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Critical and Anticritical Edges in Perfect Graphs

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Abstract

We call an edge e of a perfect graph G critical if $G - e$ is imperfect and say further that e is anticritical with respect to the complementary graph $\overline{G}$. We ask in which perfect graphs critical and anticritical edges occur and how to find critical and anticritical edges in perfect graphs. Finally, we study whether we can order the edges of certain perfect graphs such that deleting all the edges yields a sequence of perfect graphs ending up with a stable set.

1 Introduction

The present paper is devoted to the investigation of critical and anticritical edges with respect to perfectness, a rich and well-studied graph property. Berge proposed to call a graph $G = (V, E)$ **perfect** if, for each of its induced subgraphs $G' \subseteq G$, the chromatic number $\chi(G')$ equals the clique number $\omega(G')$. That is, for every induced subgraph G' of a perfect graph G we need as many stable sets to cover all nodes of G' as a maximum clique of G' has nodes (a set $V' \subseteq V$ is a clique (stable set) of G if the nodes in V' are pairwise (non-)adjacent).

Berge [1] conjectured two characterizations of perfect graphs. His first conjecture was that a graph G is perfect if and only if the clique covering number $\overline{\chi}(G')$ equals the stability number $\alpha(G')$ for all $G' \subseteq G$ (i.e., that we need as many cliques to cover all nodes of G' as a maximum stable set of G' has nodes). Since complementation transforms stable sets into cliques and colorings into clique coverings, we have $\alpha(G) = \omega(\overline{G})$ and $\chi(G) = \overline{\chi}(\overline{G})$ where $\overline{G}$ denotes the complement of G. Hence, Berge conjectured that a graph G is perfect if and only if its complement $\overline{G}$ is. This was proven by Lovász [10] and is nowadays known as *Perfect Graph Theorem*.

The second Berge conjecture concerns a characterization of perfect graphs via forbidden subgraphs. It is a simple observation that chordless odd cycles C_{2k+1} with $k \geq 2$, termed **odd holes**, and their complements $\overline{C}_{2k+1}$, called **odd antiholes**, are **imperfect**. Clearly, each graph containing an odd hole or an odd antihole as induced subgraph is imperfect as well. Berge conjectured in [1] that a graph is perfect if and only if it contains neither odd holes nor odd antiholes as induced subgraphs; such graphs are nowadays called **Berge graphs**. This still open *Strong Perfect Graph Conjecture* has already been verified for several classes of F-free Berge graphs, e.g., if F is a claw (Parthasarathy and RAVINDRA [15]), a diamond (TUCKER [18]), a clique K_4 of size 4 (TUCKER [17]), or a bull (CHVÁTAL and SBIHI [4]), see Figure 1. PADBERG [14] introduced the notion of **minimally imperfect** graphs: imperfect graphs with the property that removing any of its nodes yields a perfect graph. Using this term, the Strong Perfect Graph Conjecture reads that odd holes and odd antiholes are the only minimally imperfect graphs. Therefore, minimally imperfect Berge graphs are termed **monsters**. In order to verify or falsify the Strong Perfect Graph Conjecture, many structural properties of minimally imperfect graphs have been discovered. To mention only a few of them, the following node pairs must not occur in minimally imperfect graphs G. It is well-known that G has no **comparable pair** (two nodes x and y with $N(x) - y \subseteq N(y)$, i.e., all neighbors of x except eventually y belong to the neighborhood of y). G does not admit **twins** (**antitwins**), i.e., two nodes x and y such that all remaining nodes of G are adjacent to both or to none of x and y (to either x or to y) due to the *Replacement Lemma* [10] (*Antitwin Lemma* [13]). Note that the property of being a comparable pair, twins, or antitwins does not depend on whether or not x and y are adjacent. Furthermore, no minimally imperfect graph G contains an **even pair** (two nodes x and y such that all chordless paths connecting x and y have even length) by the *Even Pair Lemma* [12] and a **star-cutset** (a cutset S containing one node that is adjacent to all remaining nodes of S) by the *Star-Cutset Lemma* [3].

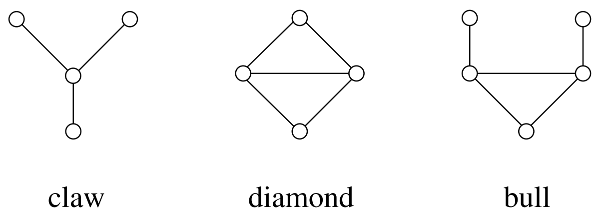


Figure 1

The starting point of the investigation of critical and anticritical edges in perfect graphs was the following. Imagine you have an arbitrary perfect graph and you consecutively delete one edge until you get a stable set, or you consecutively add one edge until a clique is reached. So you create a sequence of graphs starting and ending up with a perfect graph. But, if you choose the edges to be deleted or added randomly, most graphs of your sequence will be imperfect. The aim is to avoid the occurrence of imperfect graphs in our sequence.

Problem 1.1 *Given a certain perfect graph, is there a rule how to choose an edge to be deleted or added in order to keep perfectness?*

We call an edge e of a perfect graph G **critical** if $G-e$ is imperfect. Moreover, we say that e is an **anticritical** edge of the complement $\overline{G}$: $e \notin E(\overline{G})$ holds and $\overline{G}+e$ is imperfect due to the Perfect Graph Theorem [10]. Whenever we delete (add) a critical (anticritical) edge e of a perfect graph G, we create in $G-e$ ($G+e$) *minimally* imperfect subgraphs. In order to attack Problem 1.1, we study in Section 2 those subgraphs $G_e \subseteq G$ which yield the minimally imperfect subgraphs $G_e - e \subseteq G - e$ ($G_e + e \subseteq G + e$). One example of a critical edge is a single **short chord** of a cycle with odd length ≥ 5 (that is a chord whose endnodes have distance two on the cycle), the deletion of e yields an odd hole. Hence, one rule in the sense of Problem 1.1 is "never remove a single short chord of an odd cycle". Analogously, adding the edge between the endnodes of a chordless path P_{2k+1} with $k \geq 2$ yields an odd hole, too. "Never add an edge connecting the endnodes of a chordless path of even length ≥ 4" is, therefore, another rule in the sense of 1.1. Further results of that type are presented in Section 2.

Section 3 investigates whether it is easier to solve Problem 1.1 if the graph in question belongs to a certain *subclass* of perfect graphs. Certain perfect graphs may not possess any critical or anticritical edge, then we could choose an arbitrary edge keeping perfectness.

Problem 1.2 *In which perfect graphs do critical and anticritical edges occur at all?*

One answer gives a new characterization of **Meyniel graphs**. They have been introduced in [11] as graphs all odd cycles of length ≥ 5 of which admit at least two chords. So critical edges in form of single short chords of odd cycles do obviously not occur in Meyniel graphs. Actually, removing any edge from a Meyniel graph keeps perfectness by [8] and we show: *a graph is Meyniel if and only if it does not admit any critical edge.* In particular, every perfect but non-Meyniel graph admits at least one critical edge. We study in Section 3 for such classes of perfect graphs which minimally imperfect subgraphs may occur after deleting (adding) a critical (anticritical) edge. The large abundance of classes of perfect graphs led us to mention only results for some "classical" classes: **Strongly perfect graphs** have been introduced in [2] to be graphs all of whose subgraphs $G' \subseteq G$ admit a stable set that has a non-empty intersection with all maximal cliques of G'. **Weakly triangulated graphs** are defined to have neither holes C_k nor antiholes $\overline{C}_k$ with $k \geq 5$ as induced subgraphs [5]. Meyniel [12] called a graph G **strict quasi parity** if each of its non-complete subgraphs has an even pair and **quasi parity** if G' or $\overline{G}'$ owns an even pair for each subgraph $G' \subseteq G$.

In Section 4, we treat the following problem and present several classes of perfect graphs for which the answer is in the affirmative.

Problem 1.3 *For a certain perfect graph, is there an order of all the edges to be deleted (added) so that we get a sequence of perfect graphs ending up with a stable set (clique)?*

It turns out that it does not suffice to identify non-critical or non-anticritical edges: we can certainly remove an arbitrary edge from a Meyniel graph keeping perfectness but, at present, we do not know anything about critical edges of the resulting graph. Thus we have to look for edges the deletion or addition of which preserves membership within the corresponding *subclass* of perfect graphs. By this way, we present the studied ordering of edges to be deleted or added for, e.g., weakly triangulated graphs.

2 Critical and Anticritical Edges

For every critical (anticritical) edge e of a perfect graph G there is necessarily at least one subgraph $G_e \subseteq G$ such that $G_e - e$ $(G_e + e)$ is *minimally* imperfect. We study those subgraphs G_e in order to give an answer to Problem 1.1. If $G_e - e \subseteq G - e$ is an odd hole, then G_e is isomorphic to an odd cycle of length ≥ 5 which admits precisely one chord, namely e. Moreover, e has to be a short chord of this cycle, forming a triangle with two of its edges, since G_e must not contain an odd hole (the case $G_e - e = C_7$ is depicted in Figure 2(a) with $e = xy$). Thus, we have immediatly:

Proposition 2.1 *If $G_e - e$ is an odd hole C_{2k+1} with $k \geq 2$, then $\omega(G_e) = 3$ and $\alpha(G_e) = k$ holds. G_e contains an even hole C_{2k}. A bull and the complement of a claw and of a diamond appear in G_e if $k \geq 3$.*

In the complementary graph $\overline{G}$, e is an anticritical edge and $\overline{G}_e + e$ an odd antihole (see Figure 2(b)). Clearly, the complementary statement of Proposition 2.1 is true for $\overline{G}_e$.

Proposition 2.2 *If $G_e + e$ is an odd antihole $\overline{C}_{2k+1}$ with $k \geq 2$, then $\omega(G_e) = k$ and $\alpha(G_e) = 3$ holds. G_e contains an even antihole $\overline{C}_{2k}$. A bull, a claw, and a diamond appear in G_e if $k \geq 3$.*

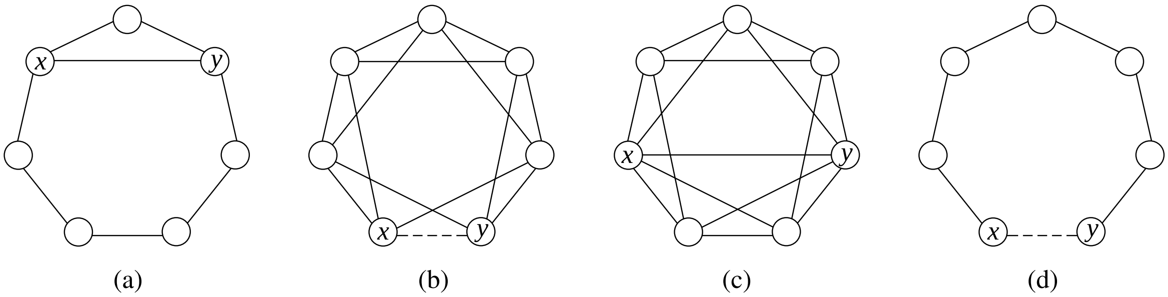


Figure 2

If $G_e - e \subseteq G - e$ is an odd antihole, then G_e is the complement of an induced path P_{2k+1} with $k \geq 2$ (the case $G_e - e = \overline{C}_7$ is depicted in Figure 2(c) with $e = xy$). We have, therefore:

Proposition 2.3 *If $G_e - e$ is an odd antihole $\overline{C}_{2k+1}$ with $k \geq 2$, then $\omega(G_e) = k + 1$ and $\alpha(G_e) = 2$ holds. G_e is isomorphic to $\overline{P}_{2k+1}$. A diamond appears in G_e if $k \geq 3$.*

In the complement $\overline{G}$, e is an anticritical edge, $\overline{G}_e + e$ an odd hole, and $\overline{G}_e$ an induced path (see Figure 2(d)). The complementary assertion of Proposition 2.3 is true for $\overline{G}_e$.

Proposition 2.4 *If $G_e + e$ is an odd hole C_{2k+1} with $k \geq 2$, then $\omega(G_e) = 2$ and $\alpha(G_e) = k + 1$ holds. G_e is isomorphic to P_{2k+1}. The complement of a diamond appears in G_e if $k \geq 3$.*

Of course, such a description cannot be given if $G_e - e$ or $G_e + e$ is supposed to be a monster and, if the Strong Perfect Graph Conjecture is true, this case does not occur at all. With look at Problem 1.1, we list some properties of G_e if e is a critical edge and $G_e - e$ a generally minimal imperfect graph.

Lemma 2.5 *Let e be a critical edge of a perfect graph G and $G_e - e \subseteq G - e$ be minimally imperfect.*

(i) *The endnodes of e belong to the intersection of all maximum cliques of G_e.*

(ii) *The endnodes of e occur in a triangle of G_e.*

(iii) *An even hole is running through e in G_e.*

(iv) *If $G_e - e$ is a monster, then e belongs to a K_5 and to a diamond of G_e.*

Proof. Consider a critical edge e of a perfect graph G and let $G_e - e \subseteq G - e$ be minimally imperfect. We know $\omega(G_e) = \chi(G_e)$ but $\omega(G_e - e) < \chi(G_e - e)$. Since the removal of an edge cannot increase the chromatic number, $\omega(G_e) > \omega(G_e - e)$ follows. That means deleting e destroys all maximum cliques of G_e and (i) is true. In particular, $\omega(G_e) = 2$ and $\omega(G_e - e) = 1$ follows from (i) if the endnodes of e do not possess any common neighbor in G_e. Hence the endnodes of e occur in a triangle of G_e and (ii) is shown.

Due to the Even Pair Lemma [12] there is an odd induced path between every two non-adjacent nodes of $G_e - e$, in particular, between the endnodes of e. Thus an even hole is running through e in G_e and we obtain (iii).

In the special case that $G_e - e$ is supposed to be a monster, we need $\omega(G_e - e) \geq 4$ due to a result of TUCKER [17]. (i) implies $\omega(G_e - e) < 4$ if e does not belong to a K_5 in G_e. To complete the proof of (iv), consider a common neighbor z of the endnodes x and y of e which exists by (ii). The nodes x, z, y induce a P_3 in $G_e - e$, hence a hole C is running through x, z, y in $G_e - e$ due to a result of HOÁNG [9]. If $G_e - e$ is a monster, C has to be an even hole of length 4 (otherwise $G_e - e$ or G_e contain an odd hole). $\Box$

As immediate consequence, the complementary results hold for $\overline{G}_e$ if e is anticritical.

Lemma 2.6 *Suppose $e = xy$ to be a critical edge of the perfect graph $\overline{G}$.*

(i) *The nodes x and y belong to a stable set of size 3 in G.*

(ii) *G admits an even antihole containing x and y.*

(iii) *If e is M-anticritical, then x and y appear in the complement of a diamond and are contained in a stable set of size 5.*

Lemma 2.5(ii),(iii) and Lemma 2.6(i),(ii) can be seen as rules which we ask for in Problem 1.1: given a perfect graph, how to identify a non-critical or non-anticritical edge? We are able to give further conditions when an edge of a perfect graph G cannot be critical.

Lemma 2.7 *Suppose $e = xy$ to be a critical edge of a perfect graph G. Then x and y do not form twins, antitwins, or a comparable pair of G. Neither x nor y is a simplicial node in G (that is a node with a clique as neighborhood).*

Proof. Consider a perfect graph G, a critical edge $e = xy$ of G, and a subgraph $G_e \subseteq G$ such that $G_e - e$ is minimally imperfect. Then x and y do not form a comparable pair, twins, and antitwins of G, else they would also be a comparable pair, twins, and antitwins in $G_e - e$ leading to a contradiction to a well-known fact, the Replacement Lemma [10], and the Antitwin Lemma [13], respectively. Now, neither x nor y can be a simplicial node, since then x and y are either twins or a comparable pair of $G_e - e$. □

If two nodes form twins, antitwins, or a comparable pair in G, they do so in the complementary graph $\overline{G}$. In addition, x and y form a **2-pair** if all induced paths connecting x and y have lenth two. We can also show that x and y must not form a 2-pair if $e \notin E(G)$ is supposed to be an anticritical edge of G.

Lemma 2.8 *Let $e = xy$ be an anticritical edge of the perfect graph G. Then x and y are neither twins, antitwins, a comparable pair, nor a 2-pair of G.*

Proof. Consider a perfect graph G, an anticritical edge $e = xy \notin E(G)$, and $G_e \subseteq G$ such that $G_e + e \subseteq G - e$ is minimally imperfect. Then Lemma 2.7 implies that x and y does not form a comparable pair, twins, and antitwins of G. Now assume x and y to form a 2-pair. In the case $N_{G_e}(x) - N_{G_e}(y) = \emptyset$ (all neighbors of x in G_e belong to the neighborhood of y), x and y would be a comparable pair in $G_e + e$, hence there is a node $x' \in N_{G_e}(x) - N_{G_e}(y)$. Since x and y are a 2-pair in G_e and x' and y are non-adjacent, all paths in $G_e + e$ connecting x' and y must contain x or a node in $N_{G_e}(x) \cap N_{G_e}(y)$ in contradiction to the Star-Cutset Lemma [3]. □

3 Which Graph Classes Admit Critical or Anticritical Edges?

This section is devoted to the investigation of Problem 1.2: In which perfect graphs do critical or anticritical edges occur at all? More precisely, are there classes of perfect graphs so that no graph in this class has a critical or anticritical edge? For some graph classes, this follows immediately using the results established in the previous section. E.g., every critical edge is contained in a triangle by Lemma 2.7(ii), so **bipartite graphs** do not possess any critical edge (since they contain no odd cycles). Lemma 2.7(iii) says that every critical edge has to occur in an even hole. Consequently, no **triangulated graph** has a critical edge (since triangulated graphs do not admit any hole of length ≥ 4). Both properties together yield the existence of an odd cycle of length ≥ 5 in the union of the triangle and the even hole. So **line-perfect graphs cannot admit critical edges**, too (since they are defined to contain no odd cycles of length ≥ 5).

The next theorem *characterizes the perfect graphs without any critical edge*. It also yields a new characterization of Meyniel graphs, defined to contain no odd cycles of length ≥ 5 with at most one chord. The proof of this theorem relies on the perfectness of a superclass of Meyniel graphs. Let $G = (V, E)$ be a Meyniel graph and $V' \subseteq V$. The **slim graph** $G(V')$ is obtained by deleting every edge of G with both endnodes in V' and $G(V')$ is perfect due to HERTZ [8].

Theorem 3.1 (Hougardy, Wagler) *A perfect graph does not admit any critical edge if and only if it is Meyniel.*

Proof. Let G be a perfect graph without any critical edge. Assume G to be not a Meyniel graph then G admits an odd cycle C with length ≥ 5 which has at most one chord. Since G is perfect, C cannot be chordless, hence C possess precisely one chord e. If e were not a short chord of C, an odd hole would be contained in G. Thus e has to be a short chord of C and so it is an H-critical edge of G in contradiction to the precondition.

On the other hand, suppose $G = (V, E)$ to be a Meyniel graph. Then, for every subset $V' \subseteq V$, the graph $G(V')$ generated from G by deleting every edge of G with both endnodes in V' is a slim graph, still perfect by Hertz [8]. Thus, if we choose an arbitrary pair of adjacent nodes in a Meyniel graph and delete the edge connecting them, we get a perfect graph again, i.e., a Meyniel graph cannot admit any critical edge. □

Making use of this new characterization of Meyniel graphs, a perfect graph does not admit any anticritical edge if and only if it is co-Meyniel, the complement of a Meyniel graph. Hence we obtain immediatly:

Corollary 3.2 *A perfect graph does neither admit any critical nor anticritical edge if and only if it is both Meyniel and co-Meyniel.*

Consequently, critical (anticritical) edges occur in all perfect graphs G not contained in the class of Meyniel (co-Meyniel) graphs. Our aim is to find out which minimally imperfect subgraphs may occur in $G - e$ $(G + e)$. The results listed in the next lemma are obtained by combining the knowledge on forbidden subgraphs in the respective graph classes and the results from the previous section.

Lemma 3.3 *Let G be a perfect graph, $e \in E(G)$ be a critical edge of G, and $e' \notin E(G)$ be an anticritical edge of G.*

(i) *If G is diamond-free Berge, then $G - e$ and $G + e'$ do not contain a $\overline{C}_{2k+1}$ with $k \geq 3$ by Proposition 2.3 and 2.2. Furthermore, $G - e$ cannot admit any monster by Lemma 2.5(iv).*

(ii) *For every K_4-free Berge graph G, Proposition 2.3 and Lemma 2.5(iv) imply that $G - e$ has odd holes as the only minimally imperfect subgraphs. $G + e'$ does not contain any $\overline{C}_{2k+1}$ with $k \geq 4$ by Proposition 2.4.*

(iii) *Is G weakly triangulated or bull-free Berge, then $G - e$ and $G + e'$ cannot contain C_{2k+1} and $\overline{C}_{2k+1}$ with $k \geq 3$ by Proposition 2.1 and 2.2, respectively.*

(iv) *Is G strict quasi parity, a slim graph, or strongly perfect, then no even antihole $\overline{C}_{2k}$ with $k \geq 3$ appears in G. Hence, $G + e$ does not admit a $\overline{C}_{2k+1}$ with $k \geq 3$ by Proposition 2.2.*

4 Perfect and Co-Perfect Edge Orders

After considering the occurrence of critical edges in several classes of perfect graphs, we turn to Problem 1.3: we are interested whether it is possible, for certain perfect graphs, to successively delete or add edges keeping perfectness until a stable set or a clique is reached. For that, we use knowledge from the previous sections.

Let $G = (V, E)$ be a perfect graph. We call a numbering $(e_1, \ldots, e_m)$ of its edge set E a **perfect edge order** if, for $G = G_0$, all graphs $G_i := G_{i-1} - e_i$ are perfect for $1 \leq i \leq m$. Clearly, e_i has to be a non-critical edge of G_{i-1} for $1 \leq i \leq m$, and G_m is a stable set. Analogously, we say that a perfect graph G admits a **co-perfect edge order** iff its complement $\overline{G}$ has a perfect edge order. Here we simply use the numbering of the edges of $\overline{G}$ for the non-edges of G and get finally a clique.

Note that it does not suffice to identify non-critical or non-anticritical edges in the perfect graphs in question. E.g., we can certainly delete an arbitrary edge of a Meyniel graph keeping perfectness by [8], but we may obtain a slim graph that is not Meyniel and do not know anything about its non-critical edges so far. Hence, we cannot provide perfect edge orders of Meyniel graphs, although the graphs in this class are even characterized that they do not contain any critical edge due to Theorem 3.1. So we mainly have to look for edges such that their deletion or addition preserves the membership to the corresponding *subclass* of perfect graphs.

In general, we have to look for **critical (anticritical)** graphs with respect to the subclass of perfect graphs under consideration: that are graphs which lose the studied property by deleting (adding) an arbitrary edge. If there exist (anti)critical graphs in a certain subclass $\mathcal{C}$ of perfect graphs, then there is no (co-)perfect edge order for all the graphs belonging to $\mathcal{C}$. Conversely, if we can ensure that no (anti)critical graph with respect to a subclass $\mathcal{C}$ exists, then we know there is a (co-)perfect edge order for all graphs in $\mathcal{C}$: every graph G in $\mathcal{C}$ admits at least one edge e such that $G - e$ $(G + e)$ still belongs to $\mathcal{C}$. It is "only" left to find that edge e. In order to answer the question whether or not Meyniel graphs admit a perfect edge order we have, therefore, to solve the following problem:

Problem 4.1 *Find a critical Meyniel graph or show that no such graph exists.*

Looking for graph classes without critical graphs, we first observe that there are no critical graphs with respect to every *monotone* class (that is a class defined by some forbidden *partial* subgraphs). The simplest example is the class of bipartite graphs having no odd cycles. Obviously, deleting an *arbitrary* edge of a bipartite graph yields a bipartite graph again. Hence, bipartite graphs admit perfect edge orders and, in particular, *every* of their edge orders is perfect. A superclass of bipartite graphs consists of all line-perfect graphs having no odd cycles of length ≥ 5 as forbidden partial subgraphs. Again, there are no critical line-perfect graphs, hence every line-perfect graph admits a perfect edge order. Moreover, we have that they are *precisely* those perfect graphs such that *every* of its edge orders is perfect.

Theorem 4.2 *A graph is line-perfect iff all edge orders are perfect.*

Proof. A line-perfect graph G does not contain any odd cycle of length ≥ 5 as partial subgraph. Obviously, $G - e$ does also not contain any odd cycle of length ≥ 5 $\forall e \in E(G)$ and, therefore, is still line-perfect. Thus every ordering of $E(G)$ is perfect.

Now, assume G to be perfect but not line-perfect. Then G admits a cycle C of odd length at least 5 as partial subgraph. C cannot be chordless and every edge order of G that deletes all chords of C before an edge of C is not perfect. □

Bipartite and line-perfect graphs are subclasses of Meyniel graphs. A further class of Meyniel graphs for which we know a perfect edge order consists off all triangulated graphs. For those graphs, we have a well-known structural result, namely, that a graph is triangulated iff every subgraph has a simplicial node. Consider a triangulated graph $G = (V, E)$ and let x be a simplicial node of G. Then no edge e incident to x is critical by Lemma 2.7. The graph $G - e$ is not only still perfect, but even still triangulated, since x is also simplicial in $G - e$. As a consequence, we obtain a perfect edge order for triangulated graphs:

Theorem 4.3 *Every triangulated graph G admits a perfect edge order $(e_1, \ldots, e_m)$ with $G = G_0$, $G_i = G_{i-1} - e_i$, and e_i incident to a simplicial node of G_{i-1} for $1 \leq i \leq m$.*

Clearly, the complementary classes of bipartite, line-perfect, and triangulated graphs admit the corresponding co-perfect edge orders. For one class of perfect (but not Meyniel) graphs, both a perfect and a co-perfect edge order are known, namely, for weakly triangulated graphs. Let G be non-complete and weakly triangulated. Then a 2-pair x, y occurs in G due to a characterization of weakly triangulated graphs given by HAYWARD, HOÀNG, and MAFFRAY in [7]. The graph $G + xy$ is not only perfect by Lemma 2.8 but still weakly triangulated by a result of SPINRAD and SRITHARAN [16]. Consequently, we obtain a co-perfect edge order for weakly triangulated graphs by consecutively adding edges between 2-pairs (called 2-pair non-edge order in [16]).

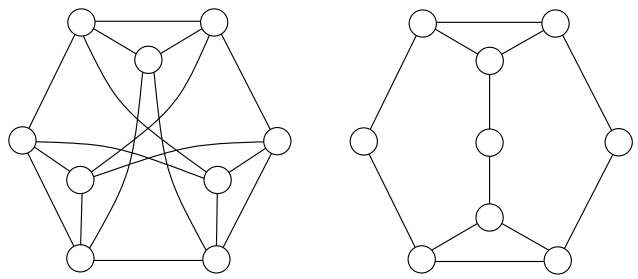
Theorem 4.4 (SPINRAD and SRITHARAN [16]) *Every weakly triangulated graph G admits a co-perfect edge order $(e_1, \ldots, e_{\overline{m}})$ with $G = G_0$, $G_i = G_{i-1} + e_i$ such that the endnodes of e_i form a 2-pair in G_{i-1} for $1 \leq i \leq \overline{m} = |E(\overline{G})|$.*

That the class of weakly triangulated graphs is closed under complementation yields particularly a perfect edge order for every weakly triangulated graph. HAYWARD proved in [6] that the following perfect edge order of a weakly triangulated graph G corresponds to the 2-pair non-edge order for $\overline{G}$ given by SPINRAD and SRITHARAN [16].

Theorem 4.5 (HAYWARD [6]) *Every weakly triangulated graph G admits a perfect edge order $(e_1, \ldots, e_m)$ with $G = G_0$, $G_i = G_{i-1} - e_i$ such that e_i is not the middle edge of any P_4 in G_i for $1 \leq i \leq m = |E(G)|$.*

Weakly triangulated graphs are defined as a common generalization of triangulated and co-triangulated graphs. Hence, the above theorems provide also a co-perfect edge order for triangulated graphs and a perfect edge order for co-triangulated graphs. In addition, the 2-pair non-edge order for weakly triangulated graphs enables us to establish a co-perfect edge order for bipartite graphs G: either there are non-adjacent nodes a, b in different color classes of G and $G+ab$ is still bipartite, or G is as complete bipartite graph weakly triangulated.

At present, perfect or co-perfect edge orders are not known to the author for other subclasses of perfect graphs. However, we can ensure that no such edge order exists for some classes $\mathcal{C}$ of perfect graphs: If $\mathcal{C}$ contains a **critically (anticritically) perfect graph** G such that deleting (adding) an arbitrary edge yields an imperfect graph, then G is in particular critical (anticritical) with respect to $\mathcal{C}$ and there is no perfect (co-perfect) edge order for the graphs in $\mathcal{C}$. The graph depicted in Figure 3(a) is critically and anticritically perfect and belongs to the classes of F-free Berge graphs where F is a claw, a diamond, or a K_4. Hence, that graph is also (anti)critical with respect to those classes and there are, therefore, neither perfect nor co-perfect edge orders for those classes. The graph in Figure 3(b) is anticritically perfect, strongly perfect, strict quasi parity and quasi parity. Consequently, there are no co-perfect edge orders for all strongly perfect, all strict quasi parity, or all quasi parity graphs.



(a) (b)

Figure 3

References

[1] C. Berge, *Färbungen von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind.* Wiss. Zeitschrift der Martin-Luther-Universität Halle-Wittenberg (1961) 114-115

[2] C. Berge and P. Duchet, *Strongly Perfect Graphs.* In: *Topics on Perfect Graphs.* (C. Berge and V. Chvátal, eds.), North Holland, Amsterdam (1984) 57-61

[3] V. Chvátal, *Star-Cutsets and Perfect Graphs.* J. Combin. Theory (B) 39 (1985) 189-199

[4] V. Chvátal and N. Sbihi, *Bull-Free Berge Graphs are Perfect.* Graphs and Combinatorics 3 (1987) 127-139

[5] R.B. HAYWARD, *Weakly Triangulated Graphs.* J. Combin. Theory (B) 39 (1985) 200-209

[6] R.B. HAYWARD, *Generating Weakly Triangulated Graphs.* J. Graph Theory 21, No.1 (1996) 67-69

[7] R.B. HAYWARD, C.T. HOÀNG, and F. MAFFRAY, *Optimizing Weakly Triangulated Graphs.* Graphs and Combinatorics 5 (1989) 339-349, erratum in 6 (1990) 33-35

[8] A. HERTZ, *Slim Graphs.* Graphs and Combinatorics 5 (1989) 149-157

[9] C.T. HOÀNG, *Some Properties of Minimal Imperfect Graphs.* Discrete Math. 160 (1996) 165-175

[10] L. LOVÁSZ, *Normal Hypergraphs and the Weak Perfect Graph Conjecture.* Discrete Math. 2 (1972) 253-267

[11] H. MEYNIEL, *On The Perfect Graph Conjecture.* Discrete Math. 16 (1976) 339-342

[12] H. MEYNIEL, *A New Property of Critical Imperfect Graphs and some Consequences.* Europ. J. Combinatorics 8 (1987) 313-316

[13] S. OLARIU, *No Antitwins in Minimal Imperfect Graphs.* J. Combin. Theory (B) 45 (1988) 255-257

[14] M.W. PADBERG, *Perfect Zero-One Matrices.* Math. Programming 6 (1974) 180-196

[15] K.R. PARTHASARATHY, G. RAVINDRA, *The Strong Perfect Graph Conjecture is True for $K_{1,3}$-free Graphs.* J. Combin. Theory (B) 21 (1976) 212-223

[16] J. SPINRAD and R. SRITHARAN, *Algorithms for Weakly Triangulated Graphs.* Discrete Appl. Math. 59 (1995) 181-191

[17] A. TUCKER, *Critical Perfect Graphs and Perfect 3-chromatic Graphs.* J. Combin. Theory (B) 23 (1977) 143-149

[18] A. TUCKER, *Coloring Perfect $(K_4$-$e)$-free Graphs.* J. Combin. Theory (B) 42 (1987) 313-318