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Stable Multi-Sets

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Abstract

In this paper we introduce a generalization of stable sets: stable multi-sets. A stable multi-set is an assignment of integers to the vertices of a graph, such that specified bounds on vertices and edges are not exceeded. In case all vertex and edge bounds equal one, stable multi-sets are equivalent to stable sets.

For the stable multi-set problem, we derive reduction rules and study the associated polytope. We state necessary and sufficient conditions for the extreme points of the linear relaxation to be integer. These conditions generalize the conditions for the stable set polytope. Moreover, the classes of odd cycle and clique inequalities for stable sets are generalized to stable multi-sets and conditions for them to be facet defining are determined.

The study of stable multi-sets is initiated by optimization problems in the field of telecommunication networks. Stable multi-sets emerge as an important substructure in the design of optical networks.

1 Introduction

Stable sets, also referred to as independent sets, cliques, or set packings, are among the most studied structures in graph theory/combinatorial optimization. A stable set in a graph is a subset of the vertices, where no two vertices are adjacent. A generalization of the notion of stability in a graph is obtained by considering multi-sets of vertices, where the multiplicities of the elements (vertices) in the multi-set are bounded by vertex and edge weights. This generalization is called *stable multi-set* here. The study of stable multi-sets is motivated by an optimization problem from telecommunication industry. In the design of optical networks, structures like stable multi-sets play an important role. For more information how stable multi-sets are used in the optimization of optical networks, we refer to [7].

This paper serves as a first step in the development of polyhedral theory for stable multi-sets. In Section 2, we formally define stable multi-sets and the associated optimization problem. Next, in Section 3 we derive rules to reduce the problem complexity. Our main purpose is to study the associated stable multi-set polytope, which is the topic of Section 4.

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We not only show that the well-known cycle and clique inequalities of the stable set polytope can be generalized to the stable multi-set polytope, but also prove necessary and sufficient conditions under which the polytope defined by the linear relaxation is integer. The paper is concluded in Section 5 with a comparison of the results obtained for stable multi-sets to stable sets and directions for further research.

2 Problem Description

Throughout this paper, we assume that all graphs are simple, i.e., contain no loops and no multiple edges. We first introduce a generalized (stable) set concept which the studied problem is based upon.

Definition 2.1 A multi-set S is a pair (V, t) consisting of a ground set V and a function $t : V \rightarrow \mathbb{Z}_0^+$, where $t(v)$ denotes the multiplicity of $v \in V$ in the multi-set S .

Definition 2.2 Given a graph $G = (V, E)$, non-negative integers α_v associated with every vertex $v \in V$, and non-negative integers β_{vw} associated with every edge $vw \in E$, a stable multi-set (SMS) is a multi-set S defined by the multiplicity function $t : V \rightarrow \mathbb{Z}_0^+$ such that $t(v) \leq \alpha_v$ for all $v \in V$ and $t(v) + t(w) \leq \beta_{vw}$ for all $vw \in E$.

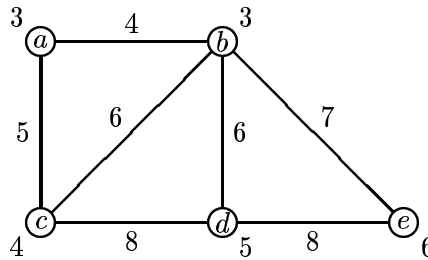


Figure 1: Example of a stable multi-set instance.

Consider the example in Figure 1. At every vertex α_v is given, whereas β_{vw} is displayed at the edges. Possible stable multi-sets are for example $(0, 1, 4, 2, 6)$, $(2, 2, 3, 4, 4)$ or $(1, 3, 3, 3, 4)$.

Note that the special case with $\alpha_v = 1$ for all $v \in V$ and $\beta_{vw} = 1$ for all $vw \in E$ represents the traditional stable set. With weights c_v for all $v \in V$, the (weighted) stable multi-set problem (short: SMS problem or SMSP) now can be defined as follows.

STABLE MULTI-SET PROBLEM

Input: A graph $G = (V, E)$, non-negative integers α_v and c_v associated with every vertex $v \in V$, non-negative integers β_{vw} associated with every edge $vw \in E$, and a positive integer $K \leq \sum_{v \in V} c_v \alpha_v$.

Question: Does there exist non-negative integers t_v such that $t_v \leq \alpha_v$ for all $v \in V$, $t_v + t_w \leq \beta_{vw}$ for all $vw \in E$, and $\sum_{v \in V} c_v t_v \geq K$?

For the example of Figure 1 and weight function $c = (1, 1, 1, 1, 1)$, the solution $(2, 2, 3, 4, 4)$ belongs to the set of optimal solutions (with value 15).

In case $\alpha_v = \beta_{vw} = 1$ for all $v \in V$ and $vw \in E$, the problem reduces to a maximum weighted stable set problem (SSP) which is well studied (cf. [1]). Since the maximum (weighted) stable set problem is well known to be \mathcal{NP} -hard and occurs as a special case, the maximum (weighted) stable multi-set problem is also \mathcal{NP} -hard.

We can formulate the problem with integer variables t_v as follows:

$$z = \max \sum_{v \in V} c_v t_v \tag{1}$$

$$\text{s.t. } t_v + t_w \leq \beta_{vw} \quad \forall vw \in E \tag{2}$$

$$t_v \leq \alpha_v \quad \forall v \in V \tag{3}$$

$$t_v \in \mathbb{Z}_0^+ \quad \forall v \in V \tag{4}$$

In the sequel, we refer to the inequalities (2) as the *edge inequalities*, whereas the *vertex inequalities* are given by (3). Together with the *non-negativity inequalities* $t_v \geq 0$, we will refer to them as the *model inequalities*.

In this paper, we primarily study the polytope associated with this formulation. However, by basic preprocessing it might be possible to reduce the graph and some of the bounds, and thus, to simplify the problem formulation. So we first point out these reduction rules before analyzing the polyhedral structure of (not further reducible) stable multi-set problems.

3 Reduction Rules for Stable Multi-Set Problems

We consider a given stable multi-set problem on a graph $G = (V, E)$ with vertex bounds α_v for all $v \in V$, edge bounds β_{vw} for all $vw \in E$, and objective values c_v for all $v \in V$. Let $N(v)$ denote the set of neighbors of $v \in V$, i.e., $N(v) := \{w \in V : vw \in E\}$. First of all we check whether $c_v > 0$ for all $v \in V$, since $c_v \leq 0$ implies $t_v = 0$, and the vertex can be removed from the problem. Next, if $\beta_{vw} \geq \alpha_v + \alpha_w$, inequality (2) will always be fulfilled so, edge vw can be removed from the instance. If $\alpha_v > \beta_{vw}$, then α_v can be set to β_{vw} . Hence, without loss of generality $\alpha_v \leq \min_{w \in N(v)} \beta_{vw}$. As a consequence, we may assume in the sequel that $\max\{\alpha_v, \alpha_w\} \leq \beta_{vw} < \alpha_v + \alpha_w$ for all $vw \in E$. Finally, we assume that $\alpha_v > 0$ for all $v \in V$ and $\beta_{vw} > 0$ for all $vw \in E$, because otherwise vertex v or both vertices v and w , respectively, can be deleted from the graph (with the appropriate vertex values fixed to 0).

Now, we can specify lower bounds on the variable values in any optimal solution independent of the objective function. Let $\gamma_v := \min_{w \in N(v)} \{\beta_{vw} - \alpha_w\}$ for $v \in V$.

Lemma 3.1 *Any optimal solution t^* meets $t_v^* \geq \gamma_v$ for all $v \in V$.*

In case there exist $\gamma_v > 0$, the values of the vectors α and β can be reduced using the lower bounds provided by Lemma 3.1 in order to simplify the problem representation.

Corollary 3.2 *For a graph $G = (V, E)$ let $P = (G, \alpha, \beta, c)$ define a SMS problem and set $\gamma_v = \min_{w \in N(v)} \{\beta_{vw} - \alpha_w\}$ for all $v \in V$. On the same graph, let $P' = (G, \alpha', \beta', c')$ be the SMS problem defined by $\alpha'_v = \alpha_v - \gamma_v$ for all $v \in V$, $\beta'_{vw} = \beta_{vw} - \gamma_v - \gamma_w$ for all $vw \in E$, and $c'_v = c_v$ for all $v \in V$.*

Then each optimal solution t' for P' corresponds to an optimal solution t for P by $t_v = t'_v + \gamma_v$. Moreover, if z' is the optimal value of P' , then $z = z' + \sum_{v \in V} c_v \gamma_v$ is the optimal value for P .

Note that all previous assumptions are unaffected by this transformation, i.e., the conditions $0 < \max\{\alpha'_v, \alpha'_w\} \leq \beta'_{vw} < \alpha'_v + \alpha'_w$ hold for all $vw \in E$. Furthermore, the reduced SMS problem P' can not be reduced again (in the same way), since $\gamma'_v = \min_{w \in N(v)} \{\beta'_{vw} - \alpha'_w\} = 0$ for all $v \in V$.

For the SMS problem of Figure 1, the reduced problem is given by Figure 2. In this case, $\gamma = (1, 1, 2, 2, 3)$.

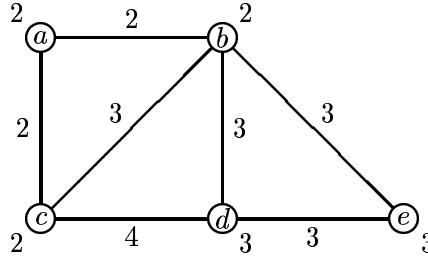


Figure 2: Reduced SMS problem of Figure 1.

Since the described transformation can be simply applied and may reduce the problem size (or at least some values), it suffices to analyze the problem only for instances which are not further reducible this way.

4 The Stable Multi-Set Polytope

In this section, we study the stable multi-set polytope (P_{SMS}). In contrast to the well-studied stable set polytope (see [1] for a survey), to our knowledge only Gerards and Schrijver [5] discuss a related polytope (in the context of the Edmonds-Johnson property).

A SMS polytope P_{SMS} is defined by the triple $(G = (V, E), \alpha, \beta)$. We assume that none of the reduction rules described in the previous section can be applied anymore, i.e., we have $0 < \max\{\alpha_v, \alpha_w\} \leq \beta_{vw} < \alpha_v + \alpha_w$ for all $vw \in E$ and $\gamma_v = \min_{w \in N(v)} \{\beta_{vw} - \alpha_w\} = 0$ for all $v \in V$. For a graph $G = (V, E)$ and vectors $\alpha \in \mathbb{Z}_+^{|V|}$ and $\beta \in \mathbb{Z}_+^{|E|}$, let $T(G, \alpha, \beta)$ denote the set of all solutions to (2)–(4), i.e.,

$$T(G, \alpha, \beta) = \{t \in \mathbb{Z}^{|V|} : 0 \leq t_v \leq \alpha_v \ \forall v \in V, t_v + t_w \leq \beta_{vw} \ \forall vw \in E\}.$$

The convex hull of this set is denoted by

$$T_{IP}(G, \alpha, \beta) = \text{conv}(T(G, \alpha, \beta)).$$

Moreover, by $T_{LP}(G, \alpha, \beta)$ we denote the polytope described by the linear relaxation of inequalities (2)–(4):

$$T_{LP}(G, \alpha, \beta) = \{t \in \mathbb{R}^{|V|} : 0 \leq t_v \leq \alpha_v \forall v \in V, t_v + t_w \leq \beta_{vw} \forall vw \in E\}$$

If there is no danger of confusion, we use T , T_{IP} , and T_{LP} as short version of $T(G, \alpha, \beta)$, $T_{IP}(G, \alpha, \beta)$, and $T_{LP}(G, \alpha, \beta)$, respectively.

Let us now start the study of the relation between T_{LP} and T_{IP} . First of all, we state the dimension and the trivial facets of T_{IP} . Let $n = |V|$ denote the number of vertices.

Proposition 4.1 (i). *The dimension of T_{IP} equals the number of vertices, $\dim T_{IP} = n$.*

(ii). *For all $v \in V$, the non-negativity inequality $t_v \geq 0$ defines a facet of T_{IP} .*

(iii). *For $v \in V$, the non-negativity inequality $t_v \leq \alpha_v$ defines a facet of T_{IP} if and only if $\beta_{vw} > \alpha_v$ for all $w \in N(v)$.*

Proof. For $v \in V$, let $e^v \in \{0, 1\}^{|V|}$ be the vertex unit vector defined by $e_v^v = 1$ and $e_w^v = 0$ for all $w \in V \setminus \{v\}$.

(i). Since $\alpha_v > 0$ for all $v \in V$, T contains all unit vectors e^v for $v \in V$. Moreover, $0 \in T$. Obviously, these vectors are affinely independent.

(ii). Except for e^v , all vectors of the previous part satisfy $t_v = 0$ at equality.

(iii). We define n affinely independent solutions that satisfy $t_v = \alpha_v$. Let the solution t^v be defined by $t_v^v = \alpha_v$, $t_w^v = 0$ for $w \neq v$. Next, for all $w \in V$, $w \neq v$, we can define the solution t^w by $t_v^w = \alpha_v$, $t_w^w = 1$, $t_u^w = 0$ for $u \notin \{v, w\}$. These solutions are valid since $\alpha_w > 0$ and $\beta_{vw} > \alpha_v$ for $w \in N(v)$. Moreover, they are affinely independent.

In case there exists $w \in N(v)$ with $\beta_{vw} = \alpha_v$, then $t_v = \alpha_v$ implies $t_w = 0$. This yields $\dim\{t \in T_{IP} : t_v = \alpha_v\} \leq n - 2$, i.e., $t_v \leq \alpha_v$ can not define a facet of T_{IP} . ■

Completing the facets defined by the model inequalities, we state a similar result for the edge inequalities.

Proposition 4.2 *For $vw \in E$, the edge inequality $t_v + t_w \leq \beta_{vw}$ defines a facet of T_{IP} if and only if for all $u \in N(v) \cup N(w)$, there exist integers $\bar{x}_v \leq \alpha_v$ and $\bar{x}_w \leq \alpha_w$ with $\bar{x}_v + \bar{x}_w = \beta_{vw}$, $\bar{x}_v < \beta_{vu}$ if $u \in N(v) \setminus \{w\}$, and $\bar{x}_w < \beta_{wu}$ if $u \in N(w) \setminus \{v\}$.*

Proof. If $t_v + t_w \leq \beta_{vw}$ defines a facet of T_{IP} , there exist n affinely independent solutions of T_{IP} on the appropriate hyperplane. For each $u \in N(v) \cup N(w)$, there is at least one solution t^u with $t_u^u > 0$. Then $\bar{x}_v = t_v^u$ and $\bar{x}_w = t_w^u$ are integers with $\bar{x}_v + \bar{x}_w = \beta_{vw}$, and

since $t_u^u > 0$, we have $\bar{x}_v \leq \beta_{uv} - t_u^u < \beta_{uv}$, if $u \in N(v) \setminus \{w\}$, and $\bar{x}_w \leq \beta_{uw} - t_u^u < \beta_{uw}$, if $u \in N(w) \setminus \{v\}$.

For sufficiency, we construct n affinely independent solutions satisfying $t_v + t_w = \beta_{vw}$. Again, for $v \in V$ let $e^v \in \{0, 1\}^{|V|}$ be the vertex unit vector defined by $e_v^v = 1$ and $e_w^v = 0$ for all $w \in V \setminus \{v\}$. Since we have $\alpha_v + \alpha_w > \beta_{vw}$ by assumption, there must be integers $0 < \bar{x}_v \leq \alpha_v$ and $0 < \bar{x}_w \leq \alpha_w$ with $\bar{x}_v + \bar{x}_w = \beta_{vw}$. Then $t^1 = \bar{x}_v e^v + \bar{x}_w e^w$ is a solution of T_{IP} . Moreover, either $\bar{x}_v < \alpha_v$ or $\bar{x}_w < \alpha_w$. Without loss of generality, we assume $\bar{x}_v < \alpha_v$, and as a consequence, $\bar{x}_w = \beta_{vw} - \bar{x}_v > \beta_{vw} - \alpha_v \geq 0$ by the problem assumptions. Thus, the vector $t^2 = (\bar{x}_v + 1)e^v + (\bar{x}_w - 1)e^w$ is another solution of T_{IP} .

Now, let $u \in V \setminus \{v, w\}$ be arbitrarily chosen. If $u \notin N(v) \cup N(w)$, then we define $t^u = t^1 + e^u$ which is clearly also a solution of T_{IP} . If $u \in N(v) \cup N(w)$, then there exist \bar{x}_v and \bar{x}_w as stated in the Proposition, and we set $t^u = \bar{x}_v e^v + \bar{x}_w e^w + e^u \in T_{IP}$.

Summarized, the vectors t^1, t^2 , and t^u for all $u \in V \setminus \{v, w\}$ are solutions of T_{IP} and satisfy $t_v + t_w \leq \beta_{vw}$ at equality. Since they are affinely independent, the edge inequality defines a facet of T_{IP} . \blacksquare

Note that in general, the inequalities $t_v \leq \alpha_v$ and $t_v + t_w \leq \beta_{vw}$ do not define facets for the stable set polytope (with $\alpha_v = \beta_{vw} = 1$ for all $v \in V$ and for all $vw \in E$). In this special case, it follows from Proposition 4.2 that $t_v + t_w \leq 1$ is a facet if and only if the vertices do not have a common neighbor, i.e., vw is a maximal clique. In the sequel, we will refer to the facets of Propositions 4.1 and 4.2 as the trivial facets.

Next, we derive conditions under which the model inequalities describe all facets of T_{IP} , i.e., the special case $T_{LP} = T_{IP}$. For the stable set polytope, the edge inequalities (2) suffice to describe T_{IP} completely if and only if the graph does not contain an odd cycle (i.e., G is bipartite) and no isolated vertices exist (see Padberg [9]). Since, T_{LP} also includes the vertex inequalities (3), the condition that the graph does not contain isolated vertices can be dropped. Hence, for the stable set polytope, $T_{IP} = T_{LP}$ if and only if G is bipartite. For the stable multi-set polytope, $T_{IP} = T_{LP}$ may also hold for non-bipartite graphs. To state conditions that are necessary and sufficient for $T_{IP} = T_{LP}$, we first have to define the cycle inequalities that are valid for T_{IP} . For a cycle C in G , let $E(C)$ denote the edges on the cycle. Moreover, let $\beta(C) = \sum_{vw \in E(C)} \beta_{vw}$ denote the sum of the edge values on the cycle. A cycle C is called even (odd), if $|C|$ is even (odd), and even-valued (odd-valued), if $\beta(C)$ is even (odd).

Definition 4.3 *Let C be a cycle in G . Then*

$$\sum_{v \in C} t_v \leq \lfloor \frac{1}{2} \beta(C) \rfloor \tag{5}$$

defines a cycle inequality.

It is easy to see that all cycle inequalities (5) are valid for T_{IP} . Moreover, every even cycle inequality cannot be violated by any solution $t \in T_{LP}$.

Lemma 4.4 *Let C be an even cycle. Then the cycle inequality (5) is satisfied by all solutions $t \in T_{LP}$.*

Proof. Let the edges of the cycle be consecutively indexed, beginning with an arbitrary edge. Then the inequality defined by the sum of the edge inequalities of either the *even-indexed* edges in the cycle or the *odd-indexed* edges has a right hand side that is at least as strong as the right hand side of (5). Thus, the cycle inequality (5) is satisfied by all solutions, since all of them satisfy all edge inequalities. ■

Also every even-valued odd cycle inequality cannot be violated by any $t \in T_{LP}$, since in this case the cycle inequality can be obtained by summing up all edge inequalities half. An even stronger result can be proved: if all odd cycles are even-valued, then the polytope is completely described by the model inequalities. This bases on a result of Gerards and Schrijver [5] (a *Chvátal-Gomory cut* is an inequality $cx \leq \lfloor \delta \rfloor$, where c integer, $c = \lambda A$ and $\delta = \lambda b$ for some vector $\lambda \geq 0$ of appropriate size):

Proposition 4.5 (Gerards and Schrijver [5]) *Let A be the vertex-edge incidence matrix of a graph $\bar{G} = (\bar{V}, \bar{E})$ and $b \in \mathbb{Z}^{|\bar{E}|}$, then the system*

$$Ax \leq b, \quad cx \leq \lfloor \delta \rfloor,$$

where $cx \leq \delta$ is implied by $Ax \leq b$ with c integral (i.e., $cx \leq \lfloor \delta \rfloor$ is a Chvátal-Gomory cut) has the same solution set as the system

$$Ax \leq b, \quad \sum_{v \in C} x_v \leq \lfloor \frac{1}{2} b(C) \rfloor \quad \text{for all odd cycles } C \text{ in } \bar{G},$$

where $b(C) = \sum_{v \in C} b_v$.

Theorem 4.6 *If all odd cycles C in G are even-valued, then $T_{IP} = T_{LP}$.*

Proof. We prove the theorem by showing that the *Chvátal rank* (cf. [2, 10]) of the polytope T_{LP} is 0. The Chvátal rank of a polytope is defined as the minimum number of times *all* Chvátal-Gomory cuts have to be added to the set of linear inequalities in order to obtain a complete description of the integer polytope. The Chvátal rank of T_{LP} equals 0 if and only if $T_{IP} = T_{LP}$. Otherwise stated, we demonstrate that every Chvátal-Gomory cut is equivalent or dominated by a linear combination of the model inequalities (2)–(3).

To do so, we apply Proposition 4.5 to a slightly modified graph \bar{G} where the vertex bounds are replaced by edge bounds. We define the graph $\bar{G} = (\bar{V}, \bar{E})$ by

$$\begin{aligned} \bar{V} &= V \cup \{\bar{v}, \bar{\bar{v}}\}, \\ \bar{E} &= E \cup \{v\bar{v} : v \in V\} \cup \{v\bar{\bar{v}} : v \in V\} \cup \{\bar{v}\bar{\bar{v}}\}, \end{aligned}$$

and

$$b_{vw} = \begin{cases} \beta_{vw} & \text{if } vw \in E, \\ \alpha_v & \text{if } v \in V, w = \bar{v} \vee w = \bar{\bar{v}}, \\ 0 & \text{if } v = \bar{v}, w = \bar{\bar{v}}. \end{cases}$$

Then every solution of T corresponds to a solution of $X := \{x \in \mathbb{Z}^{n+2} : Ax \leq b, x \geq 0\}$, and vice versa (note that $Ax \leq b$ implies $x_{\bar{v}} \leq 0$ and $x_{\bar{\bar{v}}} \leq 0$). Let $X_{IP} := \text{conv}(X)$ and

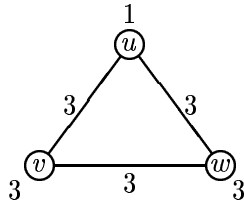


Figure 3: Example of a SMS instance with an odd-valued odd cycle and $T_{IP} = T_{LP}$.

$X_{LP} := \{x \in \mathbb{R}^{n+2} : Ax \leq b, x \geq 0\}$. To verify Chvátal rank 0 for T_{IP} , it suffices to prove that the Chvátal-Gomory cuts of rank 1 are satisfied by all solutions $x^* \in X_{LP}$. By applying the result of Gerards and Schrijver, all rank 1 Chvátal-Gomory cuts are given by the inequalities for the odd cycles \bar{C} in \bar{G} . Now, we distinguish three cases where $|\bar{C} \cap \{\bar{v}, \bar{v}\}| \in \{0, 1, 2\}$. If \bar{C} is an odd cycle in G (i.e., $|\bar{C} \cap \{\bar{v}, \bar{v}\}| = 0$), then $b(C)$ is even by the assumptions stated in the theorem, and the inequality is satisfied by every solution $x^* \in X_{LP}$ (see Lemma 4.4). Now, suppose $|\bar{C} \cap \{\bar{v}, \bar{v}\}| = 1$, then we may assume without loss of generality that $\bar{v} \in \bar{C}$, $\bar{v} \notin \bar{C}$. Let $\bar{C} = \{v_1, \dots, v_n, \bar{v}\}$ (with n even). Then the odd cycle inequality reads

$$\sum_{i=1}^n x_{v_i} \leq \left\lfloor \frac{1}{2} \left(\sum_{i=1}^{n-1} \beta_{v_i v_{i+1}} + \alpha_{v_1} + \alpha_{v_n} \right) \right\rfloor \quad (6)$$

Now, let \bar{C} be an even cycle with $\bar{C} = \{v_1, \dots, v_n, \bar{v}, \bar{v}\}$. The even cycle inequality for this cycle is identical to (6) due to the setting of b_{vw} . As a consequence, by Lemma 4.4, inequality (6) cannot be violated. Finally, if $\bar{v}, \bar{v} \in \bar{C}$, a similar idea is applied. We construct an even cycle by removing either \bar{v} or \bar{v} from the cycle. By the definition of b_{vw} both cycles result in the same cycle inequality, which cannot be violated by any $x^* \in X_{LP}$ by Lemma 4.4. This completes the proof. \blacksquare

Note that the condition of Theorem 4.6 is not necessary for $T_{IP} = T_{LP}$. Consider the instance in Figure 3. Although, it contains an odd cycle with odd value (i.e., 9), the polytope T_{IP} is completely described by the model inequalities. This is because the odd cycle inequality $x_u + x_v + x_w \leq 4$ is dominated by the sum of the model inequalities $x_u \leq 1$ and $x_v + x_w \leq 3$. For odd-valued odd cycles, this example precisely gives the condition under which an odd cycle inequality (5) cannot be violated. To state this result, as well as to prove it, we will frequently denote the vertices of an odd cycle C by $\{v_1, \dots, v_{2k+1}\}$. In general, the index j of a vertex v_j is calculated modulo $2k+1$, e.g., if $2k+1 < j \leq 4k+2$ then $j \equiv j - 2k - 1$.

Theorem 4.7 *Let $C = \{v_1, \dots, v_{2k+1}\}$ be an odd cycle in G with $\beta(C)$ odd. The odd cycle inequality (5) is dominated by a linear combination of the model inequalities if and only if*

$$\min_{i=1, \dots, 2k+1} \left\{ \alpha_{v_i} + \sum_{j=1}^k \beta_{v_{i+2j-1} v_{i+2j}} \right\} \leq \left\lfloor \frac{1}{2} \beta(C) \right\rfloor. \quad (7)$$

Proof. The proof of sufficiency is easy. Let p be an index for which the left hand side of (7) obtains its minimum. Then the inequalities $t_{v_p} \leq \alpha_{v_p}$, and $t_{v_{2j+p-1}} + t_{v_{2j+p}} \leq \beta_{v_{2j+p-1}v_{2j+p}}$ for all $j = 1, \dots, k$ sum up to an inequality that is at least as strong as the odd cycle inequality (5).

Next, we show that (7) holds whenever the odd cycle inequality (5) is dominated. It is dominated if there exists a linear combination (λ, μ) of the model inequalities that results in the same left hand side as (5), but has a right hand side that is less than or equal to the right hand side of (5). The minimum right hand side of such a linear combination (λ, μ) is calculated by the following linear program

$$z^* = \min \quad z(\lambda, \mu) := \sum_{i=1}^{2k+1} \alpha_{v_i} \lambda_{v_i} + \sum_{i=1}^{2k+1} \beta_{v_i v_{i+1}} \mu_{v_i v_{i+1}} \quad (8)$$

$$\text{s.t.} \quad \lambda_{v_i} + \mu_{v_{i-1}v_i} + \mu_{v_i v_{i+1}} = 1 \quad \forall i \in \{1, \dots, 2k+1\} \quad (9)$$

$$\lambda_{v_i} \geq 0, \mu_{v_i v_{i+1}} \geq 0 \quad \forall i \in \{1, \dots, 2k+1\} \quad (10)$$

where λ_{v_i} corresponds to the inequality $x_{v_i} \leq \alpha_{v_i}$, and $\mu_{v_i v_{i+1}}$ to the inequality $x_{v_i} + x_{v_{i+1}} \leq \beta_{v_i v_{i+1}}$. We have to prove that (7) holds, whenever $z^* \leq \lfloor \frac{1}{2} \beta(C) \rfloor$. This is done by analyzing the optimal solution of (8)-(10).

First of all, we prove that if $z^* \leq \lfloor \frac{1}{2} \beta(C) \rfloor$, then there exists an integer vector (λ, μ) with $z(\lambda, \mu) = z^*$. Suppose $\lambda_{v_i} = 0$ for all $i = 1, \dots, 2k+1$. Then the (unique) solution is given by $\mu_{v_i v_{i+1}} = \frac{1}{2}$ for all $i = 1, \dots, 2k+1$ which contradicts $z^* \leq \lfloor \frac{1}{2} \beta(C) \rfloor$, since $\beta(C)$ is odd. So, at least one $\lambda_{v_i} > 0$. Next, suppose $\lambda_{v_i} \geq \lambda_{v_{i+1}} > 0$ for some $i \in \{1, \dots, 2k+1\}$. Then the solution $(\bar{\lambda}, \bar{\mu})$ defined by

$$\bar{\lambda}_{v_j} = \begin{cases} \lambda_{v_i} - \lambda_{v_{i+1}} & \text{if } j = i, \\ 0 & \text{if } j = i+1, \\ \lambda_{v_j} & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{\mu}_{v_j v_{j+1}} = \begin{cases} \mu_{v_i v_{i+1}} + \lambda_{v_{i+1}} & \text{if } j = i, \\ \mu_{v_j v_{j+1}} & \text{otherwise,} \end{cases}$$

has objective value $z(\bar{\lambda}, \bar{\mu}) = z(\lambda, \mu) + \lambda_{v_{i+1}} (\beta_{v_i v_{i+1}} - \alpha_{v_i} - \alpha_{v_{i+1}}) < z(\lambda, \mu)$ by $\alpha_v + \alpha_w > \beta_{vw}$. So, without loss of generality $\lambda_{v_i} \geq 0$ implies $\lambda_{v_{i-1}} = \lambda_{v_{i+1}} = 0$. This idea can be generalized to obtain an integer optimal solution (λ, μ) .

Let (i, j) be an index pair with $0 < \lambda_{v_i} < 1$, $0 < \lambda_{v_j} < 1$, and $\lambda_{v_p} = 0$ for all $i < p < j$. Now, suppose $j - i = 2l$ for some $l > 0$. Moreover, without loss of generality let $\alpha_{v_i} + \sum_{p=1}^i \beta_{v_{i+2p-1}v_{i+2p}} \leq \alpha_{v_j} + \sum_{p=1}^l \beta_{v_{j-2p+1}v_{j-2p+2}}$ (otherwise i, j have to be exchanged, and the indexing direction must be reversed). Note that from the conditions of the sequence, it follows that $\mu_{v_{i+2p-2}v_{i+2p-1}} = \lambda_{v_j} + \mu_{v_j v_{j+1}}$, $\mu_{v_{i+2p-1}v_{i+2p}} = 1 - \lambda_{v_j} - \mu_{v_j v_{j+1}}$, $p = 1, \dots, l$, and $\lambda_{v_i} \leq 1 - \lambda_{v_j} - \mu_{v_j v_{j+1}}$. Then the solution $(\bar{\lambda}, \bar{\mu})$ defined by

$$\bar{\lambda}_{v_q} = \begin{cases} \lambda_{v_i} + \lambda_{v_j} & \text{if } q = i, \\ 0 & \text{if } i < q \leq j, \\ \lambda_{v_q} & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{\mu}_{v_q v_{q+1}} = \begin{cases} \mu_{v_q v_{q+1}} - \lambda_{v_j} & \text{if } q < j, q - i \text{ even,} \\ \mu_{v_q v_{q+1}} + \lambda_{v_j} & \text{if } q < j, q - i \text{ odd,} \\ \mu_{v_q v_{q+1}} & \text{otherwise,} \end{cases}$$

is valid and has objective value

$$z(\bar{\lambda}, \bar{\mu}) = z(\lambda, \mu) + \lambda_{v_j} (\alpha_{v_i} + \sum_{p=1}^l \beta_{v_{i+2p-2}v_{i+2p-1}} - \alpha_{v_j} - \sum_{p=1}^l \beta_{v_{i+2p-1}v_{i+2p}}) \leq z(\lambda, \mu)$$

where the lower estimation comes from $\alpha_{v_i} + \sum_{p=1}^l \beta_{v_{i+2p-1}v_{i+2p}} \leq \alpha_{v_j} + \sum_{p=1}^l \beta_{v_{j-2p+1}v_{j-2p+2}}$. The number of fractional variables is reduced by at least one (namely λ_{v_j}).

If $j - i = 2l + 1$ for some $l > 0$, then either the solution $(\bar{\lambda}, \bar{\mu})$ or $(\tilde{\lambda}, \tilde{\mu})$ is at least as good as (λ, μ) . Here, $(\bar{\lambda}, \bar{\mu})$ is defined by

$$\bar{\lambda}_{v_q} = \begin{cases} \lambda_{v_q} + \mu_{v_i v_{i+1}} & \text{if } q \in \{i, j\}, \\ 0 & \text{if } i < q < j, \\ \lambda_{v_q} & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{\mu}_{v_q v_{q+1}} = \begin{cases} 0 & \text{if } q < j, q - i \text{ even}, \\ 1 & \text{if } q < j, q - i \text{ odd}, \\ \mu_{v_q v_{q+1}} & \text{otherwise} \end{cases}$$

and $(\tilde{\lambda}, \tilde{\mu})$ by

$$\tilde{\lambda}_{v_q} = \begin{cases} \lambda_{v_q} - \lambda & \text{if } q \in \{i, j\}, \\ 0 & \text{if } i < q < j, \\ \lambda_{v_q} & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{\mu}_{v_q v_{q+1}} = \begin{cases} \mu_{v_q v_{q+1}} + \lambda & \text{if } q < j, q - i \text{ even}, \\ \mu_{v_q v_{q+1}} - \lambda & \text{if } q < j, q - i \text{ odd}, \\ \mu_{v_q v_{q+1}} & \text{otherwise} \end{cases}$$

with $\lambda = \min\{\lambda_{v_i}, \lambda_{v_j}\}$. Again, the number of fractional variables is reduced by this construction. This procedure can be repeated as long as there are multiple fractional λ_{v_i} variables. In the end, at most a single fractional λ_{v_i} variable remains. The existence of such a single fractional λ_{v_i} variable, however, leads to a contradiction. Let $0 < \lambda_{v_1} < 1$, whereas $\lambda_{v_i} \in \{0, 1\}$ for all $i \geq 2$. Then at least $\mu_{v_1 v_2}$ or $\mu_{v_{2k+1} v_1}$ is fractional as well. Without loss of generality, we can assume that $\mu_{v_1 v_2}$ is fractional. From $\lambda_{v_2} \in \{0, 1\}$ and $\mu_{v_1 v_2} > 0$ it follows that $\lambda_{v_2} = 0$ and $\mu_{v_2 v_3} = 1 - \mu_{v_1 v_2}$. Repeating this argument leads to $\lambda_{v_i} = 0$ for all $i \geq 2$, and $\mu_{v_{2k+1} v_1} = \mu_{v_1 v_2}$. It also holds that $\mu_{v_{2k+1} v_1} = 1 - \lambda_{v_1} - \mu_{v_1 v_2}$. This implies that $\mu_{v_{2i-1} v_{2i}} = \frac{1}{2} - \frac{1}{2}\lambda_{v_1}$ and $\mu_{v_{2i} v_{2i+1}} = \frac{1}{2} + \frac{1}{2}\lambda_{v_1}$ for $i = 1, \dots, k$. However, then (λ, μ) is a convex combination of the solutions $(\bar{\lambda}, \bar{\mu})$ and $(\tilde{\lambda}, \tilde{\mu})$ defined by

$$\bar{\lambda}_{v_i} = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \bar{\mu}_{v_i v_{i+1}} = \begin{cases} 1 & \text{if } i \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{\lambda}_{v_i} = 0 \text{ for all } i, \quad \text{and} \quad \tilde{\mu}_{v_i v_{i+1}} = \frac{1}{2} \text{ for all } i.$$

Hence, $(\lambda, \mu) = \lambda_{v_1}(\bar{\lambda}, \bar{\mu}) + (1 - \lambda_{v_1})(\tilde{\lambda}, \tilde{\mu})$. Moreover, since (λ, μ) is optimal, we get $z^* = z(\lambda, \mu) = z(\lambda, \tilde{\mu}) = \frac{1}{2}\beta(C)$, which contradicts $z^* \leq \lfloor \frac{1}{2}\beta(C) \rfloor$ for $\beta(C)$ odd.

So, after repeated application of the above described procedure, all λ_{v_i} variables are integer, with at least one non-zero. Consequently, all $\mu_{v_i v_{i+1}}$ variables are integer as well. Now, let $I_\lambda = \{i : \lambda_{v_i} = 1\}$, and let $I_\mu = \{i : \mu_{v_i v_{i+1}} = 1\}$. Note that $|I_\lambda| = 2l + 1$ for some $l \geq 0$. If $l = 0$, then the proof is completed. Otherwise, suppose that, although $z^* \leq \lfloor \frac{1}{2}\beta(C) \rfloor$, (7) does not hold. Then for all $i \in \{1, \dots, 2k + 1\}$, $\alpha_{v_i} + \sum_{j=1}^k \beta_{v_{i+2j-1}v_{i+2j}} \geq \frac{1}{2}(\beta(C) + 1)$. Summation for all $i \in I_\lambda$ results in

$$\sum_{i \in I_\lambda} \alpha_{v_i} + l\beta(C) + \sum_{i \in I_\mu} \beta_{v_i v_{i+1}} \geq \frac{2l + 1}{2}(\beta(C) + 1)$$

which yields

$$\sum_{i \in I_\lambda} \alpha_{v_i} + \sum_{i \in I_\mu} \beta_{v_i v_{i+1}} \geq \frac{1}{2}(\beta(C) + 1) + l \geq \frac{1}{2}(\beta(C) + 1) \quad (11)$$

On the other hand, however, since (λ, μ) is optimal, it holds for every pair $i, j \in I_\lambda$ with $p \notin I_\lambda$ for all $i < p < j$ that the solution $(\tilde{\lambda}, \tilde{\mu})$ defined by

$$\tilde{\lambda}_{v_q} = \begin{cases} 0 & \text{if } q \in \{i, j\}, \\ \lambda_{v_q} & \text{otherwise} \end{cases}, \quad \text{and} \quad \tilde{\mu}_{v_q v_{q+1}} = \begin{cases} 1 - \mu_{v_q v_{q+1}} & \text{if } i \leq q < j \\ \mu_{v_q v_{q+1}} & \text{otherwise} \end{cases}$$

(i.e., an exchange of the μ values between i and j) does not improve the solution value. Hence,

$$\alpha_{v_i} + \alpha_{v_j} + \sum_{p \in I_\mu: i < p < j} \beta_{v_p v_{p+1}} \leq \sum_{p \in I_\mu: i < p < j} \beta_{v_{p-1} v_p} + \beta_{v_{j-1} v_j}$$

Summation for all pairs $i, j \in I_\lambda$ with $p \notin I_\lambda$ for all $i < p < j$ results in

$$2 \sum_{i \in I_\lambda} \alpha_{v_i} + \sum_{p \in I_\mu} \beta_{v_p v_{p+1}} \leq \sum_{p \in I_\mu} \beta_{v_{p-1} v_p} + \sum_{i \in I_\lambda} \beta_{v_{i-1} v_i}$$

Addition of $\sum_{p \in I_\mu} \beta_{v_p v_{p+1}}$ to both sides gives

$$2 \sum_{i \in I_\lambda} \alpha_{v_i} + 2 \sum_{p \in I_\mu} \beta_{v_p v_{p+1}} \leq \beta(C)$$

Since the right hand side is odd by assumption, whereas the left hand side is even, it holds that

$$\sum_{i \in I_\lambda} \alpha_{v_i} + \sum_{p \in I_\mu} \beta_{v_p v_{p+1}} \leq \frac{1}{2} (\beta(C) - 1)$$

which contradicts to (11). So, (7) holds whenever $z^* \leq \lfloor \frac{1}{2} \beta(C) \rfloor$. ■

Now, Theorem 4.6 and Theorem 4.7 together result in the following statement.

Corollary 4.8 *The polytope T_{IP} is completely described by the model inequalities, i.e., $T_{IP} = T_{LP}$, if and only if for every odd cycle $C = \{v_1, \dots, v_{2k+1}\}$ in G either $\beta(C)$ is even or (7) is satisfied.*

Now, let us return to the situation $T_{IP} \neq T_{LP}$. Several classes of valid inequalities exist. A natural starting point is the class of odd cycle inequalities. Padberg [8] proved for the stable set polytope that the odd cycle inequality defines a facet in case the graph is an odd cycle. For more general graphs, cycle inequalities have to be lifted to obtain a facet. Also for stable multi-sets we have to restrict ourselves an odd cycle. Moreover, by Corollary 4.8, additional conditions are necessary. In the next proposition, we state conditions that are not only necessary but also sufficient for an odd cycle to be facet defining. Let G_C denote the subgraph restricted (not induced) to the cycle C , i.e., $G_C = (C, E(C))$.

Proposition 4.9 *Let $C = \{v_1, \dots, v_{2k+1}\}$ be an odd cycle in G with $\beta(C)$ odd. Then the odd cycle inequality (5) defines a facet of $T_{IP}(G_C, \alpha, \beta)$ if and only if*

$$\max_{i=1, \dots, 2k+1} \left\{ \sum_{j=1}^k \beta_{v_{i+2j-1} v_{i+2j}} \right\} \leq \lfloor \frac{1}{2} \beta(C) \rfloor < \min_{i=1, \dots, 2k+1} \left\{ \alpha_{v_i} + \sum_{j=1}^k \beta_{v_{i+2j-1} v_{i+2j}} \right\}.$$

Proof. From Corollary 4.8 it is clear that

$$\min_{i=1,\dots,2k+1} \left\{ \alpha_{v_i} + \sum_{j=1}^k \beta_{v_{i+2j-1}v_{i+2j}} \right\} > \lfloor \frac{1}{2}\beta(C) \rfloor \quad (12)$$

has to be satisfied for all $i \in \{1, \dots, 2k+1\}$ in order to define a non-dominated valid inequality.

The odd cycle inequality (5) defines a facet if and only if there exist $2k+1$ affinely independent solutions that satisfy (5) with equality. Since the right hand side of (5) is violated by at most $\frac{1}{2}$, it follows that every solution must satisfy all but one edge inequalities $t_{v_i} + t_{v_{i+1}} \leq \beta_{v_i v_{i+1}}$ with equality, whereas the remaining inequality has to have slack exactly one. Since C is an odd cycle (and thus the edge-node incidence matrix has full rank), only one such a solution can exist for each edge. As a consequence, we can identify the $2k+1$ solutions by the index i of the edge $v_i v_{i+1}$ that is not satisfied with equality by the specific solution: t^i . Each solution t^i is uniquely defined by

$$t_{v_j}^i + t_{v_{j+1}}^i = \begin{cases} \beta_{v_j v_{j+1}} - 1 & \text{if } j = i, \\ \beta_{v_j v_{j+1}} & \text{otherwise,} \end{cases}$$

To be feasible, these solutions have to satisfy

$$0 \leq t_{v_j}^i \leq \alpha_{v_j} \quad \forall j = 1, \dots, 2k+1$$

From the equality constraints, it follows that

$$0 \leq t_{v_j}^i = \frac{1}{2} \left(\sum_{p=0}^k \beta_{v_{j+2p}v_{j+2p+1}} - \sum_{p=1}^k \beta_{v_{j+2p-1}v_{j+2p}} + (-1)^{j-i} \right) \leq \alpha_{v_j}$$

should hold for all $i \in \{1, \dots, 2k+1\}$, $j \in \{i+1, \dots, i+2k+1\}$. Since for fixed j the difference $j-i$ is half the times even and half the times odd. This statement is equivalent to

$$1 \leq \sum_{p=0}^k \beta_{v_{j+2p}v_{j+2p+1}} - \sum_{p=1}^k \beta_{v_{j+2p-1}v_{j+2p}} \leq 2\alpha_{v_j} - 1$$

for all $j \in \{i+1, \dots, i+2k+1\}$. The right part of this estimation follows immediately from (12). The left part is equivalent to

$$\sum_{p=1}^k \beta_{v_{i+2p-1}v_{i+2p}} \leq \sum_{p=0}^k \beta_{v_{i+2p}v_{i+2p+1}} - 1$$

for all $i \in \{1, \dots, 2k+1\}$, and addition of $\sum_{p=1}^k \beta_{v_{i+2p-1}v_{i+2p}}$ to both sides and division by 2 this transforms to

$$\sum_{p=1}^k \beta_{v_{i+2p-1}v_{i+2p}} \leq \frac{1}{2}\beta(C) - \frac{1}{2} = \lfloor \frac{1}{2}\beta(C) \rfloor$$

for all $i \in \{1, \dots, 2k+1\}$. Now, the condition stated in the proposition follows immediately. \blacksquare

Chvátal [3] introduced the concept of t -perfectness of a graph. A graph is t -perfect if the associated stable set polytope is completely described by the model inequalities and the odd cycle inequalities (or equivalently, the polytope has Chvátal rank 1). However, a full characterization of t -perfect graphs is unknown, and as a consequence, the same holds for the graphs for which the stable multi-set polytope has Chvátal rank 1. Gerards and Schrijver [5] proved that for strongly t -perfect graphs, i.e., graphs that do not contain a subdivision of K_4 such that all four cycles corresponding to triangles in K_4 are odd, the model and odd cycle inequalities completely describe the stable multi-set polytope. This result is independent of the vectors α and β . For actual values α_v and β_{vw} , stronger results for Chvátal rank 1 are likely to hold (cf. the results for Chvátal rank 0, Corollary 4.8).

Moreover, without going into details, the polynomial time separation algorithm for odd cycle inequalities proposed by Grötschel, Lovász and Schrijver [6], is not applicable to the separation of odd-valued odd cycles. As a consequence, the question whether the separation of odd cycle inequalities for the SMSP can be solved in polynomial time remains open.

A second class of inequalities that is well-known for the stable set polytope are the clique inequalities. For the stable multi-set polytope, we derive a similar result for cliques with uniform edge bounds $\beta_{vw} = \beta$ for all $vw \in E$ which we call β -cliques. For suitable vertex bounds α_v we get the following β -clique inequalities defined on a clique K_n .

Proposition 4.10 *Let $G = (V, E) = K_n$ for an integer $n \geq 2$, with $\beta_{vw} = \beta$ for all $vw \in E$ and $\alpha_v \geq \lceil \frac{1}{2}\beta \rceil$ for all $v \in V$. Then the clique inequality*

$$\sum_{v \in V} t_v \leq n \lfloor \frac{1}{2}\beta \rfloor + (\beta \bmod 2) \quad (13)$$

is valid for T , and defines a facet of T_{IP} if and only if β is odd or $n = 2$.

Proof. We prove the validity of (13) by induction on n . For $n = 2$, let $K_2 = \{v_1, v_2\}$, and consider the model inequality $t_{v_1} + t_{v_2} \leq \beta$. By $\beta = 2 \lfloor \frac{1}{2}\beta \rfloor + (\beta \bmod 2)$, the inequality belongs to the class of clique inequalities (13).

Now, suppose the result holds for cliques of size $n - 1$. Then for all $u \in V$, it holds (by assumption)

$$\sum_{v \in V \setminus \{u\}} t_v \leq (n - 1) \lfloor \frac{1}{2}\beta \rfloor + (\beta \bmod 2)$$

Summation of these inequalities for all $u \in V$ yields

$$(n - 1) \sum_{v \in V} t_v \leq n(n - 1) \lfloor \frac{1}{2}\beta \rfloor + n(\beta \bmod 2)$$

Division by $n - 1$ and rounding results in

$$\begin{aligned} \sum_{v \in V} t_v &\leq \left\lfloor n \lfloor \frac{1}{2} \beta \rfloor + (\beta \bmod 2) + \frac{1}{n-1} (\beta \bmod 2) \right\rfloor \\ &= n \lfloor \frac{1}{2} \beta \rfloor + (\beta \bmod 2) + \left\lfloor \frac{1}{n-1} (\beta \bmod 2) \right\rfloor \end{aligned}$$

Because we can assume $n \geq 3$, $\frac{1}{n-1} \leq \frac{1}{2} < 1$. Moreover, since $(\beta \bmod 2) \in \{0, 1\}$, we have $\lfloor \frac{1}{n-1} (\beta \bmod 2) \rfloor = 0$, which proves that (13) is valid for n .

Finally, we prove that (13) defines a facet of T_{IP} , if and only if β is odd or $n = 2$. If $G = K_2$, the conditions of Proposition 4.2 are satisfied, and so (13) defines a facet. If β is odd, then the n solutions t^v for all $v \in V$ defined by

$$t_u^v = \begin{cases} \lfloor \frac{1}{2} \beta \rfloor + 1 & \text{if } u = v, \\ \lfloor \frac{1}{2} \beta \rfloor & \text{otherwise} \end{cases}$$

are valid, satisfy the clique inequality (13) with equality, and are affinely independent. If neither β is odd nor $n = 2$, i.e., β is even and $n \geq 3$, then (13) is a linear combination of the model inequalities $t_v + t_w \leq \beta$ for all $vw \in E$, and thus cannot define a facet. ■

For more general graphs, the clique inequality is valid for all cliques that satisfy the described properties. To be facet defining in general graphs, an additional condition has to be satisfied. Since cliques of size 2 are already treated by Proposition 4.2, we restrict to cliques with at least 3 nodes.

Proposition 4.11 *Let $G = (V, E)$ be an arbitrary graph. Let $Q \subseteq V$, $|Q| \geq 3$, be a clique in G with $\beta_{vw} = \beta$ for all $vw \in E$, and $\alpha_v \geq \lceil \frac{1}{2} \beta \rceil$ for all $v \in V$. Then the clique inequality (13) is valid for T_{IP} . Moreover, (13) defines a facet of T_{IP} if and only if β is odd and for all $u \in N(Q)$, there exists $w \in Q$ with $w \notin N(u)$ or $\beta_{uw} \geq \lceil \frac{1}{2} \beta \rceil + 1$.*

Proof. We first prove sufficiency. By the described conditions, we have $|Q|$ affinely independent solutions t^v for $v \in Q$, defined by

$$t_u^v = \begin{cases} \lfloor \frac{1}{2} \beta \rfloor + 1 & \text{if } u = v, \\ \lfloor \frac{1}{2} \beta \rfloor & \text{if } u \in Q, u \neq v, \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, by assumption there is a node $v \in Q$ for each $w \in V \setminus Q$, such that $t^w = t^v + e_w$ is valid (where e_w denotes the w -th unit vector). These in total n vectors are affinely independent and satisfy (13) with equality, implying that (13) defines a facet of T_{IP} .

Necessity is shown indirectly. By Proposition 4.10, we know that for β even, the inequality (13) can not define a facet. Moreover, if β is odd, but there exists $u \in N(Q)$ with $Q \subseteq N(u)$ and $\beta_{uw} \leq \lceil \frac{1}{2} \beta \rceil$ for all $w \in Q$, then all solutions satisfying (13) with equality also satisfy $t_u = 0$, which implies that (13) is a face of $t_u = 0$ and does not define a facet of T_{IP} . ■

Note that in case of a stable set polytope, we have $\beta = 1$, and so $\beta_{uv} \geq \lceil \frac{1}{2}\beta \rceil + 1$ cannot be satisfied. Thus, the clique inequality (13) defines a facet for the stable set polytope if and only if Q is a maximal clique, which has been shown by Fulkerson [4] and Padberg [8].

In case G is complete, T_{IP} is completely described by the clique inequalities. To prove this result we need some general properties of facet defining inequalities.

Lemma 4.12 *Let $\pi t \leq \pi_0$ be a valid inequality that defines a facet of T_{IP} . Then $\pi_0 \geq 0$, and if $\pi_0 = 0$, then $\pi t \leq 0$ is identical to a non-negativity inequality $t_v \geq 0$ for some $v \in V$. Moreover, if $\pi_0 > 0$, then $0 \leq \pi_v \leq \pi_0/\alpha_v$ for all $v \in V$.*

Proof. From $0 \in T$, it follows that $\pi_0 \geq 0$. If $\pi_0 = 0$, then the solution $t^v = e^v$, $v \in V$, implies that $\pi_v \leq 0$. But in this case, $\pi t \leq 0$ is a linear combination of the inequalities $t_v \geq 0$. Since both $t_v \geq 0$ and $\pi t \leq 0$ define facets of T_{IP} , the inequalities have to be identical.

Now, let $\pi_0 > 0$ and $\pi_v < 0$ for some $v \in V$. Since $\pi t \leq \pi_0$ defines a facet (and is not identical to a non-negativity constraint), there has to be a solution \bar{t} with $\pi \bar{t} = \pi_0$ and $\bar{t}_v > 0$. Then the solution \tilde{t} with $\tilde{t}_v = \bar{t}_v - 1$ and $\tilde{t}_w = \bar{t}_w$, for all $w \neq v$, however, violates $\pi t \leq \pi_0$. Finally, by $\alpha_v e^v \in T$ for all $v \in V$, $\pi_v \alpha_v \leq \pi_0$ for all $v \in V$. \blacksquare

Theorem 4.13 *Let $G = (V, E) = K_n$ for an integer $n \geq 2$, with $\beta_{vw} = \beta$ for all $vw \in E$ and $\alpha_v \geq \lceil \frac{1}{2}\beta \rceil$ for all $v \in V$. Then T_{IP} is completely described by the model inequalities and the k -clique inequalities (13) for all $k \leq n$.*

Proof. We have to prove that each inequality $\pi t \leq \pi_0$ that defines a facet of T_{IP} is in fact identical to one of the inequalities describing the polytope P ,

$$P = \left\{ t \in \mathbb{R}^n : \begin{array}{ll} t_v \leq \alpha_v & v \in V, \\ t_v + t_w \leq \beta_{vw} & vw \in E, \\ \sum_{v \in S} t_v \leq |S| \lceil \frac{1}{2}\beta \rceil + 1 & S \subseteq V, |S| \geq 3 \end{array} \right\}.$$

First of all, without loss of generality, we assume that all coefficients $\pi_0 \in \mathbb{Z}_0^+$ and $\pi_v \in \mathbb{Z}_0^+$, $v \in V$. By Lemma 4.12, we can restrict us to $\pi_0 > 0$, and it holds that $\pi_v \geq 0$ for all $v \in V$. Moreover, in case $\pi_v = 0$ for all vertices but one $v \in V$, the inequality must be identical to the inequality $t_v \leq \alpha_v$. So, in the sequel we may assume at least two positive coefficients π_v . We also may assume that at least two different positive coefficients exist, otherwise the inequality has to be identical to an edge or clique inequality.

Now, let $u = \arg \max_{v \in V} \pi_v$. It holds that either $\pi_u > \sum_{v \in V \setminus \{u\}} \pi_v$ or $\pi_u \leq \sum_{v \in V \setminus \{u\}} \pi_v$. Suppose that $\pi_u > \sum_{v \in V \setminus \{u\}} \pi_v$ holds. Since $\pi t \leq \pi_0$ defines a facet not identical to $t_u \leq \alpha_u$, there has to be a solution $\bar{t} \in T_{IP}$ with $\pi \bar{t} = \pi_0$ and $\bar{t}_u \leq \alpha_u - 1$. Now, construct a new solution \tilde{t} by $\tilde{t}_u = \bar{t}_u + 1$, $\tilde{t}_v = \max\{0, \bar{t}_v - 1\}$ for all $v \neq u$. Obviously $\tilde{t} \in T_{IP}$, but $\pi \tilde{t} > \pi_0$, which implies that $\pi t \leq \pi_0$ cannot be a valid inequality in this case.

So, it holds that $\pi_u \leq \sum_{v \in V \setminus \{u\}} \pi_v$. We construct a non-negative combination of the inequalities describing P , which is equivalent to or dominates $\pi t \leq \pi_0$. On the one hand, if $\pi_u = \sum_{v \in V \setminus \{u\}} \pi_v$, we can combine edge inequalities $t_u + t_w \leq \beta$ to an equivalent (or

dominating) inequality. On the other hand, if there exists a $v \in V \setminus \{u\}$ with $\pi_v = \pi_u$, a combination of clique inequalities gives the desired result. In general, we have to combine these two approaches. To do so, let $V^{(0)} = V \setminus \{u\}$ and $\mu^{(0)} = 0$. Moreover, for $i \geq 1$ let $V^{(i)} = \{v \in V^{(i-1)} : \pi_v > \mu^{(i-1)}\}$ and $\mu^{(i)} = \min_{v \in V^{(i)}} \pi_v$. Now, consider the values $f(i) := \sum_{v \in V^{(i)}} \pi_v - \pi_u$ and $g(i) := (|V^{(i)}| - 1)\mu^{(i)}$. An increase of i to $i + 1$ results in a decrease $f(i)$ by $|V^{(i)} \setminus V^{(i+1)}|\mu^{(i)}$, whereas $g(i)$ decreases by $|V^{(i)} \setminus V^{(i+1)}|\mu^{(i)} - (|V^{(i+1)}| - 1)(\mu^{(i+1)} - \mu^{(i)})$. So, the $f(i)$ decreases at least as fast as $g(i)$. Moreover, let q be the smallest integer such that $V^{(q+1)} = \emptyset$. Then $f(q) = |V^{(q)}|\mu^{(q)} - \pi_u \leq g(q)$, whereas $f(0) \geq 0 = g(0)$. As a consequence, there exists a minimal $p \in \{0, \dots, q\}$ such that $f(p) \leq g(p)$.

Note that, $|V^{(p)}| \geq 1$, since $p \leq q$. Now, suppose $|V^{(p)}| = 1$. By definition of p , we have $f(p) \leq g(p)$ and $f(p-1) > g(p-1)$. With $|V^{(p)}| = 1$ we have $\sum_{v \in V^{(p)}} \pi_v = \mu^{(p)}$ and $\sum_{v \in V^{(p-1)}} \pi_v = (|V^{(p-1)}| - 1)\mu^{(p-1)} + \mu^{(p)}$. This yields $\mu^{(p)} - \pi_u \leq 0$ and $(|V^{(p-1)}| - 1)\mu^{(p-1)} + \mu^{(p)} - \pi_u > (|V^{(p-1)}| - 1)\mu^{(p-1)}$, but both cannot be true. Hence, we have $|V^{(p)}| \geq 2$.

Now, consider the non-negative combination of the inequalities

$$\begin{aligned} \sum_{v \in V^{(i)} \cup \{u\}} t_v &\leq (|V^{(i)}| + 1) \lfloor \frac{1}{2} \beta \rfloor + 1 && \text{with weight } \mu^{(i)} - \mu^{(i-1)}, \text{ for } 1 \leq i < p \\ \sum_{v \in V^{(p)} \cup \{u\}} t_v &\leq (|V^{(p)}| + 1) \lfloor \frac{1}{2} \beta \rfloor + 1 && \text{with weight } \frac{\sum_{v \in V^{(p)}} \pi_v - \pi_u}{(|V^{(p)}| - 1)} - \mu^{(p-1)} \\ t_u + t_w &\leq \beta && \text{with weight } \pi_w - \frac{\sum_{v \in V^{(p)}} \pi_v - \pi_u}{(|V^{(p)}| - 1)}, \text{ for all } w \in V^{(p)} \end{aligned}$$

This yields to the inequality

$$\begin{aligned} \pi t &\leq \sum_{i=1}^{p-1} (\mu^{(i)} - \mu^{(i-1)}) \left\{ (|V^{(i)}| + 1) \lfloor \frac{1}{2} \beta \rfloor + 1 \right\} \\ &+ \left(\frac{\sum_{v \in V^{(p)}} \pi_v - \pi_u}{(|V^{(p)}| - 1)} - \mu^{(p-1)} \right) \left\{ (|V^{(p)}| + 1) \lfloor \frac{1}{2} \beta \rfloor + 1 \right\} \\ &+ \sum_{w \in V^{(p)}} \left(\pi_w - \frac{\sum_{v \in V^{(p)}} \pi_v - \pi_u}{(|V^{(p)}| - 1)} \right) \beta \end{aligned} \tag{14}$$

which has the same left hand side as the inequality $\pi t \leq \pi_0$. On the other hand, the solution \bar{t} defined by $\bar{t}_u = \lfloor \frac{1}{2} \beta \rfloor + 1$, $\bar{t}_v = \lfloor \frac{1}{2} \beta \rfloor$ for all $v \neq u$ satisfies (14) at equality, because it satisfies all combined inequalities at equality. So, π_0 has to be at least as large as the right hand side of (14). As a consequence, $\pi t \leq \pi_0$ is identical to or dominated by a non-negative combination of the inequalities describing P . \blacksquare

For $\beta = 1$ (stable set), Chvátal [2] showed that the maximum clique inequality defines the only non-trivial facet if G is complete. By Proposition 4.11, we know that for $\beta > 1$, in fact all clique inequalities are necessary.

For arbitrary β_{vw} , the clique inequality can be derived by Chvátal-Gomory rounding of clique inequalities of subcliques. An n -clique inequality can be derived from n k -clique in-

equalities, $k \geq \lfloor \frac{1}{2}n \rfloor$. Here, different values of k as well as different permutations of the vertices result in different right hand sides, which makes a formula for it difficult to determine. Consider, for example, the SMS problem in Figure 4. The 5-clique inequality obtained by Chvátal-Gomory rounding of the 3-clique inequalities defined on $\{a, b, c\}$, $\{b, c, d\}$, $\{c, d, e\}$, $\{a, d, e\}$, and $\{a, b, e\}$ results in a right hand side of 15, whereas Chvátal-Gomory rounding of the 3-clique inequalities defined on $\{a, c, e\}$, $\{b, c, e\}$, $\{b, d, e\}$, $\{a, b, d\}$, and $\{a, c, d\}$ results in a right hand side of 13. Moreover, the Chvátal-Gomory rounding of the 4-clique inequalities defined on $\{a, b, c, d\}$, $\{b, c, d, e\}$, $\{a, c, d, e\}$, $\{a, b, d, e\}$, and $\{a, b, c, e\}$ results in a right hand side of 14. So, it is not clear which smaller cliques have to be combined to obtain the best right hand side. Moreover, none of the above defines a facet. In fact, the 5-clique $\{a, b, c, d, e\}$ does not define a facet at all.

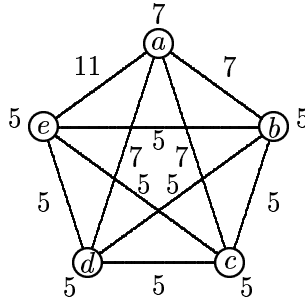


Figure 4: Example SMS clique problem.

So far, we have given some sets of valid inequalities that are generalizations of the valid inequalities derived for the stable set polytope. This section is closed with a more general result how facet defining valid inequalities for a stable multi-set polytope can be derived from the facet defining inequalities of a stable multi-set polytope with smaller α and β .

Proposition 4.14 *Let $P_{SMS} = (G = (V, E), \alpha, \beta)$ define a stable multi-set polytope and $u \in V$. Moreover, on the same graph G let $P_{SMS}^* = (G, \alpha^*, \beta^*)$ be a second stable multi-set polytope defined by*

$$\alpha_v^* = \begin{cases} \alpha_v + 1 & \text{if } v = u, \\ \alpha_v & \text{otherwise} \end{cases} \quad \text{and} \quad \beta_{vw}^* = \begin{cases} \beta_{vw} + 1 & \text{if } v = u \text{ or } w = u, \\ \beta_{vw} & \text{otherwise.} \end{cases}$$

If $\pi t \leq \pi_0$ with $\pi_u > 0$ defines a facet of P_{SMS} , then $\pi t \leq \pi_0 + \pi_u$ defines a facet of P_{SMS}^ .*

Proof. Let t^* be a solution of P_{SMS}^* with $t_u^* > 0$. Then $t^* - e^u$ (where e^u denote the unit vector with 1 at position u) is a solution of P_{SMS} , and as a consequence $\pi(t^* - e^u) \leq \pi_0$. So, $\pi t^* \leq \pi_0 + \pi_u$. If $t_u^* = 0$, then by $\alpha_v^* = \alpha_v \leq \beta_{uv} < \beta_{uv}^*$ for all $v \in N(u)$, we have that t^* is also a feasible solution for P_{SMS} . So $\pi t^* \leq \pi_0 \leq \pi_0 + \pi_u$, and it follows that $\pi t \leq \pi_0 + \pi_u$ is valid for P_{SMS}^* .

Now, by assumption let $\pi t \leq \pi_0$ be facet defining for P_{SMS} . Thus, there exist $n = \dim P_{SMS}$ affinely independent solutions t^1, \dots, t^n of P_{SMS} that satisfy $\pi t \leq \pi_0$ with equality. Then $t^1 + e^u, \dots, t^n + e^u$ are n affinely independent solutions of P_{SMS}^* that satisfy $\pi t \leq \pi_0 + \pi_u$ with equality. ■

	Stable set polytope	Stable multi-set polytope
Dimension	n	n
Trivial facets	$t_v \geq 0$	$t_v \geq 0$ $t_v \leq \alpha_v$ $t_v + t_w \leq \beta_{vw}$
Odd cycle inequalities	$\sum_{v \in C} t_v \leq \lfloor \frac{1}{2} C \rfloor$ facet for G_C	$\sum_{v \in C} t_v \leq \lfloor \frac{1}{2}\beta(C) \rfloor$ facet for G_C
Clique inequalities	$\sum_{v \in Q} t_v \leq 1$ facet for Q maximal	$\sum_{v \in Q} t_v \leq n \lfloor \frac{1}{2}\beta \rfloor + (\beta \bmod 2)$ facet for $\beta_{vw} = \beta$ odd for all $vw \in E$
$T_{IP} = T_{LP}$	G bipartite	for all cycles C , $ C $ odd, $\beta(C)$ even or (7) satisfied.
Complete description K_n	$\sum_{v \in V} t_v \leq 1$ $t_v \geq 0, \forall v \in V$	$\sum_{v \in S} t_v \leq S \lfloor \frac{1}{2}\beta \rfloor + 1, \forall S \subseteq V, S \geq 2$ $0 \leq t_v \leq \alpha_v, \forall v \in V$

Table 1: Similarities and differences stable sets vs. stable multi-sets

Note that the result does not hold in general for the reverse direction of Proposition 4.14, i.e., a decrease of the values α and β associated to some vertex u . Especially, solutions with $t_u^* = 0$ that satisfy $\pi t \leq \pi_0 + \pi_u$ at equality cannot be transformed to new solutions that satisfy $\pi t \leq \pi_0$ at equality.

5 Concluding Remarks

In this paper, we introduced and studied stable multi-sets. In particular, properties of the associated polytope were identified. In Table 1, the obtained results are compared with their analogues for the stable set polytope. Note that Table 1 does not show all conditions for the model inequalities to define facets. The table shows that many of the results obtained for stable sets have their counterpart for stable multi-sets.

As this paper is meant to be a starting point for studying the stable multi-set polytope, only a limited number of results for stable sets have been generalized, and many directions for further research exist. First of all, many other classes of valid inequalities are known for stable sets: wheel inequalities, web and anti-web inequalities, odd anti-hole inequalities, and blossom inequalities, to name a few (cf. [1]). It is likely that many of them can be generalized to the stable multi-set polytope. Another opportunity is to study whether specific classes of inequalities are separable in polynomial time. As we pointed out, for the odd cycle inequalities, the polynomial time algorithm by Grötschel, Lovász and Schrijver [6] is not applicable anymore. For the separation of clique inequalities, the separation problem is even more complex, since no general right hand side is known.

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