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Antiwebs are Rank-Perfect

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Abstract

We discuss a nested collection of three superclasses of perfect graphs: near-perfect, rank-perfect, and weakly rank-perfect graphs. For that, we start with the description of the stable set polytope for perfect graphs and allow stepwise more general facets for the stable set polytopes of the graphs in each superclass. Membership in those three classes indicates how far a graph is away from being perfect. We investigate for webs and antiwebs to which of the three classes they belong. We provide a complete description of the facets of the stable set polytope for antiwebs (with help of a result due to Shepherd on near-bipartite graphs). The main result is that antiwebs are rank-perfect.

1 Introduction

BERGE [1] proposed to call a graph $G = (V, E)$ **perfect** if, for each (node-induced) subgraph $G' \subseteq G$, the chromatic number equals the clique number. That is, for all $G' \subseteq G$, we need as many stable sets to cover all nodes of G' as a maximum clique of G' has nodes (a set $V' \subseteq V$ is a clique (stable set) if the nodes in V' are mutually (non-)adjacent; maximum cliques (stable sets) contain a maximal number of nodes).

BERGE [1] conjectured two characterizations of perfect graphs. His first conjecture was that *a graph G is perfect if and only if its complement $\overline{G}$ is* ($\overline{G}$ denotes the graph on the same node set as G where two nodes are adjacent iff they are non-adjacent in G). This conjecture was proven by LOVÁSZ [7]. The second BERGE conjecture concerns a characterization via forbidden subgraphs. It is a simple observation that chordless odd cycles C_{2k+1} with $k \geq 2$, termed **odd holes**, and their complements $\overline{C}_{2k+1}$, called **odd antiholes**, are imperfect. Clearly, each graph containing an odd hole or an odd antihole as subgraph is imperfect as well. BERGE [1] conjectured: *a*

graph is perfect if and only if it contains neither odd holes nor odd antiholes as subgraphs (Strong Perfect Graph Conjecture). This conjecture is still open and one of the most famous conjectures in graph theory.

PADBERG [9, 10] asked which graphs are "almost" perfect, i.e., which graphs are imperfect with the property that all of their proper induced subgraphs are perfect. Such graphs are nowadays called **minimally imperfect**. Using this term, the Strong Perfect Graph Conjecture reads: *odd holes and odd antiholes are the only minimally imperfect graphs.* In order to give a characterization of minimally imperfect graphs (and thereby to verify or falsify the Strong Perfect Graph Conjecture), many fascinating structures of such graphs have been discovered, e.g., properties reflecting an extraordinary amount on symmetry of their maximum cliques and stable sets [7, 9].

A common generalization of odd holes and odd antiholes is a class of graphs with circular symmetry of their maximum cliques and stable sets, so-called webs: A **web** W_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge if i and j differ by at most k (i.e., if $|i-j| \leq k \bmod n$). We assume $k \geq 1$ and $n \geq 2(k+1)$ in the following in order to exclude the degenerated cases when W_n^k is a clique. W_n^1 is the hole C_n and W_{2k+1}^{k-1} the odd antihole $\overline{C}_{2k+1}$ for $k \geq 2$. Two further examples of webs are depicted in Figure 1. (Note that webs are also known as circulant graphs C_n^k [3]. Furthermore, graphs $W(n,k)$ with $n \geq 2$, $1 \leq k \leq \frac{1}{2}n$ and $W(n,k) = \overline{W}_n^{k-1}$ were introduced in [13].) The complements of webs, called **antiwebs**, have clearly also a circular symmetry of their maximum cliques and stable sets and contain all odd holes and odd antiholes, see Figure 1 for two further examples.

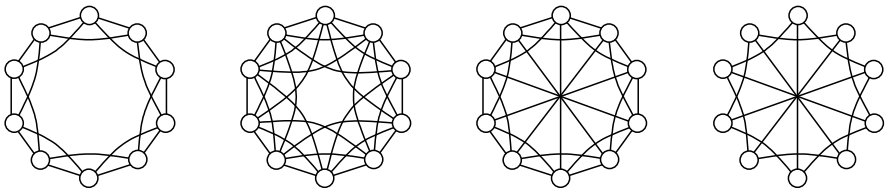

Figure 1: Examples of webs and antiwebs.

In the next section, we introduce a nested collection of superclasses of perfect graphs: near-perfect, rank-perfect, and weakly rank-perfect graphs. For that, we start with the description of the stable set polytope for perfect graphs [2, 9] and allow stepwise more general facets for the stable set polytopes of the graphs in each superclass. Hence, each superclass contains graphs which are step by step "less" perfect and membership in those three

graph classes indicates how close or how far a graph is from being perfect (see [14, 15] for more details). In [15] it was asked to which of the three superclasses webs and antiwebs belong. We discuss this question in Section 3. We give characterizations from [15] which webs and antiwebs are near-perfect. We further present known examples of rank-perfect webs from [4] but also webs which are not rank-perfect by [6]. The main contribution of the present paper is Theorem 5 which gives a complete description of the facets of the stable set polytopes for antiwebs (with help of a result due to SHEPHERD [12] that a superclass of antiwebs, so-called near-bipartite graphs, consists of weakly rank-perfect graphs only). Theorem 5 implies to which of the three superclasses of perfect graphs antiwebs belong:

Theorem 1 *Antiwebs are rank-perfect.*

2 Three Superclasses of Perfect Graphs

A well-known characterization of perfect graphs [2, 9] says that *a graph is perfect if and only if its stable set polytope coincides with its fractional stable set polytope.* The **stable set polytope** STAB(G) of a graph G is defined as the convex hull of the incidence vectors of all stable sets of G. A relaxation of STAB(G), the **fractional stable set polytope** QSTAB(G), is given by all "trivial" facets, the **nonnegativity constraints**

$$x_i \geq 0 \tag{0}$$

for all nodes i of G, and by the **clique constraints**

$$\sum_{i \in Q} x_i \leq 1 \tag{1}$$

for all cliques $Q \subseteq G$. Since a clique and a stable set have at most one node in common, QSTAB(G) contains all incidence vectors of stable sets of G, thus STAB(G) $\subseteq$ QSTAB(G) holds for all graphs G. If G is an imperfect graph, then STAB(G) $\subset$ QSTAB(G) follows by [2, 9] and QSTAB(G) has in particular also fractional extreme points. The difference between STAB(G) and QSTAB(G) can be used as a tool in order to decide how far a graph is away from being perfect. PADBERG [10] proved that a graph G is minimally imperfect if and only if QSTAB(G) has *precisely one* fractional extreme point which can be cut off by *exactly one* additional cutting plane, namely, the so-called **full rank constraint**

$$\sum_{i \in G} x_i \leq \alpha(G) \tag{2}$$

associated with G (where $\alpha(G)$ denotes the size of a maximum stable set in G). For minimally imperfect graphs G, the polytope QSTAB(G) is, therefore, the smallest possible relaxation of STAB(G).

The next relaxation of STAB(G) is the case when QSTAB(G) may have more than one fractional extreme point but, again, the full rank constraint is required as only cutting plane to cut off all those fractional extreme points. This lead SHEPHERD [11] to define **near-perfect graphs** where the stable set polytope is given by nonnegativity constraints (0), clique constraints (1), and the full rank constraint (2). Minimally imperfect graphs are obviously near-perfect. Since there is no requirement that QSTAB(G) has at least one fractional extreme point but only that all fractional extreme points are cut off by the full rank constraint, perfect graphs are near-perfect, too.

Following a suggestion of GRÖTSCHEL, LOVÁSZ, and SCHRIJVER [5] one may relax the notion of perfectness further by generalizing clique constraints to other classes of inequalities valid for the stable set polytope and then by investigating all graphs such that their stable set polytope is entirely described by nonnegativity constraints and the inequalities in question. A natural way to generalize both clique constraints and the full rank constraint is to consider *all* 0/1-inequalities, i.e., to investigate the **rank constraints**

$$\sum_{i \in G'} x_i \;\leq\; \alpha(G') \tag{3}$$

associated with arbitrary induced subgraphs $G' \subseteq G$ (note $\alpha(G') = 1$ holds if and only if G' is a clique). For convenience, we often write (3) as $x(G') \leq \alpha(G')$. We define all graphs G to be **rank-perfect** if STAB(G) is given by all nonnegativity constraints (0) and all rank constraints (3), i.e., if we need only 0/1-inequalities (3) to cut off all fractional extreme points of QSTAB(G). Every perfect, every minimally imperfect, and every near-perfect graph is obviously also rank-perfect, see [14, 15] for more examples. The main result of Section 3 is that antiwebs belong to the class of rank-perfect graphs.

For some cases, a sufficient condition is known when a rank constraint $x(G') \leq \alpha(G')$ associated with a *proper* subgraph $G' \subset G$ yields a facet of the stable set polytope of the whole graph G. PADBERG [8] showed, e.g., that clique constraints $x(Q) \leq 1$ are facet-inducing for STAB(G) iff Q is an (inclusion-wise) maximal clique of G. In general, a rank constraint associated with a proper subgraph $G' \subset G$ does not need to provide a facet of STAB(G), even if STAB(G') admits the full rank facet. However, rank constraints $x(G') \leq \alpha(G')$ with $G' \subset G$ may be strengthened to a facet

$$\sum_{i \in G'} x_i \;+ \sum_{i \in G - G'} a_i \, x_i \;\leq\; \alpha(G') \tag{4}$$

of STAB(G) using *sequential lifting* [8], i.e., by determining appropriate lifting coefficients a_i for all nodes i in $G - G'$ such that the right hand side $\alpha(G')$ of the inequality is still satisfied and that there are $|G|$ many stable sets of weight $\alpha(G')$ the incidence vectors of which are linearly independent. Every inequality (4) is a **weak rank constraint** if it is obtained by lifting a base rank constraint $x(G') \leq \alpha(G')$ which is facet-defining for STAB(G'), i.e., if G' produces the full rank facet. We define all graphs G to be **weakly rank-perfect** if STAB(G) is given by all nonnegativity constraints (0) and all weak rank constraints (4). Clearly, facet-defining rank constraints are weak rank constraints with $a_i = 0$ for $i \in G - G'$. Thus, weakly rank-perfect graphs contain all rank-perfect graphs.

We have finally obtained a nested collection of superclasses of perfect graphs: near-perfect, rank-perfect, and weakly rank-perfect graphs. The difference between QSTAB(G) and STAB(G) increases in each step, hence each superclass contains graphs which are gradually "less" perfect. Thus, membership in those three graph classes gives some information on the stage of imperfectness. We discuss in the next section to which of the three superclasses of perfect graphs webs and antiwebs belong. The main result is that antiwebs are rank-perfect.

3 Stage of Imperfectness for Webs and Antiwebs

In this section we discuss the stage of imperfectness of webs and antiwebs by treating the question: *which of the three superclasses of perfect graphs contains all webs and antiwebs?* We start our consideration with the class closest to perfect graphs: with near-perfect graphs. First, there are some perfect webs W_n^k and antiwebs $\overline{W}_n^k$, namely those with $k = 1$, n even and with $k \geq 1$, $n = 2k+2$: if n is even then W_n^1 is an even hole and $\overline{W}_n^1$ an even antihole, the antiweb $\overline{W}_{2k+2}^k$ consists of k disjoint edges. Perfect webs and antiwebs are obviously near-perfect. Next, the webs W_n^k and antiwebs $\overline{W}_n^k$ with $k = 1$, n odd and with $k \geq 1$, $n = 2k + 3$ are minimally imperfect: if n is odd then W_n^1 is an odd hole and $\overline{W}_n^1$ an odd antihole (note $n \geq 5$ holds by $n \geq 2k + 2$, odd), the webs W_{2k+3}^1 are odd antiholes and the antiwebs $\overline{W}_{2k+3}^1$ odd holes. Minimally imperfect webs and antiwebs are near-perfect by [10]. W_n^k and $\overline{W}_n^k$ contain an odd hole or odd antihole as proper subgraph if $k \geq 2$ and $n \geq 2k + 4$ by [13]. Thus, all those webs and antiwebs with $k \geq 2$ and $n \geq 2k+4$ are imperfect but not minimal. For imperfect graphs, a

necessary condition for the membership in the class of near-perfect graphs is to produce the full rank facet. From a result in [13] follows that a web W_n^{k-1} produces the full rank facet if and only if k is not a divisor of n (the two webs shown in Figure 1 satisfy this property). An antiweb $\overline{W}_n^{k-1}$ produces the full rank facet if and only if it is **prime**, i.e., if k and n are relatively prime [13]. The antiweb $\overline{W}_{10}^2$ in Figure 1 has this property but $\overline{W}_{10}^3$ not. (Recall from the definition of webs that we exclude the case of cliques). Thus, not all imperfect webs and antiwebs satisfy this necessary condition, hence webs and antiwebs are not all near-perfect. The next two theorems characterize those webs and antiwebs for which the full rank facet is the only facet of the stable set polytope besides facets of type (0) and (1).

Theorem 2 (WAGLER [15]) *A web is near-perfect if and only if it is perfect, an odd hole, W_{11}^2, or if it has stability number two.*

The definition of webs implies $\alpha(W_n^{k-1}) = \lfloor \frac{n}{k} \rfloor$. Thus $\alpha(W_n^{k-1}) = 2$ is satisfied for all webs W_n^{k-1} with $n < 3k$. Hence, there are other webs with stability number two than odd antiholes, e.g., W_8^2, W_{10}^3, and W_{11}^3. That means: there are near-perfect webs which are neither perfect nor minimally imperfect. This is not true for antiwebs:

Theorem 3 (WAGLER [15]) *An antiweb is near-perfect if and only if it is perfect, an odd hole, or an odd antihole.*

Our next question is: do webs and antiwebs belong to the class of rank-perfect graphs? DAHL [4] showed that webs W_n^2 are rank-perfect for all $n \geq 4$. But there are webs with clique number > 4 the stable set polytopes of which have non-rank facets (see KIND [6]), the web W_{25}^5 is one example. There are two ways to partition the nodes $1, \ldots, 25$ of W_{25}^5 such that the resulting two subgraphs induce a W_{15}^3 and a W_{10}^2: the node sets 1, 2, 3, 6, 7, 8, 11, 12, 13, 16, 17, 18, 21, 22, 23 and 1, 2, 4, 6, 7, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24 both induce a W_{15}^3 and the remaining nodes of W_{25}^5 induce a W_{10}^2 in both cases. The resulting facets of the form

$$x(W_{15}^3) + 2x(W_{10}^2) \leq 6$$

are obviously non-rank facets of STAB(W_{25}^5). Hence, webs are not all rank-perfect. We do not know so far which webs are rank-perfect. But scaling the above constraint to $\frac{1}{2}x(W_{15}^3) + x(W_{10}^2) \leq 3$ shows that this facet is a weak rank constraint basing on the prime antiweb constraint associated with W_{10}^2. It is not known so far whether all webs are at least weakly rank-perfect.

In the remaining part of this section, we show that antiwebs are rank-perfect. We use a result due to SHEPHERD [12] that gives a complete description of the stable set polytope of near-bipartite graphs, a superclass of antiwebs.

A graph G is **near-bipartite** if removing all neighbors of an arbitrary node leaves the graph bipartite. (That means, $G - N(v)$ can be partitioned into two stable sets for all nodes v of G.) SHEPHERD showed in [12] that *the stable set polytope of near-bipartite graphs has facets of type*

$$\sum_{i \le k} \frac{1}{\alpha(\overline{W}_i)} x(\overline{W}_i) + x(Q) \le 1 \tag{5}$$

associated with the complete join of prime antiwebs $\overline{W}_1, \ldots, \overline{W}_k$ *and a clique* Q *as only nontrivial facets.* (The **complete join** of two disjoint graphs G_1 and G_2 is obtained by joining every node of G_1 and every node of G_2 by an edge.) Note that the support graph of such facets arise by the complete join of graphs which all produce their full rank facet, i.e., we put together disjoint facet blocks. The obtained constraints can be scaled in such a way that they have the form (4) with a base rank constraint $x(G') \le \alpha(G')$ and (eventually noninteger) coefficients a_i for $i \in G - G'$. In this sense, (5) can be seen as a lifted clique constraint. That means, constraints of type (5) are weak rank constraints and near-bipartite graphs are, therefore, weakly rank-perfect.

Every antiweb is near-bipartite. (This can be easily seen since every web W_n^k has, by definition, the property that the neighborhood of every node i can be partitioned into two cliques, namely, $i-k, \ldots, i-1$ and $i+1, \ldots, i+k$ where all indices are taken modulo n). Thus, the only nontrivial facets of the stable set polytope of an antiweb are weak rank constraints (5) associated with the complete join of prime antiwebs and a clique due to [12]. TROTTER [13] proved that whenever a prime antiweb $\overline{W}_{n'}^{k'}$ occurs as subgraph of an antiweb $\overline{W}_n^k$, then STAB($\overline{W}_n^k$) has a rank facet associated with $\overline{W}_{n'}^{k'}$. We show that those facets are the only nontrivial facets of STAB($\overline{W}_n^k$) besides clique constraints by proving that no antiweb contains the complete join of two prime antiwebs or the complete join of a prime antiweb and a clique.

Lemma 4 *If an antiweb* $\overline{W}_n^k$ *contains the complete join of an antiweb* $\overline{W}_{n'}^{k'}$ *and a single node, then* $\overline{W}_{n'}^{k'}$ *is a stable set.*

Proof. Let $\overline{W}_n^k$ be an antiweb. Assume there is an antiweb $\overline{W}_{n'}^{k'} \subset \overline{W}_n^k$ completely joined to a node $i \in \overline{W}_n^k - \overline{W}_{n'}^{k'}$. Then the web W_n^k properly contains a web $W_{n'}^{k'}$ and a single node $i \in W_n^k - W_{n'}^{k'}$ such that there is no edge in W_n^k

connecting i and a node in $W^{k'}_{n'}$. By the definition of a web, $W^{k'}_{n'}$ has nodes in $W^k_n - \{i-k, \ldots, i, \ldots, i+k\}$ only (indices are taken modulo n). Let j be the node of $W^{k'}_{n'}$ with the smallest index $> i+k (\mathrm{mod}\, n)$. Then no node between $i+k$ and j is a neighbor of j in $W^{k'}_{n'}$ (by the choice of j). Thus all neighbors of j in $W^{k'}_{n'}$ belong to the set $\{j+1, \ldots, j+k\}$. Hence, the neighborhood of j in $W^{k'}_{n'}$ is a clique. From the circular symmetry of webs follows that $W^{k'}_{n'}$ is a clique and, therefore, $\overline{W}^{k'}_{n'}$ a stable set. $\Box$

The complete join of two antiwebs or of an antiweb and a (nonempty) clique always contains an antiweb completely joined to a single node in particular. By the above lemma, this antiweb has to be a stable set (i.e., a degenerated antiweb $\overline{W}^k_n$ with $k = 0$ or $n < 2(k+1)$ which we excluded by definition) and, therefore, it is no *prime* antiweb in particular. Hence, the above lemma implies: the only nontrivial facets of the stable set polytope of an antiweb are weak rank constraints which do not consist of different facet blocks but are associated with either a prime antiweb or a clique. Since both prime antiweb and clique constraints are rank constraints in particular, we have obtained:

Theorem 5 *The stable set polytope of an antiweb has as only nontrivial facets rank constraints associated either with cliques or with prime antiwebs.*

As an immediate consequence of Theorem 5 follows Theorem 1, i.e., the answer to the question to which of the three studied superclasses of perfect graphs antiwebs belong: *antiwebs are rank-perfect.*

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