gamma_i^l + \sum_{l=1}^3 \sum_{i \in D_l} \delta_i^l = 2n - k
\end{aligned}$$

The binary variables  $\gamma_i^l$  and  $\delta_i^l$  for team selection as well as the binary variables  $\lambda_i^l$ ,  $\lambda_{1i}^l$  and  $\lambda_{2i}^l$  for team classification are assigned to all teams in the whole division  $L = \bigcup_{l=1}^3 D_l$ . In the dashed block, all teams in  $D_l$  are classified into three categories: tied ( $\lambda_i^l = 0$ ), superior ( $\lambda_{1i}^l = 0$ ), or inferior ( $\lambda_{2i}^l = 0$ ) to team  $a \in D_{l'}$  in the number of wins. The teams with  $\gamma_i^l = 0$  are chosen from the superior teams with  $\lambda_{1i}^l = 0$  in the first solid block, and the teams with  $\delta_i^l = 0$  are chosen from the tied teams with  $\lambda_i^l = 0$  in the second solid block. Here

$$\begin{aligned}
& \gamma_a^{l'} = 1, \quad \lambda_a^{l'} = 0, \\
& \gamma_i^l + \delta_i^l \geq 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})
\end{aligned}$$

hold for any feasible solution. Exactly  $k$  teams are thus chosen from the whole division either as  $\gamma_i^l = 0$  or as  $\delta_i^l = 0$ .

The flow chart for finding the intra-division  $k$ -th place clinch number  $\#C^k$  of team  $a$  is given in Figure 3, where  $\bar{z}^k$  denotes the optimal objective function value of problem  $(C^k)$ .

## 4.2 Being eliminated from $k$ -th place

The  $k$ -th place intra-division elimination number for team  $a$  can be calculated by solving

**Problem  $(E^k)$ :**

$$\max_{\gamma \in X} \sum_{j \in L} x_{ja}$$

$\gamma^l, \delta^l, \lambda^l, \lambda_1^l, \lambda_2^l \in \{0, 1\}^{n_l}, l \in \{1, 2, 3\}$

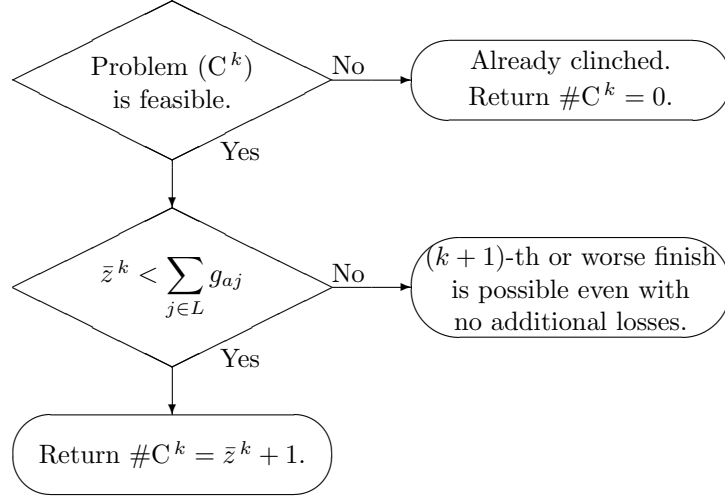


Figure 3: Finding the intra-division  $k$ -th place clinch number  $\#C^k$

subject to

$$\left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M+1) \lambda_{1i}^l - 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M+1) \lambda_{2i}^l + 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\lambda_i^l + \lambda_{1i}^l + \lambda_{2i}^l = 2 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\lambda_{2i}^l \leq \gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + x_{aj})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^l)} \geq \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + x_{ij})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^l)} - \delta_i^l$$

$$(\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\lambda_i^l \leq \delta_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}), \quad \delta_a^{l'} = 1$$

$$\sum_{l=1}^3 \sum_{i \in D_l} \gamma_i^l + \sum_{l=1}^3 \sum_{i \in D_l} \delta_i^l = n + k$$

where the binary variables as well as the dashed block for team classification are the same as in the preceding subsection, and

$$\gamma_a^{l'} = 1, \quad \lambda_a^{l'} = 0$$

$$\gamma_i^l + \delta_i^l \geq 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

hold for any feasible solution. Here, the teams with  $\gamma_i^l = 0$  are chosen from the inferior teams with  $\lambda_{2i}^l = 0$  in the first solid block, and the teams with  $\delta_i^l = 0$  are chosen from the tied teams with  $\lambda_i^l = 0$  and satisfying the reversed inequality. Exactly  $(n - k)$  teams are finally chosen from the whole division either as  $\gamma_i^l = 0$  or as  $\delta_i^l = 0$ .

The flow chart for finding the intra-division  $k$ -th place elimination number  $\#E^k$  of team  $a$  is shown in Figure 4, where  $\bar{y}^k$  in denotes the optimal objective function value of problem  $(E^k)$ .

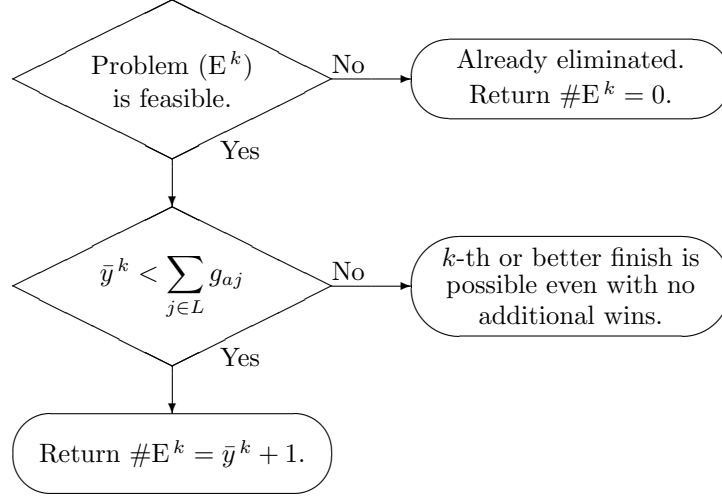


Figure 4: Finding the intra-division  $k$ -th place elimination number  $\#E^k$

### 4.3 Avoiding the B1 relegation tournament

The four B1 teams with the worst regular-season records will contest a relegation tournament. The clinch and elimination numbers for avoiding the relegation tournament are hence defined as the intra-division  $(n - 4)$ -th clinch and elimination numbers.

### 4.4 Home-ground advantage for the B1 relegation tournament

In the first round of the relegation tournament, the two teams with better regular-season records will have home-ground advantage, whose clinch and elimination numbers are then equal to the intra-division  $(n - 2)$ -th clinch and elimination numbers.

### 4.5 Avoiding the B2 relegation match

The B2 team with the worst regular-season record has to play a single match with the top B3 team. The clinch and elimination numbers for avoiding this relegation match are hence given as the intra-division  $(n - 1)$ -th clinch and elimination numbers.

## 5 Clinch and elimination for wildcard berths

Clinch and elimination calculations involving wildcard berths need special treatments. Optimization models for essentially three cases are developed in this section. Intriguingly, the models to be developed in subsection 5.4 are in *dual* relations with those in subsection 5.2.

### 5.1 Championship tournament qualification

In division B1, if a team finishes in the top two of its district or finishes within the best two records among all teams except the top two of each district, the team will qualify for the championship tournament. The latter two teams are called wildcard. Therefore, for calculating the clinch number for the championship tournament qualification, we need to find a maximal number of future wins for each team under the conditions that the team does not finish within the top two of its district and that the team does not finish within the best two records among all teams except the top two of all districts. Taking into consideration that the latter condition is equivalent to that there exist four teams with better records in one district or there exist three teams with better records in each of two districts, we will show how to obtain the clinch number for team  $a \in D_{l'}$ .

In the case where tiebreakers are additionally played among tied teams as in MLB, it is enough to consider the following simple problem.

$$\begin{aligned}
 & \max_{\substack{x \in X \\ \alpha \in \{0,1\}^{n_{l'}} \\ \gamma^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3 \\ \theta \in \{0,1\}}} \sum_{j \in L} x_{aj} \\
 & \text{subject to} \quad \boxed{\begin{aligned} & \sum_{j \in L} (w_{aj} + x_{aj}) \leq \sum_{j \in L} (w_{ij} + x_{ij}) + (M+1)\alpha_i - 1 \quad (\forall i \in D_{l'}) \\ & \sum_{i \in D_{l'}} \alpha_i = n_{l'} - 2 \end{aligned}} \\
 & \quad \boxed{\begin{aligned} & \sum_{j \in L} (w_{aj} + x_{aj}) \leq \sum_{j \in L} (w_{ij} + x_{ij}) + (M+1)\gamma_i^l - 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\ & (1-\theta) \left( \sum_{i \in D_l} \gamma_i^l - n_l + 4\sigma_l \right) = 0 \quad (\forall l \in \{1,2,3\}) \\ & (1-\theta) \left( \sum_{l=1}^3 \sigma_l - 1 \right) = 0 \\ & \theta \left( \sum_{i \in D_l} \gamma_i^l - n_l + 3\sigma_l \right) = 0 \quad (\forall l \in \{1,2,3\}) \\ & \theta \left( \sum_{l=1}^3 \sigma_l - 2 \right) = 0 \end{aligned}} \\
 & \alpha_a = \gamma_a^{l'} = 1
 \end{aligned}$$

The two constraint blocks respectively represent the conditions of not finishing in 1st place and not earning a wildcard spot. Here are four types of binary variable:  $\alpha_i$  is assigned to all teams in  $D_{l'}$ ,  $\gamma_i^l$  to all teams in  $D_l$ ,  $\sigma_i$  to all districts, and  $\theta$  is used for case selection. In the first block, exactly two teams that are superior to team  $a$  are chosen as  $\alpha_i = 0$  from the same district. In the second block, exactly four teams that are superior to team  $a$  are chosen as  $\gamma_i^l = 0$  from any single district when  $\theta = 0$ , or exactly three teams that are superior to team  $a$  are respectively chosen as  $\gamma_i^l = 0$  from any two districts when  $\theta = 1$ .

Incorporating the second tiebreaking criterion into this problem as we did in the preceding sections, we have

**Problem (C<sub>c</sub>):**

$$\begin{aligned}
 & \max_{\substack{x \in X \\ \alpha, \beta \in \{0,1\}^{n_{l'}} \\ \gamma^l, \delta^l, \lambda^l, \lambda_1^l, \lambda_2^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3 \\ \theta \in \{0,1\}}} \sum_{j \in L} x_{aj} \\
 & \text{subject to } \left[ \begin{aligned}
 & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i^l \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
 & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M+1) \lambda_{1i}^l - 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
 & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M+1) \lambda_{2i}^l + 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
 & \lambda_i^l + \lambda_{1i}^l + \lambda_{2i}^l = 2 \quad (\forall i \in D_l, \forall l \in \{1,2,3\})
 \end{aligned} \right] \\
 & \left[ \begin{aligned}
 & \lambda_{1i}^{l'} \leq \alpha_i \quad (\forall i \in D_{l'}) \\
 & \frac{\sum_{j \in D_{l'}} (w_{aj} + x_{aj})(1 - \lambda_j^{l'})}{\sum_{j \in D_{l'}} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^{l'})} \leq \frac{\sum_{j \in D_{l'}} (w_{ij} + x_{ij})(1 - \lambda_j^{l'})}{\sum_{j \in D_{l'}} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^{l'})} + \beta_i \\
 & \lambda_i^{l'} \leq \beta_i \quad (\forall i \in D_{l'}), \quad \beta_a = 1 \\
 & \sum_{i \in D_{l'}} \alpha_i + \sum_{i \in D_{l'}} \beta_i = 2n_{l'} - 2
 \end{aligned} \right]
 \end{aligned}$$



$$\begin{array}{l}
\boxed{\lambda_{1i}^l \leq \gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})} \\
\boxed{\frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + x_{aj})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^l)} \leq \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + x_{ij})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^l)} + \delta_i^l} \\
\lambda_i^l \leq \delta_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}), \quad \delta_a^{l'} = 1 \\
(1 - \theta) \left( \sum_{i \in D_l} \gamma_i^l + \sum_{i \in D_l} \delta_i^l - 2n_l + 4\sigma_l \right) = 0 \quad (\forall l \in \{1, 2, 3\}) \\
(1 - \theta) \left( \sum_{l=1}^3 \sigma_l - 1 \right) = 0 \\
\theta \left( \sum_{i \in D_l} \gamma_i^l + \sum_{i \in D_l} \delta_i^l - 2n_l + 3\sigma_l \right) = 0 \quad (\forall l \in \{1, 2, 3\}) \\
\theta \left( \sum_{l=1}^3 \sigma_l - 2 \right) = 0
\end{array}$$

The two constraint blocks now have two criteria each inside. Here are some additional binary variables. For team selection in the second criterion,  $\beta_i$  is assigned to all teams in  $D_{l'}$ , and  $\delta_i^l$  to all teams in  $D_l$ . The binary variables  $\lambda_i^l$ ,  $\lambda_{1i}^l$  and  $\lambda_{2i}^l$  are assigned to all teams in  $D_l$  for team classification. For any feasible solution, the following relations hold:

$$\begin{aligned}
\alpha_a &= \gamma_a^{l'} = 1, \quad \lambda_a^{l'} = 0, \\
\alpha_i + \beta_i &\geq 1 \quad (\forall i \in D_{l'}), \\
\gamma_i^l + \delta_i^l &\geq 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}),
\end{aligned}$$

where the last two inequalities follow from a fact that if one of the variables is zero then the other is necessarily one in each inequality. Exactly two teams in  $D_{l'}$  are chosen either as  $\alpha_i = 0$  or as  $\beta_i = 0$ , and exactly four teams (when  $\theta = 0$ ) or three teams each (when  $\theta = 1$ ) are chosen, either as  $\gamma_i^l = 0$  or as  $\delta_i^l = 0$ , respectively from a single district (when  $\theta = 0$ ) or two districts (when  $\theta = 1$ ).

The clinch number  $\#C_c$  of team  $a$  can be obtained for the championship tournament qualification as in Figure 5, where  $\bar{z}_c$  denotes the optimal objective function value of problem  $(C_c)$ .

For elimination from the championship tournament, we will find a maximal number of future losses for each team under the condition that the team either finishes within the top two of its district or earns one of the two wildcard berths. This means that we can separately calculate and combine the intra-district 2nd place elimination number and the wildcard elimination number under the respective condition. We already have problem  $(E_{l'}^2)$  for calculating the former. We will now consider how to calculate the latter. It is not difficult to see that a team can earn a wildcard berth if and only if *either* there exist  $(n_l - 4)$  teams with lower records in one district and  $(n_l - 2)$  teams with lower records in each of the other two districts *or* there exist  $(n_l - 3)$  teams with lower records

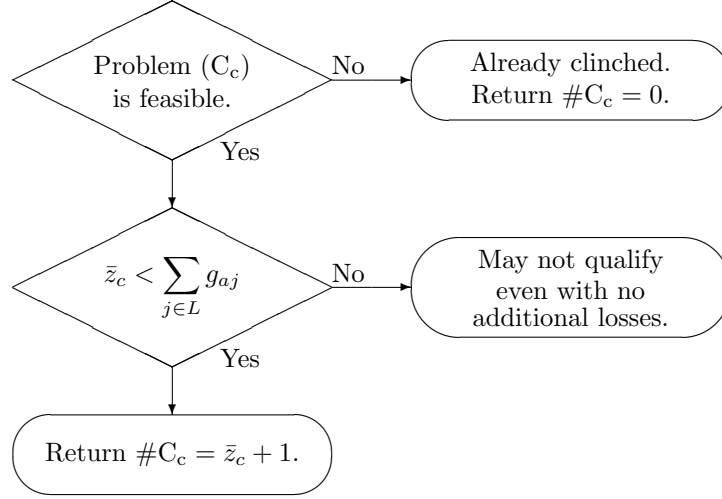


Figure 5: Finding the clinch number  $\#C_c$  for the championship tournament qualification

in each of two districts and  $(n_l - 2)$  teams with lower records in the other district. The optimization problem for calculating the wildcard elimination number of  $a \in D_{l'}$  is hence given as follows.

**Problem ( $E_w^2$ ):**

$$\begin{aligned}
 & \max_{\substack{x \in X \\ \gamma^l, \delta^l, \lambda^l, \lambda_1^l, \lambda_2^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3 \\ \theta \in \{0,1\}}} \sum_{j \in L} x_{ja} \\
 \text{subject to } & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i^l \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
 & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M+1) \lambda_{1i}^l - 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
 & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M+1) \lambda_{2i}^l + 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
 & \lambda_i^l + \lambda_{1i}^l + \lambda_{2i}^l = 2 \quad (\forall i \in D_l, \forall l \in \{1,2,3\})
 \end{aligned}$$

$$\lambda_{2i}^l \leq \gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + x_{aj})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^l)} \geq \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + x_{ij})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^l)} + \delta_i^l$$

$$(\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\lambda_i^l \leq \delta_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}), \quad \delta_a^{l'} = 1$$

$$(1 - \theta) \left( \sum_{i \in D_l} \gamma_i^l + \sum_{i \in D_l} \delta_i^l - n_l - 2 - 2\sigma_l \right) = 0 \quad (\forall l \in \{1, 2, 3\})$$

$$(1 - \theta) \left( \sum_{l=1}^3 \sigma_l - 1 \right) = 0$$

$$\theta \left( \sum_{i \in D_l} \gamma_i^l + \sum_{i \in D_l} \delta_i^l - n_l - 2 - \sigma_l \right) = 0 \quad (\forall l \in \{1, 2, 3\})$$

$$\theta \left( \sum_{l=1}^3 \sigma_l - 2 \right) = 0$$

where

$$\gamma_a^{l'} = 1, \quad \lambda_a^{l'} = 0,$$

$$\gamma_i^l + \delta_i^l \geq 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

hold for any feasible solution. When  $\theta = 0$ , exactly  $(n_l - 4)$  teams from a single district and exactly  $(n_l - 2)$  teams from each of the other two districts are chosen either as  $\gamma_i^l = 0$  or as  $\delta_i^l = 0$ . When  $\theta = 1$ , exactly  $(n_l - 3)$  teams from each of two districts and exactly  $(n_l - 2)$  teams from the other district are chosen either as  $\gamma_i^l = 0$  or as  $\delta_i^l = 0$ .

With the optimal objective function values of problem  $(E_{l'}^2)$  and  $(E_w^2)$ , three elimination numbers of team  $a$  can be obtained for intra-district 2nd-place, wildcard, and championship tournament qualification as in Figure 6, where  $\bar{y}_{l'}^2$  and  $\bar{y}_w^2$  respectively denote the optimal objective function values of problems  $(E_{l'}^2)$  and  $(E_w^2)$ . If both problem  $(E_{l'}^2)$  and problem  $(E_w^2)$  have no feasible solution, then team  $a$  is already eliminated from championship tournament qualification. If either the intra-district 2nd-place elimination number  $(\#E_{l'}^2)$  or the wildcard elimination number  $(\#E_w^2)$  does not exist, then team  $a$  has a chance of championship tournament qualification even when the team loses all remaining games and hence its elimination number does not exist. If none of above applies, then both the intra-district 2nd-place elimination number  $(\#E_{l'}^2)$  and the wildcard elimination number  $(\#E_w^2)$  exist, and hence the championship tournament qualification elimination number  $(\#E_c)$  is given by  $\#E_c = \max(\#E_{l'}^2, \#E_w^2)$ .

## 5.2 B2 playoff tournament qualification

For division B2, if a team finishes in 1st place in its district or finishes with the best record among all 2nd-place teams, the team will qualify for the B2 playoff tournament. The wildcard spot is one

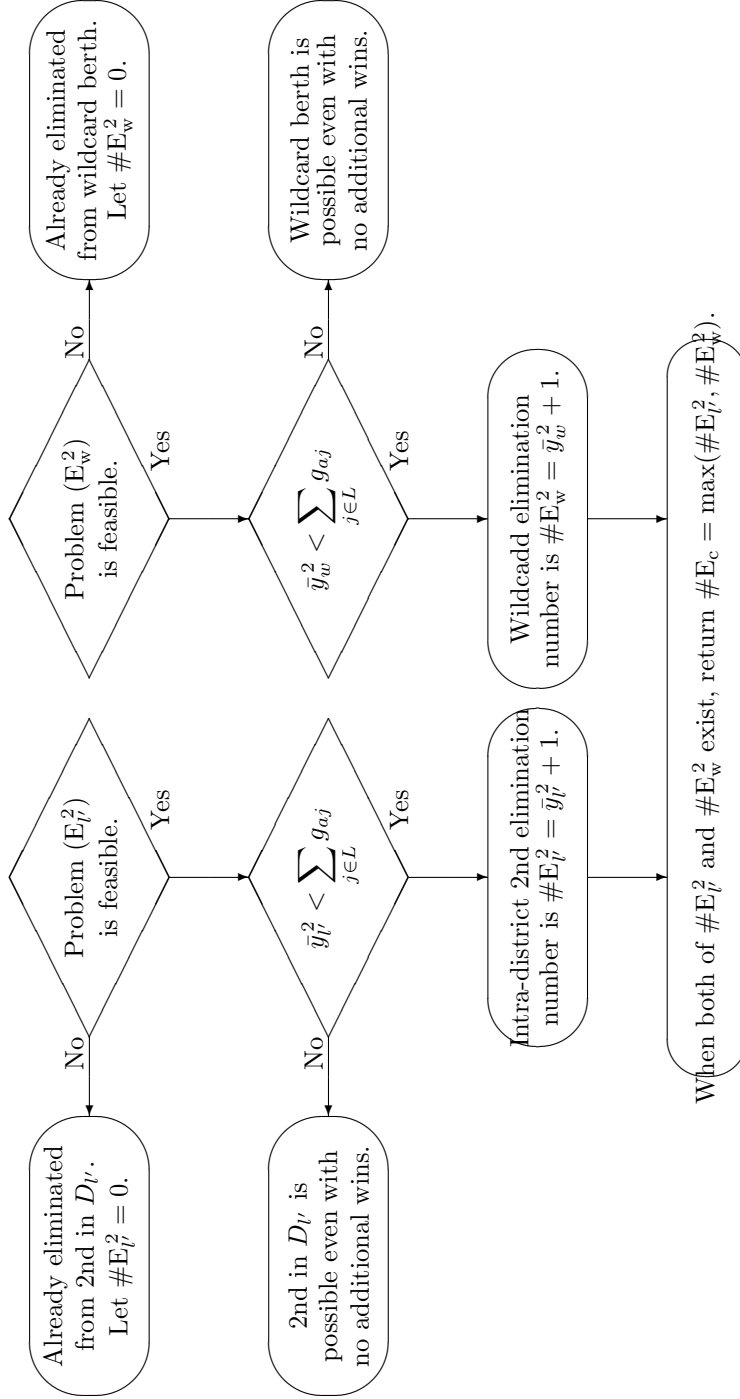


Figure 6: Finding the elimination number  $\#E_c$  for the championship tournament qualification

in this case. In order to calculate the clinch number for the B2 playoff tournament qualification, we need to find a maximal number of future wins for each team under the conditions that the team does not finish in 1st place in its district and that the team does not finish with the best record among all 2nd-place teams. Since the latter condition is equivalent to that there exist two teams with better records in a district other than its own, the clinch number of team  $a$  for the B2 playoff tournament qualification can be obtained by solving

**Problem (C<sub>p</sub>):**

$$\begin{aligned}
 & \max_{\substack{x \in X \\ \alpha, \beta \in \{0,1\}^{n_{l'}} \\ \gamma^l, \delta^l, \lambda^l, \lambda_1^l, \lambda_2^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3}} \sum_{j \in L} x_{aj} \\
 & \text{subject to } \left[ \begin{aligned}
 & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i^l \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
 & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M+1) \lambda_{1i}^l - 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
 & \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M+1) \lambda_{2i}^l + 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
 & \lambda_i^l + \lambda_{1i}^l + \lambda_{2i}^l = 2 \quad (\forall i \in D_l, \forall l \in \{1,2,3\})
 \end{aligned} \right] \\
 & \left[ \begin{aligned}
 & \lambda_{1i}^{l'} \leq \alpha_i \quad (\forall i \in D_{l'}) \\
 & \frac{\sum_{j \in D_{l'}} (w_{aj} + x_{aj})(1 - \lambda_j^{l'})}{\sum_{j \in D_{l'}} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^{l'})} \leq \frac{\sum_{j \in D_{l'}} (w_{ij} + x_{ij})(1 - \lambda_j^{l'})}{\sum_{j \in D_{l'}} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^{l'})} + \beta_i \\
 & \lambda_i^{l'} \leq \beta_i \quad (\forall i \in D_{l'}), \quad \beta_a = 1 \\
 & \sum_{i \in D_{l'}} \alpha_i + \sum_{i \in D_{l'}} \beta_i = 2n_{l'} - 1
 \end{aligned} \right]
 \end{aligned}$$

$$\begin{array}{l}
\boxed{\lambda_{1i}^l \leq \gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})} \\
\frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + x_{aj})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^l)} \leq \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + x_{ij})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^l)} + \delta_i^l \\
\quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
\lambda_i^l \leq \delta_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}), \quad \delta_a^{l'} = 1 \\
\sum_{i \in D_l} \gamma_i^l + \sum_{i \in D_l} \delta_i^l = 2n_l - \sigma_l \quad (\forall l \in \{1, 2, 3\}) \\
\sum_{l=1}^3 \sigma_l = 1, \quad \sigma_{l'} = 0
\end{array}$$

where it holds that

$$\begin{aligned}
\alpha_a &= \gamma_a^{l'} = 1, \quad \lambda_a^{l'} = 0, \\
\alpha_i + \beta_i &\geq 1 \quad (\forall i \in D_{l'}), \\
\gamma_i^l + \delta_i^l &\geq 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})
\end{aligned}$$

for any feasible solution. The binary variables for team selection and team classification are the same as in the preceding subsection. A case selection variable is not necessary in this case. See Figure 7, where  $\bar{z}_p$  denotes the optimal objective function value of problem (C<sub>p</sub>).

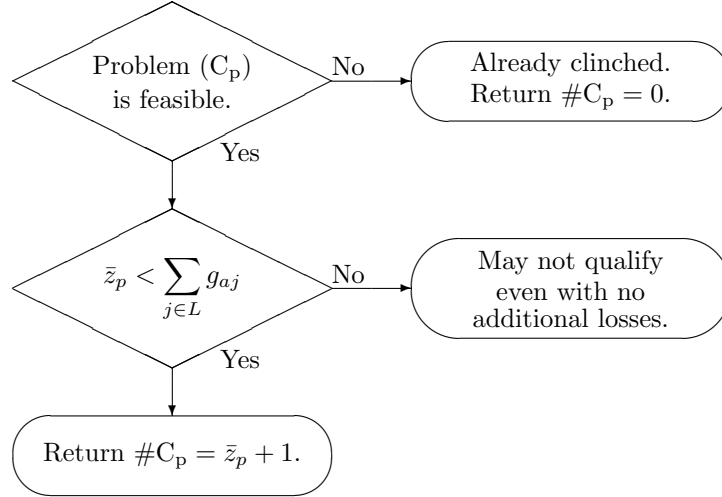


Figure 7: Finding the clinch number #C<sub>p</sub> for the B2 playoff tournament qualification

For elimination from the B2 playoff tournament, we need to find a maximal number of future losses for each team under the condition that the team win the intra-district championship or

earn a wildcard berth. We thus separately calculate and combine the intra-district championship elimination number and the wildcard elimination number under respective condition. We already have problem  $(E_{l'}^1)$ . Earning a wildcard berth for team  $a$  is equivalent to the condition that there exist  $(n_{l'} - 2)$  teams with lower records in its own district and  $(n_l - 1)$  teams with lower records in each of the other two districts. The optimization problem for finding the the wildcard elimination number is then given by

**Problem  $(E_w^1)$ :**

$$\begin{aligned}
& \max_{\substack{x \in X \\ \gamma^l, \delta^l, \lambda^l, \lambda_1^l, \lambda_2^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3}} \sum_{j \in L} x_{ja} \\
\text{subject to } & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i^l \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
& \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M+1) \lambda_{1i}^l - 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
& \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M+1) \lambda_{2i}^l + 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
& \lambda_i^l + \lambda_{1i}^l + \lambda_{2i}^l = 2 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
& \lambda_{2i}^l \leq \gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
& \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + x_{aj})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^l)} \geq \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + x_{ij})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^l)} - \delta_i^l \\
& \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
& \lambda_i^l \leq \delta_i^l \quad (\forall i \in D_l, \forall l \in \{1,2,3\}), \quad \delta_a^{l'} = 1 \\
& \sum_{i \in D_l} \gamma_i^l + \sum_{i \in D_l} \delta_i^l = n_l + 1 + \sigma_l \quad (\forall l \in \{1,2,3\}) \\
& \sum_{l=1}^3 \sigma_l = 1, \quad \sigma_{l'} = 1
\end{aligned}$$

where

$$\begin{aligned}
& \gamma_a^{l'} = 1, \quad \lambda_a^{l'} = 0, \\
& \gamma_i^l + \delta_i^l \geq 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\})
\end{aligned}$$

hold for any feasible solution.

With the optimal objective function values of problem  $(E_{l'}^1)$  and  $(E_w^1)$ , three elimination numbers of team  $a$  can be obtained for intra-district championship, wildcard, and B2 playoff tournament

qualification as in Figure 8, where  $\bar{y}_{l'}^1$  and  $\bar{y}_w^1$  respectively denote the optimal objective function values of problems  $(E_{l'}^1)$  and  $(E_w^1)$ . If both problem  $(E_{l'}^1)$  and problem  $(E_w^1)$  have no feasible solution, then team  $a$  is already eliminated from B2 playoff tournament qualification. If either the intra-district championship elimination number ( $\#E_{l'}^1$ ) or the wildcard elimination number ( $\#E_w^1$ ) does not exist, then team  $a$  has a chance of B2 playoff tournament qualification even when the team loses all remaining games and hence its elimination number does not exist. If none of above applies, then both the intra-district championship elimination number ( $\#E_{l'}^1$ ) and the wildcard elimination number ( $\#E_w^1$ ) exist, and hence the B2 playoff tournament qualification elimination number ( $\#E_p$ ) is given by  $\#E_p = \max(\#E_{l'}^1, \#E_w^1)$ .

### 5.3 Home-ground advantage for the championship tournament (B1)

In the first round of the championship tournament, the three district champions and a team with the best record among all 2nd-place teams will have home-ground advantage, whose clinch and elimination numbers are hence calculated in the same manner as those for the B2 playoff tournament qualification.

### 5.4 Home-ground advantage for the B2 playoff tournament

The optimization models developed here are in some sense *dual* to the B2 playoff tournament qualification models in subsection 5.2. In the first round of the B2 playoff tournament, the two teams with the best records among the district champions will have home-ground advantage. In other words, if a team finishes in 1st place in its district and has a better record than all teams in a different district, then the team will have home-ground advantage. In order to calculate the clinch number for the advantage, we need to find a maximal number of future wins for each team under the condition that the team does not finish in 1st place in its district or there exists a team with a better record both in the other two districts. Then we can separately calculate and combine the intra-district championship clinch number and the clinch number for having a better record than all teams in a different district. For the former, we already have problem  $(C_{l'}^1)$ . For the latter, we solve

**Problem  $(C_w^-)$ :**

$$\begin{aligned}
& \max_{\substack{x \in X \\ \gamma^l, \delta^l, \lambda^l, \lambda_1^l, \lambda_2^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3}} \sum_{j \in L} x_{aj} \\
\text{subject to } & \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i^l \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
& \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M+1) \lambda_{1i}^l - 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
& \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M+1) \lambda_{2i}^l + 1 \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\
& \lambda_i^l + \lambda_{1i}^l + \lambda_{2i}^l = 2 \quad (\forall i \in D_l, \forall l \in \{1,2,3\})
\end{aligned}$$



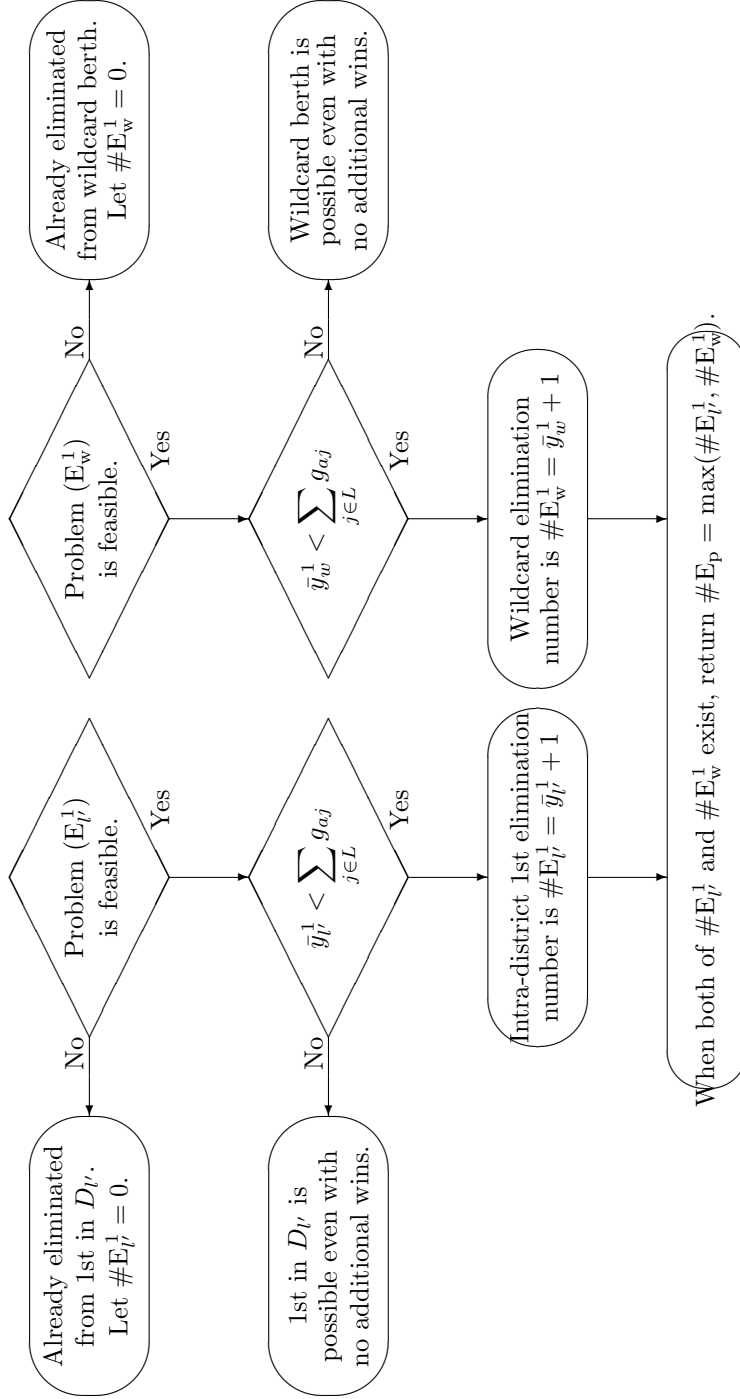


Figure 8: Finding the elimination number  $\#E_p$  for the B2 playoff tournament qualification

$$\lambda_{1i}^l \leq \gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + x_{aj})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^l)} \leq \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + x_{ij})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^l)} + \delta_i^l$$

$$(\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\lambda_i^l \leq \delta_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}), \quad \delta_a^{l'} = 1$$

$$\sum_{i \in D_l} \gamma_i^l + \sum_{i \in D_l} \delta_i^l = 2n_l - \sigma_l \quad (\forall l \in \{1, 2, 3\})$$

$$\sum_{l=1}^3 \sigma_l = 2, \quad \sigma_{l'} = 0$$

where

$$\gamma_a^{l'} = 1, \quad \lambda_a^{l'} = 0,$$

$$\gamma_i^l + \delta_i^l \geq 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

for any feasible solution. We used the notation  $(C_w^-)$  here since the problem seeks the clinch number for a *negative* wildcard spot.

With the optimal objective function values of problem  $(C_{l'}^1)$  and  $(C_w^-)$ , three clinch numbers of team  $a$  can be obtained for intra-district championship, having a better record than all teams in a different district, and home-ground advantage as in Figure 9, where  $\bar{z}_{l'}^1$  and  $\bar{z}_w^-$  respectively denote the optimal objective function values of problems  $(C_{l'}^1)$  and  $(C_w^-)$ . If both problem  $(C_{l'}^1)$  and problem  $(C_w^-)$  have no feasible solution, then team  $a$  already clinched the home-ground advantage. If either the intra-district championship clinch number  $(\#C_{l'}^1)$  or the clinch number for having a better record than all teams in a different district  $(\#C_w^-)$  does not exist, then team  $a$  has a chance of being eliminated from the home-ground advantage even when the team wins all remaining games and hence the home-ground advantage clinch number does not exist. If none of above applies, then both  $\#C_{l'}^1$  and  $\#C_w^-$  exist, and hence the home-ground advantage clinch number  $(\#C_{ph})$  is given by  $\#C_{ph} = \max(\#C_{l'}^1, \#C_w^-)$ .

For elimination from the home-ground advantage, we will find a maximal number of future losses for each team under the conditions that the team finishes in 1st place in its district and has a better record than all teams in a different district. The optimization problem to be solved is then given by

**Problem  $(E_{ph})$ :**

$$\max_{\substack{x \in X \\ \alpha, \beta \in \{0,1\}^{n_{l'}} \\ \gamma^l, \delta^l, \lambda^l, \lambda_1^l, \lambda_2^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3}} \sum_{j \in L} x_{ja}$$

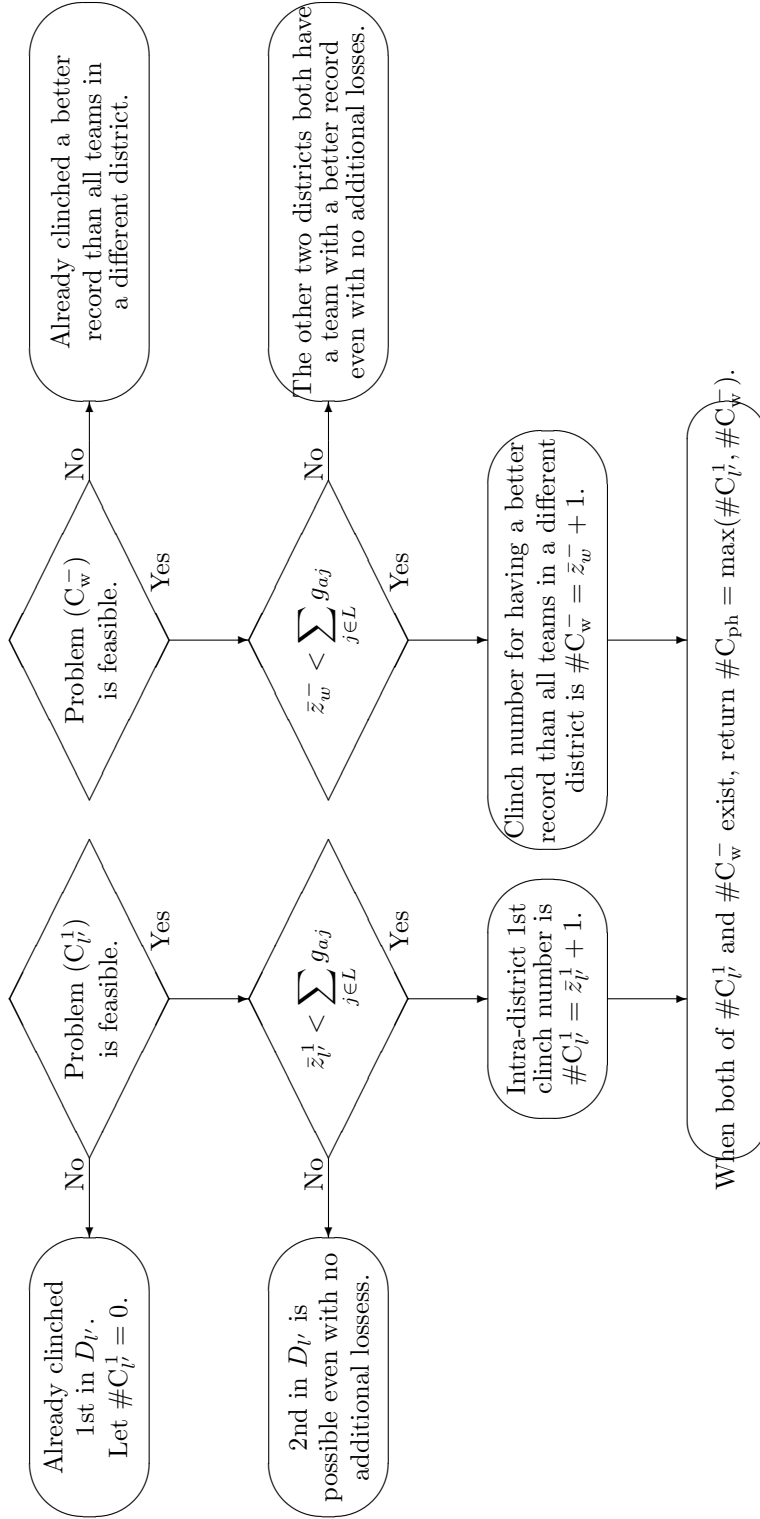


Figure 9: Finding the clinch number  $\#C_{ph}$  of home-ground advantage for the B2 playoff tournament

subject to

$$\begin{aligned}
& \left| \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \right| \leq M \lambda_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
& \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \leq (M+1) \lambda_{1i}^l - 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
& \sum_{j \in L} (w_{aj} + x_{aj}) - \sum_{j \in L} (w_{ij} + x_{ij}) \geq -(M+1) \lambda_{2i}^l + 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
& \lambda_i^l + \lambda_{1i}^l + \lambda_{2i}^l = 2 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})
\end{aligned}$$

$$\lambda_{2i}^{l'} \leq \alpha_i \quad (\forall i \in D_{l'})$$

$$\frac{\sum_{j \in D_{l'}} (w_{aj} + x_{aj})(1 - \lambda_j^{l'})}{\sum_{j \in D_{l'}} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^{l'})} \geq \frac{\sum_{j \in D_{l'}} (w_{ij} + x_{ij})(1 - \lambda_j^{l'})}{\sum_{j \in D_{l'}} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^{l'})} - \beta_i$$

( $\forall i \in D_{l'}$ )

$$\lambda_i^{l'} \leq \beta_i \quad (\forall i \in D_{l'}), \quad \beta_a = 1$$

$$\sum_{i \in D_{l'}} \alpha_i + \sum_{i \in D_{l'}} \beta_i = n_{l'} + 1$$

$$\lambda_{2i}^l \leq \gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})$$

$$\frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + x_{aj})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j^l)} \geq \frac{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + x_{ij})(1 - \lambda_j^l)}{\sum_{l=1}^3 \sum_{j \in D_l} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j^l)} - \delta_i^l$$

( $\forall i \in D_l, \forall l \in \{1, 2, 3\}$ )

$$\lambda_i^l \leq \delta_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}), \quad \delta_a^{l'} = 1$$

$$\sum_{i \in D_l} \gamma_i^l + \sum_{i \in D_l} \delta_i^l = (2 - \sigma_l) n_l \quad (\forall l \in \{1, 2, 3\})$$

$$\sum_{l=1}^3 \sigma_l = 1, \quad \sigma_{l'} = 0$$

where

$$\begin{aligned}
& \gamma_a^{l'} = 1, \quad \lambda_a^{l'} = 0, \\
& \gamma_i^l + \delta_i^l \geq 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\})
\end{aligned}$$

hold for any feasible solution. See Figure 10, where  $\bar{y}_{ph}$  denotes the optimal objective function value of problem (E<sub>ph</sub>).

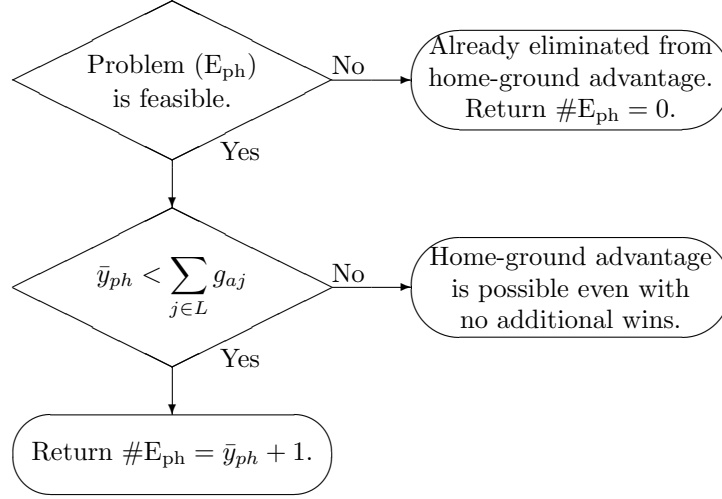


Figure 10: Finding the elimination number  $\#E_{ph}$  of home-ground advantage for the B2 playoff tournament

## 6 Upper and lower bounds

Almost all models we have constructed so far have two constraint blocks, which respectively correspond to the first and second criteria (see ① and ② in page 3). These criteria are expressed as inequalities in the constraint blocks of each problem. It should be noted here that the first constraint blocks do not *logically* include equal signs but the second constraint blocks do, which means that ties are not allowed in the first criteria but are allowed in the second criteria. The second criteria are only used in case of ties in the first criteria, which is why if equal signs are logically added to the first blocks, then the second blocks essentially disappear and the resulting problems respectively provide upper bounds to the original problems. For example, if we replace the first constraint block of the first optimization problem in subsection 3.1 by

$$\sum_{j \in L} (w_{aj} + x_{aj}) \leq \sum_{j \in L} (w_{ij} + x_{ij}) + M\alpha_i \quad (\forall i \in D_{l'}),$$

then equivalently we have

**Problem  $(\hat{C}_{l'}^k)$ :**

$$\begin{aligned} & \max_{\substack{x \in X \\ \alpha \in \{0,1\}^{n_{l'}}}} \sum_{j \in L} x_{aj} \\ & \text{subject to} \quad \sum_{j \in L} (w_{aj} + x_{aj}) \leq \sum_{j \in L} (w_{ij} + x_{ij}) + M\alpha_i \quad (\forall i \in D_{l'}) \\ & \quad \quad \quad \sum_{i \in D_{l'}} \alpha_i = n_{l'} - k \end{aligned}$$

which gives an upper bound of problem  $(C_{l'}^k)$ .

On the other hand, if we add an additional constraint that the binary variables for team selection in the second criteria must be one, then this makes the second constraint blocks invisible and the resulting problems respectively provide lower bounds to the original problems. For example, adding the constraint

$$\sum_{i \in D_{l'}} \beta_i = n_{l'},$$

namely  $\beta_i = 1$  for all  $i \in D_{l'}$ , to problem  $(C_{l'}^k)$  yields

**Problem  $(\check{C}_{l'}^k)$ :**

$$\begin{aligned} & \max_{\substack{x \in X \\ \alpha \in \{0,1\}^{n_{l'}}}} \sum_{j \in L} x_{aj} \\ \text{subject to } & \sum_{j \in L} (w_{aj} + x_{aj}) \leq \sum_{j \in L} (w_{ij} + x_{ij}) + (M+1)\alpha_i - 1 \quad (\forall i \in D_{l'}) \\ & \sum_{i \in D_{l'}} \alpha_i = n_{l'} - k \end{aligned}$$

which gives a lower bound of problem  $(C_{l'}^k)$ .

This is also the case with elimination. Upper and lower bounds of problem  $(E_{l'}^k)$  are respectively given by solving the following two problems.

**Problem  $(\hat{E}_{l'}^k)$ :**

$$\begin{aligned} & \max_{\substack{x \in X \\ \alpha \in \{0,1\}^{n_{l'}}}} \sum_{j \in L} x_{ja} \\ \text{subject to } & \sum_{j \in L} (w_{aj} + x_{aj}) \geq \sum_{j \in L} (w_{ij} + x_{ij}) - M\alpha_i \quad (\forall i \in D_{l'}) \\ & \sum_{i \in D_{l'}} \alpha_i = k \end{aligned}$$

**Problem  $(\check{E}_{l'}^k)$ :**

$$\begin{aligned} & \max_{\substack{x \in X \\ \alpha \in \{0,1\}^{n_{l'}}}} \sum_{j \in L} x_{ja} \\ \text{subject to } & \sum_{j \in L} (w_{aj} + x_{aj}) \geq \sum_{j \in L} (w_{ij} + x_{ij}) - (M+1)\alpha_i + 1 \quad (\forall i \in D_{l'}) \\ & \sum_{i \in D_{l'}} \alpha_i = k \end{aligned}$$

Every optimization problem with constraint blocks thus have an adjoint pair of problems only with the first constraint block with a nonstrict or strict inequality, from which we can get upper and lower bounds. If the problem for finding an upper bound (the outer problem) has no feasible solution, then the original problem is infeasible. Suppose that the problem for finding an lower

bound (the inner problem) has an optimal solution. If the upper and lower bounds coincide, then we do not need to solve the original problem, otherwise we can use the optimal solution of the inner problem as an initial feasible solution to the original problem.

We present here two more adjoint pairs of outer and inner problems, which will be used later for numerical experiments in subsection 7.2. The next two optimization problems ( $\hat{C}_c$ ) and ( $\check{C}_c$ ) respectively give upper and lower bounds of the optimal objective function value of problem ( $C_c$ ).

**Problem ( $\hat{C}_c$ ):**

$$\begin{aligned}
& \max_{\substack{x \in X \\ \alpha \in \{0,1\}^{n_{l'}} \\ \gamma^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3 \\ \theta \in \{0,1\}}} \sum_{j \in L} x_{aj} \\
& \text{subject to} \quad \boxed{\begin{aligned} \sum_{j \in L} (w_{aj} + x_{aj}) &\leq \sum_{j \in L} (w_{ij} + x_{ij}) + M\alpha_i \quad (\forall i \in D_{l'}) \\ \sum_{i \in D_{l'}} \alpha_i &= n_{l'} - 2 \end{aligned}} \\
& \quad \boxed{\begin{aligned} \sum_{j \in L} (w_{aj} + x_{aj}) &\leq \sum_{j \in L} (w_{ij} + x_{ij}) + M\gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1,2,3\}) \\ (1-\theta) \left( \sum_{i \in D_l} \gamma_i^l - n_l + 4\sigma_l \right) &= 0 \quad (\forall l \in \{1,2,3\}) \\ (1-\theta) \left( \sum_{l=1}^3 \sigma_l - 1 \right) &= 0 \\ \theta \left( \sum_{i \in D_l} \gamma_i^l - n_l + 3\sigma_l \right) &= 0 \quad (\forall l \in \{1,2,3\}) \\ \theta \left( \sum_{l=1}^3 \sigma_l - 2 \right) &= 0 \end{aligned}} \\
& \alpha_a = \gamma_a^{l'} = 1
\end{aligned}$$

**Problem ( $\check{C}_c$ ):**

$$\begin{aligned}
& \max_{\substack{x \in X \\ \alpha \in \{0,1\}^{n_{l'}} \\ \gamma^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3 \\ \theta \in \{0,1\}}} \sum_{j \in L} x_{aj} \\
& \text{subject to} \quad \boxed{\begin{aligned} \sum_{j \in L} (w_{aj} + x_{aj}) &\leq \sum_{j \in L} (w_{ij} + x_{ij}) + (M+1)\alpha_i - 1 \quad (\forall i \in D_{l'}) \\ \sum_{i \in D_{l'}} \alpha_i &= n_{l'} - 2 \end{aligned}}
\end{aligned}$$

$$\begin{aligned}
\sum_{j \in L} (w_{aj} + x_{aj}) &\leq \sum_{j \in L} (w_{ij} + x_{ij}) + (M+1)\gamma_i^l - 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
(1-\theta) \left( \sum_{i \in D_l} \gamma_i^l - n_l + 4\sigma_l \right) &= 0 \quad (\forall l \in \{1, 2, 3\}) \\
(1-\theta) \left( \sum_{l=1}^3 \sigma_l - 1 \right) &= 0 \\
\theta \left( \sum_{i \in D_l} \gamma_i^l - n_l + 3\sigma_l \right) &= 0 \quad (\forall l \in \{1, 2, 3\}) \\
\theta \left( \sum_{l=1}^3 \sigma_l - 2 \right) &= 0
\end{aligned}$$

The former problem ( $\hat{C}_c$ ) is exactly the same as the first problem appeared in subsection 5.1. In the latter problem ( $\check{C}_c$ ), the condition  $\alpha_a = \gamma_a^{l'} = 1$  holds for any feasible solution.

Upper and lower bounds of the optimal objective function value of problem ( $E_w^2$ ) are respectively given by solving the following two problems ( $\hat{E}_w^2$ ) and ( $\check{E}_w^2$ ).

**Problem ( $\hat{E}_w^2$ ):**

$$\begin{aligned}
&\max_{\substack{x \in X \\ \gamma^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3 \\ \theta \in \{0,1\}}} \sum_{j \in L} x_{ja} \\
\text{subject to } &\sum_{j \in L} (w_{aj} + x_{aj}) \geq \sum_{j \in L} (w_{ij} + x_{ij}) - M\gamma_i^l \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
&(1-\theta) \left( \sum_{i \in D_l} \gamma_i^l - 2 - 2\sigma_l \right) = 0 \quad (\forall l \in \{1, 2, 3\}) \\
&(1-\theta) \left( \sum_{l=1}^3 \sigma_l - 1 \right) = 0 \\
&\theta \left( \sum_{i \in D_l} \gamma_i^l - 2 - \sigma_l \right) = 0 \quad (\forall l \in \{1, 2, 3\}) \\
&\theta \left( \sum_{l=1}^3 \sigma_l - 2 \right) = 0 \\
&\gamma_a^{l'} = 1
\end{aligned}$$

**Problem ( $\check{E}_w^2$ ):**

$$\max_{\substack{x \in X \\ \gamma^l \in \{0,1\}^{n_l}, l \in \{1,2,3\} \\ \sigma \in \{0,1\}^3 \\ \theta \in \{0,1\}}} \sum_{j \in L} x_{ja}$$



$$\begin{aligned}
\text{subject to } & \sum_{j \in L} (w_{aj} + x_{aj}) \geq \sum_{j \in L} (w_{ij} + x_{ij}) - (M+1)\gamma_i^l + 1 \quad (\forall i \in D_l, \forall l \in \{1, 2, 3\}) \\
& (1 - \theta) \left( \sum_{i \in D_l} \gamma_i^l - 2 - 2\sigma_l \right) = 0 \quad (\forall l \in \{1, 2, 3\}) \\
& (1 - \theta) \left( \sum_{l=1}^3 \sigma_l - 1 \right) = 0 \\
& \theta \left( \sum_{i \in D_l} \gamma_i^l - 2 - \sigma_l \right) = 0 \quad (\forall l \in \{1, 2, 3\}) \\
& \theta \left( \sum_{l=1}^3 \sigma_l - 2 \right) = 0
\end{aligned}$$

In the latter problem ( $\check{E}_w^2$ ), the condition  $\gamma_a^{l'} = 1$  holds for any feasible solution.

The optimization problems that have multiple constraint blocks with tiebreaking criteria must be solved separately for each team, but some of their adjoints can be easily solved as a single optimization problem of finding a threshold (see Adler et al. [1]). For example, instead of solving problem  $(\hat{C}_{l'}^k)$  for all teams in a district, we can consider the max-min problem

$$\max_{\substack{x \in X \\ \kappa \in \{0,1\}^{n_{l'}}}} \min_{i \in D_{l'}} \sum_{j \in L} (w_{ij} + x_{ij}) + M\kappa_i$$

or equivalently

$$\begin{aligned}
& \max_{\substack{x \in X \\ \kappa \in \{0,1\}^{n_{l'}} \\ u \in \mathbb{Z}}} u \\
\text{subject to } & u \leq \sum_{j \in L} (w_{ij} + x_{ij}) + M\kappa_i \quad (\forall i \in D_{l'}) \\
& \sum_{i \in D_{l'}} \kappa_i = n_{l'} - k - 1
\end{aligned}$$

which is a problem of finding a maximal number of total wins that can be reached by a  $(k+1)$ -th place team at the season end. With this threshold, the current number of wins and the number of remaining games, one can readily find the optimal objective value of problem  $(\hat{C}_{l'}^k)$  for each team if it exists. For problem  $(\hat{E}_{l'}^k)$ , we can solve the min-max problem

$$\min_{\substack{x \in X \\ \kappa \in \{0,1\}^{n_{l'}}}} \max_{i \in D_{l'}} \sum_{j \in L} (w_{ij} + x_{ij}) - M\kappa_i$$

or

$$\begin{aligned}
& \min_{\substack{x \in X \\ \kappa \in \{0,1\}^{n_{l'}} \\ v \in \mathbb{Z}}} v \\
\text{subject to } & v \geq \sum_{j \in L} (w_{ij} + x_{ij}) - M\kappa_i \quad (\forall i \in D_{l'})
\end{aligned}$$

$$\sum_{i \in D_{l'}} \kappa_i = k - 1$$

instead in order to find a minimal number of total wins that can be attained by a  $k$ -th place team at the season end. The situation is also the same with the inner problems. When  $k = 1$  for clinch calculations or when  $k = n_{l'} - 1$  for elimination calculations, we can determine the upper and lower bounds via simple arithmetic calculations without resorting to optimization (see also [1]).

We do not go further into details here about thresholds (including those for wildcard spots) since solving outer and inner problems itself is not relatively time-consuming at least in the latter half of the season as we see later in our experiments. However, utilizing some thresholds may help if one starts the calculation at an earlier stage of the season.

## 7 Computational results

B.LEAGUE's regular season starts in late September or early October and ends in early May. Using the data of division B1 during the last 50 game days (starting January 1) of the 2016–2017 season, we did several computational experiments. Some of those are presented here in this section. All experiments were performed with SCIP 6.0.1 as well as SoPlex 4.0.1, CppAD 20180000.0, ZLIB 1.2.11, GMP 6.1.2 and ZIMPL 3.3.6 on an Intel Core i7 2.8 GHz processor (4 cores, 16 GB memory). A Python script was written to read data, to generate intermediate ZIMPL files for the optimization models, and to perform algorithms to determine the clinch and the elimination numbers based on the results of SCIP calculations of those optimization models.

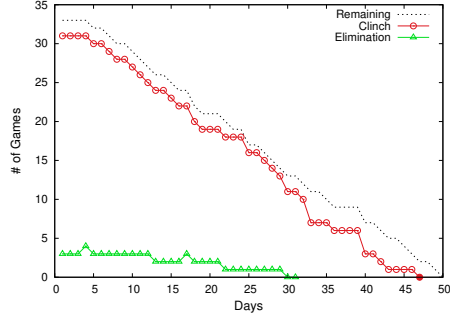
### 7.1 Intra-district championship

We start with the calculation of clinch and elimination numbers for intra-district championship in division B1. Calculated clinch and elimination numbers in three districts are separately shown in Figures 11–13. The six teams in each district are displayed in the order of final standing; namely, the team displayed in subfigure (a) was the district champion and the team in (f) was the tailender of the district in the 2016–2017 season. Each subfigure shows clinch and/or elimination numbers of the team, if exists, with circular or triangular dots. It should be noted that the elimination number is expressed here as a maximal number of future wins needed to be eliminated from the intra-district championship, which is defined as

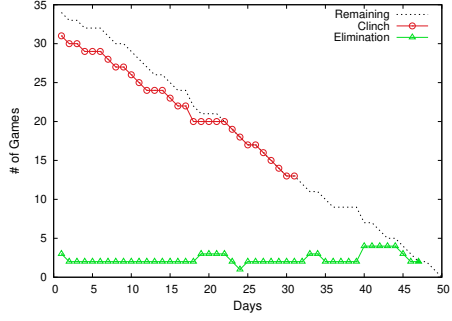
$$\sum_{j \in L} g_{aj} - \#E_{l'}^1,$$

in order to compare with the clinch number in terms of wins (see also the explanation in page 9). Every sequence of elimination numbers is nonincreasing in terms of losses but may not be so in terms of wins. The dotted line denotes the number of remaining games for the team, which looks similar but is different for each team. A team clinches when the circular dot intersects the Days axis, and is eliminated when the triangular dot touches the dotted line. Filled circular or triangular dot respectively denotes that the team clinched or is eliminated from intra-district championship. Once a team clinch or is eliminated, the following filled dots in its subfigure are omitted for visibility.

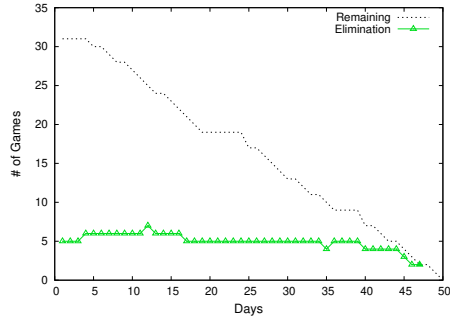
The initial values of clinch and elimination numbers depend on initial allocation of games (see section 1), but the clinch and elimination numbers for intra-district championship respectively start from 58 wins and 43 losses (17 wins) at the beginning of the 2016–2017 season. Since we only used



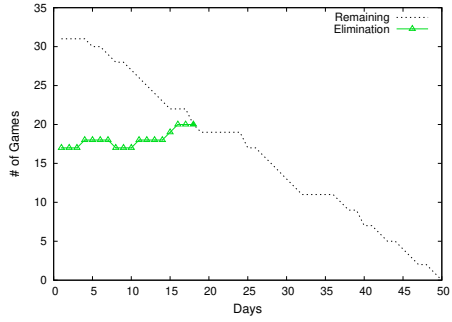
(a) Tochigi Brex



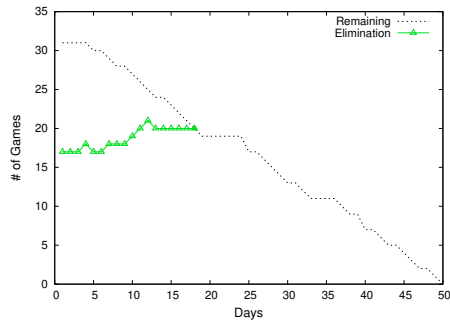
(b) Alvark Tokyo



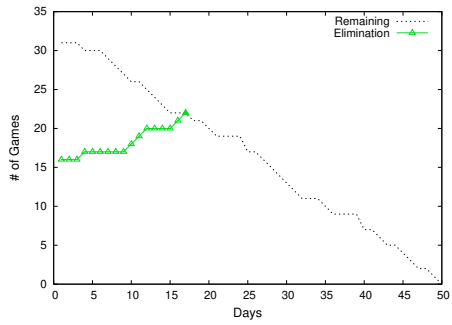
(c) Chiba Jets Funabashi



(d) Levanga Hokkaido

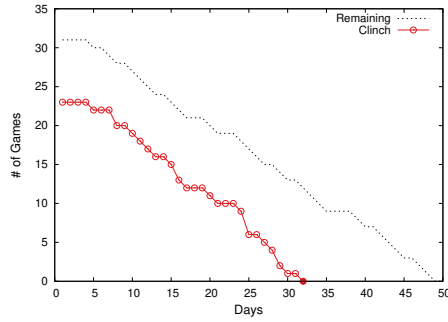


(e) Akita Northern Happinets

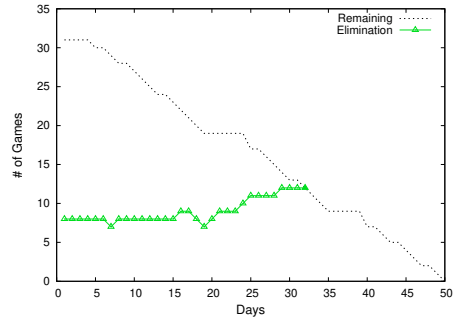


(f) Sendai 89ers

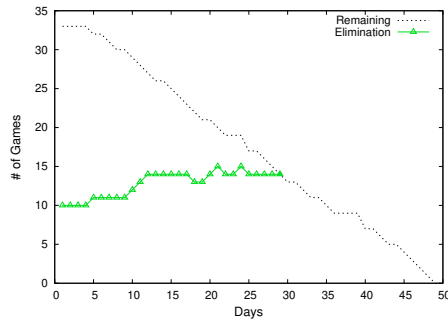
Figure 11: Intra-district championship clinch and elimination numbers (Division B1, East District)



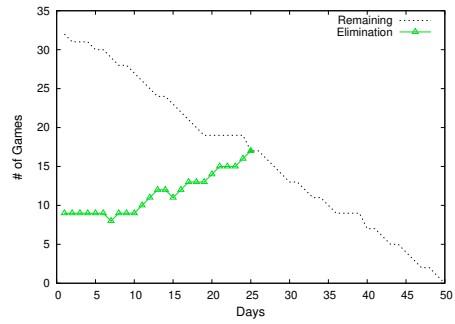
(a) Kawasaki Brave Thunders



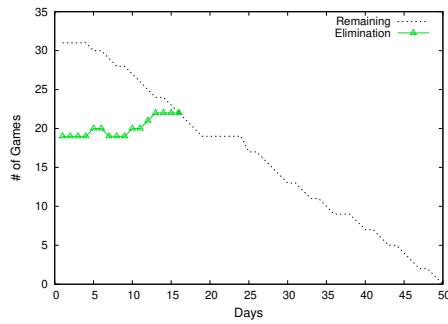
(b) San-en Neophoenix



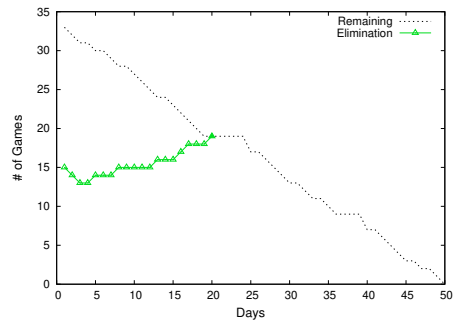
(c) Sunrockers Shibuya



(d) Niigata Albirex BB

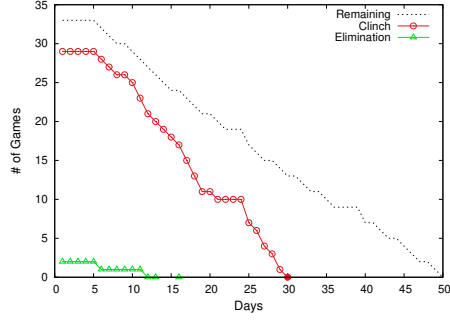


(e) Toyama Grouses

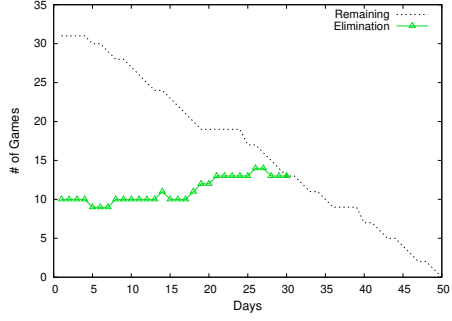


(f) Yokohama B-Corsairs

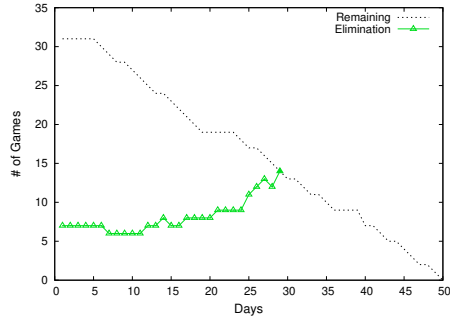
Figure 12: Intra-district championship clinch and elimination numbers (Division B1, Middle District)



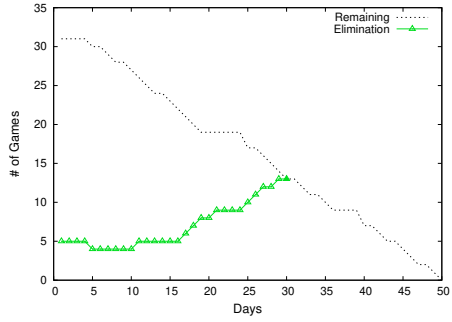
(a) Seahorses Mikawa



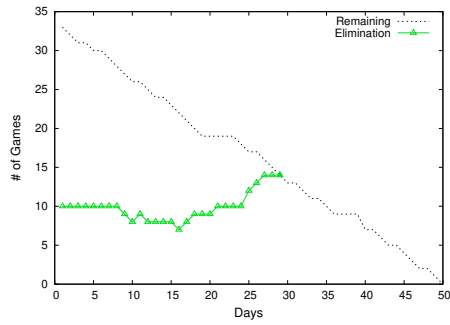
(b) Ryukyu Golden Kings



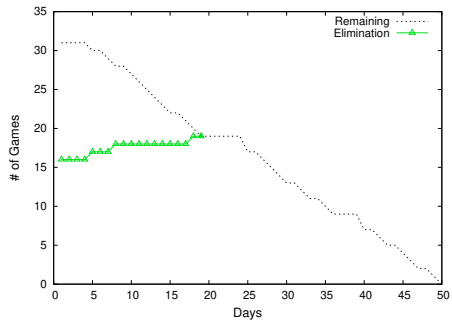
(c) Osaka Evessa



(d) Nagoya Diamond Dolphins



(e) Kyoto Hannarys



(f) Shiga Lakestars

Figure 13: Intra-district championship clinch and elimination numbers (Division B1, West District)

the last four months data, some teams do not have clinch or elimination number at day one. We can see from Figure 11 that Alvark Tokyo and Chiba Jets Funabashi were both eliminated on the day when Tochigi Brex clinched the championship in the East district. There were keen competition for the 2nd place in all districts though it is not perfectly clear until the 2nd place clinch and elimination numbers are calculated.

The computation times for the intra-district championship clinch and elimination numbers are respectively shown as box plots of 18 instances on each day in Figures 14 and 15. The ends of the whiskers represent the shortest time within 1.5 IQR (interquartile range) of the lower quartile and the longest time within 1.5 IQR of the upper quartile, and any data outside the whiskers are plotted as an outlier with a dot. Computation finished within a second in all instances. Calculation of elimination numbers needs more time than that of clinch numbers since  $k$  is small. When  $k$  is 5 instead, the situation is reversed.

## 7.2 Championship tournament qualification

Let us next check the computational results for championship tournament qualification in division B1. Calculated clinch and elimination numbers are similarly shown in Figures 16–18. Since two to four teams may advance to the championship tournament from each district, the appearance of circular dots dominates that of triangular dots when compared to Figures 11–13 for intra-district championship.

It is not easy to hand-calculate the initial values of the clinch and elimination number sequences in this case, but the clinch number for championship tournament qualification starts from 48, while the intra-district 2nd place clinch number starts from 55. For elimination, the wildcard elimination starts from 43, the intra-district 2nd place elimination starts from 46, and hence the latter gives the whole elimination number (14 in terms of wins) for championship tournament qualification. Chiba Jets Funabashi and Sunrockers Shibuya finished the season in two wildcard spots respectively from the East and Middle districts. These two teams clinched the qualification respectively on days 33 and 46, but at these moments, they still had chances to be the 2nd place in their districts. Indeed, except for Ryukyu Golden Kings, all teams that finally advanced to the championship tournament clinched the qualification as a wildcard. In the East district, Alvark Tokyo finally defeated Chiba Jets Funabashi with the second criterion. In the Middle and West districts, San-en Neophoenix and Ryukyu Golden Kings won the 2nd place by one point.

The computation times for championship tournament qualification clinch and elimination numbers are shown again as box plots in Figures 19 and 20. Subfigures (a) show the results of solving  $(C_c)$  or  $(E_w^2)$  from scratch, while subfigures (b) show the results with using upper and lower bounds of  $(C_c)$  or  $(E_w^2)$ . Calculation of elimination numbers is again several times more difficult than clinch numbers. Exploiting upper and lower bounds seems to be effective in shortening the computation time of extremely time-consuming cases and hence in shortening the average computation time. This is more visible in elimination number calculations.

For clinch numbers, in 549 out of 900 instances, either problem  $(\hat{C}_c)$  is infeasible or the upper and lower bounds of problem  $(C_c)$  coincide, and hence the problem  $(C_c)$  itself did not need to be solved. Each calculation did not take a second except one instance. In the remaining 351 instances, problem  $(C_c)$  was actually solved using the upper and lower bounds. For elimination numbers, exactly in a half of 900 instances, either problem  $(\hat{E}_w^2)$  is infeasible or the two bounds of problem  $(E_w^2)$  coincide, and hence the problem  $(E_w^2)$  itself did not need to be solved. Each calculation finished within a second except three instances. Problem  $(E_w^2)$  was actually solved

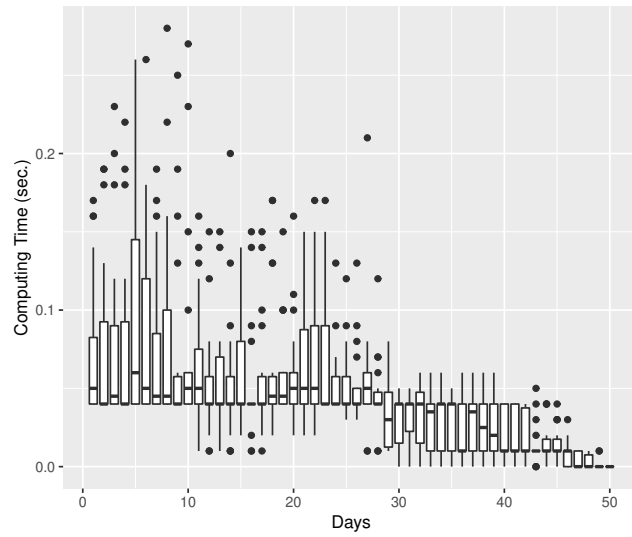


Figure 14: Computation time for intra-district championship clinch numbers (Division B1)

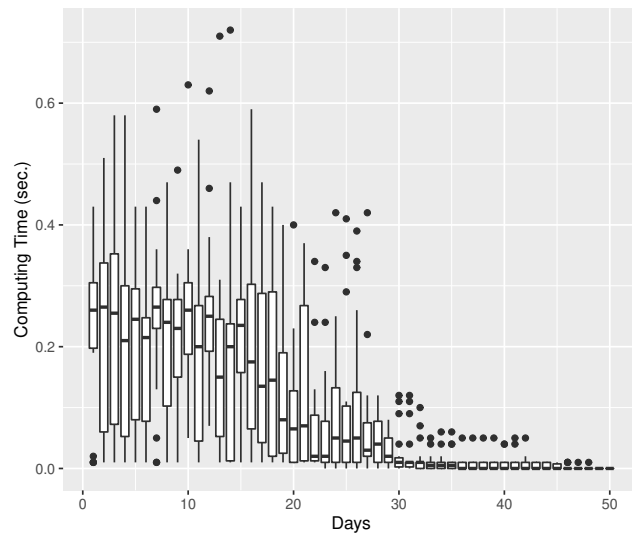
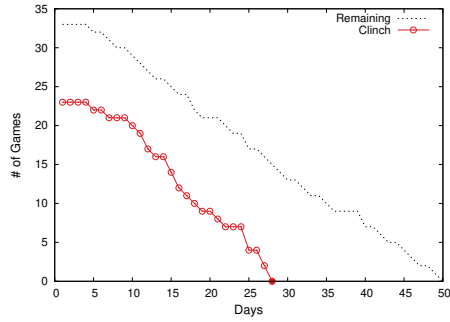
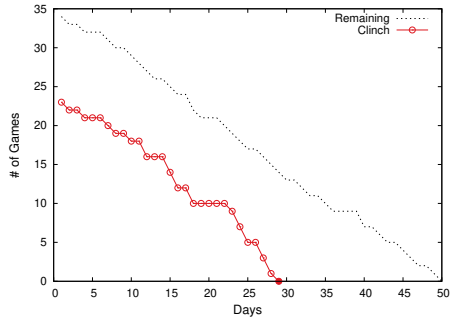


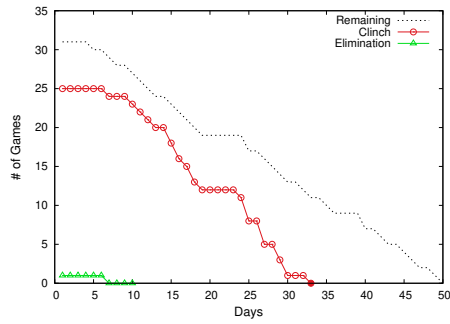
Figure 15: Computation time for intra-district championship elimination numbers (Division B1)



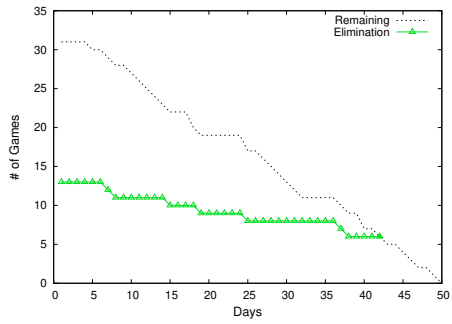
(a) Tochigi Brex



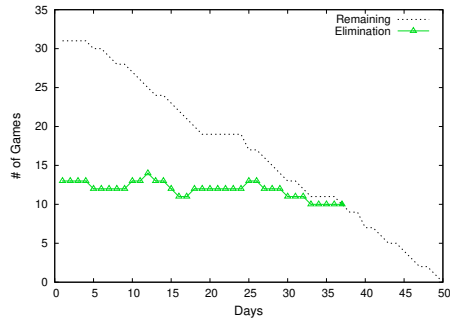
(b) Alvark Tokyo



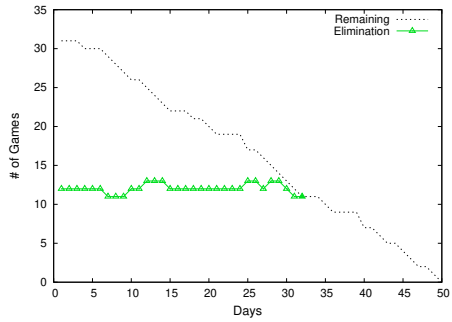
(c) Chiba Jets Funabashi



(d) Levanga Hokkaido



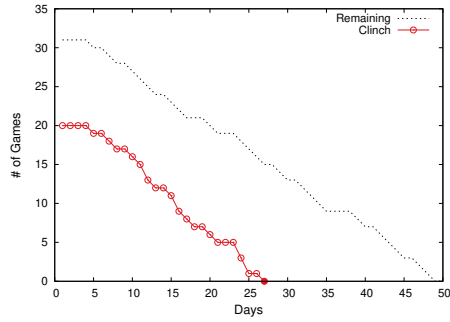
(e) Akita Northern Happinets



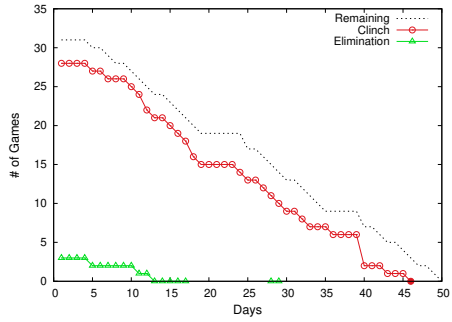
(f) Sendai 89ers

Figure 16: Championship tournament qualification clinch and elimination numbers (Division B1, East District)

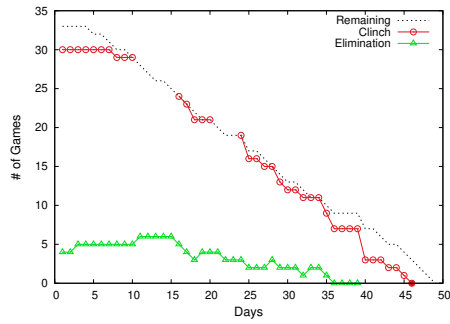




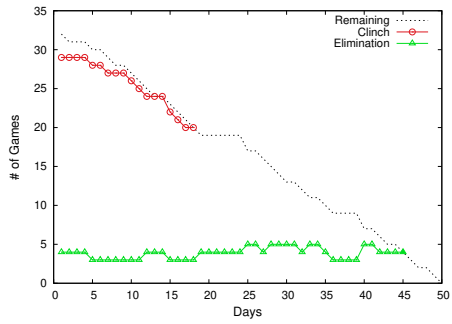
(a) Kawasaki Brave Thunders



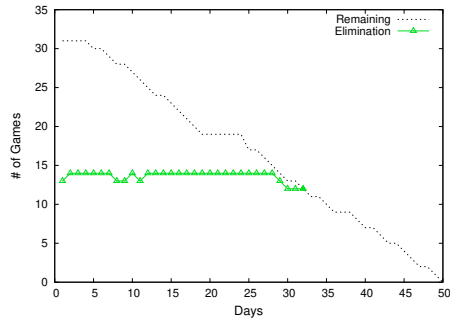
(b) San-en Neophoenix



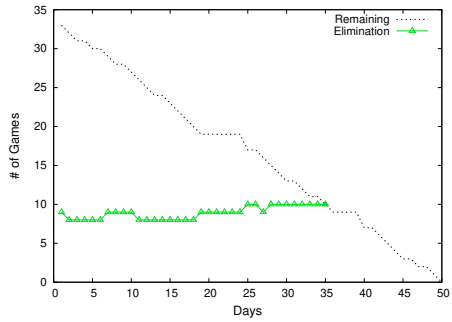
(c) Sunrockers Shibuya



(d) Niigata Albirex BB

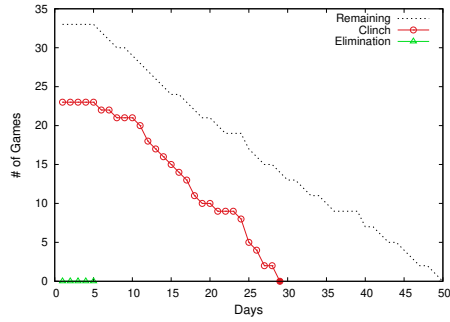


(e) Toyama Grouses

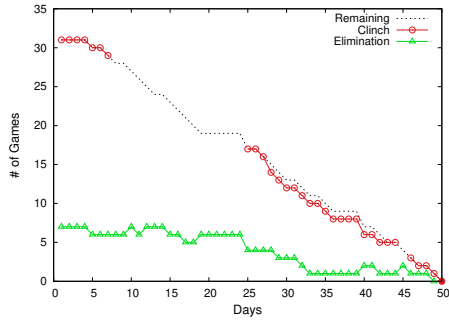


(f) Yokohama B-Corsairs

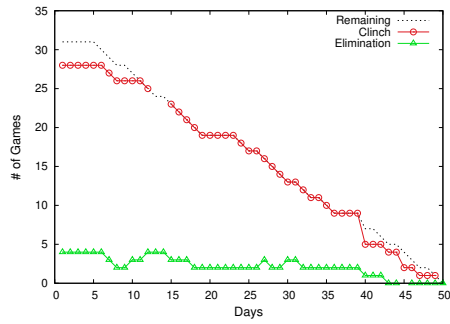
Figure 17: Championship tournament qualification clinch and elimination numbers (Division B1, Middle District)



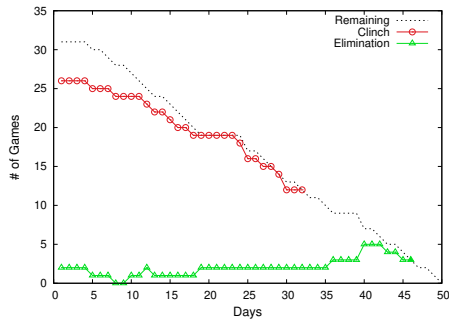
(a) Seahorses Mikawa



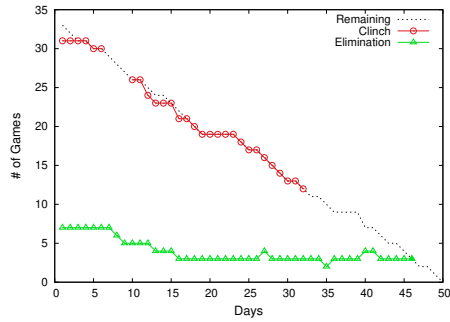
(b) Ryukyu Golden Kings



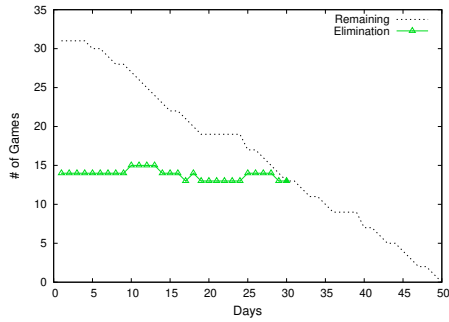
(c) Osaka Evessa



(d) Nagoya Diamond Dolphins



(e) Kyoto Hannarys



(f) Shiga Lakestars

Figure 18: Championship tournament qualification clinch and elimination numbers (Division B1, West District)

using the bounds for the other half. Figures 21 and 22 respectively show the computing time in each case for clinch and elimination number calculations.

## 8 Concluding remarks

We have formulated several optimization models for calculating clinch and elimination numbers especially in the presence of predefined tiebreaking rules without playing additional tiebreakers. Many sports leagues have tiebreaking criteria that are not based on simple win-loss records, and it is then natural not to take such criteria into consideration in general. Indeed, we defined the clinch and elimination numbers without considering the scoring differential that appears in the third criterion for the Japanese professional men's basketball league. However, at the very last stage of the season, it may become necessary to take this into consideration. In that case, we can consider another lower bound of each problem. For example, when we replace the inequality for the second tiebreaking criterion in the second block of problem  $(C_{l'}^k)$  by a logically strict inequality

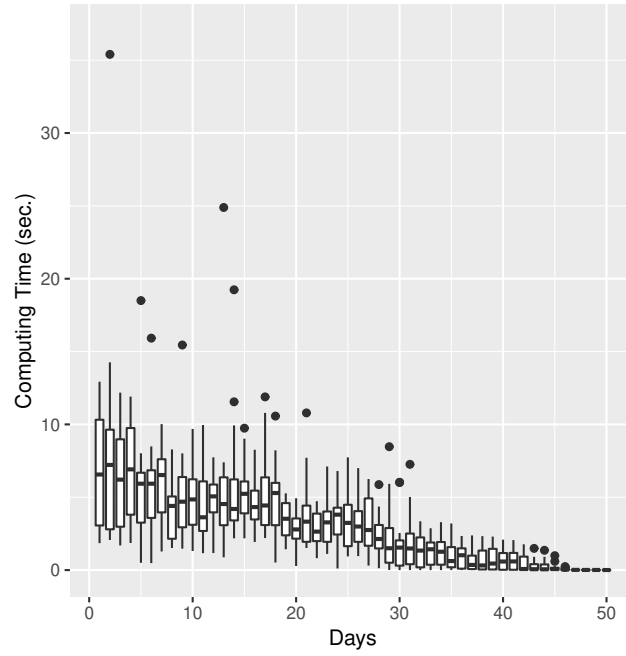
$$\frac{\sum_{j \in D_{l'}} (w_{aj} + x_{aj})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{aj} + w_{ja} + g_{aj})(1 - \lambda_j)} \leq \frac{\sum_{j \in D_{l'}} (w_{ij} + x_{ij})(1 - \lambda_j)}{\sum_{j \in D_{l'}} (w_{ij} + w_{ji} + g_{ij})(1 - \lambda_j)} + \frac{2(M-1)M+1}{2(M-1)M} \beta_i - \frac{1}{2(M-1)M} \quad (\forall i \in D_{l'})$$

we have a tighter lower bound than problem  $(\check{C}_{l'}^k)$  by solving the resulting problem. If the optimal objective function value of this problem is strictly less than that of problem  $(C_{l'}^k)$ , then there is a chance of clinching the intra-district  $k$ -th place with a less number of additional wins by the third or later criterion. This is also the case with elimination and other models.

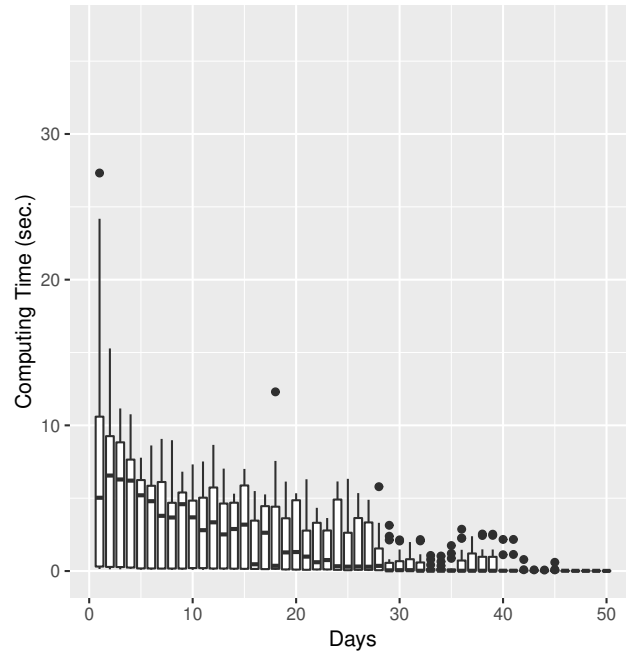
Suppose in general that first  $m$  out of multiple tiebreaking criteria depend only on win-loss records in some sports league. Then we can construct optimization models with  $m$  constraint blocks corresponding to the  $m$  criteria. The first  $(m-1)$  criteria are given as logically strict inequalities in their respective blocks, and only the  $m$ -th criterion is expressed as a nonstrict inequality in the  $m$ -th block. Replacing each of the first  $(m-1)$  criteria by a nonstrict inequality, we have  $(m-1)$  outer problems with inclusive relations. By adding an additional constraint that the binary variable for team selection must be one backward step by step from the  $m$ -th block to the second block, namely by removing the last several from the  $m$  blocks, we have  $(m-1)$  inner problems with inclusive relations. See Figure 23. The outermost outer problem and the innermost inner problem only have one constraint block and are easiest to solve in general, while the innermost outer problem and the outermost inner problem still have  $(m-1)$  constraint blocks and may be difficult to solve, but these problems provide us with the tightest upper and lower bounds. One can utilize this multilayered structure in order to solve the original optimization problem faster.

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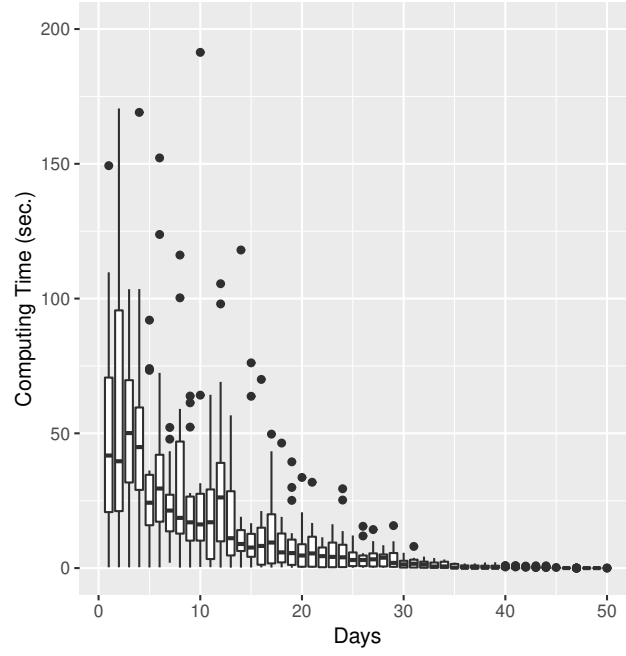


(a) Results from scratch

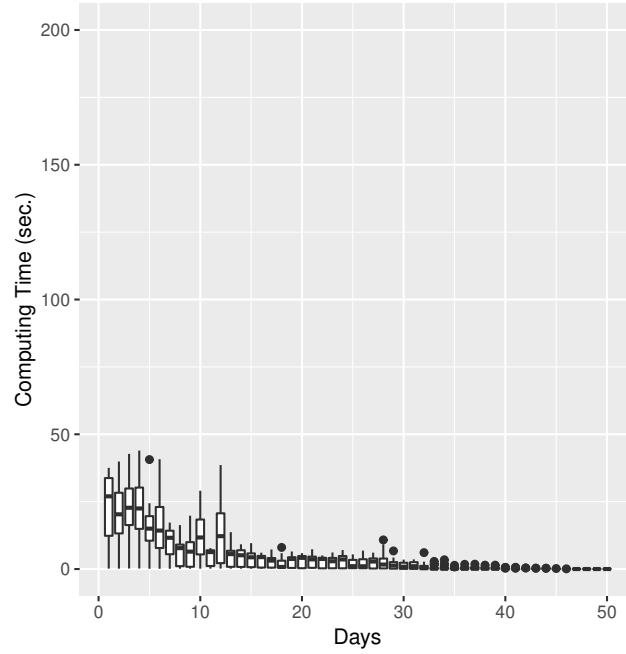


(b) Results with using upper and lower bounds

Figure 19: Computation time for championship tournament qualification clinch numbers (Division B1)

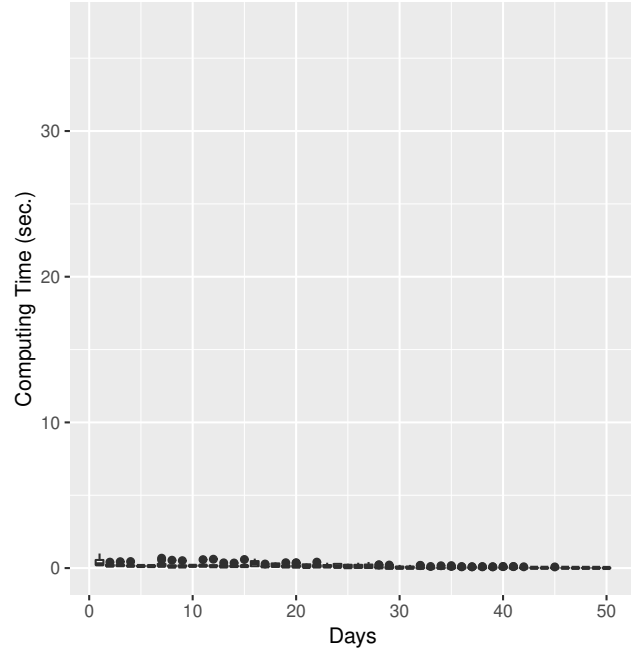


(a) Results from scratch

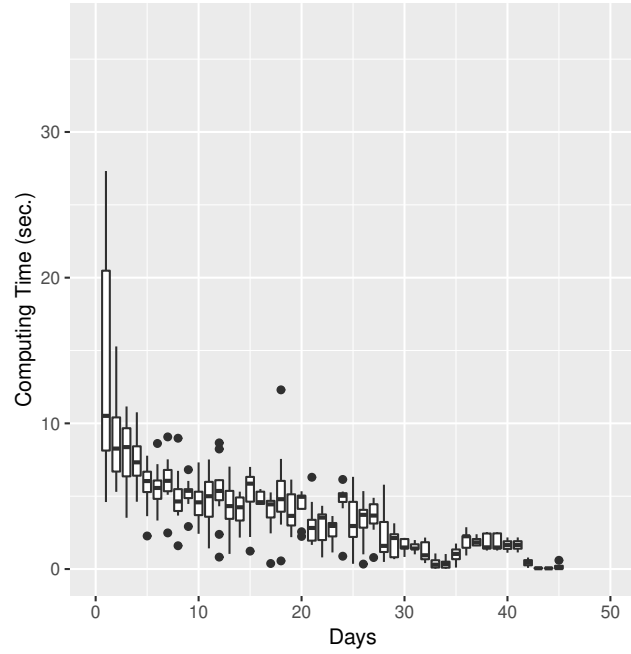


(b) Results with using upper and lower bounds

Figure 20: Computation time for championship tournament qualification elimination numbers (Division B1)

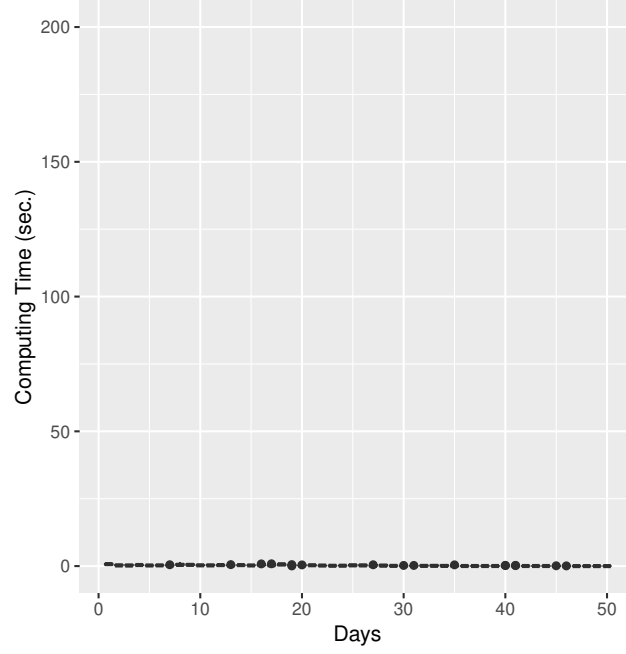


(a) Results for the case without solving problem ( $C_c$ )

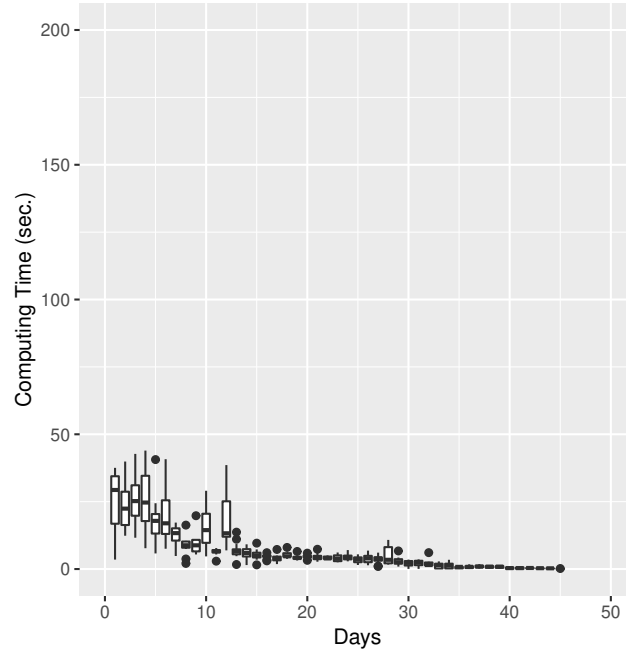


(b) Results for the case with solving problem ( $C_c$ )

Figure 21: Two cases for championship tournament qualification clinch number computations



(a) Results for the case without solving problem ( $E_w^2$ )



(b) Results for the case with solving problem ( $E_w^2$ )

Figure 22: Two cases for championship tournament qualification elimination number computations

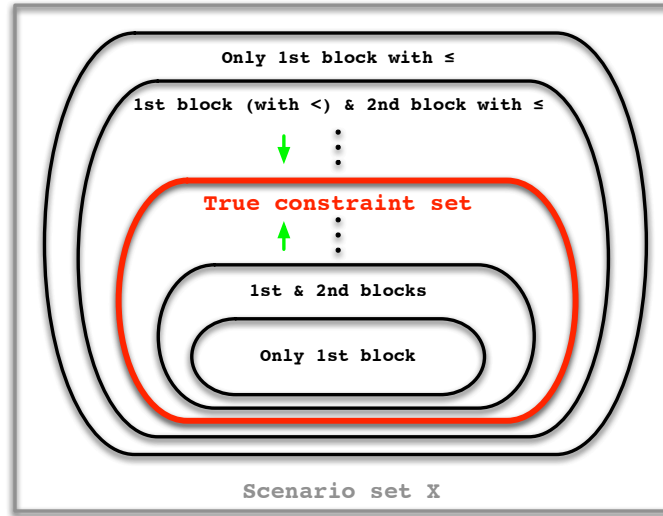


Figure 23: Multilayered structure of outer and inner problems

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