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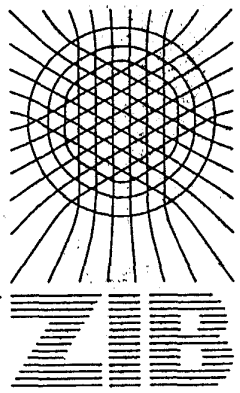

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On the Difference Between Real and Complex Arrangements

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On the Difference Between Real and Complex Arrangements

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Abstract. If $\mathcal{B}$ is an arrangement of linear complex hyperplanes in $\mathbb{C}^d$, then the following can be constructed from knowledge of its intersection lattice:
(a) the cohomology groups of the complement [Br],
(b) the cohomology algebra of the complement [OS],
(c) the fundamental group of the complement, if $d \leq 2$,
(d) the singularity link up to homeomorphism, if $d \leq 3$,
(e) the singularity link up to homotopy type [ZZ].
If $\mathcal{B}'$ is, more generally, a 2-arrangement in $\mathbb{R}^{2d}$ (an arrangement of real subspaces of codimension 2 with even-dimensional intersections), then the intersection lattice still determines (a) the cohomology groups of the complement [GM] and (e) the homotopy type of the singularity link [ZZ].
We show, however, that for 2-arrangements the data (b), (c) and (d) are not determined by the intersection lattice. They require the knowledge of extra information on sign patterns, which can be computed as determinants of linear relations, or (equivalently) as linking coefficients in the sense of knot theory.

1. Introduction

Let $\mathcal{B} = \{H_1, \ldots, H_n\}$ be an arrangement of complex hyperplanes in $\mathbb{C}^d = \mathbb{R}^{2d}$. We will only consider arrangements that are linear (the hyperplanes are vector subspaces) and essential (the intersection of all the hyperplanes is $\{0\}$).

The principal combinatorial structure associated with a complex arrangement is the *intersection lattice* $L_{\mathcal{B}} := \{\bigcap_{a\in A} H_a : A \subseteq \{1, \ldots, n\}\}$ of all intersections of hyperplanes, ordered by reversed inclusion. This is a geometric lattice (matroid), whose rank function is given by complex codimension, $r(A) = \operatorname{codim}_{\mathbb{C}}(\bigcap_{a\in A} H_a)$.

Let $D_{\mathcal{B}} := S^{2d-1} \cap \bigcup \mathcal{B}$ denote the *singularity link* and let $C_{\mathcal{B}} := \mathbb{C}^d \setminus \bigcup \mathcal{B}$ denote the *complement* of the arrangement. A by now classical result of Arnol'd, Brieskorn and Orlik & Solomon asserts that a presentation of the cohomology algebra of $C_{\mathcal{B}}$ can be constructed from the data that are encoded by the intersection lattice $L_{\mathcal{B}}$, as follows.

Theorem 1.1. [A][Br][OS] *Let $\mathcal{B} = \{H_1, \ldots, H_n\}$ be a complex arrangement in $\mathbb{C}^d$. For every $H_a \in \mathcal{B}$ choose a linear form $l_a \in (\mathbb{C}^d)^*$ that defines it, such that $\ker(l_a) = H_a$ $(1 \le a \le n)$. Then the integral cohomology algebra of the complement is generated by the classes*

$$\omega_a := \frac{1}{2\pi i}\frac{\mathrm{d}l_a}{l_a},$$

for $1 \le a \le n$. It has a presentation of the form

$$0 \longrightarrow I \longrightarrow \Lambda^*\mathbb{Z}^n \xrightarrow{\pi} \mathrm{H}^*(C_{\mathcal{B}}; \mathbb{Z}) \longrightarrow 0,$$

defined by $\pi(e_a) := [\omega_a]$, where $\{e_1, \ldots, e_n\}$ denotes a basis of $\mathbb{Z}^n$. The relation ideal I is generated by the elements

$$\sum_{i=0}^{k} (-1)^i \, e_{a_0} \wedge \ldots \wedge \widehat{e_{a_i}} \wedge \ldots \wedge e_{a_k},$$

for circuits $A = \{a_0, \ldots, a_k\}$ of L, that is, for the minimal subsets $A \subseteq \{1, \ldots, n\}$ with $r(A) < |A|$.

Goresky & MacPherson [GM, p. 257], whose section title we have used for this paper, suggest to study the following (seemingly) mild generalization. A *2-arrangement* is a finite set $\mathcal{B}' = \{H_1, \ldots, H_n\}$ of real vector subspaces of codimension 2 in $\mathbb{R}^{2d}$ so that every intersection $\bigcap_{a\in A} H_a$ has even codimension in $\mathbb{R}^{2d}$. Again we assume $\bigcap \mathcal{B}' = \{0\}$. The combinatorial essence of a "complex structure" can be studied by comparing the structure of 2-arrangements with that of complex arrangements.

The intersection lattice of a 2-arrangement is again a geometric lattice, where real codimension corresponds to twice the lattice rank: $2{\cdot}r(A) = \operatorname{codim}_{\mathbb{R}}(\bigcap_{a\in A} H_a)$.

The cohomology groups of the complements of 2-arrangements were computed by Goresky & MacPherson using Stratified Morse Theory. An alternative approach to the

computation, via spectral sequences, is provided by Vassiliev [Va] and Jewell, Orlik & Shapiro [JOS]. A third proof, with homotopy methods, is given by Ziegler & Živaljević [ZZ]. However, the algebra structure is not supplied by either approach. The combinatorial method of Björner & Ziegler [BZ] yields the following information about it.

Theorem 1.1′. [GM][BZ] *Let $\mathcal{B} = \{H_1, \ldots, H_n\}$ be a 2-arrangement in $\mathbb{R}^{2d}$. For every $H_a \in \mathcal{B}'$ choose two linear forms $l_a, l'_a \in (\mathbb{R}^{2d})^*$ that define it, such that $\ker(l_a) \cap \ker(l'_a) = H_a$ $(1 \leq a \leq n)$. Then the integral cohomology algebra of the complement $C_{\mathcal{B}}$ is generated by the 1-dimensional classes*

$$\omega(l_a, l'_a) := \frac{1}{2\pi} \frac{-l'_a \mathrm{d}l_a + l_a \mathrm{d}l'_a}{{l_a}^2 + {l'_a}^2},$$

for $1 \leq a \leq n$. It has a presentation of the form

$$0 \longrightarrow I \longrightarrow \Lambda^* \mathbb{Z}^n \xrightarrow{\pi} \mathrm{H}^*(C_{\mathcal{B}}; \mathbb{Z}) \longrightarrow 0,$$

defined by $\pi(e_a) := [\omega(l_a, l'_a)]$, where $\{e_1, \ldots, e_n\}$ denotes a basis of $\mathbb{Z}^n$. The relation ideal I is generated by elements of the form

$$\sum_{i=0}^{k} \epsilon_i \cdot e_{a_0} \wedge \ldots \wedge \widehat{e_{a_i}} \wedge \ldots \wedge e_{a_k},$$

for the circuits $A = \{a_0, \ldots, a_k\}$ of L, with $\epsilon_i \in \{+1, -1\}$.

In the following, we will show that the inability of Theorem 1.1′ to determine the precise form of the presentation of $\mathrm{H}^*(C_{\mathcal{B}'}; \mathbb{Z})$ from the combinatorial data is not a weakness of stratified Morse theory of [GM] and of the combinatorial set-up of [BZ]. In fact, the cohomology algebra $\mathrm{H}^*(C_{\mathcal{B}'}; \mathbb{Z})$, and hence the homotopy type of the complement of a 2-arrangement, is not determined by the combinatorial data!

Theorem 1.2. *There are two different 2-arrangements $\mathcal{B}$ and $\mathcal{B}'$ of 2-dimensional linear subspaces in $\mathbb{R}^4$ whose intersection lattices coincide (the corresponding matroid is the uniform matroid $U_{2,4}$), but whose complements have non-isomorphic cohomology algebras.*

In the following section we will give an extensive analysis of the topology of the 2-arrangements of 4 transversal 2-subspaces in $\mathbb{R}^4$ (corresponding to the uniform matroid $U_{2,4}$), and show how they can sometimes be distinguished by the cohomology algebras of their complements. In Section 3 the implications for the singularity links of 2-arrangements are derived. In Section 4 we obtain a general method to compute a presentation of $\mathrm{H}^*(C_{\mathcal{B}'}; \mathbb{Z})$ once equations for $\mathcal{B}'$ are chosen. Section 5 discusses the relation to the study of knots and links in S^3.

In the following we will denote 2-arrangements by $\mathcal{B}'$, and only drop the prime in the case of a complex arrangement.

2. Example

In this section we consider 2-arrangements in $\mathbb{R}^4$: arrangements of 2-dimensional linear subspaces $\mathcal{B}' = \{H_a : 1 \leq a \leq n\}$ in $\mathbb{R}^4$ that are *transversal*, that is, have pairwise intersection $\{0\}$. They represent the uniform matroid $M = U_{2,n}$. We will use coordinates u, v, x, y on $\mathbb{R}^4$, which we abbreviate as $w = u + iv$, $z = x + iy$ when the usual complex structure (identification of $\mathbb{C}^2$ and $\mathbb{R}^4$) is chosen.

In suitable coordinates we can assume that

$$\begin{aligned} H_1 &= \{(w,z) \in \mathbb{R}^4 : w = 0\}, \\ H_2 &= \{(w,z) \in \mathbb{R}^4 : z = 0\}, \\ H_3 &= \{(w,z) \in \mathbb{R}^4 : z = w\}. \end{aligned}$$

We note here that the projection $\pi : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$, which maps $(u,v,x,y) \longmapsto (u,v)$, $(w,z) \longmapsto w$, makes the complement $C_{\mathcal{B}'}$ into a fiber bundle over $\mathbb{C}^*$, whose fiber is $\mathbb{C}$ minus $n-1$ points. As a consequence of this we deduce that every 2-arrangement in $\mathbb{R}^4$ is a $K(\pi,1)$-arrangement: the long exact homotopy sequence of the fiber bundle shows that the higher homotopy groups of $C_{\mathcal{B}'}$ vanish in this case.

The fiber bundle is trivial if $\mathcal{B}$ is a complex arrangement [Or, Prop. 5.3]: assume that the hyperplanes are $H_1 = \{(w,z) : w = 0\}$ and $H_a = \{(z,w) : z = \lambda_a w\}$ for $2 \leq a \leq n$, then

$$\begin{aligned} \mu : C_{\mathcal{B}} &\longrightarrow \mathbb{C}^* \times \mathbb{C} \backslash \{\lambda_2, \ldots, \lambda_n\} \\ (w,z) &\longmapsto (w, \frac{z}{w}) \end{aligned}$$

trivializes the bundle. The bundle is usually non-trivial for 2-arrangements in $\mathbb{R}^4$. Our results of this section will imply that it is not in general homotopy equivalent to a product space with a factor $\mathbb{C}^*$.

In the complex case, even more can be said. For this, note that the singularity link of a complex arrangement in $\mathbb{C}^2$ is a disjoint union of circles, and each circle has a natural orientation, given by multiplication with e^{it}.

Proposition 2.1. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two arrangements of n hyperplanes (1-dimensional complex subspaces) in $\mathbb{C}^2$, with singularity links D_1 and D_2. Then every orientation-preserving homeomorphism $D_1 \longrightarrow D_2$ can be extended to a homeomorphism $(S^3, D_1) \longrightarrow (S^3, D_2)$.*

Proof. Assume that $\mathcal{B}_i$ is given by $l_{i1}(w,z) = z$ and $l_{ij}(w,z) = w - \lambda_{ij} z$ for $2 \leq j \leq n$. Then any homeomorphism of the Riemann sphere that fixes infinity and maps λ_{1j} to λ_{2j} for all j yields a homeomorphism $(S^3, D_1) \longrightarrow (S^3, D_2)$. The fact that the initial homeomorphism $D_1 \longrightarrow D_2$ can be prescribed arbitrarily now follows from surgery along a tubular neighborhood of D_1 resp. D_2. □

Now we will restrict our attention to the case $n = 4$. It is not hard to see [VD, p. 1038] that there are three isotopy classes of arrangements. One class contains the

complex arrangements, which we denote by $\mathcal{B}$. The image of a complex arrangement $\mathcal{B}$ after a reflection in $\mathbb{R}^4$ is not isotopic to a complex arrangement, but clearly isomorphic as a 2-arrangement. The third class is represented by the 2-arrangement $\mathcal{B}'$ in the second case of the following example.

Example 2.2. As the **first case** we consider the arrangement $\mathcal{B}$

$$\mathcal{B}: \begin{cases} H_1 = & \{(w,z) \in \mathbb{R}^4 : \quad w = 0\} \\ H_2 = & \{(w,z) \in \mathbb{R}^4 : \quad z = 0\} \\ H_3 = & \{(w,z) \in \mathbb{R}^4 : \quad z = w\} \\ H_4 = & \{(w,z) \in \mathbb{R}^4 : \quad z = 2w\}. \end{cases}$$

This is a complex (in fact: complexified real) arrangement. Its cohomology algebra has by Theorem 1.1 the following presentation:

$$\mathrm{H}^*(C_{\mathcal{B}};\mathbb{Z}) \cong \Lambda^*\mathbb{Z}^4 / \left\langle \begin{matrix} +e_{12} - e_{13} + e_{23} \\ +e_{12} - e_{14} + e_{24} \\ +e_{13} - e_{14} + e_{34} \\ +e_{23} - e_{24} + e_{34} \end{matrix} \right\rangle$$

where the last relation is a consequence of the first three.

As the **second case** we consider the arrangement $\mathcal{B}'$

$$\mathcal{B}': \begin{cases} H_1 = & \{(w,z) \in \mathbb{R}^4 : \quad w = 0\} \\ H_2 = & \{(w,z) \in \mathbb{R}^4 : \quad z = 0\} \\ H_3 = & \{(w,z) \in \mathbb{R}^4 : \quad z = w\} \\ H_4 = & \{(w,z) \in \mathbb{R}^4 : \quad z = 2\overline{w}\}. \end{cases}$$

This arrangement is not linearly isomorphic to a complex one. However, its cohomology algebra has by Theorem 1.1′ a very similar presentation. With the method of Theorem 4.1 below, one can determine the signs in the presentation:

$$\mathrm{H}^*(C_{\mathcal{B}'};\mathbb{Z}) \cong \Lambda^*\mathbb{Z}^4 / \left\langle \begin{matrix} +e_{12} - e_{13} + e_{23} \\ +e_{12} + e_{14} + e_{24} \\ -e_{13} - e_{14} + e_{34} \\ +e_{23} - e_{24} - e_{34} \end{matrix} \right\rangle$$

where the last relation is a consequence of the first three.

In both cases the broken circuit complex

$$\mathrm{BC}(U_{2,4}) = \{\emptyset, 1, 2, 3, 4, 12, 13, 14\}$$

indexes a basis of the cohomology module, that is, the (classes of) e_1, e_2, e_3, e_4 induce a $\mathbb{Z}$-basis of H^1, while e_{12}, e_{13}, e_{14} induce a $\mathbb{Z}$-basis of H^2, and $\mathrm{H}^3 = \mathrm{H}^4 = 0$, see [BZ, Sect. 7]. In particular, the cohomology modules $\mathrm{H}^*(C_{\mathcal{B}};\mathbb{Z})$ and $\mathrm{H}^*(C_{\mathcal{B}'};\mathbb{Z})$ are linearly isomorphic. Their difference is hidden in the multiplicative structure. For the following theorem we do not assume that an isomorphism maps generators to generators, but argue with an invariant construction. It was inspired by [F1], although Falk's invariants do not suffice to distinguish the algebras $\mathrm{H}^*(C_{\mathcal{B}};\mathbb{Z})$ and $\mathrm{H}^*(C_{\mathcal{B}'};\mathbb{Z})$.

Theorem 2.3. *The cohomology algebras* $\mathrm{H}^*(C_{\mathcal{B}};\mathbb{Z})$ *and* $\mathrm{H}^*(C_{\mathcal{B}'};\mathbb{Z})$ *are not isomorphic as graded* $\mathbb{Z}$*-algebras.*

Proof. Let A denote any of the two algebras and let A^1 be its 1-dimensional part. Then A has a presentation of the form

$$0 \longrightarrow I \longrightarrow \Lambda^* A^1 \longrightarrow A \longrightarrow 0,$$

where I is again a graded ideal. Here $I^1 = 0$ by construction, while I^2 has rank 3. We consider the map

$$\kappa : I^2 \otimes I^2 \longrightarrow \Lambda^4 A^1$$

induced by multiplication in $\Lambda^* A^1$. In the case we are considering $A^1 \cong \mathbb{Z}^4$, so $\Lambda^4 A^1 \cong \mathbb{Z}$, and κ defines a symmetric bilinear form on I^2.

A direct calculation shows that κ vanishes identically for $\mathrm{H}^*(C_{\mathcal{B}};\mathbb{Z})$. This can also be derived from the Künneth formula, since $C_{\mathcal{B}}$ is a product space.

However, for $\mathcal{B}'$ the bilinear form κ has rank 2: with respect to the basis $\{e_{12}, e_{13}, e_{14}\}$ it is represented by the matrix

$$\begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}.$$

This proves $\mathrm{H}^*(C_{\mathcal{B}};\mathbb{Q}) \ncong \mathrm{H}^*(C_{\mathcal{B}'};\mathbb{Q})$. □

3. Links of 2-arrangements

The singularity link of any 2-arrangement is homotopy equivalent to a wedge of spheres if $d \geq 3$ [BZ, Thm. 6.6] [ZZ, Cor. 3.3] and it is a disjoint union of circles if $d = 2$.

In fact, there is a certain plausibility to the conjecture that for complex arrangements the singularity links are determined up to homeomorphism by the intersection lattices, for all d. This is a very strong conjecture, which would (with the ideas below) imply the same for the complements of complex arrangements, which is much stronger than the notorious conjecture [Or] for the homotopy type. In this section, we will prove this fact for $d \leq 3$, and then use the example of Section 2 to disprove a similar statement in the case of 2-arrangements.

Theorem 3.1. *The intersection lattice determines the singularity link of a complex arrangement in* $\mathbb{C}^3$ *up to homeomorphism.*

Proof. Let $\mathcal{B}$ be a complex arrangement of n 2-dimensional subspaces in $\mathbb{C}^3$, with intersection lattice L. The singular set of D is a disjoint union of k circles, where k is the number of coatoms (elements of rank 2) in L. Thus we construct $\mathrm{sing}(D)$ as a set of k disjoint oriented circles, where the orientation is supposed to be the natural one corresponding to multiplication with e^{it}. Now we glue n 3-spheres into the given set of oriented circles. The attaching maps exist and are unique by Proposition 2.1. □

Theorem 3.2. *The intersection lattice does not determine the singularity link of a 2-arrangement in* $\mathbb{R}^6$ *up to homeomorphism.*

Proof. We consider generic 2-arrangements $\hat{\mathcal{B}}' = \{H_1, H_2, H_3, H_4, H_5\}$ representing the matroid $U_{3,5}$, that is, arrangements of five 4-dimensional subspaces in $\mathbb{R}^6$ so that the intersection of any three of them is $\{0\}$. Their singularity links are unions of five copies $S_i := H_i \cap S^5$ of S^3, pairwise intersecting in circles. We notice that the non-singular parts $S_i^\circ := S_i \backslash \bigcup_{j \neq i} S_j$ are easily identified by local cohomology. Each of these parts S_i° is homeomorphic to the complement of the restriction $\hat{\mathcal{B}}'|H_i$, which is a 2-arrangement of four 2-subspaces in $\mathbb{R}^4$, as discussed in Section 2. If $\hat{\mathcal{B}}'$ is a complex arrangement, then the restrictions $\hat{\mathcal{B}}'|H_i$ are complex as well.

However, for $\hat{\mathcal{B}}'$ given by the equations

$$\begin{aligned} H_1 &: z_1 = 0, \\ H_2 &: z_2 = 0, \\ H_3 &: z_3 = 0, \\ H_4 &: z_1 - z_2 + z_3 = 0, \\ H_5 &: z_1 - 2\overline{z}_2 + 3z_3 = 0, \end{aligned}$$

we find that $\hat{\mathcal{B}}'|H_3$ is isomorphic to the arrangement $\mathcal{B}'$ considered in Example 2.2, so S_3° is homeomorphic to $C_{\mathcal{B}'}$, and hence it is not homeomorphic to a non-singular part of the singularity link of a complex arrangement. □

4. Cohomology of 2-arrangements

In this section we describe a method to compute the relations in the cohomology algebra of any 2-arrangement. It relies on the representation of cohomology classes by the corresponding differential forms of real deRham theory, and it exploits the passage to complex deRham theory in the case of subarrangements that have a complex structure, like those corresponding to the circuits of the matroid. It seems desirable to derive a presentation in the combinatorial framework and generality of [BZ]; however, this has not yet been achieved.

We will need the relation between the real and the complex differential form representing the cohomology class of a complex hyperplane. For this assume that coordinates have been chosen so that the hyperplane H is represented by $z = 0$, which with $z = x+iy$ corresponds to real equations $x = y = 0$. Then straightforward computations show that

$$\frac{1}{2\pi i}\left(\frac{\mathrm{d}z}{z} - \frac{\mathrm{d}\overline{z}}{\overline{z}}\right) = \frac{1}{2\pi i}\mathrm{d}\log(x^2+y^2),$$

which is an exact form, while

$$\frac{1}{2\pi i}\left(\frac{\mathrm{d}z}{z} + \frac{\mathrm{d}\overline{z}}{\overline{z}}\right) = \frac{1}{\pi}\frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2},$$

which is twice the real differential form that represents the cohomology class of $\mathbb{C}^d \backslash H$. Thus

$$\left[\frac{1}{2\pi i}\frac{\mathrm{d}z}{z}\right] = \left[\frac{1}{2\pi i}\frac{\mathrm{d}\overline{z}}{\overline{z}}\right] = \left[\frac{1}{2\pi}\frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}\right].$$

Theorem 4.1. *Let a 2-arrangement $\mathcal{B}' = \{H_1, \ldots, H_n\}$ in $\mathbb{R}^{2d}$ be given by*

$$H_a = \{\mathbf{x} \in \mathbb{R}^{2d} : l_a(\mathbf{x}) = l'_a(\mathbf{x}) = 0\},$$

where the $l_a, l'_a : \mathbb{R}^{2d} \longrightarrow \mathbb{R}$ are linear forms so that

$$\operatorname{rank}\{l_a, l'_a : a \in A\} = \begin{cases} \text{even} & \text{for all } A \subseteq \{1,\ldots,n\}, \\ 2 & \text{for all } A = \{a\}, \\ 4 & \text{for all } A = \{a,b\}, \\ 2d & \text{for } A = \{1,\ldots,n\}. \end{cases}$$

To every $H_a \in \mathcal{B}'$ associate the differential form

$$\omega(l_a, l'_a) := \frac{1}{2\pi}\frac{-l'_a \mathrm{d}l_a + l_a \mathrm{d}l'_a}{{l_a}^2 + {l'_a}^2},$$

which is a closed form on $\mathbb{R}^{2d} \backslash H_a$ that is normalized to have residue ± 1. The relations between the corresponding cohomology classes can be constructed as follows. Let $A = \{a_0, a_1, \ldots, a_k\}$ be a circuit of $L_{\mathcal{B}'}$, so there are two real linear dependencies of the form

$$\sum_{j=0}^{k} \alpha_j l_{a_j} + \beta_j l'_{a_j} = 0,$$
$$\sum_{j=0}^{k} \gamma_j l_{a_j} + \delta_j l'_{a_j} = 0,$$

with $\alpha_0 = \delta_0 = -1$, $\beta_0 = \gamma_0 = 0$. These induce the relation

$$\sum_{j=0}^{k} (-1)^j \operatorname{sign}\begin{vmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{vmatrix} \omega(l_{a_1}, l'_{a_1}) \wedge \ldots \wedge \omega(\widehat{l_{a_j}, l'_{a_j}}) \wedge \ldots \wedge \omega(l_{a_k}, l'_{a_k}) \sim 0$$

in the cohomology algebra $\mathrm{H}^*(C_{\mathcal{B}'}; \mathbb{Z})$.

Proof. The conditions on the forms l_a, l'_a assure that they define a 2-arrangement. The differential forms $\omega(l_a, l'_a)$ generate $\mathrm{H}^*(C_{\mathcal{B}'}; \mathbb{Z})$, by Theorem 1.1′.

To derive the relations we construct coordinates x_j, y_j for $\mathbb{R}^{2d}$ so that

$$x_j = \alpha_j l_{a_j} + \beta_j l'_{a_j}$$
$$y_j = \gamma_j l_{a_j} + \delta_j l'_{a_j}$$

for $1 \le j \le k$ — this is possible since A is a circuit, so $\{l_a, l'_a : a \in A\backslash a_0\}$ is linearly independent. The even rank condition furthermore guarantees $\begin{vmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{vmatrix} \neq 0$. Now we observe, by computing the residues, that

$$\omega(x_{a_j}, y_{a_j}) \sim \operatorname{sign} \begin{vmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{vmatrix} \cdot \omega(l_{a_j}, l'_{a_j}).$$

Introducing complex coordinates $z_j := x_j + iy_j$, we get that H_{a_0} has the (complex!) equation $z_1 + \ldots + z_k = 0$, while H_{a_j} is given by $z_j = 0$ for $1 \le j \le k$, and thus

$$\omega_j := \frac{1}{2\pi i} \frac{\mathrm{d}z_j}{z_j} \sim \omega(x_j, y_j) \sim \operatorname{sign} \begin{vmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{vmatrix} \cdot \omega(l_{a_j}, l'_{a_j}).$$

Thus the relation

$$\sum_{j=0}^{k} (-1)^j \, \omega_0 \wedge \ldots \wedge \widehat{\omega_i} \wedge \ldots \wedge \omega_k \; = \; 0$$

for the complex arrangement $\{H_{a_0}, H_{a_1}, \ldots, H_{a_k}\}$ translates into the desired formula. □

Note that the formula of Theorem 4.1 specializes to the Orlik-Solomon relations in the case of a complex arrangement: for a complex arrangement we can write the defining forms as $l_a + il'_a$, and the relation corresponding to a circuit takes the form

$$\sum_{j=0}^{k} (\alpha_j + i\beta_j)(l_{a_j} + il'_{a_j}) = 0 \quad \Longleftrightarrow \quad \begin{cases} \sum_{j=0}^{k} \alpha_j l_{a_j} + \beta_j l'_{a_j} = 0 \\ \sum_{j=0}^{k} -\beta_j l_{a_j} + \alpha_j l'_{a_j} = 0 \end{cases}$$

Thus for the formula of Theorem 4.1 we get the special case $\gamma_j = -\beta_j$ and $\delta_j = \alpha_j$, so that $\operatorname{sign} \begin{vmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{vmatrix} = \operatorname{sign}(\alpha_j^2 + \beta_j^2) = +1$.

5. Link invariants

The classification of 2-arrangements in $\mathbb{R}^4$ is clearly equivalent to the study of
- arrangements of disjoint great circles in S^3,
- arrangements of skew lines in $\mathbb{RP}^3$,
- arrangements of affine skew lines in $\mathbb{R}^3$,

as is e.g. stressed by Viro [V] and by Viro & Drobotukhina [VD, p. 1046]. The corresponding equivalence relation on line arrangements is there called *rigid isotopy*.

Considering arrangements of circles in S^3 as *links*, one is lead to study to what extent link invariants can distinguish equivalence classes of 2-arrangements in $\mathbb{R}^4$. In particular, the (2×2)-determinants derived in Section 4, which determine the sign pattern of the

relations in cohomology, are just linking numbers of the corresponding (oriented) circles. In the description as links in the 3-sphere it is well-known [VD, p. 1034] that for every triple of circles we get a *linking coefficient* ± 1 that does not depend on the order or the chosen orientations. These linking coefficients of triples are sufficient to distinguish the arrangements $\mathcal{B}$ and $\mathcal{B}'$ of Example 2.2. We refer to [V], [VD] and [M] for this approach.

The key problem here is that links do not in general determine the homotopy type of their complements in S^3 (see e.g. [R, p. 62]), although this might be true for the special type of links that correspond to arrangements. Therefore the results of [V], [VD] and [M] do not immediately distinguish complements of 2-arrangements. In fact, the relation between the "new" link invariants and "classical" data like the fundamental group of the complement is still obscure [Bi, p. 59].

A presntation of the fundamental group of the complement of a 2-arrangement can easily be computed — the standard method due to Wirtinger (see [R]) derives it from a planar projection; since the links we consider are closed braids, an equivalent (but more systematic) way is given by Artin's approach [Ar]. However, even for the simple case of the 2-arrangement of four subspaces in $\mathbb{R}^4$ the corresponding links have projections with 12 crossings, so these methods become unwieldy. From the description as a fiber bundle, one sees that the fundamental group π in the case of a complex arrangement is a product of $\mathbb{Z}$ with the free group $F\langle t_1, t_2, t_3\rangle$ on three generators. In the case of the arrangement $\mathcal{B}'$, we find that π' is a non-trivial solution of the extension problem

$$F\langle t_1, t_2, t_3\rangle \longrightarrow \pi' \longrightarrow \mathbb{Z} \longrightarrow 0.$$

However, there seems to be no simple or direct way to describe the homotopy group π'.

Corollary 5.1. *The fundamental groups π, π' of the complements of the arrangements $\mathcal{B}$ and $\mathcal{B}'$ of Example 2.2 are not isomorphic.*

Proof. We have seen that $C_{\mathcal{B}} \simeq K(\pi, 1)$ and $C_{\mathcal{B}'} \simeq K(\pi', 1)$ in Section 2, and that these spaces have non-isomorphic cohomology algebras in Theorem 2.3. Hence $\pi \ncong \pi'$. □

Example 5.2. [M, Ass. 3] [VD, p. 1043] There are two 2-arrangements $\mathcal{B}'$ and $\mathcal{B}''$ of six two-dimensional transversal subspaces in $\mathbb{R}^4$ with the following properties:

- the cohomology algebras $\widetilde{H}^*(C_{\mathcal{B}'}; \mathbb{Z})$ and $\widetilde{H}^*(C_{\mathcal{B}''}; \mathbb{Z})$ are isomorphic, because the subspaces in the arrangements can be labeled and oriented in such a way that the pairwise linking numbers coincide,
- the pairs (S^3, D') and (S^3, D'') are not homeomorphic, since they represent inequivalent links in S^3 that can be distinguished by link polynomials.

We do not know whether the complements $S^3 \backslash D'' \simeq C_{\mathcal{B}''}$ and $S^3 \backslash D' \simeq C_{\mathcal{B}'}$ are homotopy equivalent or, equivalently (by the argument of Corollary 5.1), whether the fundamental groups coincide.

More generally, we do not know whether the complements of two 2-arrangements must be homotopy equivalent once their cohomology algebras are isomorphic.

It seems that the analysis of the cohomology algebra (as in Theorem 2.3) is simpler than any computation of the fundamental group of a 2-arrangement. However, we are not aware of any systematic study of the cohomology algebra of the complement of a link (compare e.g. [R, p. 50]). The linear structure of this algebra is determined by the number of components of the link, because of Alexander duality. But, as we have seen in Section 2, the multiplicative structure encodes non-trivial information.

However, we note that the complement of every 2-arrangement is *formal* in the sense of rational homotopy theory [GrH, p. 158]. In fact, by Theorem 1.1' the cohomology algebra $\widetilde{H}^*(C_{\mathcal{B}'};\mathbb{Z})$ can be represented by a subalgebra of the real deRham complex on $C_{\mathcal{B}'}$. With the argument of [F2, p. 546] this implies that $C_{\mathcal{B}'}$ is a formal space. In particular, there is no "higher order" cohomology information (like the Massey products used in [GrM, Sect. IIIX.C]) contained in the real deRham complex of $C_{\mathcal{B}'}$.

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