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# **Individual Overflow and Freed Carried Traffics for a Link with Trunk Reservation**

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# Individual Overflow and Freed Carried Traffics for a Link with Trunk Reservation<sup>1</sup>

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## Abstract

Two traffic streams  $\Phi_1, \Phi_2$  are offered a link. The calls of  $\Phi_i$  require exponential holding times with parameter  $\mu$  and are accepted if less than  $C_i$  trunks are occupied. Approximating the  $\Phi_i$  by appropriate renewal processes meeting their first two moments, defined as the moments of the numbers of calls in virtual links of infinite capacity to which the traffic streams as freed traffics are virtually directed and where the calls get fresh exponential i.i.d. holding times with parameter  $\mu$ , stable recursive algorithms of complexity  $O(\max(C_1, C_2))$  are derived for the first two defined as above moments of the individual overflow and freed carried traffics. The results offer a unified handling of both overflow and carried traffics in circuit switching networks with trunk reservation, providing a basis for new two-moment network dimensioning algorithms.

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**Keywords:** trunk reservation; overflow traffic; freed carried traffic; factorial moments; peakedness; two-moment method; circuit switching network.

## 1 Introduction and model description

In teletraffic engineering a traffic stream arriving at a link in a circuit switching network can be described as *freed traffic* (ft) by a pair  $(\Phi, F(t))$ , where the first component is a stationary point process  $\Phi = \{T_n\}_{n=-\infty}^{\infty}$  on the real line  $\mathbb{R}$ , where  $\dots < T_0 \leq 0 < T_1 < \dots$  denote the arrival instants, cf. e.g. [BFL], and the second component is the distribution function of the i.i.d. holding times. Directing  $(\Phi, F(t))$  virtually to a virtual link of

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infinite capacity, where the holding times are i.i.d. sampled with distribution  $F(t)$ , i.e. considering a  $G/GI/\infty$  system, the factorial moments  $M_{(k)}$ ,  $k \in \mathbb{N} := \{1, 2, \dots\}$ , of the number of occupied trunks give the moment characterization of  $(\Phi, F(t))$ . In particular, the mean  $M$  and peakedness  $Z$ , defined by  $M := M_{(1)}$ ,  $Z := 1 - M_{(1)} + M_{(2)}/M_{(1)}$  (variance-to-mean ratio), give the two-moment characterization of  $(\Phi, F(t))$ . It is well-known that link blocking probabilities of peaky traffic ( $Z > 1$ ) can be substantially larger than those seen by Poisson traffic ( $Z = 1$ ) with the same mean. The opposite occurs in case of smooth traffic ( $Z < 1$ ). Thus there is a need of working with two-moment characterizations of traffic streams.

In case of a *freed overflow traffic* (fot)  $(\Phi', F(t))$ , whose arrival times are the arrival times of blocked by a link calls, this notion of peakedness  $Z'$  corresponds to the commonly used one. In case of a *freed carried traffic* (fct)  $(\Phi'', F(t))$ , cf. [v.Do], whose arrival times are the arrival times of accepted by a link calls, this notion of peakedness  $Z''$  differs from the variance-to-mean ratio of the number of busy trunks in the link considered,<sup>1</sup> a quantity which is normally used, cf. [Wilk], [Katz], [Desc], [Gira], and which corresponds to the description of a traffic stream by a stationary marked point process, where the marks are the holding times.

It were [HH] who observed the incommensurability of both concepts of traffics for the  $GI/M/C/0$  system (renewal arrival process, exponential holding times), when working with two-moment characterizations of traffic streams, and that the corresponding peakedness factors may differ considerably even in case of the carried traffic of a Poisson arrival process. In their paper a numerical algorithm for  $Z''$  is developed, but it is numerically instable for larger links and of rather high complexity, cf. [HH] or [BB2] Remark 3.1. These problems prevented for a long time the rigorous use of the concept of ft in network analysis algorithms, although this would be desirable as mentioned in [Gira] p.111. Therefore in practice widely a mixed concept is used (the same holding times are used for the carried traffic, but not in the whole network), cf. [Katz], [Desc]. However, recently the authors derived an explicit representation of  $Z''$  for the  $GI/M/C/0$  system offering a unified handling of both overflow and carried traffics in networks, cf. [BB2].

In the following we assume that the holding times are exponential, and by choosing the mean holding time as time unit, with parameter  $\mu = 1$ , i.e.  $F(t) := 1 - \exp(-t)$ ,  $t \in \mathbb{R}_+$ , and we suppress the second component  $F(t)$  in the description of a ft. Because of exponential holding times, for a given

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<sup>1</sup>Note that, although carried arrivals of calls are accepted simultaneously at the link considered and at the virtual link, the departure times are different in general.

mean  $M$ , the peakedness  $Z$  varies within the interval  $[Z_{min}(M), \infty)$ , where  $Z_{min}(M) := (1 - \exp(-M^{-1}))^{-1} - M$ , cf. [v.Do] p. 3, [FK]. If moreover the ft  $\Phi$  is a renewal process with inter-arrival time distribution  $A(t)$ , then its factorial moments  $M_{(k)}$  are given by, cf. e.g. [Taka],

$$M_{(k)} = \frac{1}{EA} \prod_{j=1}^{k-1} \frac{jA^*(j)}{1-A^*(j)}, \quad k \in \mathbb{N}, \quad (1.1)$$

where  $A^*(s)$  denotes the LST of  $A(t)$  and  $EA = -A^{*\prime}(0)$  is the mean inter-arrival time.

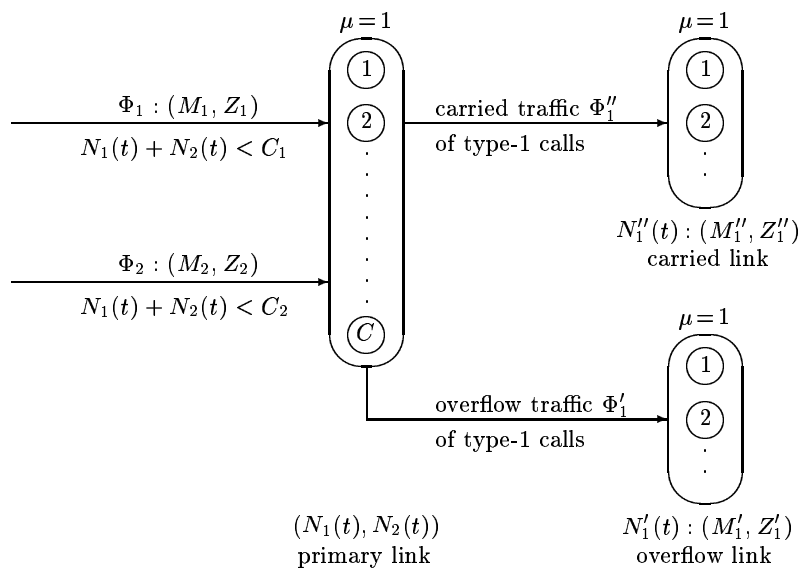


Figure 1.1: *Link of capacity  $C = \max(C_1, C_2)$  and trunk reservation level  $|C_1 - C_2|$ . Each accepted (non-accepted) type-1 call generates a virtual call, which arrives at a virtual carried (overflow) link of infinite capacity, where the holding time is independent of the required holding time of its generating call. The holding times of all calls are exponential with mean 1.*

The basic model considered in this paper, cf. Figure 1.1, is a link of capacity (trunk size)  $C$  offered two ft  $\Phi_i$  of calls (type- $i$  calls) which are characterized by their mean  $M_i$  and peakedness  $Z_i$ ,  $i \in \{1, 2\}$ . The calls require one trunk and are accepted according to a *trunk reservation* (tr) strategy, which is a very efficient and simple admission strategy for limiting excessive alternative routing, prioritizing direct traffic and preventing network instabilities, cf. e.g. [Krup], [Robe], [Reim], [BGZ], [Ross], [BM], [BB1]

and the references therein. In particular, an arriving type- $i$  call will be accepted as long as less than  $C_i$  trunks are occupied, where  $C = \max(C_1, C_2)$  and  $|C_1 - C_2|$  is the tr level. The protected stream ( $C_i = C$ ) can be considered a primary or direct traffic. The other stream, representing fot and fct from other links in a network, is handicapped by tr. For  $i \in \{1, 2\}$ , the factorial moments of the number  $N_i'(t)$  of type- $i$  calls in the virtual overflow link at time  $t$ , in particular the mean  $M_i'$  and peakedness  $Z_i'$ , and the factorial moments of the number  $N_i''(t)$  of type- $i$  calls in the virtual carried link at time  $t$ , in particular the mean  $M_i''$  and peakedness  $Z_i''$ , give moment characterizations of the fot  $\Phi_i'$  and of the fct  $\Phi_i''$  of type- $i$  calls, respectively.

The aim of this paper is to derive numerically stable and efficient algorithms for computing the means and peakedness factors of the individual fot and fct. The results provide a basis for net-planning tools with tr strategy links based on a fitting of the first two moments and on a unified handling of all call streams.

The paper is organized as follows. In Section 2 the arrival streams  $\Phi_i$ ,  $i \in \{1, 2\}$ , are modeled by renewal processes with Coxian inter-arrival time distributions meeting their first two moments. In Section 3 stable non linear recursive algorithms of complexity  $O(C^2)$  for the  $k$ th factorial moments,  $k \in \mathbb{N}$ , of the individual fot and fct are developed. By paralleling the algorithms with respect to  $k$ , a numerically stable algorithm of complexity  $O(C)$  for the means and peakedness factors of the individual fot and fct is constructed. Some numerical results are presented in Section 4.

## 2 Modeling by a Markov process

Consider the model of a link with tr in steady state given in Figure 1.1. Let us model the arrival stream  $\Phi_i$ ,  $i \in \{1, 2\}$ , by a renewal process with a two-phases Coxian inter-arrival time distribution

$$A_i(t) := (1 - r_i)E_{\lambda_i}(t) + r_i(E_{\lambda_i} * E_{\alpha_i})(t), \quad t \in \mathbb{R}_+, \quad (2.1)$$

with parameters  $\lambda_i, \alpha_i \in (0, \infty)$ ,  $r_i \in (0, 1]$ , where  $E_\gamma(t) := 1 - \exp(-\gamma t)$  and  $*$  denotes the convolution. Then after elementary algebra from (1.1) for the mean  $M_i$  and peakedness  $Z_i$  of the ft  $\Phi_i$  we obtain

$$M_i = \frac{\alpha_i \lambda_i}{\alpha_i + \lambda_i r_i}, \quad Z_i = 1 + \frac{\lambda_i r_i (\lambda_i (1 - r_i) - \alpha_i)}{(\alpha_i + \lambda_i r_i)(1 + \alpha_i + \lambda_i r_i)}. \quad (2.2)$$

For given positive  $M_i$  and  $Z_i$  of the arrival stream  $\Phi_i$ , for the parameters  $\lambda_i$ ,  $\alpha_i$  and  $r_i$  we find the necessary and sufficient conditions

$$\lambda_i > \max(M_i/Z_i, M_i + Z_i - 1), \quad \lambda_i^2(1 - Z_i) \leq (\lambda_i - M_i)(M_i + Z_i - 1),$$

$$r_i = \frac{(\lambda_i - M_i)(\lambda_i - (M_i + Z_i - 1))}{\lambda_i(\lambda_i Z_i - M_i)}, \quad \alpha_i = \frac{M_i \lambda_i r_i}{\lambda_i - M_i}. \quad (2.3)$$

The range of mean and peakedness, for which corresponding parameters  $\lambda_i$ ,  $\alpha_i$  and  $r_i$  exist, is precisely

$$M_i > 0, \quad Z_i \geq Z_{MIN}(M_i) := (3M_i + 1)/(4M_i + 1). \quad (2.4)$$

In the following we assume that (2.4) is satisfied. (If in practical situations  $Z_i < Z_{MIN}(M_i)$  arises, then a reasonable approximation is to replace  $Z_i$  by  $Z_{MIN}(M_i)$ .) For the parameter  $\lambda_i$  we choose

$$\lambda_i := (M_i + 1)Z_i + \frac{5M_i^2 - 2M_i - 1}{4M_i + 1}, \quad (2.5)$$

covering the whole feasible region of  $(M_i, Z_i)$  given by (2.4). In case of  $Z_i > 1$  the renewal processes defined by (2.1) are the interrupted Poisson processes, cf. [Kucz]. Thus in this case instead of (2.5) also Rapp's approximation  $\lambda_i(1 - r_i) = M_i Z_i + 3Z_i(Z_i - 1)$  could be applied, cf. [Gira] p. 115.

For constructing a Markov process for the system dynamics the following stochastically equivalent description of the  $\Phi_i$  by random switches is used. The state  $M_i(t)$  of switch  $i$  at time  $t$ ,  $i \in \{1, 2\}$ , may take the value 1 or 0. If  $M_i(t) = 1$ , then type- $i$  calls are generated according to a Poisson process of intensity  $\lambda_i$ , else no type- $i$  calls are generated. After each generation of a type- $i$  call with probability  $r_i$  switch  $i$  changes into state 0. The switch stays in state 0 for an exponential period with mean  $1/\alpha_i$  and changes then into state 1. Obviously, switch  $i$  generates a renewal process with inter-arrival time distribution (2.1).

Additionally to the variables  $N'_i(t)$  and  $N''_i(t)$  defined in Section 1, let us introduce the vectors  $M(t) := (M_1(t), M_2(t))$  of the states of the switches and  $N(t) := (N_1(t), N_2(t))$  of the numbers  $N_i(t)$  of type- $i$  calls in the primary link at time  $t$ . Since in the following without loss of generality we consider only the fot and fct of type-1 calls, let

$$\begin{aligned} N'(t) &:= N'_1(t), & M' &:= M'_1, & Z' &:= Z'_1, \\ N''(t) &:= N''_1(t), & M'' &:= M''_1, & Z'' &:= Z''_1 \end{aligned} \quad (2.6)$$

for notational convenience. Further let

$$\varphi_i(0) := r_i \lambda_i, \quad \varphi_i(1) := (1 - r_i) \lambda_i, \quad i \in \{1, 2\}. \quad (2.7)$$

The process  $N(t)$  is non Markovian with the state space

$$\Omega := \{n = (n_1, n_2) \in \mathbb{Z}_+^2 : n_1 \leq C_1, n_2 \leq C_2, n_1 + n_2 \leq C\}.$$

However,  $(M(t), N(t), N'(t))$  and  $(M(t), N(t), N''(t))$  are Markov processes with the joint state space  $\mathbb{X} := \{0, 1\}^2 \times \Omega \times \mathbb{Z}_+$ . Let

$$p'(m, n, l) := P(M(t) = m, N(t) = n, N'(t) = l), \quad (m, n, l) \in \mathbb{X}, \quad (2.8)$$

$$p''(m, n, l) := P(M(t) = m, N(t) = n, N''(t) = l), \quad (m, n, l) \in \mathbb{X} \quad (2.9)$$

its stationary distributions, where  $p'(m, n, l) := p''(m, n, l) := 0$  for all infeasible states  $(m, n, l) \in \{0, 1\}^2 \times (\mathbb{Z}^3 \setminus (\Omega \times \mathbb{Z}_+))$ .

The balance equations for  $p'(m, n, l)$  read

$$\begin{aligned} & \left( \sum_{i=1}^2 \left( \mathbb{I}\{m_i = 0\} \alpha_i + \mathbb{I}\{m_i = 1\} (\varphi_i(0) + \varphi_i(1)) + n_i \right) + l \right) p'(m, n, l) \\ &= \sum_{i=1}^2 \mathbb{I}\{m_i = 1\} \alpha_i p'(\pi_i(m), n, l) + \sum_{i=1}^2 (n_i + 1) p'(m, n + e_i, l) \\ & \quad + \sum_{i=1}^2 \varphi_i(m_i) \mathbb{I}\{|n| \leq C_i\} p'(\pi_i(m) + e_i, n - e_i, l) \\ & \quad + \sum_{i=1}^2 \varphi_i(m_i) \mathbb{I}\{|n| \geq C_i\} p'(\pi_i(m) + e_i, n, l - \mathbb{I}\{i = 1\}) \\ & \quad + (l + 1) p'(m, n, l + 1), \quad (m, n, l) \in \mathbb{X}, \end{aligned} \quad (2.10)$$

where  $e_i$  are the unit vectors in  $\mathbb{R}^2$ ,  $\pi_i(m) := m - m_i e_i$  and  $|n| := n_1 + n_2$ . Analogously, the balance equations for  $p''(m, n, l)$  read

$$\begin{aligned} & \left( \sum_{i=1}^2 \left( \mathbb{I}\{m_i = 0\} \alpha_i + \mathbb{I}\{m_i = 1\} (\varphi_i(0) + \varphi_i(1)) + n_i \right) + l \right) p''(m, n, l) \\ &= \sum_{i=1}^2 \mathbb{I}\{m_i = 1\} \alpha_i p''(\pi_i(m), n, l) + \sum_{i=1}^2 (n_i + 1) p''(m, n + e_i, l) \\ & \quad + \sum_{i=1}^2 \varphi_i(m_i) \mathbb{I}\{|n| \leq C_i\} p''(\pi_i(m) + e_i, n - e_i, l - \mathbb{I}\{i = 1\}) \\ & \quad + \sum_{i=1}^2 \varphi_i(m_i) \mathbb{I}\{|n| \geq C_i\} p''(\pi_i(m) + e_i, n, l) \\ & \quad + (l + 1) p''(m, n, l + 1), \quad (m, n, l) \in \mathbb{X}. \end{aligned} \quad (2.11)$$

Note that (2.10) and (2.11) remain valid for  $(m, n, l) \in \{0, 1\}^2 \times \mathbb{Z}^3$ . The normalizing conditions read

$$\sum_{(m, n, l) \in \mathbb{X}} p'(m, n, l) = 1, \quad \sum_{(m, n, l) \in \mathbb{X}} p''(m, n, l) = 1. \quad (2.12)$$

The mean  $M'$  and peakedness  $Z'$  of the fot of type-1 calls are given by

$$M' = \sum_{(m,n,l) \in \mathbb{X}} lp'(m, n, l), \quad (2.13)$$

$$Z' = 1 - M' + \frac{1}{M'} \sum_{(m,n,l) \in \mathbb{X}} l(l-1)p'(m, n, l) \quad (2.14)$$

and the mean  $M''$  and peakedness  $Z''$  of the fct of type-1 calls by

$$M'' = \sum_{(m,n,l) \in \mathbb{X}} lp''(m, n, l), \quad (2.15)$$

$$Z'' = 1 - M'' + \frac{1}{M''} \sum_{(m,n,l) \in \mathbb{X}} l(l-1)p''(m, n, l). \quad (2.16)$$

### 3 Algorithms for the moments of the fot and fct

In this section we develop numerically stable recursive algorithms for the factorial moments of the fot and fct of type-1 calls. For  $m \in \{0, 1\}^2$ ,  $\ell \in \mathbb{Z}$ ,  $k \in \mathbb{Z}_+$  let

$$q'_k(m, \ell) := (-1)^{|m|} \sum_{(n,l) \in \Omega \times \mathbb{Z}_+, |n|=\ell} (l+1-k)_k p'(m, n, l), \quad (3.1)$$

$$q''_k(m, \ell) := (-1)^{|m|} \sum_{(n,l) \in \Omega \times \mathbb{Z}_+, |n|=\ell} (l+1-k)_k p''(m, n, l) \quad (3.2)$$

be signed  $k$ th partial factorial moments of  $N'(t)$  and  $N''(t)$  on the condition  $\{M(t) = m, N_1(t) + N_2(t) = \ell\}$ , respectively. Obviously, the signed zeroth partial factorial moments of  $N'(t)$  and  $N''(t)$  coincide. Moreover, we have

$$q'_k(m, \ell) = q''_k(m, \ell) = 0, \quad m \in \{0, 1\}^2, \quad \ell \in \mathbb{Z} \setminus \{0, 1, \dots, C\}. \quad (3.3)$$

#### 3.1 Stable algorithm of complexity $O(C^2)$ for the fot

In this subsection we develop a numerically stable recursive algorithm for the partial factorial moments  $q'_k(m, \ell)$  of the fot of type-1 calls, whose complexity is  $O(C^2)$  in case of fixed order  $k$ .

For fixed  $m \in \{0, 1\}^2$ ,  $\ell \in \mathbb{Z}_+$ ,  $k \in \mathbb{Z}_+$ , multiplying the balance equations (2.10) by  $(-1)^{|m|}(l+1-k)_k$ , summing over  $(n, l) \in \mathbb{Z}^3$  with  $|n| \leq \ell$ , using

$$(l+2-k)_k = (l+1-k)_k + k(l+2-k)_{k-1} \quad (3.4)$$



and some algebra yield

$$\begin{aligned}
(\ell+1)q'_k(m, \ell+1) &= \sum_{i=1}^2 \mathbb{I}\{\ell < C_i, m_i = 1\} \varphi_i(1) q'_k(m, \ell) + k \sum_{j=0}^{\ell} q'_k(m, j) \\
&+ \sum_{i=1}^2 \alpha_i \sum_{j=0}^{\ell} q'_k(\pi_i(m), j) + \sum_{i=1}^2 \varphi_i(0) \sum_{j=0}^{\ell - \mathbb{I}\{\ell < C_i, m_i = 0\}} q'_k(\pi_i(m) + e_i, j) \\
&+ k(-1)^{m_1} \varphi_1(m_1) \sum_{j=C_1}^{\max(\ell, C_1-1)} q'_{k-1}(\pi_1(m) + e_1, j). \tag{3.5}
\end{aligned}$$

Note that for  $\ell = 0, 1, \dots, C_1 - 1$  the Recursion (3.5) is a homogeneous one. Because of  $q'_k(m, C+1) = 0$ ,  $m \in \{0, 1\}^2$ , from (3.5) for  $\ell = C$  it follows

$$\begin{aligned}
kq'_k(m) + \sum_{i=1}^2 \left( \alpha_i q'_k(\pi_i(m)) + \varphi_i(0) q'_k(\pi_i(m) + e_i) \right) \\
= k(-1)^{1-m_1} \varphi_1(m_1) \sum_{j=C_1}^C q'_{k-1}(\pi_1(m) + e_1, j), \quad m \in \{0, 1\}^2, \tag{3.6}
\end{aligned}$$

where

$$q'_k(m) := \sum_{j=0}^C q'_k(m, j), \quad m \in \{0, 1\}^2, \tag{3.7}$$

are corresponding signed  $k$ th partial factorial moments of  $N'(t)$  on the condition  $\{M(t) = m\}$ . In case of  $k = 0$  from (3.6) and from the normalizing condition

$$\sum_{m \in \{0, 1\}^2} (-1)^{|m|} q'_0(m) = 1, \tag{3.8}$$

cf. (2.12), we obtain

$$q'_0(m) = \prod_{i=1}^2 \frac{m_i \alpha_i - (1 - m_i) \varphi_i(0)}{\alpha_i + \varphi_i(0)}, \quad m \in \{0, 1\}^2. \tag{3.9}$$

In case of  $k \in \mathbb{N}$  the quantities  $q'_k(m)$ ,  $m \in \{0, 1\}^2$ , can be computed by solving the linear system of equations (3.6) if the right-hand side of (3.6) is given.

In view of (2.7) and (3.1), multiplying (3.6) by  $(-1)^{|m|}$  and summing over  $m \in \{0, 1\}^2$  yields

$$\begin{aligned} \sum_{(m,n,l) \in \mathbb{X}} (l+1-k)_k p'(m, n, l) &= \sum_{m \in \{0,1\}^2} (-1)^{|m|} q'_k(m) \\ &= \lambda_1 \sum_{j=C_1}^C \left( q'_{k-1}((1, 1), j) - q'_{k-1}((1, 0), j) \right), \quad k \in \mathbb{N}. \end{aligned} \quad (3.10)$$

From (2.13), (2.14) and (3.10) for  $M'$  and  $Z'$  we obtain

$$M' = \lambda_1 \sum_{j=C_1}^C \left( q'_0((1, 1), j) - q'_0((1, 0), j) \right), \quad (3.11)$$

$$Z' = 1 - M' + \frac{\lambda_1}{M'} \sum_{j=C_1}^C \left( q'_1((1, 1), j) - q'_1((1, 0), j) \right). \quad (3.12)$$

The time congestion  $\bar{B}$  of the type-1 calls is given by the probability that at least  $C_1$  trunks are occupied, i.e., it holds

$$\bar{B} = \sum_{j=C_1}^C \left( q'_0((1, 1), j) - q'_0((1, 0), j) - q'_0((0, 1), j) + q'_0((0, 0), j) \right). \quad (3.13)$$

In view of (3.5), the signed  $k$ th partial factorial moments  $q'_k(m, \ell)$  of  $N'(t)$  can be computed recursively with respect to  $k$ . For fixed  $k \in \mathbb{Z}_+$ , the signed  $k$ th partial factorial moments of  $N'(t)$  can be computed numerically stable recursively with respect to  $\ell \in \{0, 1, \dots, C\}$  as follows:

**Algorithm 3.1** *Let  $k \in \mathbb{Z}_+$  be fixed. For fixed  $L \in \{0, \dots, C\}$ ,  $m' \in \{0, 1\}^2$  let  $(q'_{k,m',L}(m, \ell))_{\ell=0}^{L+1}$  be the four-dimensional sequence that is defined by the homogenized recursion (3.5) and by the condition*

$$\sum_{j=0}^L q'_{k,m',L}(m, j) = \mathbb{I}\{m = m'\}, \quad m \in \{0, 1\}^2. \quad (3.14)$$

*Epecially, from (3.5), (3.14) it follows that for  $L \in \{0, \dots, C\}$ ,  $m' \in \{0, 1\}^2$*

$$\begin{aligned} q'_{k,m',L}(m, L+1) &= \frac{1}{L+1} \left( \mathbb{I}\{m' = m\} k \right. \\ &\quad + \sum_{i=1}^2 \mathbb{I}\{m' = \pi_i(m)\} \alpha_i + \sum_{i=1}^2 \mathbb{I}\{m' = \pi_i(m) + e_i\} \varphi_i(0) \\ &\quad \left. + \sum_{i=1}^2 \mathbb{I}\{L < C_i\} (-1)^{1-m_i} \varphi_i(m_i) q'_{k,m',L}(\pi_i(m) + e_i, L) \right). \end{aligned} \quad (3.15)$$

Since the four sequences  $(q'_{k,m',L}(m,\ell))_{\ell=0}^{L+1}$ ,  $m' \in \{0,1\}^2$ , are linearly independent, there are uniquely determined coefficients  $\beta_{m',m''}(k,L)$ , such that

$$q'_{k,m',L}(m,\ell) = \sum_{m'' \in \{0,1\}^2} \beta_{m',m''}(k,L) q'_{k,m'',L+1}(m,\ell), \quad \ell \in \{0, \dots, L+1\}, \quad m' \in \{0,1\}^2. \quad (3.16)$$

In view of (3.14), summing over  $\ell \in \{0, \dots, L+1\}$  yields

$$\beta_{m',m''}(k,L) = \mathbb{I}\{m' = m''\} + q'_{k,m',L}(m'', L+1). \quad (3.17)$$

Using the inverse  $(\beta_{m',m''}^*(k,L))$  of the matrix  $(\beta_{m',m''}(k,L))$  from (3.16) it follows

$$q'_{k,m',L+1}(m,\ell) = \sum_{m'' \in \{0,1\}^2} \beta_{m',m''}^*(k,L) q'_{k,m'',L}(m,\ell), \quad \ell \in \{0, \dots, L+1\}, \quad m' \in \{0,1\}^2, \quad (3.18)$$

i.e., the sequences  $(q'_{k,m',L+1}(m,\ell))_{\ell=0}^{L+1}$ ,  $m' \in \{0,1\}^2$ , can be computed from  $(q'_{k,m',L}(m,\ell))_{\ell=0}^{L+1}$ ,  $m' \in \{0,1\}^2$ . In particular, in view of (3.14), summing over  $\ell \in \{0, \dots, L\}$  yields

$$q'_{k,m',L+1}(m, L+1) = \mathbb{I}\{m' = m\} - \beta_{m',m}^*(k,L), \quad m' \in \{0,1\}^2. \quad (3.19)$$

Subsequently,  $q'_{k,m',L+1}(m, L+2)$ ,  $m' \in \{0,1\}^2$ , can be computed via (3.15).

For fixed  $L \in \{0, \dots, C\}$  let  $(q'_{k,*,L}(m,\ell))_{\ell=0}^{L+1}$  be the four-dimensional sequence that is defined by (3.5) and by the condition

$$\sum_{j=0}^L q'_{k,*,L}(m,j) = 0, \quad m \in \{0,1\}^2, \quad (3.20)$$

in case of  $k=0$  or  $L < C_1$  being the sequence of zero-vectors. In particular, from (3.5), (3.20) it follows that for  $L \in \{0, \dots, C\}$ ,  $m' \in \{0,1\}^2$

$$q'_{k,*,L}(m, L+1) = \frac{1}{L+1} \left( k(-1)^{m_1} \varphi_1(m_1) \sum_{j=C_1}^{\max(L, C_1-1)} q'_{k-1}(\pi_1(m) + e_1, j) + \sum_{i=1}^2 \mathbb{I}\{L < C_i\} (-1)^{1-m_i} \varphi_i(m_i) q'_{k,*,L}(\pi_i(m) + e_i, L) \right). \quad (3.21)$$

Since the right-hand sides of

$$q'_{k,*,L+1}(m,\ell) = q'_{k,*,L}(m,\ell) - \sum_{m' \in \{0,1\}^2} q'_{k,*,L}(m', L+1) q'_{k,m',L+1}(m,\ell), \quad \ell \in \{0, \dots, L+1\}, \quad (3.22)$$

satisfy the recursion (3.5) and, in view of (3.14), (3.20), sum up to zero, it holds (3.22), and  $q'_{k,*,L+1}(m, L+2)$  is given by (3.21). Finally, from (3.5), (3.7), (3.14), (3.20) for  $m \in \{0, 1\}^2$ ,  $\ell \in \{0, \dots, C\}$  we obtain

$$q'_k(m, \ell) = q'_{k,*,C}(m, \ell) + \sum_{m' \in \{0,1\}^2} q'_k(m') q'_{k,m',C}(m, \ell). \quad (3.23)$$

**Remark 3.1** Because of the non-initial condition (3.14), the existence of the four-dimensional sequences  $(q'_{k,m',L}(m, \ell))_{\ell=0}^{L+1}$  has to be proven. Assume that there exist  $L \in \{0, \dots, C\}$  and  $m' \in \{0, 1\}^2$  such that the corresponding sequence does not exist. Replacing  $C_i$  by  $\min(C_i, L)$ ,  $i \in \{1, 2\}$ , and thus  $C$  by  $\min(C, L)$ , in view of (3.5), only the definition of the last item of this sequence changes. Therefore without loss of generality we may assume that  $L = C$ . Let  $\mathcal{S}$  denote the vector space of the four-dimensional sequences  $(\tilde{q}_k(m, \ell))_{\ell=0}^{C+1}$  satisfying the homogenized recursion (3.5). Since the four sequences from  $\mathcal{S}$  defined by the unit vectors of  $\mathbb{R}^4$  as first items form a basis in  $\mathcal{S}$ , the dimension of  $\mathcal{S}$  is four. Summing up the first  $C+1$  items of the sequences from  $\mathcal{S}$  defines a linear mapping  $\mathcal{H} : \mathcal{S} \rightarrow \mathbb{R}^4$ . As at least one of the unit vectors of  $\mathbb{R}^4$  does not belong to the image of  $\mathcal{H}$ , the dimension of the image of  $\mathcal{H}$  is less than four. Thus there exists a nontrivial sequence  $(\tilde{q}_k^*(m, \ell))_{\ell=0}^{C+1} \in \mathcal{S}$  in the kernel of  $\mathcal{H}$ , i.e.

$$\sum_{j=0}^C \tilde{q}_k^*(m, j) = 0, \quad m \in \{0, 1\}^2. \quad (3.24)$$

Because of (3.24), from the homogenized recursion (3.5) for  $\ell = C$  it follows

$$\tilde{q}_k^*(m, C+1) = 0, \quad m \in \{0, 1\}^2. \quad (3.25)$$

Taking in the homogenized recursion (3.5) the differences of successive equations, because of (3.25), it follows that  $\tilde{q}_k^*(m, \ell)$  is a nontrivial solution of the homogeneous linear system of equations

$$\begin{aligned} & \left( \sum_{i=1}^2 \left( \mathbb{I}\{m_i=0\} \alpha_i + \mathbb{I}\{m_i=1\} (\varphi_i(0) + \varphi_i(1)) \right) + (\ell + k) \right) \tilde{q}_k^*(m, \ell) \\ &= - \sum_{i=1}^2 \mathbb{I}\{m_i=1\} \alpha_i \tilde{q}_k^*(\pi_i(m), \ell) + \mathbb{I}\{\ell < C\} (\ell + 1) \tilde{q}_k^*(m, \ell + 1) \\ & \quad + \sum_{i=1}^2 (-1)^{1-m_i} \varphi_i(m_i) \mathbb{I}\{0 < \ell \leq C_i\} \tilde{q}_k^*(\pi_i(m) + e_i, \ell - 1) \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^2 (-1)^{1-m_i} \varphi_i(m_i) \mathbb{I}\{\ell \geq C_i\} \tilde{q}_k^*(\pi_i(m) + e_i, \ell), \\
& \qquad (m, \ell) \in \{0, 1\}^2 \times \{0, \dots, C\}. \tag{3.26}
\end{aligned}$$

Adding  $\tilde{q}_k^*(m, \ell)$  on both sides of (3.26), taking the absolute values on both sides and applying then the triangle inequality to the right-hand side yields

$$\begin{aligned}
& \left( \sum_{i=1}^2 \left( \mathbb{I}\{m_i = 0\} \alpha_i + \mathbb{I}\{m_i = 1\} (\varphi_i(0) + \varphi_i(1)) \right) + (\ell + k) \right) |\tilde{q}_k^*(m, \ell)| \\
& \leq \sum_{i=1}^2 \mathbb{I}\{m_i = 1\} \alpha_i |\tilde{q}_k^*(\pi_i(m), \ell)| + \mathbb{I}\{\ell < C\} (\ell + 1) |\tilde{q}_k^*(m, \ell + 1)| \\
& \quad + \sum_{i=1}^2 \varphi_i(m_i) \mathbb{I}\{0 < \ell \leq C_i\} |\tilde{q}_k^*(\pi_i(m) + e_i, \ell - 1)| \\
& \quad + \sum_{i=1}^2 \varphi_i(m_i) \mathbb{I}\{\ell \geq C_i\} |\tilde{q}_k^*(\pi_i(m) + e_i, \ell)|, \\
& \qquad (m, \ell) \in \{0, 1\}^2 \times \{0, \dots, C\}, \tag{3.27}
\end{aligned}$$

where equality in (3.27) for fixed  $(m, \ell) \in \{0, 1\}^2 \times \{0, \dots, C-1\}$  would imply  $\tilde{q}_k^*(m, \ell) \tilde{q}_k^*(m, \ell + 1) \geq 0$ , and  $\tilde{q}_k^*(m, \ell) = 0$  would imply  $\tilde{q}_k^*(m, \ell + 1) = 0$  in case of equality in (3.27). Therefore, because of (3.24), equality in all the inequalities (3.27) would provide  $\tilde{q}_k^*(m, \ell) = 0$ ,  $(m, \ell) \in \{0, 1\}^2 \times \{0, \dots, C\}$ , being a contradiction. Thus at least one of the inequalities in (3.27) is strict. However, summing up the inequalities (3.27) yields the contradiction

$$\sum_{(m, \ell) \in \{0, 1\}^2 \times \{0, \dots, C\}} k |\tilde{q}_k^*(m, \ell)| < 0.$$

### 3.2 Stable algorithm of complexity $O(C^2)$ for the fct

In this subsection we develop a numerically stable recursive algorithm for the partial factorial moments  $q_k''(m, \ell)$  of the fct of type-1 calls, whose complexity is  $O(C^2)$  in case of fixed order  $k$ . We proceed analogously to Subsection 3.1, where some results obtained there are used.

For fixed  $m \in \{0, 1\}^2$ ,  $\ell \in \mathbb{Z}_+$ ,  $k \in \mathbb{Z}_+$ , multiplying the balance equations (2.11) by  $(-1)^{|m|} (\ell + 1 - k)_k$ , summing over  $(n, l) \in \mathbb{Z}^3$  with  $|n| \leq \ell$ , using (3.4) and some algebra yield

$$(\ell + 1) q_k''(m, \ell + 1) = \sum_{i=1}^2 \mathbb{I}\{\ell < C_i, m_i = 1\} \varphi_i(1) q_k''(m, \ell) + k \sum_{j=0}^{\ell} q_k''(m, j)$$

$$\begin{aligned}
& + \sum_{i=1}^2 \alpha_i \sum_{j=0}^{\ell} q_k''(\pi_i(m), j) + \sum_{i=1}^2 \varphi_i(0) \sum_{j=0}^{\ell - \mathbb{1}\{\ell < C_i, m_i=0\}} q_k''(\pi_i(m) + e_i, j) \\
& + k(-1)^{m_1} \varphi_1(m_1) \sum_{j=0}^{\min(\ell, C_1)-1} q_{k-1}''(\pi_1(m) + e_1, j). \tag{3.28}
\end{aligned}$$

Because of  $q_k''(m, C+1) = 0$ ,  $m \in \{0, 1\}^2$ , from (3.28) for  $\ell = C$  it follows

$$\begin{aligned}
& kq_k''(m) + \sum_{i=1}^2 \left( \alpha_i q_k''(\pi_i(m)) + \varphi_i(0) q_k''(\pi_i(m) + e_i) \right) \\
& = k(-1)^{1-m_1} \varphi_1(m_1) \left( q_{k-1}''(\pi_1(m) + e_1) \right. \\
& \quad \left. - \sum_{j=C_1}^C q_{k-1}''(\pi_1(m) + e_1, j) \right), \quad m \in \{0, 1\}^2, \tag{3.29}
\end{aligned}$$

where

$$q_k''(m) := \sum_{j=0}^C q_k''(m, j), \quad m \in \{0, 1\}^2, \tag{3.30}$$

are corresponding signed  $k$ th partial factorial moments of  $N''(t)$  on the condition  $\{M(t) = m\}$ . Obviously, we have  $q_0''(m) = q_0'(m)$ ,  $m \in \{0, 1\}^2$ . In case of  $k \in \mathbb{N}$  the quantities  $q_k''(m)$ ,  $m \in \{0, 1\}^2$ , can be computed by solving the linear system of equations (3.29) if the right-hand side of (3.29) is given.

In view of (2.7) and (3.2), multiplying (3.29) by  $(-1)^{|m|}$  and summing over  $m \in \{0, 1\}^2$  yields

$$\begin{aligned}
& \sum_{(m,n,l) \in \mathbb{X}} (l+1-k)_k p''(m, n, l) = \sum_{m \in \{0,1\}^2} (-1)^{|m|} q_k''(m) \\
& = \lambda_1 \sum_{j=0}^{C_1-1} \left( q_{k-1}''((1, 1), j) - q_{k-1}''((1, 0), j) \right), \quad k \in \mathbb{N}. \tag{3.31}
\end{aligned}$$

From (2.15), (2.16) and (3.31) for  $M''$  and  $Z''$  we obtain

$$M'' = \lambda_1 \sum_{j=0}^{C_1-1} \left( q_0''((1, 1), j) - q_0''((1, 0), j) \right), \tag{3.32}$$

$$Z'' = 1 - M'' + \frac{\lambda_1}{M''} \sum_{j=0}^{C_1-1} \left( q_1''((1, 1), j) - q_1''((1, 0), j) \right). \tag{3.33}$$

Because of (3.7), (3.9), (3.11), (2.7) and (2.3), from (3.32) it follows

$$\begin{aligned} M'' &= \lambda_1 \left( q_0'((1, 1)) - q_0'((1, 0)) \right) - \lambda_1 \sum_{j=C_1}^C \left( q_0'((1, 1), j) - q_0'((1, 0), j) \right) \\ &= \lambda_1 \frac{\alpha_1}{\alpha_1 + \varphi_1(0)} - M' = M_1 - M'. \end{aligned} \quad (3.34)$$

In view of (3.32), (2.7) and (2.3), from (3.29) for  $k=1$  we obtain

$$\begin{aligned} & q_1''((1, 1)) - q_1''((1, 0)) \\ &= \frac{\alpha_1(\varphi_1(0) + \varphi_1(1)) + \varphi_1(1)}{\alpha_1 + \varphi_1(0) + 1} \sum_{j=0}^{C_1-1} \left( q_0''((1, 1), j) - q_0''((1, 0), j) \right) \\ &= (Z_1 + M_1 - 1) \frac{M''}{\lambda_1}. \end{aligned} \quad (3.35)$$

Thus because of (3.30), (3.34), from (3.33) it follows

$$\begin{aligned} Z'' &= 1 - M'' + \frac{\lambda_1}{M''} \left( q_1''((1, 1)) - q_1''((1, 0)) \right) \\ &\quad - \frac{\lambda_1}{M''} \sum_{j=C_1}^C \left( q_1''((1, 1), j) - q_1''((1, 0), j) \right) \\ &= Z_1 + M' - \frac{\lambda_1}{M''} \sum_{j=C_1}^C \left( q_1''((1, 1), j) - q_1''((1, 0), j) \right). \end{aligned} \quad (3.36)$$

In view of (3.28), the signed  $k$ th partial factorial moments  $q_k''(m, \ell)$  of  $N''(t)$  can be computed recursively with respect to  $k$ , where the initial values are given by  $q_0''(m, \ell) = q_0'(m, \ell)$ ,  $(m, \ell) \in \{0, 1\}^2 \times \{0, 1, \dots, C\}$ . For fixed  $k \in \mathbb{N}$ , the signed  $k$ th partial factorial moments of  $N''(t)$  can be computed recursively with respect to  $\ell$ . Using the fact that Equations (3.5) and (3.28) represent two recursions with respect to  $\ell \in \mathbb{Z}_+$  which have the same homogenization, we can compute the signed  $k$ th partial factorial moments of  $N''(t)$  numerically stable recursively with respect to  $\ell \in \{0, 1, \dots, C\}$  as follows:

**Algorithm 3.2** *Let  $k \in \mathbb{N}$  be fixed. Further, for fixed  $L \in \{0, \dots, C\}$  let  $(q_{k,*,L}''(m, \ell))_{\ell=0}^{L+1}$  be the four-dimensional sequence that is defined by (3.28) and by the condition*

$$\sum_{j=0}^L q_{k,*,L}''(m, j) = 0, \quad m \in \{0, 1\}^2. \quad (3.37)$$

Especially, from (3.28), (3.37) it follows that for  $L \in \{0, \dots, C\}$ ,  $m' \in \{0, 1\}^2$

$$\begin{aligned} q''_{k,*,L}(m, L+1) &= \frac{1}{L+1} \left( k(-1)^{m_1} \varphi_1(m_1) \sum_{j=0}^{\min(L, C_1)-1} q''_{k-1}(\pi_1(m) + e_1, j) \right. \\ &\quad \left. + \sum_{i=1}^2 \mathbb{I}\{L < C_i\} (-1)^{1-m_i} \varphi_i(m_i) q''_{k,*,L}(\pi_i(m) + e_i, L) \right). \end{aligned} \quad (3.38)$$

Since the right-hand sides of

$$\begin{aligned} q''_{k,*,L+1}(m, \ell) &= q''_{k,*,L}(m, \ell) - \sum_{m' \in \{0,1\}^2} q''_{k,*,L}(m', L+1) q'_{k,m',L+1}(m, \ell), \\ &\quad \ell \in \{0, \dots, L+1\}, \end{aligned} \quad (3.39)$$

satisfy the recursion (3.28), and, in view of (3.14), (3.37), sum up to zero, it holds (3.39). After computing  $q''_{k,*,L+1}(m, \ell)$  for  $\ell \in \{0, \dots, L+1\}$  via (3.39),  $q''_{k,*,L+1}(m, L+2)$  can be computed via (3.38). Finally, from (3.28), (3.30), (3.37), (3.14) for  $m \in \{0, 1\}^2$ ,  $\ell \in \{0, \dots, C\}$  we obtain

$$q''_k(m, \ell) = q''_{k,*,C}(m, \ell) + \sum_{m' \in \{0,1\}^2} q''_k(m') q'_{k,m',C}(m, \ell). \quad (3.40)$$

### 3.3 Parallelized stable algorithm of complexity $O(C)$

By paralleling the Algorithms 3.1 and 3.2 with respect to  $k \in \{0, 1\}$ , in this subsection we set up a numerically stable recursive algorithm of complexity  $O(C)$  for the means and peakedness factors of the fot and fct of type-1 calls.

For  $k \in \mathbb{Z}_+$ ,  $L \in \{0, \dots, C\}$ ,  $m' \in \{0, 1\}^2$  let

$$s'_{k,m',L}(m) := \sum_{j=C_1}^{\max(L, C_1-1)} q'_{k,m',L}(m, j), \quad (3.41)$$

in case of  $L < C_1$  being the four-dimensional zero-vector. From (3.18) for  $L < C$  it follows

$$\begin{aligned} s'_{k,m',L+1}(m) &= \mathbb{I}\{L+1 \geq C_1\} q'_{k,m',L+1}(m, L+1) \\ &\quad + \sum_{m'' \in \{0,1\}^2} \beta_{m',m''}^*(k, L) s'_{k,m'',L}(m). \end{aligned} \quad (3.42)$$

For fixed  $L \in \{0, \dots, C\}$ ,  $m' \in \{0, 1\}^2$  let  $(q'_{m',L}(m, \ell))_{\ell=0}^{L+1}$  be the four-dimensional sequence that is defined by the recursion

$$(\ell+1)q'_{m',L}(m, \ell+1) = \sum_{i=1}^2 \mathbb{I}\{\ell < C_i, m_i = 1\} \varphi_i(1) q'_{m',L}(m, \ell)$$



$$\begin{aligned}
& + \sum_{j=0}^{\ell} q'_{m',L}(m, j) + \sum_{i=1}^2 \alpha_i \sum_{j=0}^{\ell} q'_{m',L}(\pi_i(m), j) \\
& + \sum_{i=1}^2 \varphi_i(0) \sum_{j=0}^{\ell - \mathbb{1}\{\ell < C_i, m_i=0\}} q'_{m',L}(\pi_i(m) + e_i, j) \\
& + (-1)^{m_1} \varphi_1(m_1) \sum_{j=C_1}^{\max(\ell, C_1-1)} q'_{0,m',L}(\pi_1(m) + e_1, j), \\
& \ell \in \{0, \dots, L\}, \quad m \in \{0, 1\}^2, \quad (3.43)
\end{aligned}$$

and by the condition

$$\sum_{j=0}^L q'_{m',L}(m, j) = 0, \quad m \in \{0, 1\}^2, \quad (3.44)$$

in case of  $L < C_1$  being the sequence of zero-vectors. In particular, from (3.41), (3.43), (3.44) it follows that for  $L \in \{0, \dots, C\}$ ,  $m \in \{0, 1\}^2$

$$\begin{aligned}
q'_{m',L}(m, L+1) & = \frac{1}{L+1} \left( (-1)^{m_1} \varphi_1(m_1) s'_{0,m',L}(\pi_1(m) + e_1) \right. \\
& \left. + \sum_{i=1}^2 \mathbb{1}\{L < C_i\} (-1)^{1-m_i} \varphi_i(m_i) q'_{m',L}(\pi_i(m) + e_i, L) \right). \quad (3.45)
\end{aligned}$$

Since, because of (3.5), (3.18) and (3.43), the right-hand sides of

$$\begin{aligned}
q'_{m',L+1}(m, \ell) & = \sum_{m'' \in \{0,1\}^2} \beta_{m',m''}^*(0, L) \\
& \left( q'_{m'',L}(m, \ell) - \sum_{m''' \in \{0,1\}^2} q'_{m'',L}(m''', L+1) q'_{1,m''',L+1}(m, \ell) \right), \\
& \ell \in \{0, \dots, L+1\}, \quad m \in \{0, 1\}^2, \quad (3.46)
\end{aligned}$$

satisfy the modified recursion (3.43), where  $L$  is replaced by  $L+1$ , and, in view of (3.14), (3.44), their sum over  $\ell \in \{0, \dots, L+1\}$  equals zero, it holds (3.46) for  $L \in \{0, \dots, C-1\}$ . In particular, in view of (3.19), it follows

$$\begin{aligned}
q'_{m',L+1}(m, L+1) & = \sum_{m'', m''' \in \{0,1\}^2} \beta_{m',m''}^*(0, L) \beta_{m''',m}^*(1, L) \\
& q'_{m'',L}(m''', L+1), \quad m \in \{0, 1\}^2. \quad (3.47)
\end{aligned}$$

Moreover, since, because of (3.5), (3.23), (3.43) and  $q'_{0,*,C}(m, \ell) = 0$  for  $(m, \ell) \in \{0, 1\}^2 \times \{0, \dots, C\}$ , the right-hand sides of

$$q'_1(m, \ell) = \sum_{m' \in \{0, 1\}^2} q'_0(m') q'_{m', C}(m, \ell) + \sum_{m' \in \{0, 1\}^2} q'_1(m') q'_{1, m', C}(m, \ell),$$

$$\ell \in \{0, \dots, C\}, \quad m \in \{0, 1\}^2, \quad (3.48)$$

satisfy the recursion (3.5) for  $k=1$ , and, in view of (3.14), (3.44) and (3.7), their sum over  $\ell \in \{0, \dots, C\}$  equals the corresponding sum for the left-hand sides, it holds (3.48). For  $L \in \{0, \dots, C\}$ ,  $m' \in \{0, 1\}^2$  let

$$s'_{m', L}(m) := \sum_{j=C_1}^{\max(L, C_1-1)} q'_{m', L}(m, j), \quad (3.49)$$

in case of  $L < C_1$  being the four-dimensional zero-vector. From (3.46), (3.41) for  $L < C$  we obtain

$$s'_{m', L+1}(m) = \sum_{m'' \in \{0, 1\}^2} \beta_{m', m''}^*(0, L) \left( \mathbb{I}\{L+1 \geq C_1\} q'_{m'', L}(m, L+1) \right. \\ \left. + s'_{m'', L}(m) - \sum_{m''' \in \{0, 1\}^2} q'_{m'', L}(m''', L+1) s'_{1, m''', L+1}(m) \right). \quad (3.50)$$

Analogously, for fixed  $L \in \{0, \dots, C\}$ ,  $m' \in \{0, 1\}^2$  let  $(q''_{m', L}(m, \ell))_{\ell=0}^{L+1}$  be the four-dimensional sequence that is defined by the recursion

$$(\ell+1)q''_{m', L}(m, \ell+1) = \sum_{i=1}^2 \mathbb{I}\{\ell < C_i, m_i = 1\} \varphi_i(1) q''_{m', L}(m, \ell) \\ + \sum_{j=0}^{\ell} q''_{m', L}(m, j) + \sum_{i=1}^2 \alpha_i \sum_{j=0}^{\ell} q''_{m', L}(\pi_i(m), j) \\ + \sum_{i=1}^2 \varphi_i(0) \sum_{j=0}^{\ell - \mathbb{I}\{\ell < C_i, m_i = 0\}} q''_{m', L}(\pi_i(m) + e_i, j) \\ + (-1)^{m_1} \varphi_1(m_1) \sum_{j=0}^{\min(\ell, C_1)-1} q'_{0, m', L}(\pi_1(m) + e_1, j),$$

$$\ell \in \{0, \dots, L\}, \quad m \in \{0, 1\}^2, \quad (3.51)$$

and by the condition

$$\sum_{j=0}^L q''_{m', L}(m, j) = 0, \quad m \in \{0, 1\}^2. \quad (3.52)$$

In particular, because of (3.14), (3.41), (3.51) and (3.52), it follows that for  $L \in \{0, \dots, C\}$ ,  $m \in \{0, 1\}^2$

$$\begin{aligned} q''_{m',L}(m, L+1) &= \frac{1}{L+1} \left( (-1)^{m_1} \varphi_1(m_1) \mathbb{I}\{m' = \pi_1(m) + e_1\} \right. \\ &\quad + (-1)^{1-m_1} \varphi_1(m_1) \mathbb{I}\{L < C_1\} q'_{0,m',L}(\pi_1(m) + e_1, L) \\ &\quad + (-1)^{1-m_1} \varphi_1(m_1) \mathbb{I}\{L \geq C_1\} s'_{0,m',L}(\pi_1(m) + e_1) \\ &\quad \left. + \sum_{i=1}^2 \mathbb{I}\{L < C_i\} (-1)^{1-m_i} \varphi_i(m_i) q''_{m',L}(\pi_i(m) + e_i, L) \right). \end{aligned} \quad (3.53)$$

Since, because of (3.5), (3.18) and (3.51), the right-hand sides of

$$\begin{aligned} q''_{m',L+1}(m, \ell) &= \sum_{m'' \in \{0,1\}^2} \beta_{m',m''}^*(0, L) \\ &\quad \left( q''_{m'',L}(m, \ell) - \sum_{m''' \in \{0,1\}^2} q''_{m'',L}(m''', L+1) q'_{1,m''',L+1}(m, \ell) \right), \\ &\quad \ell \in \{0, \dots, L+1\}, \quad m \in \{0, 1\}^2, \end{aligned} \quad (3.54)$$

satisfy the modified recursion (3.51), where  $L$  is replaced by  $L+1$ , and, in view of (3.14), (3.52), their sum over  $\ell \in \{0, \dots, L+1\}$  equals zero, it holds (3.54) for  $L \in \{0, \dots, C-1\}$ . In particular, in view of (3.19), it follows

$$\begin{aligned} q''_{m',L+1}(m, L+1) &= \sum_{m'', m''' \in \{0,1\}^2} \beta_{m',m''}^*(0, L) \beta_{m''',m}^*(1, L) \\ &\quad q''_{m'',L}(m''', L+1), \quad m \in \{0, 1\}^2. \end{aligned} \quad (3.55)$$

Moreover, since, because of (3.5), (3.23) and (3.51) as well as  $q'_{0,*,C}(m, \ell) = 0$  and  $q'_0(m, \ell) = q''_0(m, \ell)$  for  $(m, \ell) \in \{0, 1\}^2 \times \{0, \dots, C\}$ , the right-hand sides of

$$\begin{aligned} q''_1(m, \ell) &= \sum_{m' \in \{0,1\}^2} q'_0(m') q''_{m',C}(m, \ell) + \sum_{m' \in \{0,1\}^2} q''_1(m') q'_{1,m',C}(m, \ell), \\ &\quad \ell \in \{0, \dots, C\}, \quad m \in \{0, 1\}^2, \end{aligned} \quad (3.56)$$

satisfy the recursion (3.28) for  $k=1$ , and, in view of (3.14), (3.52) and (3.30), their sum over  $\ell \in \{0, \dots, C\}$  equals the corresponding sum for the left-hand sides, it holds (3.56). For  $L \in \{0, \dots, C\}$ ,  $m' \in \{0, 1\}^2$  let

$$s''_{m',L}(m) := \sum_{j=C_1}^{\max(L, C_1-1)} q''_{m',L}(m, j), \quad (3.57)$$

in case of  $L < C_1$  being the four-dimensional zero-vector. From (3.54), (3.41) for  $L < C$  we obtain

$$\begin{aligned} s''_{m',L+1}(m) &= \sum_{m'' \in \{0,1\}^2} \beta_{m',m''}^*(0,L) \left( \mathbb{I}\{L+1 \geq C_1\} q''_{m'',L}(m, L+1) \right. \\ &\quad \left. + s''_{m'',L}(m) - \sum_{m''' \in \{0,1\}^2} q''_{m'',L}(m''', L+1) s'_{1,m''',L+1}(m) \right). \end{aligned} \quad (3.58)$$

Because of (3.23), (3.41), it holds

$$s'_0(m) := \sum_{j=C_1}^C q'_0(m, j) = \sum_{m' \in \{0,1\}^2} q'_0(m') s'_{0,m',C}(m). \quad (3.59)$$

In view of (3.41), (3.49), from (3.48) it follows

$$\begin{aligned} s'_1(m) &:= \sum_{j=C_1}^C q'_1(m, j) \\ &= \sum_{m' \in \{0,1\}^2} \left( q'_0(m') s'_{m',C}(m) + q'_1(m') s'_{1,m',C}(m) \right) \end{aligned} \quad (3.60)$$

and, in view of (3.41), (3.57), from (3.56)

$$\begin{aligned} s''_1(m) &:= \sum_{j=C_1}^C q''_1(m, j) \\ &= \sum_{m' \in \{0,1\}^2} \left( q'_0(m') s''_{m',C}(m) + q''_1(m') s'_{1,m',C}(m) \right). \end{aligned} \quad (3.61)$$

Finally, from (3.11), (3.12), (3.13), (3.34), (3.36), (3.59), (3.60) and (3.61) we obtain the representations

$$\bar{B} = s'_0((1,1)) - s'_0((1,0)) - s'_0((0,1)) + s'_0((0,0)), \quad (3.62)$$

$$M' = \lambda_1 \left( s'_0((1,1)) - s'_0((1,0)) \right), \quad (3.63)$$

$$Z' = 1 - M' + \frac{\lambda_1}{M'} \left( s'_1((1,1)) - s'_1((1,0)) \right), \quad (3.64)$$

$$M'' = M_1 - M', \quad (3.65)$$

$$Z'' = Z_1 + M' - \frac{\lambda_1}{M''} \left( s''_1((1,1)) - s''_1((1,0)) \right). \quad (3.66)$$

**Algorithm 3.3** *The sequence of the 128-dimensional vectors*

$$\begin{aligned} & (q'_{0,m',L}(m, L), q'_{1,m',L}(m, L), q'_{m',L}(m, L), q''_{m',L}(m, L), \\ & s'_{0,m',L}(m), s'_{1,m',L}(m), s'_{m',L}(m), s''_{m',L}(m)) \end{aligned} \quad (3.67)$$

can be computed recursively with respect to  $L \in \{0, \dots, C\}$  as follows:

From (3.14), (3.44), (3.52), (3.41), (3.49), (3.57) we obtain the vector (3.67) for  $L=0$ :

$$\begin{aligned} & (\mathbb{I}\{m=m'\}, \mathbb{I}\{m=m'\}, \vec{0}, \vec{0}, \\ & \mathbb{I}\{C_1=0, m=m'\}, \mathbb{I}\{C_1=0, m=m'\}, \vec{0}, \vec{0}). \end{aligned} \quad (3.68)$$

Given the vector (3.67) for any  $L \in \{0, \dots, C-1\}$ , then  $q'_{m',L}(m, L+1)$  can be computed via (3.45),  $q''_{m',L}(m, L+1)$  via (3.53). Moreover,  $q'_{0,m',L}(m, L+1)$  and  $q'_{1,m',L}(m, L+1)$  can be computed via (3.15). Subsequently,  $\beta_{m',m''}^*(0, L)$ ,  $\beta_{m',m''}^*(1, L)$  can be determined from (3.17) and then  $q'_{0,m',L+1}(m, L+1)$ ,  $q'_{1,m',L+1}(m, L+1)$  from (3.19). The sums  $s'_{0,m',L+1}(m)$ ,  $s'_{1,m',L+1}(m)$  can be computed from (3.42) and then  $s'_{m',L+1}(m)$ ,  $s''_{m',L+1}(m)$  from (3.50) and (3.58), respectively. The quantities  $q'_{m',L+1}(m, L+1)$  and  $q''_{m',L+1}(m, L+1)$  are given by (3.47) and (3.55), respectively.

Because of (3.59) and (3.9), the sum  $s'_0(m)$  can be determined from  $s'_{0,m',C}(m)$ ,  $m' \in \{0, 1\}^2$ . Since thus the right-hand sides of (3.6) and (3.29) are given, the quantities  $q'_1(m)$ ,  $q''_1(m)$  can be computed by solving the linear systems of equations (3.6) and (3.29), respectively. Finally, the sums  $s'_1(m)$  and  $s''_1(m)$  can be computed using (3.60) and (3.61), respectively.

Algorithm 3.3 is numerically stable. It is non linear as matrices are inverted. Its complexity is  $O(C)$ , its memory requirement  $O(1)$ . The recursive computation of the 128-dimensional sequence (3.67) is visualized in Figure 3.1.

**Remark 3.2** *A more precise analysis of Algorithm 3.3 shows that the length of vector (3.67) can be reduced to 92 by omitting the following entries:*

$$\begin{aligned} & q'_{1,m',L}((0, 0), L), q'_{m',L}((0, 0), L), q''_{m',L}((0, 0), L), s'_{1,m',L}((0, 1)), \\ & s'_{1,m',L}((0, 0)), s'_{m',L}((0, 1)), s'_{m',L}((0, 0)), s''_{m',L}((0, 1)), s''_{m',L}((0, 0)). \end{aligned}$$

Moreover, in case of  $L < C_1$  it holds

$$s'_{0,m',L}(m) = 0, s'_{1,m',L}(m) = 0, s'_{m',L}(m) = 0, s''_{m',L}(m) = 0.$$

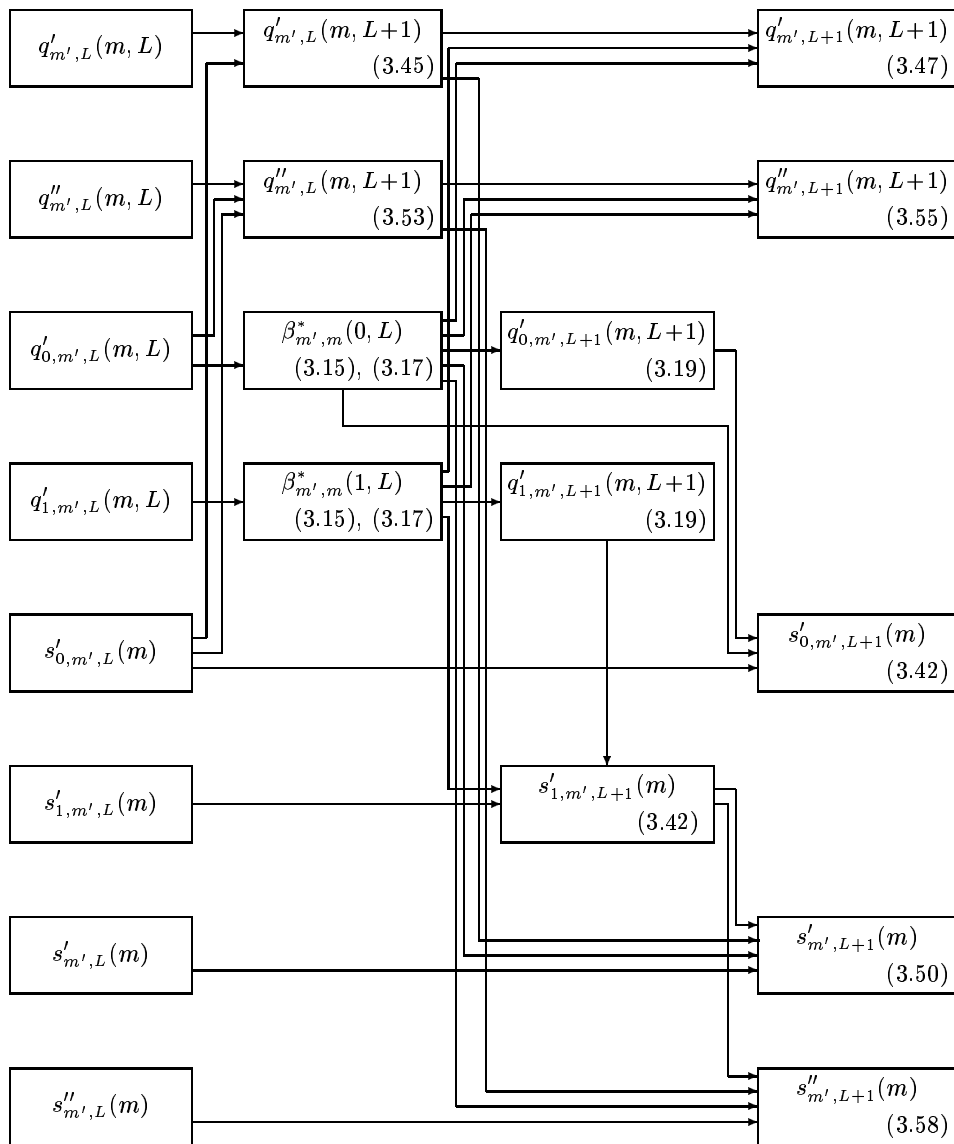


Figure 3.1: Recursive computation of the 128-dimensional sequence (3.67).

Thus in case of  $C_1 = C$  the length of vector (3.67) can additionally be reduced to 52 as we have

$$\begin{aligned} s'_{0,m',C}(m) &= q'_{0,m',C}(m, C), & s'_{1,m',C}(m) &= q'_{1,m',C}(m, C), \\ s'_{m',C}(m) &= q'_{m',C}(m, C), & s''_{m',C}(m) &= q''_{m',C}(m, C). \end{aligned}$$

Paralleling the computation for the type-1 and type-2 calls thus we have to handle a common vector of length 144. However, the length of this common vector can additionally be reduced to 116 as the quantities  $q'_{0,m',L}(m, L)$  and  $q'_{1,m',L}(m, L)$  are independent of the type.

## 4 Numerical results

The Algorithm 3.3 has been implemented in a C-program, which allows up to a link of capacity  $10^9$  for given  $M_i$ ,  $Z_i$  and  $C_i$ ,  $i \in \{1, 2\}$ , a numerically stable and very efficient computation of the means and peakedness factors of the individual fot and fct and of the individual time congestion for both traffic streams. In case of a Pentium processor with 200 MHz the computation time amounts to approximately 22 ms for  $C = 1\,000$ .

Numerical experiments not reported here have shown that there is no great difference in the means and peakedness factors of the departure streams if in case of  $Z_i > 1$  Rapp's well-known approximation for the parameters of the interrupted Poisson process is used instead of (2.5), justifying the use of (2.5) also in this case. In particular, (2.5) is used throughout Table 4.1, where a numerical example is given for a link of capacity  $C = 1\,000$ . The first arrival stream with mean  $M_1 = 600$  is protected, i.e.  $C_1 = C$ , the other input parameters  $Z_1$ ,  $M_2$ ,  $Z_2$  and  $C_2$  vary accordingly. For the used parameters the link works in the domain of critical loading, which seems to be the most interesting case from a practical point of view. The results in Table 4.1 illustrate the strong dependence of  $M_i''$ ,  $Z_i''$  and  $M_i'$ ,  $Z_i'$ ,  $i \in \{1, 2\}$ , on the tr parameter  $C_2$ , in particular the protection of the first input stream. The last block of data in Table 4.1, where for comparison Poisson arrival processes are chosen, illustrates the strong impact of the peakedness of arrival processes on the means of the departure streams and hence the advantage of two-moment over one-moment methods.

Table 4.1: *Link of capacity  $C_1 = C = 1000$  with given mean  $M_1 = 600$  of the protected first arrival stream.*

$Z_1$	$M_2$	$Z_2$	$C_2$	$M_1''$	$Z_1''$	$M_1'$	$Z_1'$	$M_2''$	$Z_2''$	$M_2'$	$Z_2'$
0.8	300	0.8	990	600.00	0.80	0.00	2.10	299.96	0.80	0.04	8.28
0.8	400	0.8	990	599.97	0.80	0.03	2.10	372.03	1.36	27.97	12.53
0.8	500	0.8	990	599.91	0.80	0.09	2.09	384.85	2.00	115.15	6.58
0.8	300	0.8	995	600.00	0.80	0.00	2.24	299.98	0.80	0.02	7.60
0.8	400	0.8	995	599.41	0.79	0.59	2.29	375.92	1.25	24.08	12.51
0.8	500	0.8	995	597.87	0.78	2.13	2.10	391.53	1.88	108.47	6.58
0.8	300	0.8	1000	599.99	0.80	0.01	5.60	300.00	0.80	0.00	3.31
0.8	400	0.8	1000	586.73	0.79	13.27	8.32	391.03	0.80	8.97	5.91
0.8	500	0.8	1000	541.80	0.88	58.20	4.23	451.20	0.87	48.80	3.69
0.8	300	8.0	990	600.00	0.80	0.00	2.10	294.16	6.54	5.84	28.75
0.8	400	8.0	990	599.98	0.80	0.02	2.10	348.48	3.59	51.52	30.48
0.8	500	8.0	990	599.96	0.80	0.04	2.09	368.25	2.61	131.75	24.72
0.8	300	8.0	995	599.95	0.80	0.05	2.27	295.00	6.69	5.00	28.20
0.8	400	8.0	995	599.60	0.80	0.40	2.24	351.91	3.68	48.09	30.64
0.8	500	8.0	995	599.09	0.79	0.91	2.17	373.25	2.60	126.75	24.97
0.8	300	8.0	1000	598.75	0.80	1.25	8.04	296.69	7.03	3.31	21.53
0.8	400	8.0	1000	588.98	0.86	11.02	7.68	364.61	4.11	35.39	24.78
0.8	500	8.0	1000	573.63	0.92	26.37	5.69	401.61	2.67	98.39	21.07
8.0	300	8.0	990	591.28	6.69	8.72	25.65	288.11	6.89	11.89	27.39
8.0	400	8.0	990	575.06	5.62	24.94	22.52	353.30	6.03	46.70	29.47
8.0	500	8.0	990	556.56	5.04	43.44	18.74	396.28	5.63	103.72	26.97
8.0	300	8.0	995	589.27	6.50	10.73	28.65	290.85	6.95	9.15	23.55
8.0	400	8.0	995	567.91	5.30	32.09	25.17	362.11	5.85	37.89	26.17
8.0	500	8.0	995	542.35	4.67	57.65	20.45	412.81	5.14	87.19	24.64
8.0	300	8.0	1000	586.52	6.30	13.48	33.52	294.15	7.15	5.85	17.65
8.0	400	8.0	1000	557.64	5.04	42.36	29.84	373.68	5.93	26.32	20.77
8.0	500	8.0	1000	521.03	4.47	78.97	23.92	436.02	4.94	63.98	20.55
1.0	300	1.0	990	600.00	1.00	0.00	2.53	299.88	1.00	0.12	10.07
1.0	400	1.0	990	599.89	1.00	0.11	2.53	369.78	1.54	30.22	14.04
1.0	500	1.0	990	599.65	0.99	0.35	2.49	383.97	2.22	116.03	7.90
1.0	300	1.0	995	600.00	1.00	0.00	2.85	299.93	1.00	0.07	9.03
1.0	400	1.0	995	598.73	0.97	1.27	2.93	374.18	1.40	25.82	13.66
1.0	500	1.0	995	595.76	0.95	4.24	2.54	392.38	2.01	107.62	7.72
1.0	300	1.0	1000	599.96	1.00	0.04	6.90	299.98	1.00	0.02	3.95
1.0	400	1.0	1000	585.11	0.92	14.89	9.36	390.08	0.94	9.92	6.57
1.0	500	1.0	1000	540.82	0.97	59.18	5.06	450.69	0.97	49.31	4.39



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