Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

STEFAN HOUGARDY* ANNEGRET WAGLER

Perfectness is an Elusive Graph Property

*Humboldt-Universität zu Berlin, Institut für Informatik, Unter den Linden 6, D-10099 Berlin, Germany, hougardy@informatik.hu-berlin.de

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Perfectness is an Elusive Graph Property

Stefan Hougardy[†] Annegret Wagler[‡]

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Abstract

A graph property is called elusive (or evasive) if every algorithm for testing this property has to read in the worst case $\binom{n}{2}$ entries of the adjacency matrix of the given graph. Several graph properties have been shown to be elusive, e.g. planarity [2] or k-colorability [4]. A famous conjecture of Karp [11] says that every non-trivial monotone graph property is elusive. We prove that a non-monotone but hereditary graph property is elusive: perfectness.

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Key words. Perfect graphs, elusiveness, graph property testing

1 Introduction

Given a graph property, consider the following two-players game to define elusiveness. Player **A** wants to know whether an unknown simple graph on a given node set has the graph property in question by asking Player **B** one by one, whether a certain pair of nodes is an edge. At each stage Player **A** makes full use of the information of edges and non-edges he has up to that point in order to decide whether the graph has the property or not. Player **A** wants to minimize the number of his questions, Player **B** wants to force him to ask as many questions as possible. The number of questions needed for the decision if both players play optimally from their point of view is the recognition complexity of the studied graph property. The property is said to be **elusive** (or also evasive) if there is a strategy enabling Player **B** to force

[†]Humboldt-Universität zu Berlin, Institut für Informatik, Unter den Linden 6, D-10099 Berlin, Germany, hougardy@informatik.hu-berlin.de
[‡]Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB), Takustr. 7, D-14195 Berlin, Germany, wagler@zib.de

Player **A** to test *every* pair of nodes resp. to ask all possible $\binom{n}{2}$ questions before coming to a decision. (More precisely, such a strategy has to exist for all non-trivial cases, i.e., for all n such that there are graphs on n nodes with and without the studied property.)

Several graph properties are known to be elusive, e.g., having a clique of a certain size or a coloring with a certain number of color classes (Bollobás [3]) and being planar for graphs on ≥ 5 nodes (Best et al. [2]), see [4, 12] for more examples. On the other hand, it is well-known that there exist non-trivial graph properties that need only $O(n)$ questions, see [2, 4].

Aanderaa and Rosenberg conjectured [2] that there exists some $\gamma > 0$ such that the complexity of every non-trivial monotone graph property (i.e., a property preserved under deleting edges) is at least γn^2. This conjecture has been proved by Rivest and Vuillemin [10] for $\gamma = \frac{1}{16}$. The value of γ has been improved over the years. Currently the largest value of γ for which the conjecture of Aanderaa and Rosenberg is known to be true is $\frac{1}{4} - o(1)$. This result was established by Kahn, Saks and Sturtevant [7]. A sharpened version of the Aanderaa-Rosenberg conjecture is due to Karp [11]. He conjectures that every non-trivial monotone graph property is elusive.

The subject of the present paper is to prove:

Theorem 1 *Perfectness is an elusive graph property.*

Perfectness is a property which is not monotone but hereditary (preserved under deleting nodes) and concerned to the relation of maximum cliques and optimal colorings. Perfect graphs behave nicely from an algorithmic point of view [6] and have interesting relationships to surprisingly many other fields of scientific enquiry [9]. However, the recognition problem for perfect graphs is unsolved and its complexity is even unknown. The present paper contributes some information by showing: *perfectness is a graph property as complex as possible, namely, elusive.*

Berge [1] proposed to call a graph G **perfect** if, for each of its (node-) induced subgraphs $G' \subseteq G$, the chromatic number equals the clique number (i.e., if we need as many stable sets to cover all nodes of G' as a maximum clique of G' has nodes). This means that identifying one induced imperfect subgraph would enable Player **A** to make the final decision: the graph in question is not perfect. For that, so-called **minimally imperfect graphs** are of particular interest (that are imperfect graphs all of their proper induced subgraphs are perfect). The only known examples of minimally imperfect graphs are chordless odd cycles of length ≥ 5, termed **odd holes**,

and their complements, called **odd antiholes**. (A famous conjecture due to Berge [1] says that odd holes and odd antiholes are the only minimally imperfect graphs.) Consequently, Player **B** has to answer in such a way that no minimally imperfect induced subgraph appears until Player **A** asks the last question but that the last answer can create a minimally imperfect induced subgraph.

The odd hole of length five is the smallest imperfect graph. Hence, the cases with $n \leq 4$ nodes are trivial: Player **A** knows without asking any question that the studied graph is perfect. In order to show that perfectness is an elusive graph property we have to treat the non-trivial cases $n \geq 5$.

The idea for providing a strategy to Player **B** goes as follows. Find perfect graphs such that you cannot reach another perfect graph by deleting or adding one edge. We call an edge e of a perfect graph G **critical** if $G - e$ is imperfect. Analogously, we call an edge e not contained in a perfect graph G **anticritical** if $G + e$ is imperfect. A perfect graph G is **critical** if G has only critical edges. The complement of a critically perfect graph is again perfect (due to Lovász [8]) and has the property that adding an edge not contained in the graph so far yields an imperfect graph. We call the complements of critically perfect graphs **anticritically perfect**. We look for **bicritically perfect graphs** which are both critically *and* anticritically perfect: the deletion and addition of an *arbitrary* edge yields an imperfect graph.

If there exists a bicritically perfect graph G_n, then Player **B** has only to answer all but the last question "$ij \in E$?" of Player **A** as in G_n. I.e., Player **B** has only to apply the following strategy for graphs on n nodes.

Strategy 1 Let G_n be a bicritically perfect graph on n nodes.
For questions 1 to $\binom{n}{2} - 1$:
Answer "$ij \in E$?" with YES if $ij \in E(G_n)$, NO otherwise.

Then no induced imperfect subgraph appears during the first $\binom{n}{2} - 1$ questions, and the answer to the last question yields the decision:

$$\text{Answer "}ij \in E\text{?"with } \begin{cases} \text{YES} & \text{if } ij \in E(G_n) \quad \text{then the graph is perfect} \\ \text{NO} & \text{if } ij \in E(G_n) \quad \text{then the graph is imperfect} \\ \text{YES} & \text{if } ij \notin E(G_n) \quad \text{then the graph is imperfect} \\ \text{NO} & \text{if } ij \notin E(G_n) \quad \text{then the graph is perfect} \end{cases}$$

In order to prove Theorem 1, our task is:

Problem 1 *Find, for as many n as possible, a bicritically perfect graph G_n on n nodes.*

A first step towards Problem 1 was a computer search enumerating which perfect graphs on upto 10 nodes are critically perfect.

Theorem 2 *No critically perfect graphs with fewer than* 9 *nodes exist. On* 9 *and* 10 *nodes there are precisely* 3 *and* 10 *critically perfect graphs, resp.*

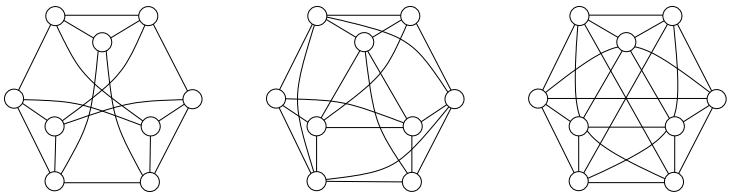

Figure 1: The three smallest critically perfect graphs.

Clearly, Theorem 2 remains true if "critically perfect" is replaced by "anti-critically perfect". Figure 1 shows the three critically perfect graphs on nine nodes. The first graph is self-complementary and, therefore, also anticritical. I.e., it is our first example G_9 of a bicritically perfect graph. The other two graphs are not anticritical, but only their complements are. Every of the critically perfect graphs with ten nodes is not anticritical (see next section). That means particularly: there are no bicritically perfect graphs G_n with $n \leq 8$ and $n = 10$. In Section 2, we present a technique of constructing examples of bicritically perfect graphs that bases on the characterization of critically and anticritically perfect line graphs. In Section 3, we apply the knowledge from the previous section to construct the studied bicritically perfect graphs G_n if $n \geq 12$. Section 4 provides a slightly different strategy for the cases $n = 10, 11$.

The cases $5 \leq n \leq 8$ are treated as follows. The odd hole C_5 is the *only* imperfect graph on five nodes (note: the C_5 is self-complementary, hence also the odd antihole on five nodes). Consequently, one cannot reach another imperfect graph from the C_5 by deleting or adding one edge. Thus, the C_5 is bicritically imperfect and Strategy 1 does also work for $n = 5$ with choosing $G_n = C_5$. For $6 \leq n \leq 8$ there is no bicritically imperfect graph with n nodes. In order to treat these cases we do not provide an explicit strategy but we show in Section 5 that there *exists* a strategy: we prove elusiveness for $6 \leq n \leq 8$ with the help of a result of Rivest and Vuillemin [10] by using a parity argument and doing some computer search.

In summary, we show the existence of a strategy in all non-trivial cases $n \geq 5$ which proves Theorem 1: *perfectness is an elusive graph property.*

2 Bicritically Perfect Line Graphs

This section provides characterizations of critically or anticritically perfect line graphs established in [13, 14]. We obtain the **line graph** $L(F)$ of a graph F by taking the edges of F as nodes of $L(F)$ and joining two nodes of $L(F)$ by an edge iff the corresponding edges of F are incident. It is well-known that the line graph $L(F)$ of a graph F is perfect iff F is **line-perfect**, i.e., iff F does not contain any odd cycle of length at least 5 as a weak subgraph.

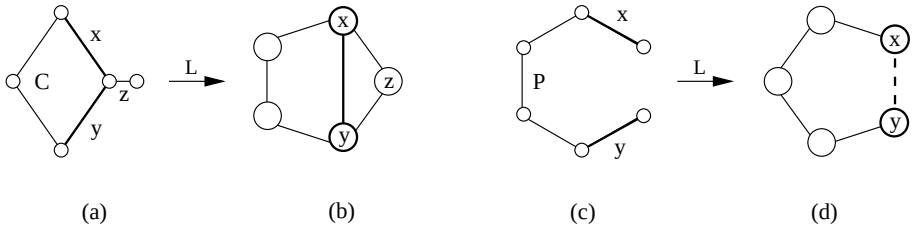

Figure 2: Definition of H-pairs and A-pairs.

In order to characterize critical and anticritical edges in $L(F)$, we define two structures in F. We say that two incident edges x and y form an **H-pair** in F if there is an edge z incident to the common node of x and y and if there is an even cycle C containing x and y but only one endnode of z (see Figure 2(a)). $L(C)$ is an even hole and the node in $L(F)$ corresponding to z has precisely two neighbors on $L(C)$, namely x and y (see Figure 2(b)). Two non-incident edges x and y are called an **A-pair** if they are the endedges of an odd path P with length at least five (see Figure 2(c)). $L(P)$ is an even, chordless path of length at least four with endnodes x and y (see Figure 2(d)). It is straightforward that deleting and adding the edge xy in $L(C \cup z)$ and $L(P)$, respectively, yields an odd hole. In [13, 14] it is established that xy is a critical (anticritical) edge in $L(F)$ *only if* x and y form an H-pair (A-pair) in F.

Theorem 3 [13, 14] *Let G be the line graph of a line-perfect graph F. An edge xy of G (not contained in G) is critical (anticritical) if and only if x and y form an H-pair (A-pair) in F.*

Consequently, if $L(F)$ is intended to be critically (anticritically) perfect, *every* pair of incident (non-incident) edges in F must form an H-pair (A-pair). We define a graph with at least two incident (non-incident) edges to be an **H-graph** (**A-graph**) if each pair of incident (non-incident) edges

forms an H-pair (A-pair). It is an immediate consequence of Theorem 3 that F has to be a line-perfect H-graph (A-graph) if $L(F)$ is intended to be critically (anticritically) perfect. In [13, 14] it was shown that F must be *bipartite* in both cases (i.e. without any odd cycle).

Finding examples of critically (anticritically) perfect line graphs means, therefore, to look for bipartite H-graphs (A-graphs). The three smallest critically perfect graphs are the complements of the line graphs of the three bipartite A-graphs presented in Figure 3. A_1 is also an H-graph, hence $L(A_1)$ is bicritical (it is in fact self-complementary). Furthermore, A_1 is the only bipartite H-graph with 3 nodes in each color class. If there are 4 nodes in one color class, then an H-graph has at least 12 edges since it has minimum degree 3 by definition. Hence, the second smallest H-graph admits 12 edges and there cannot be any bicritically perfect *line* graph on 10 or 11 nodes.

Remark 4 All anticritically perfect graphs on 10 nodes are line graphs of bipartite A-graphs (which arise from A_1, A_2, and A_3 by adding one edge). Thus, there is no bicritically perfect graph on 10 nodes. It is unknown so far whether there exist bicritically perfect graphs on 11 nodes but it is clear that there is no such line graph.

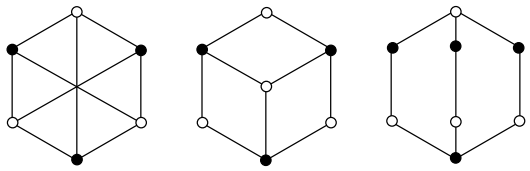

Figure 3: The three smallest bipartite A-graphs.

The following sufficient condition for a bipartite graph F to be an H-graph *and* an A-graph is established in [14].

Lemma 5 [14] *Every simple, 3-connected, bipartite graph is an H-graph as well as an A-graph.*

Remark 6 Note that duplicating edges preserves the property of being an A-graph (since no new pair of non-incident edges occurs) while it does not preserve the property of being an H-graph (since parallel edges are incident but never form an H-pair).

3 Construction of the graphs G_n for $n \geq 12$

In order to treat Problem 1, this section is intended to present a bicritically perfect graph G_n for each $n \geq 12$. Lemma 5 ensures that $L(F)$ is bicritically perfect whenever F is a simple, 3-connected, bipartite graph. Hence we will construct simple, 3-connected, bipartite graphs F_n with $n \geq 12$ edges to obtain the studied bicritically perfect graphs $G_n = L(F_n)$ on $n \geq 12$ nodes. Consider the graphs $F_{3k} = (A \cup B, E_1 \cup E_2)$ with $k \geq 3$ and

$$\begin{aligned} A &= \{1, 3, \ldots, 2k-1\} \\ B &= \{2, 4, \ldots, 2k\} \\ E_1 &= \{ii+1 : 1 \leq i \leq 2k \bmod(2k)\} \\ E_2 &= \{ii+3 : i \in A\} \end{aligned}$$

The three smallest examples of graphs F_{3k} for $k \in \{3, 4, 5\}$ are shown in Figure 4 (note $A_1 = F_9$). F_{3k} is an even cycle $(A \cup B, E_1)$ on its $2k$ nodes with k chords E_2 outgoing from a node in A with odd index and ending in a node in B with even index. Thus, the graphs F_{3k} are bipartite and simple by construction. We have to show that they are 3-connected.

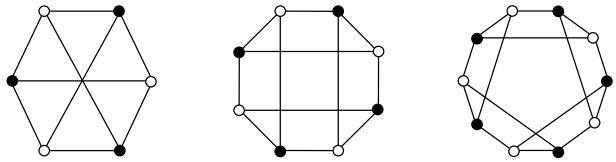

Figure 4: The graphs F_{3k} with $k = 3, 4, 5$.

Lemma 7 *The graphs F_{3k} are 3-connected for $k \geq 3$.*

Proof. We have to show that the graph obtained from F_{3k} by removing two arbitrary nodes i and j is still connected. Let $i < j$. Recall that $F_{3k} = (A \cup B, E_1 \cup E_2)$ has a Hamilton cycle $C = (A \cup B, E_1)$ and additional chords $ii+3 \in E_2$ with $i \in A$ odd, $i+3 \in B$ even. If i and j are neighbors on C (i.e., $j = i+1$) then the remaining nodes $i+2 = j+1, \ldots, 2k, 1, \ldots, i-1$ of F_{3k} are connected by a path with edges in E_1. Otherwise, removing i and j divides the cycle C into two paths $P_1 = i+1, \ldots, j-1$ and $P_2 = j+1, \ldots, 2k, 1, \ldots, i-1$. It is easy to see that there is always an edge $e \in E_2$ which connects P_1 and P_2: If i is odd, then $i+1$ is even and

$i-2i+1 \in E_2$. We have $e = i-2i+1$ as the studied edge connecting P_1 and P_2 if $i-2$ is a node of P_2 or else $V(P_2) = \{i-1\}$ and $j = i-2$ holds. But then $i-4$ is a node of P_1 (since $k \geq 3$) and we obtain $e = i-4i-1$ (all indices are taken modulo $2k$). Analogously, if i is even, then $i-1i+2 \in E_2$. We have $e = i-1i+2$ if $i+2$ is a node of P_1 or else $V(P_1) = \{i+1\}$ and $j = i+2$. But then $i+4$ is a node of P_2 (by $k \geq 3$ again) and we get $e = i+1i+4$. Hence, the graph obtained from F_{3k} by removing two arbitrary nodes is still connected. □

Thus, we can choose G_n as the line graph of F_n whenever $n = 3k, k \geq 3$ by Lemma 5. To close the gaps with $n = 3k+1, 3k+2$ for $k \geq 4$ we use the following immediate consequence of Lemma 5.

Lemma 8 *If $F = (A \cup B, E)$ is a simple, 3-connected, bipartite graph and $ab \notin E$ with $a \in A, b \in B$, then $F + ab$ is a simple bipartite A- and H-graph.*

Thus, we obtain the studied bipartite A- and H-graphs F_n for $n = 3k+1$ and $n = 3k+2$ if $k \geq 4$ by adding one and two edges, respectively, to F_{3k} such that the resulting graph is simple and bipartite. This is possible for each F_{3k} with $k \geq 4$ (but not for the complete bipartite graph F_9). We obtain, therefore, the studied bicritically perfect graphs $G_n = L(F_n)$ for $n \geq 12$ and can apply Strategy 1 for all cases with $n \geq 12$ nodes.

4 Construction of the graphs G_n for $n = 10, 11$

Due to Remark 4 there is no bicritically perfect graph on 10 nodes and no bicritically perfect line graph on 11 nodes. We construct bipartite A-graphs with 10 and 11 edges which are closest to H-graphs as possible.

Duplicating an arbitrary edge of $F_9 = A_1$ yields the graph F_{10} shown in Figure 5. F_{10} is an A-graph but not an H-graph by Remark 6. However, $L(F_{10})$ has only one non-critical edge, namely, the edge connecting the nodes that correspond to the parallel edges of F_{10}. Next, the bipartite graph F_{11} in Figure 5 can be obtained by adding two edges to the A-graph A_2 from Figure 3. It is easy to check that F_{11} is an A-graph and that the two edges incident to the only node of degree two in F_{11} form the only non-H-pair. Theorem 3 implies that $L(F_{11})$ is anticritically perfect and has all but one critical edges, too.

Let us call a graph G **almost bicritically perfect** if G is anticritically perfect and all but one edges are critical. Then we slightly modify Strategy 1 for almost bicritically perfect graphs as follows:

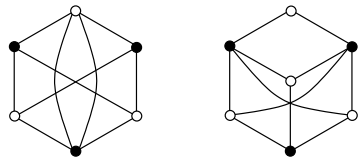

Figure 5: The graphs F_{10} and F_{11}.

Strategy 2 Let G_n be an almost bicritically perfect graph on n nodes and let uv be its only non-critical edge.
Question 1:

Answer "$ij \in E$?" with YES. Number the nodes of G_n s.t. $i = u$, $j = v$.
For questions 2 to $\binom{n}{2} - 1$:

Answer "$ij \in E$?" with YES if $ij \in E(G_n)$, NO otherwise.

Then no imperfect subgraph appears during the first $\binom{n}{2} - 1$ questions, and the answer to the last question yields the decision again. Since $L(F_{10})$ and $L(F_{11})$ are almost bicritically perfect graphs by construction, we choose $G_{10} = L(F_{10})$ and $G_{11} = L(F_{11})$ and apply Strategy 2.

5 The Remaining Cases $6 \leq n \leq 8$

To prove that perfectness is an elusive graph property for $6 \leq n \leq 8$ we use a parity argument due to Rivest and Vuillemin. In [10] they proved the following: if a property $\mathcal{P}$ is *not* elusive for graphs on n nodes then the number $G(\mathcal{P}, n, \text{even})$ of labeled graphs on n nodes with property $\mathcal{P}$ having an even number of edges equals the number $G(\mathcal{P}, n, \text{odd})$ of labeled graphs on n nodes with property $\mathcal{P}$ that have an odd number of edges. In particular, $G(\mathcal{P}, n, \text{even}) \neq G(\mathcal{P}, n, \text{odd})$ implies that $\mathcal{P}$ *is* elusive for graphs on n nodes. In Table 1 we show the numbers $G(\mathcal{P}, n, \text{even})$ and $G(\mathcal{P}, n, \text{odd})$ for perfect graphs on $6 \leq n \leq 8$ nodes which we calculated with the help of a computer program. To check whether a graph is perfect we made use of the Strong Perfect Graph Conjecture that is known to be true for all graphs with at most 25 nodes [5]. As one can see from Table 1, for $n = 8$ perfectness is an elusive graph property as the values in column 3 and column 4 differ.

For $n = 6$ and $n = 7$ we apply an extension of the previously used argument. If perfectness is not elusive for graphs on n nodes then it is also not elusive for the graphs containing one fixed edge, say ij. Therefore the

n	# perfect graphs	$G(\mathcal{P}, n, \text{even})$	$G(\mathcal{P}, n, \text{odd})$
6	30824	15412	15412
7	1741616	870808	870808
8	174494128	87264704	87229424

Table 1

number of labeled perfect graphs on n nodes which contain the edge ij and have an even number of edges must equal the number of these graphs with an odd number of edges. The last two columns in Table 2 show these numbers for $n = 6$ and $n = 7$. Note that they add up to half the number of labeled perfect graphs, as the complement of a perfect graph is again perfect [8]. As the numbers in column 3 and column 4 are different in both cases, this finishes the proof that perfectness is elusive for $n = 6, 7, 8$.

n	# perfect graphs	$G(\mathcal{P}, n, \text{even})$, $ij \in E$	$G(\mathcal{P}, n, \text{odd})$, $ij \in E$
6	30824	7712	7700
7	1741616	435284	435524

Table 2

6 Summary

In order to figure out whether perfectness is an elusive graph property, we used as main idea: Find, for as many numbers n of nodes as possible, a bicritically perfect graph G_n (Problem 1). Since one cannot reach another perfect graph from G_n by deleting or adding one edge, there is a simple strategy for Player **B** in that case: Answer all but the last question as in the bicritically perfect graph G_n (Strategy 1). We constructed bicritically perfect graphs G_n with $n = 9$ and $n \geq 12$ (Section 3) and almost bicritically perfect graphs G_{10} and G_{11} (Section 4) where a slightly different strategy has to be used (Strategy 2). Moreover, the C_5 is bicritically imperfect and Strategy 1 does also work for $n = 5$ with choosing $G_n = C_5$. Consequently, our main idea works for $n = 5$ and for all cases with $n \geq 9$ nodes. We used a parity argument from [10] in order to show the desired result for the remaining cases with $6 \leq n \leq 8$ nodes (Section 5).

In summary, we showed the *existence* of a strategy for Player **B** in all non-trivial cases $n \geq 5$ which finally proves Theorem 1: *Perfectness is an elusive graph property.*

References

[1] C. Berge, *Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind*, Wiss. Zeitschrift der Martin-Luther-Universität Halle-Wittenberg (1961) pp. 114-115

[2] M.R. Best, P. van Emde Boas, and H.W. Lenstra, *A sharpened version of the Aanderaa-Rosenberg Conjecture*, Mathematisch Centrum Amsterdam, Afd. Zuivere Wisk. (1974)

[3] B. Bollobás, *Complete Subgraphs are Elusive*, J. Comb. Theory B 21 (1976) pp. 1-7

[4] B. Bollobás, *Extremal Graph Theory*, Academic Press (1978)

[5] E. Boros, V. Gurvich, and S. Hougardy, *Recursive generation of partitionable graphs*, to appear in J. Graph Theory

[6] M. Grötschel, L. Lovász, and A. Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer-Verlag (1988)

[7] J. Kahn, M. Saks, and D. Sturtevant, *A topological approach to evasiveness*, Combinatorica 4 (1984) pp. 297–306

[8] L. Lovász, *Normal Hypergraphs and the Weak Perfect Graph Conjecture*, Discrete Math. 2 (1972) pp. 253-267

[9] B. Reed and J. Ramirez-Alfonsin, *Perfect Graphs*, Wiley (2001)

[10] R.L. Rivest and J. Vuillemin, *On recognizing graph properties from adjacency matrices*, Theor. Comput. Sci. 3 (1976/77) pp. 371–384

[11] A.L. Rosenberg, *On the time required to recognize properties of graphs: A problem*, SIGACT News (1973) pp. 15–16

[12] E. Triesch, *Some Results on Elusive Graph Properties*, SIAM J. Comput. 23 (1994) pp. 247-254

[13] A. Wagler, *On Critically Perfect Graphs*, J. Graph Theory 32 (1999) pp. 394-404

[14] A. Wagler, *Critical Edges in Perfect Graphs*, PhD thesis, TU Berlin (2000)