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under Preemptive Resume and
Impatience of the Prioritized Customers**

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Abstract

The paper deals with the two-class priority $M/M/1$ system, where the prioritized class-1 customers are served under FCFS preemptive resume discipline and may become impatient during their waiting for service with generally distributed maximal waiting times but finite expectation. The class-2 customers have no impatience. The required mean service times may depend on the class of the customer. As the dynamics of class-1 customers are related to the well analyzed $M/M/1 + GI$ system, our aim is to derive characteristics for class-2 customers and for the whole system. The solution of the balance equations for the partial probability generating functions of the detailed system state process is given in terms of the weak solution of a family of boundary value problems for ordinary differential equations. By means of this solution formulae for the joint occupancy distribution and for the sojourn and waiting times of class-2 customers are derived generalizing results recently obtained by Choi et al. in case of deterministic maximal waiting times. For deterministic maximal waiting times partially new explicit formulae are given.

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1 Introduction and model description

In this paper we consider the two-class priority $M/M/1$ system, where the customers of higher priority are served under preemptive resume but where

additionally they may become impatient during their waiting for service. More precisely, at a single server with a waiting room of infinite capacity there arrive two Poisson streams of customers (classes of customers) with positive intensity λ_i , $i \in \{1, 2\}$, which require exponential service times with mean $1/\mu_i$, $i \in \{1, 2\}$, respectively. The class-1 customers are served with a preemptive priority discipline over the class-2 customers, i.e., an arriving class-1 customer enters service immediately if a class-2 customer is in service, the preempted service of the class-2 customer is resumed from that time instant where no class-1 customers are present in the system, cf. e.g. [GH]. For both classes we have a FCFS queuing discipline. The class-1 customers waiting in the queue for service may become impatient according to the following mechanism: each class-1 customer arriving at the system has a random maximal waiting time I . If the offered waiting time, i.e., the time which a class-1 customer would have to wait for accessing the server if it were sufficiently patient, exceeds I , then the customer departs from the system after having waited time I . The maximal waiting times are assumed to be i.i.d. with a general distribution $C(u) := P(I \leq u)$, $u \in \mathbb{R}_+$, where its expectation is finite.

In view of the preemptive discipline, the dynamics of class-1 customers are not affected by class-2 customers and hence correspond to the dynamics of a $M/M/1$ system with impatient customers, denoted by $M/M/1 + GI$, which is well investigated by several authors, partly as special cases of more general models and/or particular $C(u)$, cf. [BBH], [BH], [BKL], [Ba1], [Ba2], [BB1], [BB3], [Br1], [Br2], [Dal], [GKö], [GKo], [HS], [Ju1], [KBL], [Mov], [Pal] and the references therein. In contrast, the dynamics of class-2 customers are extremely influenced by class-1 customers. They can be served only if no class-1 customers are in the system. The model is of interest in the framework of telecommunication models if one thinks of class-1 customers as time critical jobs which get lost or are routed to another system if they have to wait too long for service and of class-2 customers as less time critical jobs and if both classes of jobs are served by the same processor. Besides the references given above for special cases, there are many papers dealing with priority models and impatience mechanisms, cf. [GH], [Mil], [Jai] and the references given in [GKö]. However, the most relevant paper to ours seems to be [CKC], where – besides other results – in case of deterministic maximal waiting times the stability condition, the probability generating function (PGF) of the occupancy distribution and the LST of the sojourn time of class-2 customers are given. The aim of our paper is to derive formulae for performance characteristics in case of generally distributed maximal waiting times with finite expectation. It turns out that some of them are

also new in the special case of deterministic maximal waiting times.

In the literature, there are several other mechanisms where customers leave the system due to impatience: If a customer can calculate its prospective waiting time at its arrival instant then it leaves immediately if this time exceeds its maximal waiting time. This strategy yields a better utilization of the waiting places because they will not be occupied by customers which later abandon due to impatience. Also, impatience may act on the sojourn times. In this case not all work is useful because a customer may leave the system due to impatience during its service. For references and other more general models with impatience mechanisms we refer to [CKC], [BBH], [BH], [Ju2], [Sin], [Teg], [BB2] and the references therein.

The paper is organized as follows. In Section 2 firstly a system of balance equations is derived for the density of the stationary vector process of the numbers of class-1 and class-2 customers and of the residual and original maximal waiting times of the class-1 customers waiting for service in the system. The state description is analogously to that in [BB1] where the $M(n)/M(n)/s + GI$ system is analyzed. However, in the present paper we use another mathematical technique. The idea is to derive from the balance equations a system of equations for the partial PGF's of the vector process, which can be considered as a generalization of the approach given in [CKC] in the special case of $GI = D$. Then an explicit solution for these equations in terms of the weak solution of a family of boundary value problems for ordinary differential equations is derived. Unfortunately, for the latter only existence and uniqueness of the solution is shown, an explicit expression does not exist in the general case. By means of the partial PGF's, in Section 3 a formula for the joint occupancy distribution is derived. In Section 4 expressions for the LST's of the sojourn and waiting time distribution of class-2 customers are given, where for the latter the LST of the busy period distribution for the associated $M/M/1 + GI$ system is needed, which seems to be known only for $GI = D$. As an application, in Section 5 we specialize our general formulae to $GI = D$ providing partially new results for this special case.

2 A system of balance equations and its solution

If n class-1 customers are in the system then $\ell := (n - 1)_+$ of them are waiting for service in the queue. The notation $\ell := (n - 1)_+$ will be used also in the following. We number the waiting class-1 customers according to their positions in the queue. By the FCFS preemptive resume discipline

the first class-1 customer waiting in the queue will be potentially the next for service. In this section we assume that the system is stable (the stability condition will be given later), that $E I$ is finite and, if not stated otherwise, that $C(u)$ has a continuous density $c(u)$. Let

- $\rho_i := \lambda_i/\mu_i$ – offered load of class- i customers, $i \in \{1, 2\}$,
- $N_i(t)$ – number of class- i customers in the system at time t , $i \in \{1, 2\}$,
- $L(t) := (N_1(t) - 1)_+$ – number of waiting class-1 customers at time t ,
- $(X_1(t), \dots, X_{L(t)}(t))$ – vector of the residual maximal waiting times of waiting class-1 customers ordered according to their positions in the queue at time t ,
- $(I_1(t), \dots, I_{L(t)}(t))$ – vector of the original maximal waiting times of waiting class-1 customers ordered according to their positions in the queue at time t .

For the stationary distribution of the vector process

$$(N_1(t), N_2(t); X_1(t), \dots, X_{L(t)}(t); I_1(t), \dots, I_{L(t)}(t)), \quad t \in \mathbb{R},$$

we want to solve the corresponding balance equations. Let

$$P(n, m; x_1, \dots, x_\ell; u_1, \dots, u_\ell) := P(N_1(t) = n, N_2(t) = m; \\ X_1(t) \leq x_1, \dots, X_\ell(t) \leq x_\ell; I_1(t) \leq u_1, \dots, I_\ell(t) \leq u_\ell) \\ \text{– stationary distribution of the state process on} \\ \{(N_1(t), N_2(t)) = (n, m)\},$$

$$p(n, m) := P(N_1(t) = n, N_2(t) = m) \\ \text{– stationary distribution of the vector of the} \\ \text{numbers of customers in the system.}$$

Then

$$P(n; x_1, \dots, x_\ell; u_1, \dots, u_\ell) := \sum_{m=0}^{\infty} P(n, m; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \\ = P(N_1(t) = n; X_1(t) \leq x_1, \dots, X_\ell(t) \leq x_\ell; I_1(t) \leq u_1, \dots, I_\ell(t) \leq u_\ell)$$

is the stationary distribution of $(X_1(t), \dots, X_{L(t)}(t); I_1(t), \dots, I_{L(t)}(t))$ on $\{N_1(t) = n\}$ and

$$p(n) := \sum_{m=0}^{\infty} p(n, m) = P(N_1(t) = n)$$

is the stationary distribution of the number of class-1 customers in the system. Obviously, for fixed $n \in \mathbb{N} \setminus \{1\}$, where $\mathbb{N} := \mathbb{Z}_+ \setminus \{0\}$, and $m \in \mathbb{Z}_+$ the support of $P(n, m; x_1, \dots, x_\ell; u_1, \dots, u_\ell)$ is contained in

$$\Omega_\ell := \{(x_1, \dots, x_\ell; u_1, \dots, u_\ell) \in \mathbb{R}_+^{2\ell} : u_1 - x_1 \geq \dots \geq u_\ell - x_\ell \geq 0\}. \quad (2.1)$$

In view of the assumptions on $C(u)$, the densities

$$\begin{aligned} & p(n, m; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \\ & := \frac{\partial^{2\ell}}{\partial x_1 \dots \partial x_\ell \partial u_1 \dots \partial u_\ell} P(n, m; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \end{aligned} \quad (2.2)$$

and

$$p(n; x_1, \dots, x_\ell; u_1, \dots, u_\ell) := \sum_{m=0}^{\infty} p(n, m; x_1, \dots, x_\ell; u_1, \dots, u_\ell)$$

are continuous on Ω_ℓ .

Since the dynamics of the class-1 customers correspond to those of a $M/M/1 + GI$ system with parameters $\lambda_1, \mu_1, C(u)$, cf. Section 1, for the marginal system of class-1 customers we obtain immediately explicit results by specializing results for the general $M(n)/M(n)/s + GI$ system given in [BB1] to the case $s := 1$. (For results concerning $M(n)/M/s + GI$ see [Mov].) From [BB1] equations (2.17), (3.2), (3.3) by choosing $s := 1, \lambda_n := \lambda_1, n \in \mathbb{Z}_+, \mu_n := \mu_1, n \in \mathbb{N}$, for $n \in \mathbb{N} \setminus \{1\}$ we find

$$\begin{aligned} & p(n; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \\ & = \mathbb{I}\{(x_1, \dots, x_\ell; u_1, \dots, u_\ell) \in \Omega_\ell\} g \left(\prod_{i=1}^{\ell} \lambda_1 c(u_i) \right) e^{-\mu_1(u_1 - x_1)}, \end{aligned} \quad (2.3)$$

where

$$g^{-1} := \varrho_1^{-1} + \int_{\mathbb{R}_+} e^{\lambda_1 F(\xi) - \xi} d\xi, \quad (2.4)$$

$$F(\xi) := \int_0^{\xi/\mu_1} (1 - C(\eta)) d\eta, \quad \xi \in \mathbb{R}_+. \quad (2.5)$$

Further, from [BB1] Theorem 3.1 it follows that for a general distribution $C(u)$ with $E I < \infty$ the marginal system of class-1 customers is always stable and that the occupancy distribution of class-1 customers is given by

$$p(0) = g \varrho_1^{-1}, \quad p(n) = \frac{g}{(n-1)!} \int_{\mathbb{R}_+} (\lambda_1 F(\xi))^{n-1} e^{-\xi} d\xi, \quad n \in \mathbb{N}. \quad (2.6)$$

The stability condition concerning the class-2 customers can be obtained as follows: Observe that $1 - p(0)$ is the time fraction where class-1 customers are served. Further, the process of class-2 customers in the system is stable iff all class-2 customers are served in which case ϱ_2 is the time fraction where class-2 customers are served by the server, and since $1 - p(0, 0)$ is the time fraction where the server is busy with serving class-1 or class-2 customers, in case of a stable system it follows

$$1 - p(0) + \varrho_2 = 1 - p(0, 0),$$

or equivalently, cf. (2.6), (2.4),

$$p(0, 0) = p(0) - \varrho_2 = \left(1 + \varrho_1 \int_{\mathbb{R}_+} e^{\lambda_1 F(\xi) - \xi} d\xi\right)^{-1} - \varrho_2. \quad (2.7)$$

Hence the process of class-2 customers is stable iff the r.h.s. of (2.7) is positive, i.e., iff

$$\varrho_2 < \left(1 + \varrho_1 \int_{\mathbb{R}_+} e^{\lambda_1 F(\xi) - \xi} d\xi\right)^{-1}. \quad (2.8)$$

Thus we obtain the following stability condition for the whole system.

Theorem 2.1 *For a general distribution $C(u)$ of the maximal waiting times I with $E I < \infty$ the system is stable iff the stability condition (2.8) is fulfilled.*

Remark 2.1 *Denoting by p_I the probability that an arriving class-1 customer will leave the system due to impatience later, by the conservation principle it follows that $(1 - p_I)\varrho_1 = 1 - p(0)$, cf. also [BB1] formulae (3.6), (3.8) with $s := 1$, $\Lambda := \lambda_1$, $\lambda_n := \lambda_1$, $n \in \mathbb{Z}_+$, $\mu_n := \mu_1$, $n \in \mathbb{N}$, and thus (2.8) is equivalent to*

$$(1 - p_I)\varrho_1 + \varrho_2 < 1, \quad (2.9)$$

which is nothing else that the total served load has to be smaller than 1. It is clear that this approach combined with the results of [BB1] provides the stability condition for the many-server system, too, i.e., the stability condition for the two-class s -server system reads $(1 - p_I)\varrho_1 + \varrho_2 < s$.

In the following we assume that the stability condition (2.8) is fulfilled. Now, let us deal with the stationary distribution of the two-class model. In case of $n \in \{0, 1\}$, $m \in \mathbb{Z}_+$ we have the balance equations

$$\begin{aligned} (\lambda_1 + \lambda_2 + \mathbb{I}\{m > 0\}\mu_2)p(0, m) &= \lambda_2 p(0, m-1) \\ &+ \mu_1 p(1, m) + \mu_2 p(0, m+1), \end{aligned} \quad (2.10)$$

$$\begin{aligned}
(\lambda_1 + \lambda_2 + \mu_1)p(1, m) &= \lambda_1 p(0, m) + \lambda_2 p(1, m-1) \\
&+ \int_{\mathbb{R}_+} p(2, m; 0; u) du + \mu_1 \int_{\mathbb{R}_+^2} p(2, m; x; u) dx du, \tag{2.11}
\end{aligned}$$

where $p(n, m) := 0$ and $p(2, m; x; u) := 0$ for all infeasible states. In case of $n \in \mathbb{N} \setminus \{1\}$, $m \in \mathbb{Z}_+$ and $(x_1, \dots, x_\ell; u_1, \dots, u_\ell) \in \Omega_\ell$, which in view of (2.1) especially implies $0 \leq x_\ell \leq u_\ell$, we have the balance conditions

$$\begin{aligned}
&p(n, m; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \\
&= p(n, m; x_1 + h, \dots, x_\ell + h; u_1, \dots, u_\ell) (1 - h(\lambda_1 + \lambda_2 + \mu_1)) \\
&+ h \sum_{i=1}^{\ell+1} \int_{\mathbb{R}_+} p(n+1, m; x_1, \dots, x_{i-1}, 0, x_i, \dots, x_\ell; \\
&\quad u_1, \dots, u_{i-1}, u, u_i, \dots, u_\ell) du \\
&+ h \mu_1 \int_{\mathbb{R}_+^2} p(n+1, m; x, x_1, \dots, x_\ell; u, u_1, \dots, u_\ell) dx du \\
&+ h \lambda_2 p(n, m-1; x_1, \dots, x_\ell; u_1, \dots, u_\ell) + o(h), \\
&\quad h > 0, \quad x_\ell < u_\ell,
\end{aligned}$$

or equivalently

$$\begin{aligned}
&p(n, m; x_1 + h, \dots, x_\ell + h; u_1, \dots, u_\ell) - p(n, m; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \\
&= (\lambda_1 + \lambda_2 + \mu_1) \int_0^h p(n, m; x_1 + \xi, \dots, x_\ell + \xi; u_1, \dots, u_\ell) d\xi \\
&- \sum_{i=1}^{\ell+1} \int_0^h \int_{\mathbb{R}_+} p(n+1, m; x_1 + \xi, \dots, x_{i-1} + \xi, 0, x_i + \xi, \dots, x_\ell + \xi; \\
&\quad u_1, \dots, u_{i-1}, u, u_i, \dots, u_\ell) du d\xi \\
&- \mu_1 \int_0^h \int_{\mathbb{R}_+^2} p(n+1, m; x, x_1 + \xi, \dots, x_\ell + \xi; u, u_1, \dots, u_\ell) dx du d\xi \\
&- \lambda_2 \int_0^h p(n, m-1; x_1 + \xi, \dots, x_\ell + \xi; u_1, \dots, u_\ell) d\xi, \\
&\quad 0 < h < u_\ell - x_\ell, \tag{2.12}
\end{aligned}$$

and

$$\begin{aligned} p(n, m; x_1, \dots, x_{\ell-1}, u_\ell; u_1, \dots, u_\ell) \\ = \lambda_1 p(n-1, m; x_1, \dots, x_{\ell-1}; u_1, \dots, u_{\ell-1}) c(u_\ell), \end{aligned} \quad (2.13)$$

where all corresponding quantities for infeasible states are assigned zero.

For solving (2.10)–(2.13) it is convenient to introduce the partial PGF's

$$\begin{aligned} Q_n(z) &:= \sum_{m=0}^{\infty} p(n, m) z^m, \quad z \in [-1, 1], \quad n \in \{0, 1\}, \\ Q_n(z; x_1, \dots, x_\ell; u_1, \dots, u_\ell) &:= \sum_{m=0}^{\infty} p(n, m; x_1, \dots, x_\ell; u_1, \dots, u_\ell) z^m, \\ & \quad z \in [-1, 1], \quad n \in \mathbb{N} \setminus \{1\}. \end{aligned}$$

In view of (2.3) and (2.6), it follows that $Q_n(1) = p(n)$, $n \in \{0, 1\}$, and $Q_n(1; x_1, \dots, x_\ell; u_1, \dots, u_\ell) = p(n; x_1, \dots, x_\ell; u_1, \dots, u_\ell)$, $n \in \mathbb{N} \setminus \{1\}$, are known. Now, multiplying each of the equations (2.10)–(2.13) by z^m and summing over $m \in \mathbb{Z}_+$, after some algebra for $n \in \{0, 1\}$ we obtain

$$\left(\lambda_1 + \lambda_2(1-z) + \mu_2 \frac{z-1}{z} \right) Q_0(z) - \mu_2 \frac{z-1}{z} p(0, 0) = \mu_1 Q_1(z), \quad (2.14)$$

$$\begin{aligned} (\lambda_1 + \lambda_2(1-z) + \mu_1) Q_1(z) &= \lambda_1 Q_0(z) \\ &+ \int_{\mathbb{R}_+} Q_2(z; 0; u) du + \mu_1 \int_{\mathbb{R}_+^2} Q_2(z; x; u) dx du, \end{aligned} \quad (2.15)$$

and for $n \in \mathbb{N} \setminus \{1\}$ it follows

$$\begin{aligned} Q_n(z; x_1+h, \dots, x_\ell+h; u_1, \dots, u_\ell) &- Q_n(z; x_1, \dots, x_\ell; u_1, \dots, u_\ell) \\ &= (\lambda_1 + \lambda_2(1-z) + \mu_1) \int_0^h Q_n(z; x_1+\xi, \dots, x_\ell+\xi; u_1, \dots, u_\ell) d\xi \\ &- \sum_{i=1}^{\ell+1} \int_0^h \int_{\mathbb{R}_+} Q_{n+1}(z; x_1+\xi, \dots, x_{i-1}+\xi, 0, x_i+\xi, \dots, x_\ell+\xi; \\ & \quad u_1, \dots, u_{i-1}, u, u_i, \dots, u_\ell) du d\xi \\ &- \mu_1 \int_0^h \int_{\mathbb{R}_+^2} Q_{n+1}(z; x, x_1+\xi, \dots, x_\ell+\xi; u, u_1, \dots, u_\ell) dx du d\xi, \\ & \quad 0 < h < u_\ell - x_\ell, \end{aligned} \quad (2.16)$$

$$\begin{aligned}
& Q_n(z; x_1, \dots, x_{\ell-1}, u_\ell; u_1, \dots, u_\ell) \\
&= \lambda_1 Q_{n-1}(z; x_1, \dots, x_{\ell-1}; u_1, \dots, u_{\ell-1}) c(u_\ell). \tag{2.17}
\end{aligned}$$

For $z \in [-1, 1]$ and $n \in \mathbb{N} \setminus \{1\}$ we try the substitution

$$\begin{aligned}
Q_n(z; x_1, \dots, x_\ell; u_1, \dots, u_\ell) &= \mathbb{I}\{(x_1, \dots, x_\ell; u_1, \dots, u_\ell) \in \Omega_\ell\} \\
& Q_1(z) \left(\prod_{i=1}^{\ell} \lambda_1 c(u_i) \right) f(z, u_1 - x_1), \tag{2.18}
\end{aligned}$$

cf. (2.3) and (2.6) for $n = 1$ in case of $z = 1$, where $f(z, \cdot)$ let be a continuously differentiable function on \mathbb{R}_+ satisfying the boundary conditions

$$f(z, 0) = 1, \quad \lim_{\xi \rightarrow \infty} f(z, \xi) = \lim_{\xi \rightarrow \infty} \frac{\partial f(z, \xi)}{\partial \xi} = 0 \tag{2.19}$$

for fixed $z \in [-1, 1]$. Obviously, the r.h.s. of (2.18) satisfies (2.17). Inserting (2.18) into (2.16), by unifying the intervals of integration and some algebra we obtain that (2.16) is fulfilled iff for $0 < h < u_\ell - x_\ell \leq u_1 - x_1 =: \xi$

$$\begin{aligned}
0 &= f(z, \xi - h) - f(z, \xi) - (\lambda_1 + \lambda_2(1-z) + \mu_1) \int_0^h f(z, \xi - \eta) d\eta \\
&+ \lambda_1 \int_0^h \int_0^\xi f(z, \xi - \eta) c(u - \eta) du d\eta + \lambda_1 \int_0^h \int_\xi^\infty f(z, u - \eta) c(u - \eta) du d\eta \\
&+ \mu_1 \lambda_1 \int_0^h \int_\xi^\infty \left(\int_0^{u-\xi} f(z, u - \eta - x) dx \right) c(u - \eta) du d\eta,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
0 &= -\frac{\partial f(z, \xi)}{\partial \xi} - (\lambda_2(1-z) + \mu_1) f(z, \xi) \\
&+ \lambda_1 \int_\xi^\infty \left(f(z, u) - f(z, \xi) + \mu_1 \int_\xi^u f(z, \eta) d\eta \right) c(u) du, \quad \xi \in \mathbb{R}_+. \tag{2.20}
\end{aligned}$$

Integration by parts provides the equivalent integro-differential equation

$$\begin{aligned}
0 &= -\frac{\partial f(z, \xi)}{\partial \xi} - (\lambda_2(1-z) + \mu_1) f(z, \xi) \\
&+ \lambda_1 \int_\xi^\infty \left(\frac{\partial f(z, \eta)}{\partial \eta} + \mu_1 f(z, \eta) \right) (1 - C(\eta)) d\eta, \quad \xi \in \mathbb{R}_+. \tag{2.21}
\end{aligned}$$

Lemma 2.1 *For a general distribution $C(u)$ of the maximal waiting times I with $E I < \infty$ and fixed $z \in (-\infty, 1]$ there exists a uniquely determined solution $f(z, \cdot)$ of (2.21), (2.19) in $C^{(1)}(\mathbb{R}_+)$. Moreover, there exists an open disk \mathbb{D} with center at $z = 1$ in the complex plane \mathbb{C} such that for fixed $z \in \mathbb{D}$ there exists a uniquely determined solution $f(z, \cdot)$ of (2.21), (2.19) in $C^{(1)}(\mathbb{R}_+, \mathbb{C})$, and $f(\cdot, \xi)$, $(\partial/\partial\xi)f(\cdot, \xi)$ are holomorphic functions in \mathbb{D} for fixed $\xi \in \mathbb{R}_+$. It holds*

$$\left. \frac{\partial f(z, \xi)}{\partial \xi} \right|_{\xi=0} \in (-\infty, -\mu_1], \quad z \in (-\infty, 1]. \quad (2.22)$$

If $C(u)$ is continuous within an interval $(\alpha, \beta) \subset \mathbb{R}_+$, then $f(z, \cdot)$ is twice continuously differentiable in (α, β) and satisfies the ordinary differential equation

$$\begin{aligned} \frac{\partial^2 f(z, \xi)}{\partial \xi^2} + (\lambda_1(1-C(\xi)) + \lambda_2(1-z) + \mu_1) \frac{\partial f(z, \xi)}{\partial \xi} \\ + \lambda_1 \mu_1 (1-C(\xi)) f(z, \xi) = 0, \quad \xi \in (\alpha, \beta), \end{aligned} \quad (2.23)$$

for fixed $z \in (-\infty, 1] \cup \mathbb{D}$.

Proof. For fixed $z \in \mathbb{C}$, the substitution

$$f(z, \xi) = e^{-\mu_1 \xi} + \int_0^\xi e^{\mu_1(\eta-\xi)} \varphi(z, \eta) d\eta, \quad \xi \in \mathbb{R}_+, \quad (2.24)$$

provides that (2.21), (2.19) is equivalent to

$$\begin{aligned} \varphi(z, \xi) = -\lambda_2(1-z)e^{-\mu_1 \xi} + \int_\xi^\infty \lambda_1(1-C(\eta))\varphi(z, \eta) d\eta \\ - \int_0^\xi \lambda_2(1-z)e^{\mu_1(\eta-\xi)}\varphi(z, \eta) d\eta, \quad \xi \in \mathbb{R}_+, \end{aligned} \quad (2.25)$$

where $\varphi(z, \cdot)$ is a continuous function on \mathbb{R}_+ with

$$\lim_{\xi \rightarrow \infty} \varphi(z, \xi) = 0. \quad (2.26)$$

Note that $\varphi(1, \xi) \equiv 0$ is a solution of (2.25), (2.26) in case of $z = 1$.

For fixed $z \in \mathbb{C}$ let $\varphi_1(z, \cdot)$, $\varphi_2(z, \cdot)$ be the solutions of the integral equations

$$\varphi_1(z, \xi) = 1 - \int_0^\xi (\lambda_1(1-C(\eta)) + \lambda_2(1-z)e^{\mu_1(\eta-\xi)})\varphi_1(z, \eta)d\eta, \quad (2.27)$$

$$\varphi_2(z, \xi) = (1 - e^{-\mu_1\xi}) - \int_0^\xi (\lambda_1(1-C(\eta)) + \lambda_2(1-z)e^{\mu_1(\eta-\xi)})\varphi_2(z, \eta)d\eta \quad (2.28)$$

for $\xi \in \mathbb{R}_+$, respectively, where the existence and uniqueness follows from Banach's fixed point theorem using the norm

$$\|\varphi\|_1 := \sup_{\xi \in \mathbb{R}_+} |e^{-\kappa\xi}\varphi(\xi)|$$

for sufficiently large $\kappa \in \mathbb{R}_+$ in the subset of $C(\mathbb{R}_+, \mathbb{C})$ defined by $\|\varphi\|_1 < \infty$. Considering the sequence of intervals $[0, n]$, $n \in \mathbb{N}$, instead of \mathbb{R}_+ , one finds that uniqueness even holds in the whole class $C(\mathbb{R}_+, \mathbb{C})$. Choosing the norm

$$\|\varphi\|_2 := \sup_{\xi \in \mathbb{R}_+} |e^{-\kappa \min(\xi, \xi_*)}\varphi(\xi)|,$$

where $\kappa, \xi_* \in \mathbb{R}_+$ are sufficiently large, then Banach's fixed point theorem provides that the solutions of (2.27) and (2.28) are uniformly bounded in $\mathbb{A} \times \mathbb{R}_+$ for any compact subset \mathbb{A} of $\mathbb{D}_0 := \{z \in \mathbb{C} : |z-1| < \mu_1/\lambda_2\}$. The $\varphi_j(\cdot, \xi)$ are holomorphic functions in \mathbb{C} for fixed $\xi \in \mathbb{R}_+$, $j \in \{1, 2\}$, as the sequences of iterations given by Banach's fixed point theorem if one starts from the zero function are sequences of polynomials w.r.t. z which converge locally uniformly in \mathbb{C} for fixed $\xi \in \mathbb{R}_+$.

Obviously, $\varphi_1(z, \xi)$ is positive for sufficiently small $\xi \in \mathbb{R}_+$ and fixed $z \in (-\infty, 1)$. If for some $z \in (-\infty, 1)$ there exists a zero and thus a smallest zero $\xi_0 \in \mathbb{R}_+$ of $\varphi_1(z, \xi)$ then it follows

$$\left. \frac{\partial \varphi_1(z, \xi)}{\partial \xi} \right|_{\xi=\xi_0} = \mu_1 \int_0^{\xi_0} \lambda_2(1-z)e^{\mu_1(\eta-\xi_0)}\varphi_1(z, \eta)d\eta > 0,$$

providing a contradiction. By arguments of continuity thus $\varphi_1(z, \xi)$ is non-negative for $(z, \xi) \in (-\infty, 1] \times \mathbb{R}_+$. In view of (2.27), for $z \in (-\infty, 1]$ hence there exists the limit of

$$\psi_1(z, \xi) := 1 - \int_0^\xi \lambda_1(1-C(\eta))\varphi_1(z, \eta)d\eta$$

for $\xi \rightarrow \infty$. As $\varphi_2(z, \xi)$ is positive for sufficiently small $\xi \in (0, \infty)$ and fixed $z \in (-\infty, 1)$, analogously it follows that $\varphi_2(z, \xi)$ is non-negative for $(z, \xi) \in (-\infty, 1] \times \mathbb{R}_+$, and in view of (2.28), for $z \in (-\infty, 1]$ hence there exists the limit of

$$\psi_2(z, \xi) := (1 - e^{-\mu_1 \xi}) - \int_0^\xi \lambda_1(1 - C(\eta))\varphi_2(z, \eta)d\eta$$

for $\xi \rightarrow \infty$. As the $\varphi_j(z, \cdot)$ are bounded functions, locally uniformly for $z \in \mathbb{D}_0$, $j \in \{1, 2\}$, for $z \in \mathbb{D}_0$ there also exist the limits of $\psi_j(z, \xi)$ for $\xi \rightarrow \infty$, $j \in \{1, 2\}$, and these limits are holomorphic functions of $z \in \mathbb{D}_0$. Because of

$$\varphi_j(z, \xi) = \psi_j(z, \xi) - \int_0^\xi \lambda_2(1 - z)e^{\mu_1(\eta - \xi)}\varphi_j(z, \eta)d\eta, \quad j \in \{1, 2\},$$

which implies

$$\varphi_j(z, \xi) = \psi_j(z, \xi) - \int_0^\xi \lambda_2(1 - z)e^{(\mu_1 + \lambda_2(1 - z))(\eta - \xi)}\psi_j(z, \eta)d\eta,$$

thus there exist the limits

$$\alpha_j(z) := \lim_{\xi \rightarrow \infty} \varphi_j(z, \xi), \quad z \in (-\infty, 1] \cup \mathbb{D}_0, \quad j \in \{1, 2\}.$$

The $\alpha_j(z)$ are holomorphic functions for $z \in \mathbb{D}_0$, and it holds $\alpha_j(z) \in [0, 1]$ for $z \in (-\infty, 1]$. Assume that $\alpha_1(z) = 0$ for some $z \in (-\infty, 1]$. Then from (2.27) it follows

$$\int_0^\infty \lambda_1(1 - C(\eta))\varphi_1(z, \eta)d\eta = 1,$$

and thus from (2.27) we obtain

$$\begin{aligned} \max_{\theta \in [\xi, \infty)} \varphi_1(z, \theta) &= \varphi_1(z, \xi_*) \\ &= \int_{\xi_*}^\infty \lambda_1(1 - C(\eta))\varphi_1(z, \eta)d\eta - \int_0^{\xi_*} \lambda_2(1 - z)e^{\mu_1(\eta - \xi_*)}\varphi_1(z, \eta)d\eta \\ &\leq \max_{\theta \in [\xi, \infty)} \varphi_1(z, \theta) \int_\xi^{\xi_*} \lambda_1(1 - C(\eta))d\eta \end{aligned}$$

for $0 \leq \xi < \xi^* \leq \infty$ and suitable $\xi_* \in [\xi, \xi^*)$ if $\varphi_1(z, \eta) = 0$ for $\eta \geq \xi^*$, providing a contradiction by choosing ξ^* minimal and $\xi \in [0, \xi^*)$ close to ξ^* , i.e., $\alpha_1(z) \neq 0$ for $z \in (-\infty, 1]$. Thus there exists an open disk $\mathbb{D} \subseteq \mathbb{D}_0$ with center at $z = 1$ in the complex plane \mathbb{C} such that $\alpha_1(z) \neq 0$ for $z \in \mathbb{D}$. For $z \in (-\infty, 1] \cup \mathbb{D}$ let

$$\varphi(z, \xi) := \lambda_2(1-z) \left(\varphi_2(z, \xi) - \frac{\alpha_2(z)}{\alpha_1(z)} \varphi_1(z, \xi) \right), \quad \xi \in \mathbb{R}_+. \quad (2.29)$$

Due to the definition of $\alpha_j(z)$, $j \in \{1, 2\}$, it holds (2.26). From (2.27), (2.28) it follows

$$\begin{aligned} \varphi(z, \xi) &= -\lambda_2(1-z)e^{-\mu_1\xi} + \lambda_2(1-z) \left(1 - \frac{\alpha_2(z)}{\alpha_1(z)} \right) \\ &\quad - \int_0^\xi (\lambda_1(1-C(\eta)) + \lambda_2(1-z)e^{\mu_1(\eta-\xi)}) \varphi(z, \eta) d\eta, \quad \xi \in \mathbb{R}_+. \end{aligned} \quad (2.30)$$

Thus (2.26) yields

$$\int_0^\infty \lambda_1(1-C(\eta)) \varphi(z, \eta) d\eta = \lambda_2(1-z) \left(1 - \frac{\alpha_2(z)}{\alpha_1(z)} \right),$$

and hence it holds (2.25). As $\varphi(z, \xi)$ is a holomorphic function in \mathbb{D} for fixed $\xi \in \mathbb{R}_+$, from (2.24) it follows that $f(\cdot, \xi)$ and $(\partial/\partial\xi)f(\cdot, \xi)$ are holomorphic functions in \mathbb{D} for fixed $\xi \in \mathbb{R}_+$.

Assume that there exist two different solutions of (2.25), (2.26) for some $z \in (-\infty, 1] \cup \mathbb{D}$. Then its difference $\varphi_*(z, \cdot)$ is a nontrivial solution of the homogenized equation

$$\begin{aligned} \varphi_*(z, \xi) &= \int_\xi^\infty \lambda_1(1-C(\eta)) \varphi_*(z, \eta) d\eta - \int_0^\xi \lambda_2(1-z)e^{\mu_1(\eta-\xi)} \varphi_*(z, \eta) d\eta, \\ &\quad \xi \in \mathbb{R}_+, \end{aligned} \quad (2.31)$$

$$\lim_{\xi \rightarrow \infty} \varphi_*(z, \xi) = 0. \quad (2.32)$$

From (2.31) it follows

$$\begin{aligned} \varphi_*(z, \xi) &= \int_0^\infty \lambda_1(1-C(\eta)) \varphi_*(z, \eta) d\eta \\ &\quad - \int_0^\xi (\lambda_1(1-C(\eta)) + \lambda_2(1-z)e^{\mu_1(\eta-\xi)}) \varphi_*(z, \eta) d\eta, \end{aligned}$$

yielding

$$\varphi_*(z, \xi) = \varphi_1(z, \xi) \int_0^\infty \lambda_1(1-C(\eta))\varphi_*(z, \eta)d\eta, \quad \xi \in \mathbb{R}_+,$$

as the solution $\varphi_1(z, \xi)$ of (2.27) is uniquely determined. Hence (2.32) and $\lim_{\xi \rightarrow \infty} \varphi_1(z, \xi) = \alpha_1(z) \neq 0$ provide the contradiction $\varphi_*(z, \xi) = 0$, $\xi \in \mathbb{R}_+$.

For fixed $z \in (-\infty, 1]$ from (2.24) and (2.30) we obtain

$$\left. \frac{\partial f(z, \xi)}{\partial \xi} \right|_{\xi=0} = \varphi(z, 0) - \mu_1 = -\lambda_2(1-z) \frac{\alpha_2(z)}{\alpha_1(z)} - \mu_1.$$

Thus (2.22) results from $\alpha_1(z) > 0$ and $\alpha_2(z) \geq 0$ for $z \in (-\infty, 1]$.

If $C(u)$ is continuous within the interval $(\alpha, \beta) \subset \mathbb{R}_+$, then from (2.21) it follows that $f(z, \cdot)$ is twice continuously differentiable in (α, β) and satisfies the differential equation (2.23) there for fixed $z \in (-\infty, 1] \cup \mathbb{D}$.

□

In the following we assume that $f(z, \cdot)$ is the solution of (2.21), (2.19). For $z = 1$ from $\varphi(1, \xi) \equiv 0$ and (2.24) it follows $f(1, \xi) = e^{-\mu_1 \xi}$. In the general case of $z \in (-\infty, 1] \cup \mathbb{D}$ only for particular $C(u)$, cf. Section 5, explicit solutions of (2.21), (2.19) can be given. However, the following lemma gives more insight into the structure of f and provides the partial derivatives with respect to z at $z = 1$, in principle.

Lemma 2.2 *It holds*

$$f(z, \xi) = \sum_{j=0}^{\infty} f_j(\xi)(z-1)^j, \quad (z, \xi) \in \mathbb{D} \times \mathbb{R}_+, \quad (2.33)$$

$$\frac{\partial f(z, \xi)}{\partial \xi} = \sum_{j=0}^{\infty} f'_j(\xi)(z-1)^j, \quad (z, \xi) \in \mathbb{D} \times \mathbb{R}_+, \quad (2.34)$$

where $f_0(\xi) = e^{-\mu_1 \xi}$, $\xi \in \mathbb{R}_+$, and for $j \in \mathbb{N}$ the functions f_j are recursively given by

$$f_j(\xi) = -\lambda_2 \int_0^\xi e^{\mu_1(\eta-\xi)} \int_\eta^\infty e^{\lambda_1(F(\mu_1\theta)-F(\mu_1\eta))} f'_{j-1}(\theta) d\theta d\eta, \quad \xi \in \mathbb{R}_+. \quad (2.35)$$

In particular, it holds

$$f_1'(0) = \lambda_2 \int_0^{\infty} e^{\lambda_1 F(\xi) - \xi} d\xi, \quad (2.36)$$

$$f_2'(0) = \frac{\lambda_2^2}{\mu_1} \int_0^{\infty} \int_{\xi}^{\infty} e^{\lambda_1 F(\eta) - \eta} d\eta \left(\int_{\xi}^{\infty} e^{\lambda_1 F(\eta) - \eta} d\eta e^{-(\lambda_1 F(\xi) - \xi)} - 1 \right) d\xi. \quad (2.37)$$

Proof. For fixed $\xi \in \mathbb{R}_+$ the r.h.s. of (2.33) are the Taylor series of $f(z, \xi)$ at $z = 1$, where the coefficients are denoted by $f_j(\xi)$. As $(\partial/\partial\xi)f(z, \xi)$ is holomorphic in \mathbb{D} for fixed $\xi \in \mathbb{R}_+$, (2.34) follows from (2.33). $f(1, \xi) = e^{-\mu_1 \xi}$ yields $f_0(\xi) = e^{-\mu_1 \xi}$. Inserting (2.33), (2.34) into (2.21) and comparing the coefficients of $(z - 1)^j$, we find that the $f_j(\xi)$ satisfy the linear system of integro-differential equations

$$-f_j'(\xi) - \mu_1 f_j(\xi) + \lambda_1 \int_{\xi}^{\infty} (f_j'(\eta) + \mu_1 f_j(\eta))(1 - C(\eta)) d\eta = -\lambda_2 f_{j-1}(\xi), \quad \xi \in \mathbb{R}_+, \quad j \in \mathbb{N}, \quad (2.38)$$

and (2.19) yields the boundary conditions

$$f_j(0) = 0, \quad \lim_{\xi \rightarrow \infty} f_j(\xi) = \lim_{\xi \rightarrow \infty} f_j'(\xi) = 0, \quad j \in \mathbb{N}. \quad (2.39)$$

As the solution of (2.21), (2.19) is uniquely determined in case of $z = 1$, the $f_j(\xi)$ are uniquely determined by (2.38), (2.39) and $f_{j-1}(\xi)$. By induction using integration by parts it follows that the r.h.s. of (2.35) is well defined and satisfies the boundary conditions (2.39). In view of (2.5) and (2.39), inserting (2.35) into the l.h.s. of (2.38) and integrating by parts provides that the r.h.s. of (2.35) satisfies (2.38), too. From (2.35) for $j = 1$ and $f_0(\xi) = e^{-\mu_1 \xi}$ it follows

$$f_1(\xi) = \lambda_2 \mu_1 e^{-\mu_1 \xi} \int_0^{\xi} \int_{\eta}^{\infty} e^{(\lambda_1 F(\mu_1 \theta) - \mu_1 \theta) - (\lambda_1 F(\mu_1 \eta) - \mu_1 \eta)} d\theta d\eta \quad (2.40)$$

yielding (2.36). Equation (2.37) can be obtained along the following steps: (i) taking the first derivative of (2.35) for $j = 2$ at $\xi = 0$, (ii) integrating by parts and using (2.39), (iii) inserting $f_1(\theta)$ according to (2.40) and applying

Fubini's theorem.

□

By means of the preceding results of this section we are in a position to determine $Q_n(z)$, $n \in \{0, 1\}$, and hence the r.h.s. of (2.18), providing the solution of the system of equations (2.14)–(2.17).

Theorem 2.2 *For the partial PGF's $Q_n(z)$, $n \in \{0, 1\}$, it holds*

$$Q_0(z) = \frac{\mu_2(1-z)}{\mu_2(1-z) - \lambda_1 z(1-G(z)) - \lambda_2 z(1-z)} p(0, 0), \quad z \in [-1, 1], \quad (2.41)$$

$$Q_1(z) = \varrho_1 G(z) Q_0(z), \quad z \in [-1, 1], \quad (2.42)$$

where $Q_0(1) = p(0)$ is given by (2.6), $p(0, 0)$ is given by (2.7) and

$$G(z) := -\mu_1 \left(\frac{\partial f(z, \xi)}{\partial \xi} \Big|_{\xi=0} \right)^{-1} \in (0, 1], \quad z \in [-1, 1]. \quad (2.43)$$

The partial PGF's $Q_n(z; x_1, \dots, x_\ell; u_1, \dots, u_\ell)$, $z \in [-1, 1]$, $n \in \mathbb{N} \setminus \{1\}$, are given by (2.18), where $Q_1(z)$ is given by (2.42) and $f(z, \cdot)$ is the unique solution of (2.21), (2.19) in $C^{(1)}(\mathbb{R}_+)$.

Proof. Because of (2.22), $G(z)$ is well defined with $G(z) \in (0, 1]$, $z \in [-1, 1]$. It remains to prove (2.41) and (2.42). With the substitution (2.18) and in view of (2.19), equation (2.15) is equivalent to

$$\begin{aligned} (\lambda_2(1-z) + \mu_1) Q_1(z) &= \lambda_1 Q_0(z) \\ &+ \lambda_1 Q_1(z) \int_{\mathbb{R}_+} \left(f(z, u) - f(z, 0) + \mu_1 \int_0^u f(z, \xi) d\xi \right) c(u) du. \end{aligned}$$

Thus (2.20) for $\xi = 0$ yields

$$\begin{aligned} (\lambda_2(1-z) + \mu_1) Q_1(z) &= \lambda_1 Q_0(z) \\ &+ Q_1(z) \left(\frac{\partial f(z, \xi)}{\partial \xi} + (\lambda_2(1-z) + \mu_1) f(z, \xi) \right) \Big|_{\xi=0}, \end{aligned}$$

and we obtain (2.42) in view of (2.19), (2.43). Equation (2.41) follows from (2.14) and (2.42) immediately.

□

Remark 2.2 Note that Lemma 2.2 gives the analytic continuation of $G(z)$ and hence also of $Q_n(z)$, $n \in \{0, 1\}$, in a neighborhood of $z = 1$. Thus the identity (2.7) could be obtained, alternatively to the load arguments before Theorem 2.1, also from (2.41) by letting $z \rightarrow 1$ and using (2.43), (2.34), (2.36), (2.6), (2.4) as well as $Q_0(1) = p(0)$.

3 The PGF of the joint occupancy distribution

In this section again we assume that the system is stable, that the expectation of the maximal waiting times is finite and, if not stated otherwise, that its distribution $C(u)$ has a continuous density $c(u)$. First we deal with the PGF of $N_2(t)$ on $\{N_1(t) = n\}$. Let $Q_n(z) := E[z^{N_2(t)} \mathbb{1}\{N_1(t) = n\}]$, $z \in [-1, 1]$, $n \in \mathbb{N} \setminus \{1\}$. From (2.18) it follows

$$\begin{aligned} Q_n(z) &= \int_{\mathbb{R}_+^{2\ell}} Q_n(z; x_1, \dots, x_\ell; u_1, \dots, u_\ell) dx_1 \dots dx_\ell du_1 \dots du_\ell \\ &= Q_1(z) \lambda_1^\ell \int_{\Omega_\ell} \left(\prod_{i=1}^{\ell} c(u_i) \right) f(z, u_1 - x_1) dx_1 \dots dx_\ell du_1 \dots du_\ell \end{aligned}$$

for $n \in \mathbb{N} \setminus \{1\}$, where $\ell := n - 1$. In view of the definition (2.1) of Ω_ℓ , the substitution $u_i = \xi_i + x_i$ yields

$$\begin{aligned} Q_n(z) &= Q_1(z) \lambda_1^\ell \int_{\mathbb{R}_+^{2\ell}} \mathbb{1}\{\xi_1 \geq \dots \geq \xi_\ell\} \left(\prod_{i=1}^{\ell} c(\xi_i + x_i) \right) f(z, \xi_1) \\ &\quad dx_1 \dots dx_\ell d\xi_1 \dots d\xi_\ell \\ &= Q_1(z) \lambda_1^\ell \int_{\mathbb{R}_+} \frac{1}{(\ell-1)!} \left(\int_0^\xi (1-C(\eta)) d\eta \right)^{\ell-1} (1-C(\xi)) f(z, \xi) d\xi \\ &= -Q_1(z) \frac{\lambda_1^\ell}{\ell!} \int_{\mathbb{R}_+} \left(\int_0^\xi (1-C(\eta)) d\eta \right)^\ell \frac{\partial f(z, \xi)}{\partial \xi} d\xi, \end{aligned}$$

where the last equality follows from integration by parts and (2.19). Hence from (2.42), (2.5) and (2.19) finally we obtain

$$Q_n(z) = Q_0(z) \frac{\varrho_1}{(n-1)!} \int_{\mathbb{R}_+} (\lambda_1 F(\xi))^{n-1} h(z, \xi) d\xi, \quad n \in \mathbb{N}, \quad (3.1)$$

where

$$h(z, \xi) := -G(z) \frac{\partial f(z, \xi/\mu_1)}{\partial \xi}. \quad (3.2)$$

Note that $h(z, 0) = 1$. Now, the case of a general distribution $C(u)$ of the maximal waiting times with finite expectation is obtained by considering $C(u)$ as the limit in distribution of a suitable sequence of distributions $C_\nu(u)$ with continuous density and by arguments of continuity.

For the PGF

$$Q(w, z) = E[w^{N_1(t)} z^{N_2(t)}], \quad (w, z) \in [-1, 1]^2,$$

of the vector of the numbers of customers in the system from the definition of $Q_n(z)$ and (3.1) after elementary algebra it follows

$$\begin{aligned} Q(w, z) &= Q_0(z) + \sum_{n=1}^{\infty} Q_n(z) w^n \\ &= Q_0(z) \left(1 + \varrho_1 w \int_{\mathbb{R}_+} e^{\lambda_1 w F(\xi)} h(z, \xi) d\xi \right). \end{aligned} \quad (3.3)$$

Choosing $w = 1$, for the PGF $R(z) := E[z^{N_2(t)}]$ we obtain

$$R(z) = Q_0(z) \left(1 + \varrho_1 \int_{\mathbb{R}_+} e^{\lambda_1 F(\xi)} h(z, \xi) d\xi \right). \quad (3.4)$$

In view of (3.2), (2.5), integration by parts and (2.19) provide

$$\int_{\mathbb{R}_+} e^{\lambda_1 F(\xi)} h(z, \xi) d\xi = G(z) + \lambda_1 G(z) \int_{\mathbb{R}_+} e^{\lambda_1 F(\mu_1 \eta)} (1 - C(\eta)) f(z, \eta) d\eta.$$

Replacing on the r.h.s. $\lambda_2(1 - z)f(z, \eta)$ by the term obtained from (2.21), it follows that the integrand is just the derivative of a product function, and hence the integral can be evaluated. Applying now (2.21) for $\xi = 0$ as well as (2.19), (2.43) yields

$$\int_{\mathbb{R}_+} e^{\lambda_1 F(\xi)} h(z, \xi) d\xi = \frac{\mu_1(1 - G(z))}{\lambda_2(1 - z)}.$$

Thus from (3.4), (2.41) we find

$$R(z) = \frac{\lambda_1(1 - G(z)) + \lambda_2(1 - z)}{\mu_2(1 - z) - \lambda_1 z(1 - G(z)) - \lambda_2 z(1 - z)} \frac{\mu_2 p(0, 0)}{\lambda_2}, \quad z \in [-1, 1]. \quad (3.5)$$

Since the factorial moments $M_{2,k} := E[N_2(t)(N_2(t) - 1) \dots (N_2(t) - k + 1)]$, $k \in \mathbb{Z}_+$, of $N_2(t)$ are given by

$$M_{2,k} = \lim_{z \rightarrow 1} \frac{d^k}{dz^k} R(z), \quad (3.6)$$

from (3.5) one can derive explicit formulae for the $M_{2,k}$, in principle. In order to compute the r.h.s. of (3.6) one has to determine $f'_j(0)$ for $j = 1, \dots, k + 1$. In case of $k = 1$, the mean number $E N_2(t) = M_{2,1}$ of class-2 customers is given by the limit of the logarithmic derivative of $R(z)$ for $z \rightarrow 1$. Thus from (3.5), (2.43), (2.34) and $f_0(\xi) = e^{-\mu_1 \xi}$ after some algebra one obtains

$$E N_2(t) = \frac{\lambda_1 \mu_2 \left(f_1'^2(0) + \mu_1 f_2'(0) \right) + \left(\mu_1 \lambda_2 + \lambda_1 f_1'(0) \right)^2}{\left(\mu_1 \mu_2 - \left(\mu_1 \lambda_2 + \lambda_1 f_1'(0) \right) \right) \left(\mu_1 \lambda_2 + \lambda_1 f_1'(0) \right)}, \quad (3.7)$$

where the $f'_j(0)$, $j = 1, 2$, are given by (2.36) and (2.37), respectively.

Remark 3.1 In [CKC] Section 3.1 a performance analysis is given in case of deterministic maximal waiting times. In our terminology the starting point there is the Markov process $Z(t)$ defined by

$$Z(t) := \begin{cases} (N_1(t), 0, N_2(t)), & N_1(t) \in \{0, 1\}, \\ (2, I_1(t) - X_1(t), N_2(t)), & N_1(t) \in \mathbb{N} \setminus \{1\}. \end{cases}$$

Then via Kolmogorov's balance equations for $Q_n(z)$, $n \in \{0, 1\}$, and the partial PGF's

$$K(x, z) := \sum_{k=0}^{\infty} \frac{d}{dx} P(N_1(t) \geq 2, I_1(t) - X_1(t) \leq x, N_2(t) = k) z^k \quad (3.8)$$

a system of equations is derived and solved explicitly, providing the basis for determining further performance measures, cf. [CKC] (3.12), (3.13). Note that the approach in [CKC] indeed is limited to the case of deterministic maximal waiting times.

Using (2.18) for general distributions $C(u)$ with continuous density $c(u)$ and $E I < \infty$, by a similar calculation yielding (3.1) we find

$$\begin{aligned} K(x, z) &= \sum_{n=2}^{\infty} \int_{\mathbb{R}_+^{2\ell-1}} Q_n(z; x_1, \dots, x_\ell; x_1 + x, u_2, \dots, u_\ell) \\ &\quad dx_1 dx_2 \dots dx_\ell du_2 \dots du_\ell \\ &= Q_1(z) f(z, x) (1 - C(x)) \lambda_1 e^{\lambda_1 F(\mu_1 x)}, \end{aligned} \quad (3.9)$$

generalizing the corresponding results of [CKC] to generally distributed maximal waiting times with finite expectation.

4 Sojourn and waiting times of class-2 customers

Since the dynamics of class-1 customers correspond to the dynamics of a $M/M/1 + GI$ system, cf. Section 1, the well-known results for sojourn and waiting times for this type of queuing systems can be applied, cf. e.g. [BB1], [BH], [Dal], [Ju1], [Mov]. Thus we concentrate on the sojourn and waiting times of class-2 customers. Let

- V_2 – stationary sojourn time of a class-2 customer in the system, i.e., the time from its arrival until its service completion,
- N_2^a – stationary number of class-2 customers in the system seen by an arriving class-2 customer,
- N_2^d – stationary number of class-2 customers in the system immediately after a departure of a class-2 customer.

From the PASTA property and the conservation principle it follows

$$N_2(t) \stackrel{D}{=} N_2^a \stackrel{D}{=} N_2^d. \quad (4.1)$$

Analogously to the standard arguments for deriving the LST for the sojourn time in $M/GI/1/\infty$ queues, cf. e.g. [GH], we observe that in view of the FCFS discipline immediately after the departure instant of a class-2 customer there are just those class-2 customers in the system which arrived during the sojourn time of the departing customer. This and the properties of a Poisson process imply

$$P(N_2^d = n) = \int_{\mathbb{R}_+} \frac{(\lambda_2 t)^n}{n!} e^{-\lambda_2 t} dV_2(t), \quad (4.2)$$

where $V_2(t) := P(V_2 \leq t)$ denotes the distribution function of V_2 . From (4.1), (4.2) for $R(z)$ we obtain

$$R(z) = \sum_{n=0}^{\infty} \int_{\mathbb{R}_+} \frac{(\lambda_2 tz)^n}{n!} e^{-\lambda_2 t} dV_2(t) = V_2^*(\lambda_2(1-z)),$$

where $V_2^*(s) = E[e^{-sV_2}]$ is the LST of $V_2(t)$ and $R(z)$ is explicitly given by (3.5) for $z \in [-1, 1)$, in principle. Transformation of variables yields

$$V_2^*(s) = R(1-s/\lambda_2), \quad (4.3)$$

cf. also [CKC] Section 3. Taking the k th derivative on both sides of (4.3) and letting $s \downarrow 0$, by taking into account (3.6) we find that

$$E[V_2^k] = \lambda_2^{-k} M_{2,k}, \quad k \in \mathbb{Z}_+. \quad (4.4)$$

In particular, for $k = 1$ Little's formula is obtained.

Now, we will deal with the stationary waiting time W_2 of a class-2 customer in the queue until its service starts for the first time. Note that its distribution $W_2(t) := P(W_2 \leq t)$ has an atom at $t = 0$. In the standard $M/GI/1/\infty$ analysis the stationary waiting time distribution is obtained via the observation that the sojourn time is the sum of the waiting and service time. In view of the preemptive service discipline this argument fails for the class-2 customers because the service of a class-2 customer may be interrupted by serving class-1 customers. However, we can modify the arguments by taking into account the service interruption by class-1 customers as follows: Obviously, the sojourn time V_2 of an arriving class-2 customer is given by

$$V_2 = W_2 + U, \quad (4.5)$$

where U is the time from the beginning of service until finishing service of the class-2 customer. The r.v.'s W_2 and U are stochastically independent. Let us consider U and its distribution $U(t) := P(U \leq t)$ in more detail. At the beginning of service for the class-2 customer, there are no class-1 customers in the system. The service can be continued as long as no class-1 customers arrive. If a class-1 customer arrives before service completion then the service will be interrupted as long as there are still class-1 customers in the system. The duration B of this interruption corresponds to the busy period in the corresponding $M/M/1 + GI$ system of class-1 customers with parameters $\lambda_1, \mu_1, C(u)$. After time B the service of the class-2 customer will be continued, and, since the residual service time is again exponentially distributed with parameter μ_2 , we are stochastically in the same situation as at the beginning of service. Let $B(t) := P(B \leq t)$ be the busy period distribution. The above considerations lead to the following renewal type equation

$$U(t) = \frac{\lambda_1}{\lambda_1 + \mu_2} (E_{\lambda_1 + \mu_2} * B * U)(t) + \frac{\mu_2}{\lambda_1 + \mu_2} E_{\lambda_1 + \mu_2}(t),$$

where $E_\alpha(t)$ denotes the exponential distribution with parameter α and $*$ the convolution operator. For the LST's $U^*(s) = E[e^{-sU}]$, $B^*(s) = E[e^{-sB}]$ thus we find

$$U^*(s) = \frac{\mu_2}{\lambda_1(1 - B^*(s)) + \mu_2 + s}. \quad (4.6)$$

From (4.5) and (4.6) finally for the LST $W_2^*(s) = E[e^{-sW_2}]$ we obtain

$$W_2^*(s) = \frac{1}{\mu_2} (\lambda_1(1 - B^*(s)) + \mu_2 + s) V_2^*(s), \quad (4.7)$$

where $V_2^*(s)$ is given by (4.3), (3.5) for $s \in (0, 2\lambda_2]$, in principle, and the LST $B^*(s)$ of the busy period of the marginal $M/M/1+GI$ system of class-1 customers has to be given. Taking the k th derivative of both sides of (4.7) and letting $s \downarrow 0$ provides

$$E[W_2^k] = E[V_2^k] - \frac{k}{\mu_2} (1 + \lambda_1 E B) E[V_2^{k-1}] - \frac{\lambda_1}{\mu_2} \sum_{\ell=2}^k \binom{k}{\ell} E[B^\ell] E[V_2^{k-\ell}], \quad k \in \mathbb{N}. \quad (4.8)$$

Unfortunately, for the authors best knowledge, it seems that there are no explicit results available for $B^*(s)$ or for higher moments $E[B^\ell]$, $\ell \in \mathbb{N} \setminus \{1\}$, for general GI . In case of deterministic maximal waiting times, $B^*(s)$ is given explicitly even for the more general $M/GI/1+D$ system in [KBL] Theorem 3.3. It remains an unsolved problem to find corresponding expressions in the general case in order to exploit (4.7) or (4.8). However, in case of $k = 1$ we are able to determine EW_2 in terms of $EN_2(t)$ as follows: In the associated $M/M/1+GI$ system of class-1 customers the time epochs where the system becomes empty, i.e., where $N_2(t-0) = 1$, $N_2(t+0) = 0$, are regeneration points of $N_2(t)$, $t \geq 0$. A typical cycle of the regenerative process $N_2(t)$, $t \geq 0$, has length $Y = Z + B$, where Z is exponentially distributed with parameter λ_1 , B is a busy period of the $M/M/1+GI$ system and Z and B are independent. From the theory of regenerative processes, cf. e.g. [Asm] p. 126, it follows

$$p(0) = \frac{EZ}{EY} = \frac{1}{1 + \lambda_1 EB}, \quad (4.9)$$

where $p(0)$ is given by (2.6). This, (4.8) for $k = 1$ and Little's formula yield

$$EW_2 = EV_2 - \frac{1}{\mu_2 p(0)} = \frac{1}{\lambda_2} EN_2(t) - \frac{1}{\mu_2 p(0)}. \quad (4.10)$$

5 Deterministic maximal waiting times

Assume that the maximal waiting times are constant $\tau \in (0, \infty)$, i.e., that $C(u) := \mathbb{I}\{u \geq \tau\}$, $u \in \mathbb{R}_+$. Moreover, we assume that the stability condition (2.8) is fulfilled and that $\varrho_1 \neq 1$. (The case of $\varrho_1 = 1$ is obtained by a limiting process.)

In view of (2.5), the stability condition (2.8) reads

$$\varrho_2 < \frac{1 - \varrho_1}{1 - d}, \quad (5.1)$$

where

$$d := \varrho_1^2 e^{(\varrho_1 - 1)\mu_1 \tau}, \quad (5.2)$$

and $p(0)$ is given by the r.h.s. of (5.1), cf. [CKC].

For $z \in (-\infty, 1]$ and $\xi \in (0, \tau)$ the differential equation (2.23) simplifies to

$$\frac{\partial^2 f(z, \xi)}{\partial \xi^2} + (\lambda_1 + \lambda_2(1-z) + \mu_1) \frac{\partial f(z, \xi)}{\partial \xi} + \lambda_1 \mu_1 f(z, \xi) = 0$$

with the condition $f(z, 0) = 1$. Its general solution is

$$f(z, \xi) = a_1(z) e^{\kappa_1(z)\xi} + (1 - a_1(z)) e^{\kappa_2(z)\xi}, \quad (5.3)$$

where

$$\kappa_{1/2}(z) := -\frac{1}{2}(\lambda_1 + \lambda_2(1-z) + \mu_1) \pm \frac{1}{2}\sqrt{(\lambda_1 + \lambda_2(1-z) + \mu_1)^2 - 4\lambda_1\mu_1}. \quad (5.4)$$

For $z \in (-\infty, 1]$ and $\xi \in (\tau, \infty)$ the differential equation (2.23) reduces to

$$\frac{\partial^2 f(z, \xi)}{\partial \xi^2} + (\lambda_2(1-z) + \mu_1) \frac{\partial f(z, \xi)}{\partial \xi} = 0$$

with the condition

$$\lim_{\xi \rightarrow \infty} f(z, \xi) = \lim_{\xi \rightarrow \infty} \frac{\partial f(z, \xi)}{\partial \xi} = 0,$$

whose general solution is given by

$$f(z, \xi) = a_2(z) e^{-(\lambda_2(1-z) + \mu_1)\xi}. \quad (5.5)$$

The functions $a_j(z)$, $z \in (-\infty, 1]$, $j \in \{1, 2\}$, have to be chosen such that the function $f(z, \cdot)$ is continuously differentiable at $\xi = \tau$, implying

$$\begin{aligned} a_1(z) &= \frac{b_2(z) e^{b_2(z)}}{b_2(z) e^{b_2(z)} - b_1(z) e^{b_1(z)}}, \\ a_2(z) &= \frac{(b_2(z) - b_1(z)) e^{b_2(z)} e^{b_1(z)}}{b_2(z) e^{b_2(z)} - b_1(z) e^{b_1(z)}}, \end{aligned} \quad (5.6)$$

where

$$b_j(z) := (\kappa_j(z) + \lambda_2(1-z) + \mu_1)\tau, \quad j \in \{1, 2\}. \quad (5.7)$$

For the mean number $EN_2(t)$ of class-2 customers from (2.5), (2.36), (2.37) and (3.7) after some algebra one obtains

$$EN_2(t) = \frac{\mu_2 \varrho_2 (\varrho_1 - (1 - \varrho_1) ((1 + \varrho_1) \mu_1 \tau + 3) d - d^2)}{\mu_1 (1 - \varrho_1) ((1 - \varrho_1) - \varrho_2 (1 - d)) (1 - d)} + \frac{\varrho_2 (1 - d)}{(1 - \varrho_1) - \varrho_2 (1 - d)}, \quad (5.8)$$

where d is defined by (5.2). Finally, EV_2 and EW_2 are given by Little's formula and (4.10), respectively.

References

- [Asm] Asmussen S., *Applied Probability and Queues*. Wiley, Chichester, 1987.
- [BBH] Baccelli, F., Boyer, P., Hebuterne, G., Single server queues with impatient customers. *Adv. Appl. Probab.* 16 (1984) 887–905.
- [BH] Baccelli, F., Hebuterne, G., On queues with impatient customers. In: F.J. Kylstra (ed.), *Performance '81*. North-Holland, Amsterdam, 1981, 159–179.
- [BKL] Bae, J., Kim, S., Lee, E.Y., The virtual waiting time of the $M/G/1$ queue with impatient customers. *Queueing Systems* 38 (2001) 485–494.
- [Ba1] Barrer, D.Y., Queueing with impatient customers and indifferent clerks. *Oper. Res.* 5 (1957) 644–649.
- [Ba2] Barrer, D.Y., Queueing with impatient customers and ordered service. *Oper. Res.* 5 (1957) 650–656.
- [BB1] Brandt, A., Brandt, M., On the $M(n)/M(n)/s$ queue with impatient calls. *Performance Evaluation* 35 (1999) 1–18. Preprint SC 97-19, Konrad-Zuse-Zentrum für Informationstechnik Berlin, available by <http://www.zib.de/PaperWeb/abstracts/SC-97-19>.
- [BB2] Brandt, A., Brandt, M., On a two-queue priority system with impatience and its application to a call center. *Methodology and Computing in Applied Probability* 1 (1999) 191–210. Preprint

SC 98-21, Konrad-Zuse-Zentrum für Informationstechnik Berlin,
available by
<http://www.zib.de/PaperWeb/abstracts/SC-98-21>.

- [BB3] Brandt, A., Brandt, M., Asymptotic results and a Markovian approximation for the $M(n)/M(n)/s+GI$ system. *Queueing Systems* 41 (2002) 73–94. ZIB-Report 00-12, Konrad-Zuse-Zentrum für Informationstechnik Berlin, available by
<http://www.zib.de/PaperWeb/abstracts/ZR-00-12>.
- [Br1] Brodi, S.M., On an integro-differential equation for systems with τ -waiting. *Dop. AN URSSR* 6 (1959) 571–573 (in Ukrainian).
- [Br2] Brodi, S.M., On a service problem. In: *Tr. V. Vsesojuznogo Sovetskaniya po Teorii Veroyatnostej i Matematičeskoj Statistike*. Izd-vo AN Arm. SSR, Erevan, 1960, 143–147 (in Russian).
- [CKC] Choi, B.D., Kim, B., Chung, J., $M/M/1$ queue with impatient customers of higher priority. *Queueing Systems* 38 (2001) 49–66.
- [Dal] Daley, D.J., General customer impatience in the queue $GI/G/1$. *J. Appl. Probab.* 2 (1965) 186–205.
- [GKö] Gnedenko, B.W., König, D., *Handbuch der Bedienungstheorie*. Vol. II, Akademie-Verlag, Berlin, 1984.
- [GKo] Gnedenko, B.W., Kowalenko, I.N., *Einführung in die Bedienungstheorie*. Akademie-Verlag, Berlin, 1974 (1st ed. in Russian, Nauka, Moscow, 1966).
- [GH] Gross, D., Harris, C.M., *Fundamentals of Queueing Theory*. Wiley, New York, 1998.
- [HS] Haugen, F.R.B., Skogan, E., Queueing systems with stochastic time out. *IEEE Trans. Commun.* COM-28 (1980) 1984–1989.
- [Jai] Jaiswal, N., *Priority Queues*. Academic Press, New York, 1968.
- [Ju1] Jurkevič, O.M., On many-server systems with stochastic bounds for the waiting time. *Izv. Akad. Nauk SSSR Techničeskaja Kibernetika* 4 (1971) 39–46 (in Russian).
- [Ju2] Jurkevič, O.M., On many-server systems with general service time distribution of "impatient customers". *Izv. Akad. Nauk SSSR Techničeskaja Kibernetika* 6 (1973) 80–89 (in Russian).

- [KBL] Kim, S., Bae, J., Lee, E.Y., Busy periods of Poisson arrival queues with loss. *Queueing Systems* 39 (2001) 201–212.
- [Mil] Miller, R.G., Priority queues. *Ann. Math. Statist.* 31 (1960) 86–103.
- [Mov] Movaghar, A., On queueing with customer impatience until the beginning of service. *Queueing Systems* 29 (1998) 337–350.
- [Pal] Palm, C., Methods of judging the annoyance caused by congestion. *Tele 2* (1953) 1–20.
- [Sin] Singh, M., Steady state behaviour of serial queueing processes with impatient customers. *Math. Operationsforsch. Ser. Statist.* 15 (2) (1984) 289–298.
- [Teg] Teghem, J., Use of discrete transforms for the study of a $GI/M/s$ queue with impatient customer phenomena. *Z. Oper. Res.* 23 (1979) 95–106.