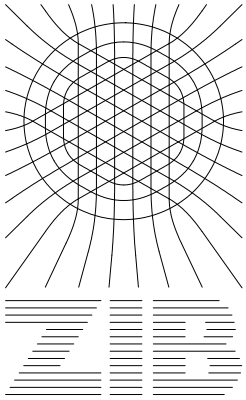

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# Polyhedral Results for the Edge Capacity Polytope

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# Polyhedral Results for the Edge Capacity Polytope

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**Abstract**

Network loading problems occur in the design of telecommunication networks, in many different settings. The polyhedral structure of this problem is important in developing solution methods for the problem. In this paper we investigate the polytope of the problem restricted to one edge of the network (the edge capacity problem).

We describe classes of strong valid inequalities for the edge capacity polytope, and we derive conditions under which these constraints define facets. As the edge capacity problem is a relaxation of the network loading problem, their polytopes are intimately related. We, therefore, also give conditions under which the inequalities of the edge capacity polytope define facets of the network loading polytope. Furthermore, some structural properties are derived, such as the relation of the edge capacity polytope to the knapsack polytope. We conclude the theoretical part of this paper with some lifting theorems, where we show that this problem is polynomially solvable for most of our classes of valid inequalities.

The derived inequalities are tested on (i) the edge capacity problem itself and (ii) a variant of the network loading problem. The results show that the inequalities substantially reduce the number of nodes needed in a branch-and-cut approach. Moreover, they show the importance of the edge subproblem for solving network loading problems.

## 1 Introduction

The network loading problem (NLP) occurs in telecommunications problems where demand for capacity for multiple commodities is to be realized by inserting capacity into a given network. The capacity can be placed in different sizes, usually multiples of each other. We restrict ourselves to a single capacity size, although many of the ideas presented in this paper can be extended in case multiple capacity sizes are available. Along with a capacity plan a routing of all commodities is to be determined to obtain a capacity plan that facilitates the demands of all commodities. This problem has been studied in many variants with respect to network layout, capacity usage, and routing possibilities. Routing of the demand can be done by reserving capacity on a subnetwork that consists of a path between the endpoints of a commodity only (non-bifurcated routing), or of a set of paths (bifurcated routing). We only consider the non-bifurcated case which is also studied by Gavish and Altinkemer [10] and Brockmüller et al. [7, 8].

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For the bifurcated case we refer to Magnanti et al. [16, 17]. With respect to capacity usage one can distinguish unidirectional and bidirectional capacity usage, i.e., if an edge contains a unit of capacity this unit can either be used in one or in both directions of the edge. In most studies the unidirectional (or undirected) case is studied. Exceptions are Bienstock and Günlük [6] and Bienstock et al. [5], who study the bidirectional case. In the sequel we consider both possibilities. Moreover, we show that the models of the corresponding NLPs have many common aspects. With respect to network lay-out we do not specialize us. For instance, we do not take reliability requirements into account. For the design of survivable networks with bifurcated routing we refer to Wessäly [20] and the references therein, for non-bifurcated routing to Van de Leensel [15]. Also generalizations like network design with multicast commodities are not considered (cf. Bienstock and Bley [4]). However, this paper considers the relaxation of NLP to a single edge of the network, the so-called edge capacity problem (ECP). We are specifically interested in the polyhedral structure of the polytope of ECP as it is an important substructure of all network loading models.

The edge capacity polytope with a single capacity type can be viewed as a 0–1 knapsack problem with a single integer variable representing the capacity of the knapsack. The closely related knapsack problem with a single *continuous* capacity variable is studied by Marchand and Wolsey [18]. They employ valid inequalities of the standard 0–1 knapsack problem (see Balas [1], Hammer et al. [13], Wolsey [21]) to obtain valid inequalities for the extended model by projection and lifting. The edge capacity polytope itself has also been studied by Brockmüller et al. [7, 8], who derive valid and facet defining inequalities for the edge capacity polytope. Magnanti et al. [16] study the version of the polytope in which the binary variables are relaxed to real variables and derive a complete description of the corresponding polytope.

In this paper we derive various new results for the edge capacity polytope. In Section 2 we discuss four non-bifurcated network loading problems and formulate the associated edge capacity models corresponding to individual capacity constraints. Although valid inequalities for individual constraint models obviously yield valid inequalities for the overall model, in Section 3 we derive a much stronger result, which implies that facet defining inequalities for the edge capacity polytope (often) correspond to facet defining inequalities for the network loading problem. In Section 4 we derive some basic results on facet defining inequalities for the edge capacity polytope, which facilitate the subsequent analysis. Section 5 introduces lower convex envelope inequalities as a general framework for valid inequalities for the edge capacity polytope. Given the interpretation of the edge capacity polytope as a knapsack polytope with variable capacity, Section 6 shows how any valid inequality for related knapsack polytopes can be transformed into a valid inequality for the edge capacity polytope by integer lifting. Integer lifting is usually a complicated process, but in Section 6 we show that for cover inequalities, this lifting process can be done efficiently. Both the class of lower convex envelope inequalities and the class of lifted knapsack covers include the so-called $c$-strong inequalities developed by Brockmüller et al. [7]. Some new properties that indicate the importance of $c$-strong inequalities in the description of the edge capacity polytope are mentioned in Section 7. In Section 8 we discuss the directed edge capacity polytope, that is the edge capacity polytope with a capacity constraint for both directions. New valid inequalities and lifting results are obtained. Finally, the computational importance of the developed theory is the subject of Section 9.

## 2 Models for Network Loading Problems

Let $G = (V, E)$ be an undirected connected graph with node set $V$ and edge set $E$. Given the graph $G$ we define the arc set $A$, which contains two directed arcs $(i, j)$ and $(j, i)$ for all edges $e = \{i, j\} \in E$. Let $Q$ be a set of demands (commodities). Each element $q \in Q$ is a triple $(s^q, t^q, d^q)$, with $s^q, t^q \in V$, $s^q \neq t^q$, representing a commodity with demand size $d^q \in \mathbb{Z}_0^+$ that must be routed from source node $s^q$ to sink node $t^q$ on a single path through the network. To route a set of commodities on an arc, sufficient capacity must be available on the corresponding edge. The capacity on an edge is determined by the number of capacity units installed on the edge, where each unit has a base capacity $\lambda \in \mathbb{Z}_0^+$. The goal is to minimize the costs of the installed capacity in the network while ensuring that all commodities can be routed from source to sink simultaneously.

We assume that for each commodity $q \in Q$ there exist at least two node-disjoint paths from source node to sink node (node-disjoint, except for the nodes $s^q$ and $t^q$). If this assumption is not satisfied, the graph $G$ contains a separating vertex, hence the problem can be decomposed into smaller problems that do satisfy the assumption.

To formulate this problem as an integer program, let $x_{ij} \in \mathbb{Z}_0^+$ be the number of capacity units installed on edge $\{i, j\}$, and let $f_{ij}^q$ be a binary variable indicating whether the commodity $q \in Q$ is routed via arc $(i, j) \in A$ or not. If $c_{ij}$ represents the costs per base capacity unit on edge $\{i, j\} \in E$, then the model reads:

$$\min \quad \sum_{\{i,j\} \in E} c_{ij} x_{ij} \tag{1}$$

$$\text{s.t.} \quad \sum_j f_{ij}^q - \sum_j f_{ji}^q = \begin{cases} 1 & \text{if } i = s^q \\ -1 & \text{if } i = t^q \\ 0 & \text{otherwise} \end{cases} \qquad \forall q \in Q, \forall i \in V \tag{2}$$

$$\lambda x_{ij} \geq \sum_{q \in Q} d^q (f_{ij}^q + f_{ji}^q) \qquad \forall \{i, j\} \in E \tag{3}$$

$$f_{ij}^q, f_{ji}^q \in \{0, 1\}, \; x_{ij} \in \mathbb{Z}_0^+ \qquad \forall q \in Q, \; \forall \{i, j\} \in E \tag{4}$$

This model is called the undirected non-bifurcated flow model, and the corresponding set of feasible solutions is denoted $UNFM$. The capacity on an edge is *undirected* because installed capacity can be used by traffic in both directions, i.e. the required capacity on an edge is determined by the sum of forward and backward flow on the edge. It is called *non-bifurcated* since the demand of a commodity has to be routed on a single path (i.e. the demand cannot be bifurcated). Finally, *flow* variables on individual arcs are used to model the routing of a commodity from source node to sink node.

Instead of using flow variables on individual edges to model routing restrictions, one can also use binary variables $z_p^q$ representing whether a certain path $p \in P^q$ (the set of all possible paths for the commodity $q$) is used to route the commodity $q$ from source node $s^q$ to sink node $t^q$. We assume that $P^q$ only contains simple paths, that is paths that visit each node at most once. If $P_{ij}^q \subseteq P^q$ denotes the set of paths for commodity $q$ that contain arc $(i, j)$, then this leads to the

following undirected non-bifurcated path model $UNPM$ with feasible solution set:

$$\min \quad \sum_{\{i,j\}\in E} c_{ij}x_{ij} \tag{5}$$

$$\text{s.t.} \quad \sum_{p\in P^q} z_p^q = 1 \qquad \forall q \in Q \tag{6}$$

$$\lambda x_{ij} \geq \sum_{q\in Q} \sum_{p\in P_{ij}^q \cup P_{ji}^q} d^q z_p^q \qquad \forall \{i,j\} \in E \tag{7}$$

$$z_p^q \in \{0,1\},\ x_{ij} \in \mathbb{Z}_0^+ \qquad \forall q \in Q,\ \forall p \in P^q,\ \forall \{i,j\} \in E \tag{8}$$

Depending on the exact application and level of aggregation, capacity that is installed on edges in the network can also be *directed*, i.e. each unit of capacity installed on an edge $\{i,j\}$ gives a capacity of $\lambda$ on both corresponding arcs $(i,j)$ and $(j,i)$, and capacity consumption is directed as well. This leads to the following directed non-bifurcated flow model $DNFM$, with feasible solution set:

$$\min \quad \sum_{\{i,j\}\in E} c_{ij}x_{ij} \tag{9}$$

$$\text{s.t.} \quad \sum_j f_{ij}^q - \sum_j f_{ji}^q = \begin{cases} 1 & \text{if } i = s^q \\ -1 & \text{if } i = t^q \\ 0 & \text{otherwise} \end{cases} \qquad \forall q \in Q, \forall i \in V \tag{10}$$

$$\lambda x_{ij} \geq \sum_{q\in Q} d^q f_{ij}^q \qquad \forall \{i,j\} \in E \tag{11}$$

$$\lambda x_{ij} \geq \sum_{q\in Q} d^q f_{ji}^q \qquad \forall \{i,j\} \in E \tag{12}$$

$$f_{ij}^q, f_{ji}^q \in \{0,1\},\ x_{ij} \in \mathbb{Z}_0^+ \qquad \forall q \in Q,\ \forall \{i,j\} \in E \tag{13}$$

Similar to the undirected case, one can model the directed case using path variables. This directed non-bifurcated path model $DNPM$, with feasible solution set reads:

$$\min \quad \sum_{\{i,j\}\in E} c_{ij}x_{ij} \tag{14}$$

$$\text{s.t.} \quad \sum_{p\in P^q} z_p^q = 1 \qquad \forall q \in Q \tag{15}$$

$$\lambda x_{ij} \geq \sum_{q\in Q} \sum_{p\in P_{ij}^q} d^q z_p^q \qquad \forall \{i,j\} \in E \tag{16}$$

$$\lambda x_{ij} \geq \sum_{q\in Q} \sum_{p\in P_{ji}^q} d^q z_p^q \qquad \forall \{i,j\} \in E \tag{17}$$

$$z_p^q \in \{0,1\},\ x_{ij} \in \mathbb{Z}_0^+ \qquad \forall q \in Q,\ \forall p \in P^q,\ \forall \{i,j\} \in E \tag{18}$$

Both the directed and the undirected version of the problem are $\mathcal{NP}$-hard. Here we will only prove our claim for the directed case, but from the proof it immediately follows that the undirected case is $\mathcal{NP}$-hard as well. We need the following problem definitions.

MINIMUM COVER (see Garey and Johnson [9])
INSTANCE: $S = \{s_1, s_2, \ldots, s_n\}$ and a family $\mathcal{C}$ of subsets of $S$, and an integer $\ell \leq |C|$.
QUESTION: Does there exist a family $\mathcal{C}'$ of $\mathcal{C}$ of at most $\ell$ sets such that $\cup_{c' \in \mathcal{C}'} c' = S$ ?

DIRECTED NON-BIFURCATED NETWORK LOADING PROBLEM
INSTANCE: Graph $G = (V, E)$, a set $Q$ of commodities $(s^q, t^q, d^q)$, with $d^q \in \mathbb{Z}_0^+$ for each $q \in Q$, integer capacity $\lambda$, cost coefficients $c_{ij} \in \mathbb{Z}_0^+$ for every $\{i, j\} \in E$, and a nonnegative integer $L$.
QUESTION: Does there exist a feasible solution for problem $DNFM$ with solution value less than or equal to $L$ ?

**Proposition 2.1** *The* DIRECTED NON-BIFURCATED NETWORK LOADING PROBLEM *is strongly* $\mathcal{NP}$*-complete.*

**Proof.** It is easy to see that the DIRECTED NON-BIFURCATED NETWORK LOADING PROBLEM is in $\mathcal{NP}$. Next, given an instance of MINIMUM COVER, construct a graph $G$ as follows. Let $V = S \cup \mathcal{C} \cup \{sink\}$. Let $\mathcal{C}_j$ be an element in $\mathcal{C}$. Introduce an edge $\{s_i, \mathcal{C}_j\}$ if $s_i \in \mathcal{C}_j$, and an edge $\{\mathcal{C}_j, sink\}$ for all $\mathcal{C}_j \in \mathcal{C}$ (see Figure 1). Next define a commodity $(s_i, sink, 1)$ for each $s_i \in S$, and

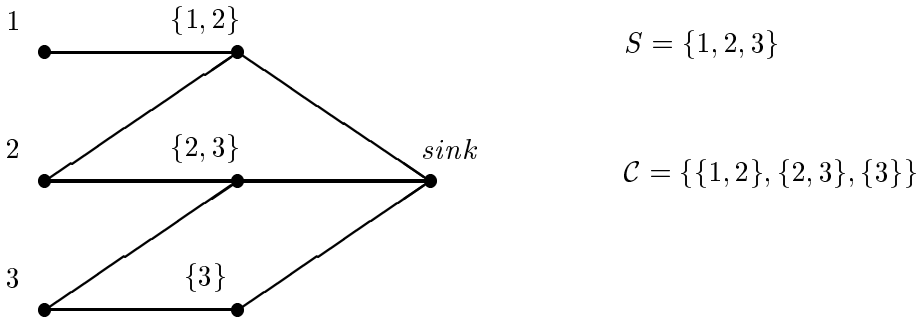


Figure 1: Transformation of an instance of MINIMUM COVER to DIRECTED NON-BIFURCATED NETWORK LOADING PROBLEM

let $\lambda = n$. The capacity costs are defined by $c_{\{s_i, \mathcal{C}_j\}} = |\mathcal{C}| + 1$ for all $i$ and $j$ and $c_{\{\mathcal{C}_j, sink\}} = 1$ for all $j$. Finally let $L = \ell + |S|(|\mathcal{C}| + 1)$. This is clearly a polynomial transformation. We leave it to the reader to verify that a problem instance for MINIMUM COVER with affirmative answer corresponds to an instance with affirmative answer for the DIRECTED NON-BIFURCATED NETWORK LOADING PROBLEM, and vice versa. ■

In this paper we focus on the polyhedral structure of the associated polytopes. More precisely, we study the convex hull of sets related to individual edge capacity constraints in the formulations. Hence, we define the following sets, which are defined by a single (denoted with $X$) or double

(denoted with $Y$) edge capacity constraint.

$$
\begin{aligned}
X_{ij}^{UF} &= \{(x,f) \in \mathbb{Z}_0^+ \times \{0,1\}^{2|Q|} \;:\; \lambda x_{ij} \geq \textstyle\sum_{q\in Q} d^q (f_{ij}^q + f_{ji}^q)\} \\
X_{ij}^{DF} &= \{(x,f) \in \mathbb{Z}_0^+ \times \{0,1\}^{|Q|} \;:\; \lambda x_{ij} \geq \textstyle\sum_{q\in Q} d^q f_{ij}^q\} \\
Y_{ij}^{DF} &= \{(x,f) \in \mathbb{Z}_0^+ \times \{0,1\}^{2|Q|} \;:\; \lambda x_{ij} \geq \textstyle\sum_{q\in Q} d^q f_{ij}^q,\ \lambda x_{ij} \geq \textstyle\sum_{q\in Q} d^q f_{ji}^q\}
\end{aligned}
$$

$$
\begin{aligned}
X_{ij}^{UP} &= \{(x,z) \in \mathbb{Z}_0^+ \times \{0,1\}^{\sum_{q\in Q} |P_{ij}^q|+|P_{ji}^q|} \;:\; \lambda x_{ij} \geq \textstyle\sum_{q\in Q} \sum_{p\in P_{ij}^q \cup P_{ji}^q} d^q z_p^q, \\
&\qquad\qquad \textstyle\sum_{p\in P_{ij}^q \cup P_{ji}^q} z_p^q \leq 1, \forall q \in Q\} \\
X_{ij}^{DP} &= \{(x,z) \in \mathbb{Z}_0^+ \times \{0,1\}^{\sum_{q\in Q} |P_{ij}^q|} \;:\; \lambda x_{ij} \geq \textstyle\sum_{q\in Q} \sum_{p\in P_{ij}^q} d^q z_p^q, \\
&\qquad\qquad \textstyle\sum_{p\in P_{ij}^q} z_p^q \leq 1, \forall q \in Q\} \\
Y_{ij}^{DP} &= \{(x,z) \in \mathbb{Z}_0^+ \times \{0,1\}^{\sum_{q\in Q} |P_{ij}^q|+|P_{ji}^q|} \;:\; \lambda x_{ij} \geq \textstyle\sum_{q\in Q} \sum_{p\in P_{ij}^q} d^q z_p^q, \\
&\qquad\qquad \lambda x_{ij} \geq \textstyle\sum_{q\in Q} \sum_{p\in P_{ji}^q} d^q z_p^q, \\
&\qquad\qquad \textstyle\sum_{p\in P_{ij}^q \cup P_{ji}^q} z_p^q \leq 1\}
\end{aligned}
$$

Obviously any valid inequality for these polytopes is valid for the corresponding original problem. In Section 3 we prove the stronger result that for the undirected models any non-trivial facet defining inequality for these polytopes is also a facet defining inequality for the corresponding original problem. For the directed models the same result holds for the edge models that incorporate capacity constraints in both directions on the edge.

# 3 The Strength of Facets of the Edge Capacity Polytope for Network Loading Problems

In this section we state some basic results about the dimension of polytopes related to the problems introduced in the previous section. More importantly, we show that non-trivial facet defining inequalities for the polytopes related to a single edge constraint are non-trivial facet defining inequalities for the polytopes corresponding to the original problem.

## 3.1 Path Formulation

**Proposition 3.1** *The dimension of both conv$(UNPM)$ and conv$(DNPM)$ are equal to* $|E| + \sum_{q\in Q}(|P^q| - 1)$.

**Proof.** The number of edge capacity variables equals $|E|$ and the number of path variables equals $\sum_{q\in Q} |P^q|$. Since the number of linearly independent equality constraints equals $|Q|$, this leads to an upper bound on the dimension of $|E| + \sum_{q\in Q}(|P^q| - 1)$. Next, we state $1 + |E| + \sum_{q\in Q}(|P^q| - 1)$ affinely independent feasible solutions, which proves our claim. In the first solution each commodity $q \in Q$ is routed via an arbitrarily chosen path $\hat{p} \in P^q$, and the capacity

equals the total flow on an edge rounded up to the nearest multiple of $\lambda$. Given this solution we can install an extra capacity unit on each edge, which yields another $|E|$ affinely independent solutions. Finally, for each commodity $q \in Q$ and each path $p \in P^q \setminus \{\hat{p}\}$ we construct a solution by keeping the routing of all other commodities fixed as in the first solution, but replacing path $\hat{p}$ by path $p$ for commodity $q$, and installing additional capacity if needed. The $\sum_{q \in Q}(|P^q| - 1)$ vectors that are obtained are affinely independent since each solution contains a path variable that is not used in any other solution vector. ■

The following lemma indicates that the number of path variables in the formulation, and therefore the dimension of the corresponding polytope, can become exponentially large in terms of the size of the graph.

**Lemma 3.1** *The number of distinct simple paths (a path without node repetition) between any pair of nodes in a complete graph on $|V|$ nodes equals $\lfloor (|V|-2)!e \rfloor$, if $|V| \geq 3$.*

**Proof.** Let $u, v \in V$, then there remain $|V| - 2$ nodes that may function as intermediate nodes on a path from $u$ to $v$. Since for a given number of $k$ intermediate nodes, the number of possible orders equals $k!$, the total number of paths between nodes $u$ and $v$ in a complete graph equals

$$\sum_{i=0}^{|V|-2} \binom{|V|-2}{i} i! = \sum_{i=0}^{|V|-2} \frac{(|V|-2)!}{(|V|-2-i)!} = \sum_{i=0}^{|V|-2} \frac{(|V|-2)!}{i!}$$

To analyze this sum, note that (for $n = |V| - 2$)

$$\sum_{i=0}^{n} \frac{n!}{i!} = n! \sum_{i=0}^{n} \frac{1}{i!} = n! \sum_{i=0}^{\infty} \frac{1}{i!} - n! \sum_{i=n+1}^{\infty} \frac{1}{i!} = n!e - n! \sum_{i=n+1}^{\infty} \frac{1}{i!}$$

Moreover, since

$$\begin{aligned} n! \sum_{i=n+1}^{\infty} \frac{1}{i!} &= \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \frac{n!}{(n+3)!} + \ldots \\ &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots \\ &= \sum_{i=1}^{\infty} \frac{1}{(n+1)^i} = \frac{1}{n} < 1, \text{ for } n \geq 1 \text{ (i.e., } V \geq 3), \end{aligned}$$

it holds that

$$n! \sum_{i=0}^{n} \frac{1}{i!} = n!e - n! \sum_{i=n+1}^{\infty} \frac{1}{i!} = \lfloor n!e \rfloor$$

This completes the proof. ■

**Proposition 3.2** *Let $q \in Q, p \in P^q$. The trivial inequalities $z_p^q \geq 0$ are valid and facet defining for conv$(UNPM)$ and conv$(DNPM)$.*

**Proof.** Let $\tilde{q} \in Q$ and $\tilde{p} \in P^{\tilde{q}}$. W.l.o.g. we assume that the path $\hat{p}$ for commodity $\tilde{q}$ as chosen in the first solution vector of the proof of Proposition 3.1 satisfies $\hat{p} \neq \tilde{p}$. But then all of the vectors but one as constructed in the proof of Proposition 3.1 satisfy $z_{\tilde{p}}^{\tilde{q}} = 0$. Hence, we have identified $|E| + \sum_{q \in Q}(|P^q| - 1)$ affinely independent vectors in the convex hull of feasible solutions that satisfy the inequality at equality, which completes the proof. ■

**Theorem 3.1** *Any non-trivial facet defining inequality for $conv(X_{ij}^{UP})$ is a non-trivial facet defining inequality for $conv(UNPM)$.*

**Proof.** Let $a_{ij}x_{ij} \geq \sum_{q\in Q}\sum_{p\in P_{ij}^q \cup P_{ji}^q} b_p^q z_p^q - c$ be a non-trivial facet defining inequality for $conv(X_{ij}^{UP})$. If $k$ denotes the dimension of $conv(X_{ij}^{UP})$ then $k = 1 + \sum_{q\in Q}(|P_{ij}^q| + |P_{ji}^q|)$, since the polytope $conv(X_{ij}^{UP})$ is full dimensional. For each $q \in Q$, let $\bar{p}^q \notin P_{ij}^q \cup P_{ji}$ be a path that does not visit arc $(i,j)$ nor $(j,i)$. Now consider the polytope

$$T \;=\; conv\left(\left\{(x,z) \in UNPM : z_p^q = 0, \forall q \in Q, \forall p \notin (P_{ij}^q \cup P_{ji}^q \cup \{\bar{p}^q\})\right\}\right).$$

This polytope is the convex hull of the set of solutions for the restricted network loading problem where a commodity $q \in Q$ can only be routed on path $\bar{p}^q$ or on a path that visits edge $\{i,j\}$. Using Proposition 3.1 the dimension of $T$ thus is $|E| + \sum_{q\in Q}(|P_{ij}^q| + |P_{ji}^q|) = k + |E| - 1$. First we show that the inequality $a_{ij}x_{ij} \geq \sum_{q\in Q}\sum_{p\in P_{ij}^q \cup P_{ji}^q} b_p^q z_p^q - c$ is also a facet defining inequality for $T$ by constructing $k+|E|-1$ affinely independent solution vectors in $T$ that satisfy the inequality at equality. Note that there exist $k$ affinely independent vectors $(\bar{x},\bar{z}) \in X_{ij}^{UP}$ that satisfy the inequality at equality. Given such a vector $(\bar{x},\bar{z})$ we define a vector $(\tilde{x},\tilde{z}) \in T$ as follows. For all $q \in Q$, let $\tilde{z}_p^q = \bar{z}_p^q$ for all $p \in P_{ij}^q \cup P_{ji}^q$, $\tilde{z}_{\bar{p}}^q = 1$ if $\sum_{p\in P_{ij}^q \cup P_{ji}^q} \bar{z}_p^q = 0$, $\tilde{z}_{\bar{p}}^q = 0$ otherwise, and $\tilde{z}_p^q = 0$ for all $p \notin (P_{ij}^q \cup P_{ji}^q \cup \{\bar{p}^q\})$. Moreover, define $\tilde{x}_{ij} = \bar{x}_{ij}$ and $\tilde{x}_{uv} = \lceil \sum_{q\in Q} d^q \rceil$, for all $\{u,v\} \neq \{i,j\}$. Then these $k$ vectors $(\tilde{x},\tilde{z}) \in T$ are also affinely independent. Moreover, for any of these given vectors, we can install one additional unit of capacity on any of the edges $\{u,v\} \neq \{i,j\}$, which leads to $|E| - 1$ additional vectors. All of these $k + |E| - 1$ vectors are affinely independent and satisfy the inequality at equality, hence, the inequality is also facet defining for the polytope $T$.

Next, we prove that maximal sequential lifting applied to a variable that is fixed to zero in the polytope $T$ yields a lifting coefficient zero, which implies that the inequality is also facet defining for the $conv(UNPM)$. Thus, let $\tilde{q} \in Q$ and let $\tilde{p} \notin (P_{ij}^{\tilde{q}} \cup P_{ji}^{\tilde{q}} \cup \{\bar{p}^{\tilde{q}}\})$. If we apply maximal lifting on the variable $z_{\tilde{p}}^{\tilde{q}}$ to obtain a valid inequality $a_{ij}x_{ij} \geq \sum_{q\in Q}\sum_{p\in P_{ij}^q \cup P_{ji}^q} b_p^q z_p^q + b_{\tilde{p}}^{\tilde{q}} z_{\tilde{p}}^{\tilde{q}} - c$, then the lifting coefficient $b_{\tilde{p}}^{\tilde{q}}$ is determined by

$$b_{\tilde{p}}^{\tilde{q}} \;=\; \min_{\substack{(x,z)\in UNPM : z_{\tilde{p}}^{\tilde{q}} = 1,\\ z_p^q = 0 \,\forall q\in Q \,\forall p \notin (P_{ij}^q \cup P_{ji}^q \cup \{\bar{p}^q, \bar{p}^{\tilde{q}}\})}} \{a_{ij}x_{ij} - \textstyle\sum_{q\in Q}\sum_{p\in P_{ij}^q \cup P_{ji}^q} b_p^q z_p^q + c\}.$$

Since the facet defining inequality under consideration is non-trivial, there exists a solution $(\tilde{x},\tilde{z}) \in T$ with $\tilde{z}_{\bar{p}}^{\tilde{q}} = 1$ that satisfies the inequality at equality. Now consider the solution vector that is obtained by replacing path $\bar{p}$ by $\tilde{p}$ for commodity $q$. This yields a solution vector that is feasible for the minimization lifting problem and has objective value zero since the coefficient of the variable $z_{\bar{p}}^{\tilde{q}}$ is zero in the facet defining inequality. Since the lifting coefficient is nonnegative it then follows that it must be zero. Repeating this argument for all remaining variables that are currently fixed to zero yields the desired result. ■

**Theorem 3.2** *Any non-trivial facet defining inequality for $conv(Y_{ij}^{DP})$ is a non-trivial facet defining inequality for $conv(DNPM)$.*

**Proof.** Similar to the proof of Theorem 3.1. ■

**Proposition 3.3** *The following statements are equivalent :*

*(i). An inequality* $ax \geq b^T z - c$ *which is a non-trivial facet defining inequality for* $conv(X_{ij}^{DP})$ *is a non-trivial facet defining inequality for* $conv(DNPM)$.

*(ii).* $\forall \hat{q} \in Q_{ji} : \exists \bar{Q} \subseteq Q_{ij} \setminus \{\hat{q}\}, \sum_{q \in \bar{Q}} b^q - c = a \max\{\lceil d^{\hat{q}} \rceil, \left\lceil \sum_{q \in \bar{Q}} d^q \right\rceil\}$

*(iii).* $\forall \hat{q} \in Q_{ji} : \exists \bar{Q} \subseteq Q_{ij} \setminus \{\hat{q}\}, \left\lceil \sum_{q \in \bar{Q}} d^q \right\rceil \geq \lceil d^{\hat{q}} \rceil$ *and* $\sum_{q \in \bar{Q}} b^q - c = a \left\lceil \sum_{q \in \bar{Q}} d^q \right\rceil$

*(iv).* $\forall \hat{q} \in Q_{ji}$*: the maximization problem*

$$\begin{array}{ll} \max & \theta \\ s.t. & a\theta = \sum_{q \in Q_{ij} \setminus \{\hat{q}\}} b^q w^q - c \\ & \theta \geq \sum_{q \in Q_{ij} \setminus \{\hat{q}\}} d^q w^q \\ & \theta \in \mathbb{Z}_0^+, w^q \in \{0,1\}, \forall q \in Q_{ij} \setminus \{\hat{q}\} \end{array}$$

*has an optimal objective value* $\theta^* \geq \lceil d^{\hat{q}} \rceil$.

**Proof.** It is fairly easy to see that $(ii) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii)$. Hence, we will restrict ourselves to prove that $(i) \iff (ii)$. Suppose (ii) holds. Similar as in the proof of Theorem 3.2 we can lift variables $z_{\hat{p}}^{\hat{q}}$ for $\hat{q} \in Q_{ji}$ and $\hat{p} \in P_{ji}^{\hat{q}}$. The maximal lifting coefficient $b_{\hat{p}}^{\hat{q}}$ for such a variable equals

$$b_{\hat{p}}^{\hat{q}} = \min_{z_{\hat{p}}^{\hat{q}}=1} \left\{ ax_{ij} - \left(\sum_{q \in Q_{ij} \setminus \{\hat{q}\}} \sum_{p \in P_{ij}^q} b_p^q z_p^q - c\right) \right\}$$

It is easy to see that $b_{\hat{p}}^{\hat{q}} \geq 0$ since otherwise the starting inequality was not valid. The conditions of (ii) now give that the minimum is indeed zero. This argument can be repeated for all variables $z_{\hat{p}}^{\hat{q}}$ for $\hat{q} \in Q_{ji}$ and $\hat{p} \in P_{ji}^{\hat{q}}$. As a consequence, the inequality is facet defining for $conv(Y_{ij}^{DP})$. Now, Theorem 3.2 gives the desired result.

The reversed claim is easy to see. If no subset $\bar{Q} \subseteq Q_{ij} \setminus \{\hat{q}\}$ satisfies the conditions as posed in (ii), then the lifting coefficient as determined by the minimization problem described in the above, will not be equal to zero. Hence, the inequality can be strengthened, and does not define a facet of conv$(DNPM)$. ■

## 3.2 Flow Formulation

**Proposition 3.4** *The dimension of both* $conv(UNFM)$ *and* $conv(DNFM)$ *are equal to* $|E| + |Q|(|A| - |V| + 1)$.

**Proof.** The number of edge capacity variables equals $|E|$ and the number of flow variables equals $|Q| \cdot |A|$. Furthermore, since for each commodity there are $|V|$ flow balance constraints, of which

$|V|-1$ are linearly independent, an upper bound on the dimension is given by $|E|+|Q|(|A|-|V|+1)$. To prove that this bound is tight we show that there exist no other implicit equalities in the model. Stated differently, if

$$\sum_{\{i,j\}\in E}\alpha_{ij}x_{ij}+\sum_{q\in Q}\sum_{\{i,j\}\in E}(\beta_{ij}f_{ij}^{q}+\beta_{ji}f_{ji}^{q})=\delta$$

is satisfied by each solution in $UNFM$, we prove that this equality is a linear combination of the model equalities.

Let $\{u,v\}\in E$, and let $(x,f)\in UNFM$. Next, define a solution $(\bar{x},\bar{f})$ as $\bar{f}=f$, $\bar{x}_{uv}=x_{uv}+1$ and $\bar{x}_{ij}=x_{ij}$ for all $\{i,j\}\neq\{u,v\}$. Because both solutions satisfy the equality, it holds that $\alpha_{uv}=0$, and since the edge was chosen arbitrarily it follows that $\alpha_{ij}=0$ for all $\{i,j\}\in E$.

Next we show that for all $q\in Q$ and for all cycles $C$ in the graph it holds that $\sum_{(i,j)\in C}\beta_{ij}^{q}=0$. Since any cycle in the graph can be decomposed into a collection of simple cycles (i.e. cycles that visit each node at most once) it follows that we only have to prove this claim for simple cycles.

Let $\hat{q}\in Q$ and $C$ a simple cycle in the graph. First we consider the case that $C$ is a 2-cycle (a cycle of two arcs, say $(u,v)$ and $(v,u)$ for some $u,v\in V$). Since there exist two node disjoint paths from $s^{\hat{q}}$ to $t^{\hat{q}}$ in the graph, there exists a path from $s^{\hat{q}}$ to $t^{\hat{q}}$ that does not contain edge $\{u,v\}$. Let $(x,f)\in UNFM$ be a solution that uses this specific path for the routing of commodity $\hat{q}$. Given this solution, let $(\bar{x},\bar{f})\in UNFM$ be a solution that employs exactly the same routing strategy for all commodities $q\in Q$, except that commodity $\hat{q}$ is additionally routed on arcs $(u,v)$ and $(v,u)$. Since both solutions satisfy the equality and $\alpha_{ij}=0$ for all $\{i,j\}\in E$ it follows that $\beta_{uv}^{\hat{q}}+\beta_{vu}^{\hat{q}}=0$.

Now we consider the case that $C$ is not a 2-cycle. Let $p$ be a simple path from $s^{\hat{q}}$ to $t^{\hat{q}}$ in the graph. If the number of nodes on the path $p$ that are also on the cycle $C$ is less than or equal to one, then we use similar arguments as before to show that $\sum_{(i,j)\in C}\beta_{ij}^{\hat{q}}=0$. Let solution $(x,f)\in UNFM$ use path $p$ for the routing of commodity $\hat{q}$. Next, define solution $(\bar{x},\bar{f})$ to be a solution that employs exactly the same routing for all commodities $q\in Q$, except that commodity $\hat{q}$ is also routed on cycle $C$. Comparing the two solutions, and using the fact that $\alpha_{ij}=0$ for all $\{i,j\}\in E$, it follows that $\sum_{(i,j)\in C}\beta_{ij}^{\hat{q}}=0$.

If the number of nodes on path $p$ that are also on the cycle $C$ is greater than or equal to 2, then define $v_1$ as the first, and $v_2$ to be the last node on the path that is also on the cycle. As a result, path $p$ can be decomposed into three parts $p_1,p_2,p_3$, where $p_1$ is a path from $s^{\hat{q}}$ to $v_1$, $p_2$ is a path from $v_1$ to $v_2$, and $p_3$ is a path from $v_2$ to $t^{\hat{q}}$. Similarly, the cycle $C$ can be decomposed into a path $C_1$ from $v_1$ to $v_2$ and a path $C_2$ from $v_2$ to $v_1$. Given these definitions, we can construct two new paths from $s^{\hat{q}}$ to $t^{\hat{q}}$ in the graph. The first path can be represented as $p_1,C_1,p_3$ and the second path as $p_1,C_2^r,p_3$, where $C_2^r$ is the reversed path of $C_2$. Let $(x,f)\in UNFM$ be a solution that uses the first path for the routing of commodity $\hat{q}$. Given this solution, define a solution $(\bar{x},\bar{f})\in UNFM$ that employs the same routing strategy for all commodities $q\in Q\backslash\{\hat{q}\}$, but uses the second path for commodity $\hat{q}$. Since both solutions satisfy the equality it follows that $\sum_{(i,j)\in C_1}\beta_{ij}^{\hat{q}}-\sum_{(i,j)\in C_2^r}\beta_{ij}^{\hat{q}}=0$. Exploiting the fact that $\beta_{ij}^{q}=-\beta_{ji}^{q}$ for all $q\in Q$ and for all $\{i,j\}\in E$, it follows that $\sum_{(i,j)\in C}\beta_{ij}^{\hat{q}}=\sum_{(i,j)\in C_1}\beta_{ij}^{\hat{q}}+\sum_{(i,j)\in C_2}\beta_{ij}^{\hat{q}}=0$, which proves our intermediate claim.

Next, for all $q \in Q$, for all $i \in V$ and a path $p$ from $s^q$ to $i$ in the graph, let $\mu_i^q = \sum_{(i,j)\in p} \beta_{ij}^q$. We claim that the value of $\mu_i^q$ is independent of the selected path $p$. To verify this claim, let $p_1, p_2$ be two paths from $s^{\hat{q}}$ to $i$ in the graph, and let $p_1^r, p_2^r$ be the reversed paths. Then $p_1 \cup p_2^r$ forms a cycle, hence, $\sum_{(i,j)\in p_1\cup p_2^r} \beta_{ij}^q = 0$. Using $\beta_{ij}^q = -\beta_{ji}^q$ it then follows that $\sum_{q\in p_1} \beta_{ij}^q = \sum_{q\in p_2} \beta_{ij}^q$, thus indeed, the value of $\mu_i^q$ is independent of the selected path from $s^q$ to $i$.

If we multiply the flow conservation equalities of the model $UNFM$ by these multipliers and add them all up, we obtain the following expression:

$$\begin{aligned} \sum_{q\in Q} \sum_{i\in V} \mu_i^q (\sum_j f_{ji}^q - \sum_j f_{ij}^q) &= \sum_{q\in Q} \sum_{\{i,j\}\in E} \{(\mu_i^q - \mu_j^q) f_{ji}^q + (\mu_j^q - \mu_i^q) f_{ij}^q\} \\ &= \sum_{q\in Q} \sum_{\{i,j\}\in E} (\beta_{ij}^q f_{ij}^q + \beta_{ji}^q f_{ji}^q) \end{aligned}$$

This implies that the equality is indeed a linear combination of the model equalities. ■

**Proposition 3.5** *Let $q \in Q$ and $(i,j) \in A$. The trivial inequalities $f_{ij}^q \geq 0$ are valid and facet defining for $conv(UNFM)$ and $conv(DNFM)$.*

**Proof.** Analogous to the proof of Proposition 3.2. ■

**Theorem 3.3** *Any non-trivial facet defining inequality for $conv(X_{ij}^{UF})$ is a non-trivial facet defining inequality for $conv(UNFM)$.*

**Proof.** Analogous to the proof of Theorem 3.1. ■

**Theorem 3.4** *Any non-trivial facet defining inequality for $conv(Y_{ij}^{DF})$ is a non-trivial facet defining inequality for $conv(DNFM)$.*

**Proof.** Analogous to the proof of Theorem 3.2. ■

**Proposition 3.6** *The following statements are equivalent :*

*(i). An inequality $ax \geq b^T f - c$ which is a non-trivial facet defining inequality for $conv(X_{ij}^{DF})$ is a non-trivial facet defining inequality for $conv(DNFM)$.*

*(ii).* $\forall \hat{q} \in Q_{ji} : \exists \bar{Q} \subseteq Q_{ij} \setminus \{\hat{q}\}, \sum_{q\in\bar{Q}} b^q - c = a \max\{\lceil d^{\hat{q}} \rceil, \left\lceil \sum_{q\in\bar{Q}} d^q \right\rceil\}$

*(iii).* $\forall \hat{q} \in Q_{ji} : \exists \bar{Q} \subseteq Q_{ij} \setminus \{\hat{q}\}, \left\lceil \sum_{q\in\bar{Q}} d^q \right\rceil \geq \lceil d^{\hat{q}} \rceil$ *and* $\sum_{q\in\bar{Q}} b^q - c = a \left\lceil \sum_{q\in\bar{Q}} d^q \right\rceil$

*(iv).* $\forall \hat{q} \in Q_{ji}$*: the maximization problem*

$$\begin{aligned} \max \quad & \theta \\ s.t. \quad & a\theta = \sum_{q\in Q_{ij}\setminus\{\hat{q}\}} b^q w^q - c \\ & \theta \geq \sum_{q\in Q_{ij}\setminus\{\hat{q}\}} d^q w^q \\ & \theta \in \mathbb{Z}_0^+, w^q \in \{0,1\}, \forall q \in Q_{ij} \setminus \{\hat{q}\} \end{aligned}$$

*has an optimal objective value $\theta^* \geq \lceil d^{\hat{q}} \rceil$.*

**Proof.** Analogous to the proof of Proposition 3.3. ■

# 4 Characteristics of the Edge Capacity Polytope

In Section 2 we introduced six different polytopes restricted to a single edge of the original model. The edge models $X_{ij}^{UF}$, $X_{ij}^{UP}$, $X_{ij}^{DF}$, $X_{ij}^{DP}$ are similar. They describe a knapsack with variable integer capacity. Since the associated polyhedra are the same, in this and the following sections we use easier notation and a redefinition of the edge capacity model that captures all of the aforementioned edge models. We show that optimizing a linear function over the polytope is an $\mathcal{NP}$-hard problem in general. Next, we state the dimension of the polytope, discuss its trivial facets and derive a general form of a facet defining inequality. The main result in this section is a shifting theorem, which indicates that the complete set of facet defining inequalities for the edge capacity polytope can be obtained from a related edge capacity polytope in which the demands satisfy $d^q \in (0, 1]$ for all $q \in Q$. Moreover, we provide bounds on the coefficients in a facet defining inequality, and derive necessary and sufficient conditions under which the model inequality yields the complete description of the edge capacity polytope.

Consider a set $Q$ of items (commodities) and let $d^q \in \mathbb{Q}^+$ represent the size (demand) for an item $q \in Q$ (normalized to the base capacity $\lambda$). Let the integer variable $x$ denote the number of capacity units selected and let the binary variables $f^q$ indicate whether or not an individual item $q$ is selected. The edge capacity set is then defined as

$$X = \left\{ (x, f) \in \mathbb{Z}_0^+ \times \{0, 1\}^{|Q|} : x \geq \textstyle\sum_{q \in Q} d^q f^q \right\} \tag{19}$$

Given an arbitrary objective function $(\delta, \gamma) \in \mathbb{Z} \times \mathbb{Z}^{|Q|}$, the problem of minimizing the objective over the set $X$ is $\mathcal{NP}$-hard in general. To formalize this statement, we define the following decision problems.

EDGE CAPACITY PROBLEM
INSTANCE: Set of items $Q$, demand size $d^q \in \mathbb{Z}_0^+$, for all $q \in Q$, objective coefficients $\gamma^q \in \mathbb{Z}$, for all $q \in Q$, objective coefficient $\delta \in \mathbb{Z}$, capacity coefficient $\lambda \in \mathbb{Z}_0^+$, and an integer $K \in \mathbb{Z}$.
QUESTION: Does there exist a vector $(x, f) \in \mathbb{Z}_0^+ \times \{0, 1\}^{|Q|}$ such that $\lambda x \geq \sum_{q \in Q} d^q f^q$ and $\delta x + \sum_{q \in Q} \gamma^q f^q \leq K$ ?

PARTITION (see Garey and Johnson [9])
INSTANCE: Set of items $A$, size $s^q \in \mathbb{Z}_0^+$, for all $q \in A$, with $\sum_{q \in A} s^q$ even.
QUESTION: Does there exist a subset $\bar{A} \subseteq A$ such that $\sum_{q \in \bar{A}} s^q = \sum_{q \in A \setminus \bar{A}} s^q$ ?

**Theorem 4.1** EDGE CAPACITY PROBLEM *is* $\mathcal{NP}$*-complete.*

**Proof.** It is easy to verify that EDGE CAPACITY PROBLEM is in $\mathcal{NP}$. Next, we show that PARTITION reduces to EDGE CAPACITY PROBLEM. Given an instance of PARTITION, define an instance of EDGE CAPACITY PROBLEM as follows. Let $Q = A \cup \{\hat{q}\}$, $d^q = s^q$, for all $q \in A$, and $d^{\hat{q}} = 1$. Next define $\gamma^q = -d^q$, for all $q \in A$, and $\gamma^{\hat{q}} = -2$. Finally, let $\lambda = \frac{1}{2} \sum_{q \in A} s^q + 1$, $\delta = \lambda$ and $K = -1$. This is obviously a polynomial transformation. Next it remains to show that an instance of PARTITION yields an affirmative answer if and only if the corresponding

instance of EDGE CAPACITY PROBLEM yields an affirmative answer. If $\bar{A} \subseteq A$ is a set of items satisfying $\sum_{q\in\bar{A}} s^q = \sum_{q\in A\setminus\bar{A}} s^q$, then the solution $(x, f)$ defined as $x = 1$, $f^q = 1, q \in \bar{A} \cup \{\hat{q}\}$, $f^q = 0$ otherwise, gives a vector satisfying the required conditions. Conversely, if there exists a vector $(x, f)$ which satisfies $\lambda x \geq \sum_{q\in Q} d^q f^q$ and $\delta x + \sum_{q\in Q} \gamma^q f^q \leq -1$, then it is easy to see that $x > 0$, since $x = 0$ would imply that $f^q = 0$ for all $q \in Q$, a contradiction with the fact that $\delta x + \gamma^T f$ has negative value. Moreover, if $x \geq 2$, then $\delta x + \gamma^T f \geq 2\lambda + \gamma^T f \geq 2\lambda - \sum_{q\in Q} \gamma^q = \sum_{q\in A} s^q + 2 - \sum_{q\in A} s^q - 2 = 0$. Hence, $x = 1$. Next, from $\delta x + \gamma^T f \leq -1$ it follows that $\gamma^T f \leq -\lambda - 1$ which implies $|\gamma^T f| \geq \lambda + 1$. From $\lambda x \geq d^T f$ it follows that $\sum_{q\in Q} d^q f^q \leq \lambda$. Since for each element in $A$ it holds that $d^q = -\gamma^q$, the only way that $|\gamma^T f|$ can be greater than $\sum_{q\in Q} d^q f^q$ is if $f^{\hat{q}} = 1$. But then it follows that both $\sum_{q\in A} \gamma^q f^q \leq -\lambda + 1$ and $\sum_{q\in A} d^q f^q \leq \lambda - 1$ which is equivalent to $\sum_{q\in A} d^q f^q = \lambda - 1 = \frac{1}{2}\sum_{q\in A} s^q$. Hence, let $\bar{A} = \{q \in A : f^q = 1\}$, then $\bar{A}$ is the required subset. ■

Let $D^q = \lceil d^q \rceil$ be the smallest integer greater than or equal to $d^q$, and let $r^q = d^q + 1 - D^q \in (0, 1]$ be the 'fractional' part of the demand. Similarly, for a subset $S \subseteq Q$, let $d(S) = \sum_{q\in S} d^q$, $D(S) = \lceil d(S) \rceil$ and $r(S) = d(S) + 1 - D(S)$. Finally, define $e^S \in \{0, 1\}^Q$ as the characteristic vector of a set $S$, i.e. $e^S_i = 1$ if $i \in S$ and zero otherwise.

**Lemma 4.1** *The dimension of conv$(X)$ is $|Q| + 1$.*

**Proof.** The vectors $(x, f) = (0, 0)$, $(x, f) = (1, 0)$ and $(x, f) = (D^q, e^{\{q\}})$ for all $q \in Q$ yield $|Q| + 2$ affinely independent vectors in $X$. ■

**Proposition 4.1** *For all $q \in Q$, the trivial inequalities $f^q \geq 0$ and $f^q \leq 1$ define facets of conv$(X)$.*

**Proof.** For $f^q \geq 0$, $|Q| + 1$ affinely independent solutions $(x, f) \in X$ are given by $(0, 0)$, $(1, 0)$ and $(D^i, e^i)$ for all $i \in Q \setminus \{q\}$. For $f^q \leq 1$, $|Q| + 1$ affinely independent solutions $(x, f) \in X$ are given by $(D^q, e^q)$, $(D^q + 1, e^q)$ and $(D(\{i, q\}), e^{\{i,q\}})$ for all $i \in Q \setminus \{q\}$. ■

**Proposition 4.2** *(Brockmüller et al. [7]) Each non-trivial facet of conv(X) can be written in the form $ax \geq \sum_{q\in Q} b^q f^q - c$, with $a, c \in \mathbb{Z}^+_0, b^q \in \mathbb{Z}^+_0$, for all $q \in Q$.*

**Proof.** A non-trivial facet defining inequality is of the form $ax \geq \sum_{q\in Q} b^q f^q - c$. The fact that $c \geq 0$ follows from the fact that the all zero solution $(x, f) = (0, 0)$ does not satisfy the inequality if $c < 0$. Next, since for any solution $(x, f) \in X$ we can increase the value of the capacity variable to an arbitrarily large integer number without violating its feasibility, it follows that $a < 0$ cannot be the case. Furthermore, if $a = 0$ then the resulting inequality $\sum_{q\in Q} b^q f^q \leq c$ is a linear combination of the trivial inequalities and the inequality $0 \leq 1$. Hence, $a > 0$. Now suppose there exists a $q' \in Q$ with $b^{q'} < 0$. Since the facet defining inequality is non-trivial there exists a solution $(\bar{x}, \bar{f}) \in X$ with $\bar{f}^{q'} = 1$ that satisfies the inequality at equality, $a\bar{x} = \sum_{q\in Q} b^q \bar{f}^q - c$. Define the solution $(\tilde{x}, \tilde{f})$ as $\tilde{x} = \bar{x}$, $\tilde{f}^{q'} = 0$, $\tilde{f}^q = \bar{f}^q$ for all $q \in Q \setminus \{q'\}$. Then $(\tilde{x}, \tilde{f}) \in X$ but $a\tilde{x} = a\bar{x} = \sum_{q\in Q} b^q \bar{f}^q - c = b^{q'}\bar{f}^{q'} + \sum_{q\in Q\setminus\{q'\}} b^q \bar{f}^q - c < b^{q'}\tilde{f}^{q'} + \sum_{q\in Q\setminus\{q'\}} b^q \tilde{f}^q - c = \sum_{q\in Q} b^q \tilde{f}^q - c$ which is in contradiction with the validity of the inequality. Hence, $b^q \geq 0$ for

all $q \in Q$. Finally, the fact that any non-trivial facet defining inequality has rational coefficients (and can therefore be written with integer coefficients) follows from the fact that all extreme points of the set are rational. ■

**Theorem 4.2** *(Shifting Theorem)*
*Let $i \in Q$ and $\mu \in \mathbb{Z}$ such that $d^i + \mu \geq 0$, and define*

$$X(i,\mu) = \left\{(x,f) \in \mathbb{Z}_0^+ \times \{0,1\}^{|Q|} : x \geq (d^i+\mu)f^i + \textstyle\sum_{q\in Q\setminus\{i\}} d^q f^q \right\}. \tag{20}$$

*Then the inequality*

$$x \geq \textstyle\sum_{q\in Q} b^q f^q - c \tag{21}$$

*is a non-trivial facet defining inequality for $conv(X)$ if and only if*

$$x \geq (b^i+\mu)f^i + \textstyle\sum_{q\in Q\setminus\{i\}} b^q f^q - c \tag{22}$$

*is a non-trivial facet defining inequality for $conv(X(i,\mu))$.*

**Proof.** Note that we only need to prove that any facet defining inequality for $conv(X)$ can be converted as indicated to a facet defining inequality for $conv(X(i,\mu))$, since the converse then directly follows for a suitable choice $\mu' = -\mu$. First, we prove validity. Let $(\bar{x},\bar{f}) \in X(i,\mu)$, then $(\bar{x}-\mu\bar{f}^i,\bar{f}) \in X$, hence $\bar{x}-\mu\bar{f}^i \geq \sum_{q\in Q} b^q f^q - c$ which implies that $\bar{x} \geq (b^i+\mu)f^i + \sum_{q\in Q\setminus\{i\}} b^q f^q - c$. Next, let $(x_1,f_1),\ldots,(x_{|Q|+1},f_{|Q|+1})$ be $|Q|+1$ affinely independent solutions of $X$ that satisfy (21) at equality. Then $(x_1+\mu f_1^i,f_1),\ldots,(x_{|Q|+1}+\mu f_{|Q|+1}^i,f_{|Q|+1})$ satisfy (22) at equality, and they are also affinely independent. ■

**Theorem 4.3** *Let $ax \geq \sum_{q\in Q} b^q f^q - c$ be a non-trivial facet defining valid inequality for $conv(X)$ with $a,c \in \mathbb{Z}_0^+$, and $b^q \in \mathbb{Z}_0^+$ for all $q \in Q$. Then $b^q = ad^q$ if $d^q$ is integer, and $b^q \in \{a(D^q-1),\ldots,aD^q\}$ if $d^q$ is not integer, for all $q \in Q$.*

**Proof.** Let $\bar{q} \in Q$ and let $(x_0,f_0) \in X$ be a solution with $f_0^{\bar{q}} = 0$ that satisfies the facet defining inequality at equality. Moreover, let $Q_0 = \{q \in Q : f_0^q = 1\}$. Then $aD(Q_0) = \sum_{q\in Q_0} b^q - c$ and since $\bar{q} \notin Q_0$ it follows from validity that $aD(Q_0 \cup \{\bar{q}\}) \geq (\sum_{q\in Q_0} b^q - c) + b^{\bar{q}} = aD(Q_0) + b^{\bar{q}}$. This yields an upper bound on $b^{\bar{q}}$ since we conclude $b^{\bar{q}} \leq aD(Q_0 \cup \{\bar{q}\}) - aD(Q_0)$.

Similarly, let $(x_1,f_1) \in X$ be a solution with $f_1^{\bar{q}} = 1$ that satisfies the facet defining inequality at equality, and define $Q_1 = \{q \in Q : f_1^q = 1\}$. Again, $aD(Q_1) = \sum_{q\in Q_1} b^q - c$ and since $\bar{q} \in Q_1$ it follows from validity that $aD(Q_1 \setminus \{\bar{q}\}) \geq (\sum_{q\in Q_1} b^q - c) - b^{\bar{q}} = aD(Q_1) - b^{\bar{q}}$. This implies a lower bound on $b^{\bar{q}}$, namely $b^{\bar{q}} \geq aD(Q_1) - aD(Q_1 \setminus \{\bar{q}\})$.

If $d^{\bar{q}}$ is integer, then both the lower and upper bound are equal to $ad^{\bar{q}}$ which proves the first part of our claim. If $d^{\bar{q}}$ is not integer, then $b^{\bar{q}} \leq aD(Q_0 \cup \{\bar{q}\}) - aD(Q_0) \leq aD^{\bar{q}}$ and $b^{\bar{q}} \geq aD(Q_1) - aD(Q_1 \setminus \{\bar{q}\}) \geq a(D^{\bar{q}} - 1)$. ■

**Theorem 4.4** *The model inequality is the unique facet defining inequality for the polyhedron $conv(X)$ if and only if $d^q \in \mathbb{Z}_0^+$, for all $q \in Q$.*

**Proof.** If $d^{\hat{q}} \notin \mathbb{Z}_0^+$ for some $\hat{q} \in Q$, then the fractional solution $(x, f) = (d^q, e^{\{\hat{q}\}})$ is an extreme point of the LP-relaxation that satisfies all the model inequalities. If $d^q \in \mathbb{Z}_0^+$ for all $q \in Q$, then by Theorem 4.3 we have that in every inequality $x \geq \sum_{q \in Q} b^q f^q - c$ that defines a non-trivial facet of $conv(X)$, $b^q = d^q$ for all $q \in Q$. If $c > 0$, then the resulting inequality is dominated by the model inequality. Hence, the model inequality $x \geq \sum_{q \in Q} d^q f^q$ defines the only non-trivial facet in this case. ■

# 5 Lower Convex Envelope Inequalities

Using the results of Section 4, the analysis of the edge capacity polytope can be greatly facilitated. By Theorem 4.3, commodities with an integer demand can be dealt with easily given a valid inequality on the remaining commodities. Furthermore, for the remaining commodities one can concentrate on the fractional part of the demand of a commodity, i.e. first assume all demands $d^q \in (0, 1)$, generate facets, and use Theorem 4.2 to obtain facets for the problem with the actual demand sizes. Stated differently, one could view the demand value as the sum of two parts, namely the integer part $D^q - 1$ and the 'fractional part' $r^q \in (0, 1]$. Likewise, the value of a coefficient of a commodity in a facet defining inequality can be viewed as the sum of two parts. Theorem 4.2 explains the part of the coefficient that originates from the integer part $D^q - 1$ of a commodity's demand. The other part of the coefficient in a facet defining inequality which needs to be explained stems from the "fractional" part of a commodity's demand, that is $r^q$. Much of the analysis in this section therefore considers demand values $d^q = r^q \in (0, 1]$ for all $q \in Q$. Still, all propositions and theorems are stated such that the results are also valid for arbitrary demand values $d^q$.

Since the values $r^q$ are somewhat comparable in size for the different $q \in Q$ ($r^q \in (0, 1]$ for all $q$), one could expect this second part of a commodity's coefficient in a facet defining inequality to be somewhat comparable in size as well. In this section we therefore introduce a class of valid inequalities called lower convex envelope inequalities, which are based on this idea. This class of valid inequalities is defined on a projection of the set $X$. We show two different types of facet defining inequalities that may arise in the class of lower convex envelope inequalities. Moreover, we show that lifting lower convex envelope inequalities to obtain valid inequalities for $X$ itself can be performed in polynomial time. We start with the definition of a projection of the edge capacity polytope.

**Definition 5.1** *Let $Q^0, Q^1 \subset Q$ be disjoint subsets of $Q$. Then $X(Q^0, Q^1)$ defined by*

$$X(Q^0, Q^1) = \{(x, f) \in X : f^q = 0 \ \forall q \in Q^0, f^q = 1 \ \forall q \in Q^1\}$$

*is the projection of $X$ on the space with $f^q = 0$ for all $q \in Q^0$ and $f^q = 1$ for all $q \in Q^1$.*

This projected edge set can be seen as a set of vectors in $(|Q| + 1)$-dimensional space. Instead of representing the set $X(Q^0, Q^1)$ in $(|Q| + 1)$-dimensional space, one can also plot all vectors in $X(Q^0, Q^1)$ in two-dimensional space, as in the example Figure 2. Let $S = Q \setminus (Q^0 \cup Q^1)$. The horizontal axis of this figure measures $\sum_{q \in S} f^q$ and the vertical axis measures the value of the capacity variable $x$. Hence, a solution $(x, f) \in X(Q^0, Q^1)$ is represented by a point with

coordinates $(\sum_{q\in S} f^q, x)$ in the two-dimensional figure. Similarly as in $(|Q|+1)$-dimensional space, an inequality $ax \geq b\sum_{q\in S} f^q - c$ that is satisfied by all solutions in this two-dimensional space yields a valid inequality for the set $X(Q^0, Q^1)$. Note that these inequalities have the same coefficient for all $q \in S$ in the inequality. We consider the strongest possible valid inequalities arising from the two-dimensional space. These inequalities describe the lower convex envelope of the set of solutions in $X(Q^0, Q^1)$, as indicated in Figure 2.

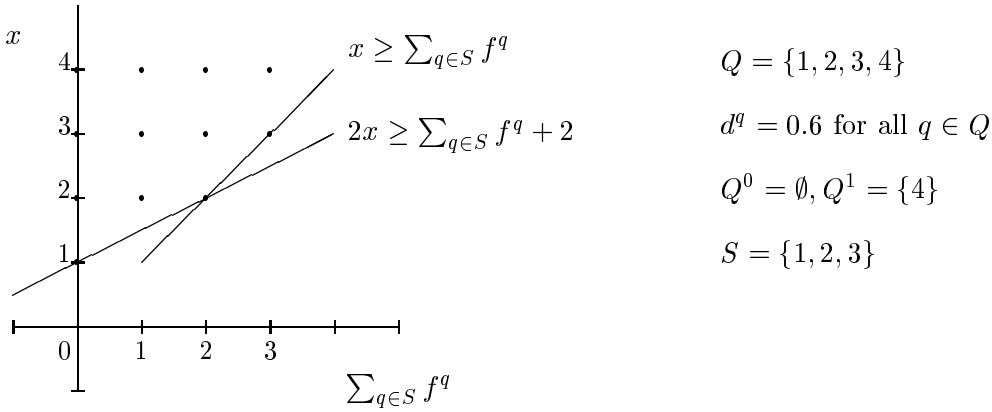


Figure 2: Lower Convex Envelope Inequalities

Assume that the commodities in $S$ are ordered such that $d^1 \leq d^2 \leq \ldots \leq d^{|S|}$, and for $k = 1,\ldots,|S|$, let $S_k = \{1,\ldots,k\}$. Then the set of lower convex envelop inequalities basically describes the lower convex envelope of the points $(k, D(Q^1 \cup S_k))$ for $k = 1,\ldots,|S|$ in two-dimensional space. The following proposition states bounds on the slope of a lower convex envelope inequality.

**Proposition 5.1** *If $ax \geq b\sum_{q\in S} f^q - c$ is a lower convex envelop inequality for the set $X(Q^0, Q^1)$ with demand values $d^q = r^q \in (0,1]$ for all $q \in Q$, then the slope $b/a$ of the lower convex envelope inequality satisfies $0 \leq b/a \leq 1$.*

**Proof.** Consider a lower convex envelope inequality $ax \geq b\sum_{q\in S} f^q - c$ for $X(Q^0, Q^1)$. The slope $b/a$ of the lower convex envelope inequality is determined by two distinct points in the two-dimensional picture that satisfy the inequality at equality. Let $(k_1, D(Q^1 \cup S_{k_1}))$ and $(k_2, D(Q^1 \cup S_{k_2}))$ be two such points with $k_1 < k_2$. Then $b/a = (D(Q^1 \cup S_{k_2}) - D(Q^1 \cup S_{k_1}))/(k_2 - k_1)$. Since $S_{k_1} \subseteq S_{k_2}$ it follows that the numerator is nonnegative, which together with $k_1 \leq k_2$ implies that $b/a \geq 0$. Furthermore, $S_{k_1} \subseteq S_{k_2}$ together with $d^q \in (0,1]$ for all $q \in S$ implies that $D(Q^1 \cup S_{k_2}) - D(Q^1 \cup S_{k_1}) \leq k_2 - k_1$. Hence, $b/a \leq 1$. ■

Next we derive necessary and sufficient conditions under which there exists a lower convex envelope inequality $ax \geq b\sum_{q\in S} f^q - c$ with slope $b/a = 1$ that is also facet defining for $conv(X(Q^0, Q^1))$. We use the following definition.

**Definition 5.2** *Let $Q^1 \subset Q$ and $S \subset Q$ with $S \cap Q^1 = \emptyset$. Then*

$$n(Q^1, S) = \sum_{q \in S} D^q + D(Q^1) - D(S \cup Q^1) \tag{23}$$

In words, $n(Q^1, S)$ measures the surplus in capacity for the set of items $Q^1 \cup S$ if $D(Q^1)$ units of capacity have already been 'installed' for the set of items $Q^1$ and an additional $D^q$ units of capacity are installed for each item $q \in S$. Note that the value of $n(Q^1, S)$ does not change if demand data $r^q \in (0, 1]$ are considered instead of the real demand data $d^q$, for all $q \in Q$.

**Theorem 5.1** *If there exists a lower convex envelope inequality $ax \geq b \sum_{q \in S} f^q - c$ with slope $b/a = 1$ for the edge capacity set with demand values $d^q = r^q \in (0, 1]$, then the inequality*

$$x \geq \sum_{q \in S} D^q f^q + D(Q^1) - n(Q^1, S) \tag{24}$$

*is valid for $conv(X(Q^0, Q^1))$ with arbitrary demand data $d^q$ for all $q \in Q$. Moreover, this is a facet defining lower convex envelope inequality for $conv(X(Q^0, Q^1))$ if and only if $D(Q^1 \cup S \setminus \{q\}) = D(Q^1 \cup S) - D^q$ for all $q \in S$.*

**Proof.** First consider $d^q = r^q \in (0, 1]$ for all $q \in S$. If there exists a lower convex envelope inequality $ax \geq \sum_{q \in S} f^q - c$ with slope $b/a = 1$, then the point $(|S|, D(Q^1 \cup S))$ satisfies that inequality at equality. Hence, for $a = b = 1$ it follows that $c = \sum_{q \in S} f^q - D(Q^1 \cup S) = \sum_{q \in S} D^q + D(Q^1) - D(Q^1) - D(Q^1 \cup S) = n(Q^1, S) - D(Q^1)$. This yields the lower convex envelope inequality $x \geq \sum_{q \in S} f^q + D(Q^1) - n(Q^1, S)$ for demand values $d^q \in (0, 1]$, for all $q \in S$. Applying Shifting Theorem 4.2 for elements $q \in S$ to obtain a valid inequality for the real demand data yields the desired inequality (24).

Now we prove that the remaining conditions are sufficient to guarantee that the inequality is facet defining for $conv(X(Q^0, Q^1))$, which has dimension $|S| + 1$. This follows from the fact that the vectors $(D(Q^1 \cup S), e^{Q^1 \cup S})$ and $(D(Q^1 \cup S \setminus \{q\}), e^{Q^1 \cup S \setminus \{q\}})$ for all $q \in S$ yield $|S| + 1$ affinely independent vectors in $X(Q^0, Q^1)$ that satisfy that inequality at equality.

Conversely, let $\hat{q} \in S$. If $D(Q^1 \cup S \setminus \{\hat{q}\}) \neq D(Q^1 \cup S) - D^{\hat{q}}$ then the solution $(x, f) = (D(Q^1 \cup S \setminus \{\hat{q}\}), e^{Q^1 \cup S \setminus \{\hat{q}\}})$ is not on the face of the valid inequality $x \geq \sum_{q \in S} D^q f^q + D(Q^1) - n(Q^1, S)$. For any set $T \subseteq S \setminus \{\hat{q}\}$ it holds that $D(Q^1 \cup S \setminus \{\hat{q}\}) \leq D(Q^1 \cup T) + \sum_{q \in S \setminus (T \cup \{\hat{q}\})} D^q$. Hence, the vector $(D(Q^1 \cup T), e^{Q^1 \cup T})$ does not satisfy the inequality at equality, since $D(Q^1 \cup T) \geq D(Q^1 \cup S \setminus \{\hat{q}\}) - \sum_{q \in S \setminus (T \cup \{\hat{q}\})} D^q > \sum_{q \in S \setminus \{\hat{q}\}} D^q - c - \sum_{q \in S \setminus (T \cup \{\hat{q}\})} D^q = \sum_{q \in T} D^q - c$. Thus there exists no solution $(x, f) \in X(Q^0, Q^1)$ with $f^{\hat{q}} = 0$ that satisfies the inequality at equality. As a result, the face of the inequality is a subset of the face defined by the inequality $f^{\hat{q}} \leq 1$, which implies that the inequality is not facet defining. ■

Under the conditions of the above theorem, there exists a facet defining lower convex envelope inequality with slope equal to one for $conv(X(Q^0, Q^1))$. In this case one can also identify conditions such that there exists a second facet defining lower convex envelope inequality for this polytope. This is the subject of the following theorem.

**Theorem 5.2** *Consider $X(Q^0, Q^1)$ for demand data $d^q = r^q \in (0, 1]$. Assume that $D(Q^1 \cup S \setminus \{q\}) = D(Q^1 \cup S) - D^q$ for all $q \in S$ such that there exists a lower convex envelope inequality with slope equal to one. If $D(Q^1 \cup S_{|S|-1}) = D(Q^1 \cup S_{|S|-2})$, then*

*(i). there exists a second lower convex envelope inequality* $\alpha x \geq \sum_{q \in S} f^q + \delta$ *through point* $(|S|-1, D(Q^1 \cup S_{|S|-1}))$ *in the two-dimensional figure for demand data* $d^q = r^q \in (0,1]$.

*(ii). the slope* $1/\alpha$ *of this lower convex envelope inequality equals*

$$1/\alpha \quad = \quad \max_{k=0,\ldots,|S|-2} \left\{ \frac{D(Q^1 \cup S) - 1 - D(Q^1 \cup S_k)}{|S| - 1 - k} \right\}$$

*for demand data* $d^q = r^q \in (0,1]$ *for all* $q \in S$.

*(iii). the constant* $\delta$ *of this lower convex envelope inequality equals* $\delta = \alpha D(Q^1 \cup S) - (|S| + \alpha - 1)$

*(iv). the inequality*

$$\alpha x \geq \textstyle\sum_{q \in S} \{\alpha(D^q - 1) + 1\} f^q + \alpha D(Q^1 \cup S) - (|S| + \alpha - 1) \tag{25}$$

*is a facet defining lower convex envelope inequality for* $conv(X(Q^0, Q^1))$ *with arbitrary demand data* $d^q$ *for all* $q \in Q$.

**Proof.** The lower convex envelope inequality as discussed in Theorem 5.1 is the most right lower convex envelope inequality of the lower convex envelop, with slope equal to one. This first inequality is guaranteed to exists due to the conditions of the theorem and the point $(|S|-1, D(Q^1 \cup S_{|S|-1})) = (|S|-1, D(Q^1 \cup S) - 1)$ is on the corresponding line. Because of the second condition $D(Q^1 \cup S_{|S|-1}) = D(Q^1 \cup S_{|S|-2})$, the point $(|S|-2, D(Q^1 \cup S_{|S|-2}))$ is not on this line. Hence, there exists a second lower convex envelope inequality $\alpha x \geq \sum_{q \in S} f^q + \delta$ through the point $(|S|-1, D(Q^1 \cup S_{|S|-1}))$. Apart from this point, there exists a second point $(k, D(Q^1 \cup S_k))$ on this second lower convex envelope inequality, for some $k \in \{0, \ldots, |S|-2\}$. Since the lower convex envelop inequality must be valid, it follows that its slope is equal to the maximal slope between point $(|S|-1, D(Q^1 \cup S) - 1)$ and $(k, D(Q^1 \cup S_k))$, over all $k \in \{0, \ldots, |S|-2\}$.

Since the point $(|S|-1, D(Q^1 \cup S_{|S|-1}))$ is on the line of this new lower convex envelope inequality and $d^q = r^q \in (0,1]$ for all $q \in S$, it holds that $\delta = \alpha D(Q^1 \cup S_{|S|-1}) - \sum_{q \in S_{|S|-1}} f^q = \alpha(D(Q^1 \cup S) - 1) - |S| + 1 = \alpha D(Q^1 \cup S) - (|S| + \alpha - 1)$. Finally, if we apply Shifting Theorem 4.2 to the items $q \in S$ then the desired inequality is obtained. The fact that this is facet defining for $conv(X(Q^0, Q^1))$ follows from the fact that the solution vectors $(D(Q^1 \cup S \setminus \{q\}), e^{Q^1 \cup S \setminus \{q\}})$ for all $q \in S$ all satisfy the inequality at equality. Moreover, the vector $(x, f) = (D(Q^1 \cup S_k), e^{Q^1 \cup S_k})$ for the value of $k \in \{0, \ldots, |S|-2\}$, such that the point $(k, D(Q^1 \cup S_k))$ is also on the line, also satisfies the inequality at equality. This yields $|S|+1$ affinely independent vectors in $X(Q^0, Q^1)$ on the face of the inequality, hence it is facet defining. ■

**Theorem 5.3** *Any lower convex envelope inequality for* $conv(X(Q^0, Q^1))$ *with demand data* $d^q = r^q \in (0,1]$ *for all* $q \in S$ *can be written in the form* $ax \geq b \sum_{q \in S} f^q - c$ *for certain* $a, b, c \in \mathbb{Z}_0^+$, *and with* $b \leq a \leq |S|$). *Maximal lifting of this inequality to obtain a valid inequality for* $X$ *can be done in polynomial time.*

**Proof.** Each lower convex envelope inequality is defined by two points, say $(k_1, D(Q^1 \cup S_{k_1}))$ and $(k_2, D(Q^1 \cup S_{k_2}))$ in the two-dimensional figure, for some $k_1, k_2 \in \{0, \ldots, |S|\}$, with $k_1 < k_2$.

The slope $b/a$ of such a line is then the quotient of $D(Q^1 \cup S_{k_2}) - D(Q^1 \cup S_{k_1})$ and $k_2 - k_1$, and both the numerator and denominator are bounded by $|S|$. Hence, the inequality can be written in a form with $a, b, c \in \mathbb{Z}_0^+$ and $a \leq |S|$. The fact that $b \leq a$ is already shown in Proposition 5.1.

Using inductive arguments, one can show that the lifting coefficients $b^q$ obtained by maximal sequential lifting of variables in $q \in Q^0 \cup Q^1$ also satisfy $b^q \leq a \leq |S|$. Now consider the lifting process for a variable $f^{\hat{q}}, \hat{q} \in Q^0$, then the value of the lifting coefficient $b^{\hat{q}}$ is given by

$$b^{\hat{q}} = \min_{(x,f) \in X(Q^0 \setminus \{\hat{q}\}, Q^1) \,:\, f^{\hat{q}} = 1} \{ax - b \sum_{q \in S} f^q + c\}$$

Note that the optimal value of the variable $x$ in this minimization problem is bounded by $|Q|$. Moreover, for a fixed value of $x$, the problem simplifies to a knapsack problem. Therefore, the problem can be solved by computing at most $|Q|$ knapsack problems. Moreover, each of these knapsack problems has an objective function in which each coefficient $b$ is bounded by $|S|$. This implies that an upper bound on the optimal value of each individual knapsack problem is given by $\mathcal{O}(|Q|^2)$, hence, they can be solved in $\mathcal{O}(|Q|^3)$ time. The lifting coefficient can thus be determined in $\mathcal{O}(|Q|^4)$ time, since it involves at most $|Q|$ knapsack problems. A similar argument holds for the lifting of a variable in $Q^1$. Since the total number of variables to be lifted is bounded by $|Q|$, the complete lifting process can be performed in $\mathcal{O}(|Q|^5)$ time. ■

# 6 Integer Lifting of Knapsack Inequalities

In this section we state a different approach to obtain the two types of valid lower convex envelope inequalities for the edge capacity polytope as mentioned in the previous section. We show that they can be obtained from valid inequalities for a related 0–1 knapsack polytope by integer lifting techniques. Two observations can be made regarding this integer lifting procedure. First of all, the computation of these inequalities, i.e. the computation of the lifting coefficients is $\mathcal{NP}$-hard in general. Secondly, any valid inequality for the knapsack polytope will in general lead to zero, one or two valid inequalities for the edge capacity polytope. For the special case of valid cover inequalities for the knapsack polytope, we show that the lifting process can be done in polynomial time. Moreover, at least one valid inequality for the edge capacity polytope will be obtained from each cover inequality. Finally, we identify necessary and sufficient conditions under which two distinct valid inequalities for the edge capacity polytope are obtained from a single cover inequality for the 0–1 knapsack polytope.

**Definition 6.1** *Let $Q^0, Q^1, S$ be a partition of $Q$, and let $\bar{x} \in \mathbb{Z}_0^+$ with $\bar{x} \geq D(Q^1 \cup \{q\})$ for all $q \in S$. Let $b = \bar{x} - d(Q^1)$, then $X(Q^0, Q^1, \bar{x})$ is a knapsack set defined by*

$$\begin{aligned} X(Q^0, Q^1, \bar{x}) &= \{(x, f) \in X : f^q = 0 \ \forall q \in Q^0, f^q = 1 \ \forall q \in Q^1, x = \bar{x}\} \\ &\simeq \{f \in \{0,1\}^{|S|} : \textstyle\sum_{q \in S} d^q f^q \leq b\} \end{aligned}$$

$X(Q^0, Q^1, \bar{x})$ is the projection of $X$ on the space with $f^q = 0$ for all $q \in Q^0$, $f^q = 1$ for all $q \in Q^1$, and $x = \bar{x}$. Note that the condition $\bar{x} \geq D(Q^1 \cup \{q\})$ for all $q \in S$ implies that $dim(conv(X(Q^0, Q^1, \bar{x}))) = |S|$, that is, the knapsack polytope is fully dimensional.

**Theorem 6.1** *(Wolsey [22]) Let $Q^0, Q^1, S$ be a partition of the set $Q$, and let $\bar{x} \in \mathbb{Z}_0^+$ be an integer with $\bar{x} \geq D(Q^1 \cup \{q\})$ for all $q \in S$. If $\sum_{q\in S} \pi^q f^q \leq \pi^0$ is a valid inequality for $X(Q^0, Q^1, \bar{x})$ then $\sum_{q\in S} \pi^q f^q \leq \pi^0 + \alpha(x - \bar{x})$ is a valid inequality for $X(Q^0, Q^1)$ if and only if*

$$\alpha^L := \max_{x\in\mathbb{Z}_0^+: x>\bar{x}} \left\{ \frac{\eta(x) - \pi^0}{x - \bar{x}} \right\} \leq \alpha \leq \min_{x\in\mathbb{Z}_0^+: D(Q^1)\leq x<\bar{x}} \left\{ \frac{\pi^0 - \eta(x)}{\bar{x} - x} \right\} =: \alpha^U$$

*where*

$$\eta(x) = \max \left\{ \textstyle\sum_{q\in S} \pi^q f^q : \sum_{q\in S} d^q f^q \leq x - d(Q^1), f^q \in \{0, 1\}, \forall q \in S \right\}.$$

*Moreover, if $\sum_{q\in S} \pi^q f^q \leq \pi^0$ is a facet defining inequality for $conv(X(Q^0, Q^1, \bar{x}))$ and $\alpha^L \leq \alpha^U$, then $\sum_{q\in S} \pi^q f^q \leq \pi^0 + \alpha^L(x - \bar{x})$ and $\sum_{q\in S} \pi^q f^q \leq \pi^0 + \alpha^U(x - \bar{x})$ are facet defining inequalities for $conv(X(Q^0, Q^1))$.*

To determine $\alpha^L$ and $\alpha^U$ we can solve the knapsack problem $\eta(x^U)$ by dynamic programming where $x^U$ is an appropriately chosen upper bound on the value of the capacity variable $x$, for instance $x^U = D(Q^1 \cup S)$. In general, $\alpha^L \leq \alpha^U$ does not necessarily hold, in which case integer lifting is not possible. Next we describe the main result of this section.

**Theorem 6.2** *Let $Q^0, Q^1, S$ be a partition of the set $Q$ such that $S \neq \emptyset$, and let $\bar{x} \in \mathbb{Z}_0^+$ satisfy $\bar{x} \geq D(Q^1 \cup \{q\})$ for all $q \in S$. If $S$ is a minimal cover for the knapsack polytope $X(Q^0, Q^1, \bar{x})$, then*

*(i). integer lifting of the minimal cover inequality can be done in polynomial time,*

*(ii). if $d^q \in (0, 1]$ for all $q \in S$, then $1 = \alpha^L \leq \alpha^U$,*

*(iii). if $d^q \in (0, 1]$ for all $q \in Q$, then $\alpha^U > 1$ if and only if $D(Q^1 \cup S_{|S|-1}) = D(Q^1 \cup S_{|S|-2})$,*

*(iv). the resulting facet defining inequalities for $conv(X(Q^0, Q^1))$ are:*

$$x \geq \textstyle\sum_{q\in S} D^q f^q + D(Q^1) - n(Q^1, S) \tag{26}$$

$$\alpha^U x \geq \textstyle\sum_{q\in S} \{\alpha^U(D^q - 1) + 1\} f^q + \alpha^U D(Q^1 \cup S) - (|S| + \alpha^U - 1). \tag{27}$$

**Proof.** The knapsack problem that needs to be solved in order to determine the lifting coefficients has the same objective coefficients for all items. Hence, a sorting algorithm can solve the knapsack problem in polynomial time. To prove the remainder of the theorem, assume that $d^q \in (0, 1]$ for all $q \in S$. Since $S$ is a minimal cover, $\sum_{q\in S\setminus\{i\}} d^q \leq \bar{x} - d(Q^1)$ for all $i \in S$. Hence, from $d^q \in (0, 1]$ for all $q \in Q$ it follows that in the special case of a minimal cover inequality $\eta(x) = |S|$ for all $x > \bar{x}$, and hence, the maximum value for $\alpha^L$ is attained for $x = \bar{x} + 1$, which yields $\alpha^L = 1$. For $x < \bar{x}$, it is easy to see that $\eta(x) \leq \eta(x + 1) - 1$. Therefore, for $x < \bar{x}$ it holds that $(\pi^0 - \eta(x))/(\bar{x} - x) \geq (\bar{x} - x)/(\bar{x} - x) = 1$, such that $\alpha^U \geq \alpha^L$.

Next, if $D(Q^1 \cup S_{|S|-1}) > D(Q^1 \cup S_{|S|-2})$ then there exist $i, j \in S$ such that $n(Q^1, S \setminus \{i, j\}) = n(Q^1, S \setminus \{i\})$, which implies $\eta(\bar{x} - 1) = |S| - 2$, and hence $\alpha^U \leq 1$. Together with $\alpha^U \geq 1$

this yields $\alpha^U = 1$. Conversely, if $n(Q^1, S \setminus \{i,j\}) \neq n(Q^1, S \setminus \{i\})$ for all $i,j \in S$, then $\eta(\bar{x}-1) \leq |S| - 3$. Hence, for $x = \bar{x} - 1$ the quotient $(\pi^0 - \eta(x))/(\bar{x} - x)$ is strictly greater than 1. Moreover, again using $\eta(x) \leq \eta(x+1) - 1$ it follows that the quotient can never attain the value 1, for $x < \bar{x}$.

Finally, since $S$ is a minimal cover and $d^q \in (0,1]$ for all $q \in S$, it follows that $\bar{x} = D(Q^1 \cup S) - 1$. Substitution of this value in the inequalities as described in Theorem 6.1 and applying the Shifting Theorem 4.2 yields the required inequalities. ■

**Example 6.1** *Consider an instance of the edge capacity polytope with three items, and let* $d^1 = 0.4, d^2 = 0.4, d^3 = 1.4$. *First we transform the data such that* $d^q \in (0,1]$ *for all* $q \in Q$ *hence we consider the data* $r^1 = r^2 = r^3 = 0.4$. *Let* $Q^0 = Q^1 = \emptyset, S = \{1,2,3\}$ *and define* $\bar{x} = 1$. *Hence, we are analyzing the set* $X(Q^0, Q^1, \bar{x})$ *with the knapsack inequality*

$$0.4f^1 + 0.4f^2 + 0.4f^3 \leq 1$$

*for which* $S$ *is a minimal cover. Applying integer lifting to the cover inequality* $f^1 + f^2 + f^3 \leq 2$ *leads to lifting coefficients* $\alpha^L = 1, \alpha^U = 2$, *and the resulting inequalities read*

$$\begin{aligned} x &\geq f^1 + f^2 + f^3 - 1 \\ 2x &\geq f^1 + f^2 + f^3 \end{aligned}$$

*which are facet defining for the edge capacity polytope on the transformed data* $r^1, r^2, r^3$. *Applying the shifting theorem to obtain inequalities which are facet defining for* $X$ *with the original data leads to the facet defining inequalities*

$$\begin{aligned} x &\geq f^1 + f^2 + 2f^3 - 1 \\ 2x &\geq f^1 + f^2 + 3f^3. \end{aligned}$$

*Note that if we had applied the same technique to the original demand data directly, and for the value* $\bar{x} = 2$, *again the set* $S$ *is a minimal cover for the associated knapsack polytope. However, if integer lifting is applied to the cover inequality* $f^1 + f^2 + f^3 \leq 2$ *in this case, then the lifting coefficients are* $\alpha^L = 1, \alpha^U = 0$, *implying that integer lifting would not be possible.*

To conclude, note that (26) and (26) are equivalent to (24) and (25), respectively. As a consequence, Theorem 5.3 holds and lifting of (26) and (26) can be done in polynomial time as well. This in contrast to the lifting problem of minimal covers for the knapsack polytope, which is still open (cf. [11]).

# 7 $C$-strong Inequalities

This section analyzes lifted cover inequalities for the value $\alpha^L = 1$. We will show that the lifting of fixed 0–1 variables in the sets $Q^0$ and $Q^1$ can be done efficiently and that the resulting inequalities are equivalent to $c$-strong inequalities as described by Brockmüller et al. [7]. Subsection 7.1 states some new properties of $c$-strong inequalities, which indicate the importance of this class of valid inequalities in the polyhedral description of the edge capacity polytope.

**Theorem 7.1** *Let $Q^0, Q^1, S, T$ be a partition of the set $Q$. If*

$$x \geq \sum_{q \in S} D^q f^q + \sum_{q \in T} (D^q - 1) f^q - c$$

*is a facet defining inequality for $conv(X(Q^0, Q^1))$ then*

*(i).* $c = n(Q^1, S) - D(Q^1)$

*(ii). if maximal lifting is applied to $\hat{q} \in Q^0$ to obtain a facet defining inequality*

$$x \geq \textstyle\sum_{q \in S} D^q f^q + \sum_{q \in T} (D^q - 1) f^q + \beta^{\hat{q}} f^{\hat{q}} - c$$

*for $conv(X(Q^0 \setminus \{\hat{q}\}, Q^1))$ then*

$$\beta^{\hat{q}} = \begin{cases} D^{\hat{q}} & \text{if } r^{\hat{q}} > 1 - r(S \cup Q^1) \\ D^{\hat{q}} - 1 & \text{otherwise} \end{cases}$$

*(iii). if maximal lifting is applied to $\hat{q} \in Q^1$ to obtain a facet defining inequality*

$$x \geq \textstyle\sum_{q \in S} D^q f^q + \sum_{q \in T} (D^q - 1) f^q + \beta^{\hat{q}} (f^{\hat{q}} - 1) - c$$

*for $conv(X(Q^0, Q^1 \setminus \{\hat{q}\}))$ then*

$$\beta^{\hat{q}} = \begin{cases} D^{\hat{q}} & \text{if } r^{\hat{q}} > 1 - r(S \cup Q^1 \setminus \{\hat{q}\}) \\ D^{\hat{q}} - 1 & \text{otherwise} \end{cases}$$

*(iv). the lifting can be done in polynomial time and leads the facet defining inequality for $conv(X)$, for a certain partition $\bar{Q}, Q \setminus \bar{Q}$ of the set $Q$:*

$$x \geq \textstyle\sum_{q \in \bar{Q}} D^q f^q + \sum_{q \in Q \setminus \bar{Q}} (D^q - 1) f^q - n(\emptyset, \bar{Q})$$

**Proof.** If the inequality is facet defining then it must be tight for some solutions. Hence,

$$\begin{aligned} c &= \max_{(x,f) \in X(Q^0, Q^1)} \left\{ \textstyle\sum_{q \in S} D^q f^q + \sum_{q \in T} (D^q - 1) f^q - x \right\} \\ &= \textstyle\sum_{q \in S} D^q - D(Q^1 \cup S) = n(Q^1, S) - D(Q^1) \end{aligned}$$

Next, if $\hat{q} \in Q^0$ is lifted, then the lifting problem reads

$$\beta^{\hat{q}} = \min_{(x,f) \in X(Q^0 \setminus \{\hat{q}\}, Q^1 \cup \{\hat{q}\})} \left\{ x - \textstyle\sum_{q \in S} D^q f^q - \sum_{q \in T} (D^q - 1) f^q + c \right\}$$

and the minimum for this problem is attained for $f^q = 1$ if $q \in S$ and $f^q = 0$ if $q \in T$, and $\bar{x} = D(Q^1 \cup S \cup \{\hat{q}\})$. This implies that

$$\begin{aligned} \beta^{\hat{q}} &= D(Q^1 \cup S \cup \{\hat{q}\}) - \textstyle\sum_{q \in S} D^q + n(Q^1, S) - D(Q^1) \\ &= D(Q^1 \cup S \cup \{\hat{q}\}) - D(Q^1 \cup S) \end{aligned}$$

which yields the required result. If $\hat{q} \in Q^1$ is lifted, a similar reasoning holds. The fact that lifting can be done in polynomial time now follows directly. ■

These lower convex envelope inequalities with slope 1 are the so-called $c$-strong inequalities as developed by Brockmüller et al. [7], which we will redefine in the next subsection.

## 7.1 Properties of $C$-strong Inequalities

In this section we list some properties of $c$-strong inequalities which indicate the importance of this class of valid inequalities in order to get a good approximation of $conv(X)$. Before we do so, we repeat some definitions and results from Brockmüller et al. [7].

**Definition 7.1** *(Brockmüller et al. [7]) A set $S \subseteq Q$ is called c-strong if $c = \sum_{q \in S} D^q - D(S)$. The set is maximal c-strong if $S \setminus \{i\}$ is c-strong, for all $i \in S$, and $S \cup \{i\}$ is not c-strong for all $i \in Q \setminus S$. Note that a set $S$ is c-strong if and only if $c \leq \sum_{q \in S}(1 - r^q) < (c+1)$. Moreover, $c = n(\emptyset, S)$.*

**Proposition 7.1** *(Brockmüller et al. [7]) Let $S \subseteq Q$ be a c-strong set. Then*

$$x \geq \sum_{q \in S} D^q f^q + \sum_{q \in Q \setminus S} (D^q - 1) f^q - c \tag{28}$$

*is a facet defining inequality for $conv(X)$ if and only if $S$ is maximal c-strong.*

**Lemma 7.1** *Let $S \subseteq Q$ be a c-strong subset, with $S \cup \{i\}$ $(c+1)$-strong for all $i \in Q \setminus S$. If $S \setminus \{\tilde{q}\}$ is $(c-1)$ strong for $\tilde{q} \in S$, then $(S \setminus \{\tilde{q}\}) \cup \{i\}$ is c-strong, for all $i \in Q \setminus S$.*

**Proof.** Let $i \in Q \setminus S$. The fact that $S \cup \{i\}$ is $(c+1)$-strong implies that $(c+1) \leq \sum_{q \in S \cup \{i\}}(1 - r^q) < (c+2)$. Similarly, since $S \setminus \{\tilde{q}\}$ is $(c-1)$-strong implies that $(c-1) \leq \sum_{q \in S \setminus \{\tilde{q}\}}(1 - r^q) < c$. Suppose that $\sum_{q \in (S \setminus \{\tilde{q}\}) \cup \{i\}}(1 - r^q) < c$, then a contradiction is obtained since $(1 - r^{\tilde{q}}) \in [0, 1)$ and $\sum_{q \in S \cup \{i\}}(1 - r^q) \geq (c+1)$. Suppose that $\sum_{q \in (S \setminus \{\tilde{q}\}) \cup \{i\}}(1 - r^q) \geq (c+1)$, then a contradiction is obtained since $(1 - r^{\tilde{q}}) \in [0, 1)$ and $\sum_{q \in S \setminus \{\tilde{q}\}}(1 - r^q) < c$. Hence, $c \leq \sum_{q \in (S \setminus \{\tilde{q}\}) \cup \{i\}}(1 - r^q) < (c+1)$, which proves our claim. ■

**Proposition 7.2** *Let $x \geq \sum_{q \in Q} b^q f^q - c$, ($c \in \mathbb{Z}_0^+$, $b^q \in \mathbb{Z}_0^+$ for all $q \in Q$) be a facet defining inequality for conv(X). Then this inequality is a c-strong inequality.*

**Proof.** Follows directly from Theorem 7.1 and Theorem 4.3. ■

**Proposition 7.3** *Each vertex of $conv(X)$ is on a face defined by a facet defining c-strong inequality.*

**Proof.** Let $(\bar{x}, \bar{f})$ be a vertex of $conv(X)$, and let $\bar{S} := \{q \in Q : \bar{f}^q = 1\}$. Let $c = \sum_{q \in \bar{S}} D^q - D(\bar{S})$, such that $\bar{S}$ is $c$-strong. Then $(\bar{x}, \bar{f})$ is on the face defined by the inequality $x \geq \sum_{q \in \bar{S}} D^q f^q + \sum_{q \in Q \setminus \bar{S}} (D^q - 1) f^q - c$. If there exists a $\hat{q} \in Q \setminus S$ such that $\bar{S} \cup \{\hat{q}\}$ is $c$-strong, add $\hat{q}$ to the set $\bar{S}$. Repeat this process until no items can be added to the set $\bar{S}$ without violating the fact that the set remains $c$-strong. Let $S_1$ represent this new set of commodities. Note that $(\bar{x}, \bar{f})$ is also on the face defined by the inequality $x \geq \sum_{q \in S_1} D^q f^q + \sum_{q \in Q \setminus S_1} (D^q - 1) f^q - c$. Next, if there exists a $\tilde{q} \in S_1$ such that $S_1 \setminus \{\tilde{q}\}$ is not $c$-strong but $(c-1)$-strong, then remove $\tilde{q}$ from $S_1$. After this removal, $(\bar{x}, \bar{f})$ is on the face defined by the inequality $x \geq$

$\sum_{q\in S_1\setminus\{\tilde{q}\}} D^q f^q + \sum_{q\in(Q\setminus S_1)\cup\{\tilde{q}\}}(D^q-1)f^q - (c-1)$. Again, repeat this process until one obtains a set $S_2$ which is $c_2$-strong and such that $S_2\setminus\{q\}$ is $c_2$-strong, for all $q\in S_2$. Lemma 7.1 implies that $S_2\cup\{q\}$ is more than $c_2$-strong, such that $S_2$ is indeed maximal $c_2$-strong. Hence the inequality $x \geq \sum_{q\in S_2} D^q f^q + \sum_{q\in Q\setminus S_2}(D^q-1)f^q - c_2$ defines a facet of $conv(X)$. ■

Moreover, in several special cases each facet of the edge capacity polytope is either a model inequality or a $c$-strong inequality.

**Proposition 7.4** *If the set $Q$ is 0-strong, then $conv(X)$ is completely described by the trivial inequalities and the 0-strong inequality $x \geq \sum_{q\in Q} D^q f^q$.*

**Proof.** Given an arbitrary objective function $(\delta,\gamma)\in\mathbb{Z}\times\mathbb{Z}^{|Q|}$ which is to be minimized over all solutions in $X$, let $M(\delta,\gamma)$ be the corresponding set of optimal solutions. We will show that for each possible vector $(\delta,\gamma)$, the set $M(\delta,\gamma)$ is a subset of a face described by either one of the trivial inequalities or the 0-strong inequality. We distinguish a number of cases.

- $\delta<0$, then the primal solution is unbounded, hence $M(\delta,\gamma)=\emptyset$.
- $\delta\geq 0, \gamma^{\hat{q}}>0$, for some $\hat{q}\in Q$. Then $M(\delta,\gamma)\subseteq\{(x,f): f^{\hat{q}}=0\}$.
- $\delta=0, \gamma^{\hat{q}}<0$, for some $\hat{q}\in Q$, then $M(\delta,\gamma)\subseteq\{(x,f): f^{\hat{q}}=1\}$.
- $\delta=0, \gamma^q=0$, for all $q\in Q$, then $M(\delta,\gamma)$ equals the set $X$ itself.
- $\delta>0, \gamma^q=0$, for all $q\in Q$. Then $M(\delta,\gamma)\subseteq\{(x,f): f^q=0\}$ for all $q\in Q$.
- $\delta>0, \gamma^q=0$ for all $q\in T$, $\gamma^q<0$ for all $q\in Q\setminus T$. Let $(\bar{x},\bar{f})\in M(\delta,\gamma)$ be an arbitrary optimal solution, and let $\bar{S}:=\{q\in Q:\bar{f}^q=1\}$. Since $Q$ is 0-strong, the same holds for the set $\bar{S}$. Moreover, since $\delta>0$, $\bar{x}=D(\bar{S})=\sum_{q\in\bar{S}} D^q$. Hence, the solution $(\bar{x},\bar{f})$ satisfies the 0-strong inequality at equality.

■

**Proposition 7.5** *If the set $Q$ is $|Q|-1$-strong, then $conv(X)$ is completely described by the trivial inequalities and the 0-strong inequalities $x\geq D^q f^q$, for all $q\in Q$.*

**Proof.** Given an arbitrary objective function $(\delta,\gamma)\in\mathbb{R}\times\mathbb{R}^{|Q|}$ which is to be minimized over all solutions in $X$, let $M(\delta,\gamma)$ be the corresponding set of optimal solutions. We will show that for each possible vector $(\delta,\gamma)$, the set $M(\delta,\gamma)$ is a subset of a face described by either one of the trivial inequalities or one of the 0-strong inequalities. We distinguish the same cases as in the proof of Proposition 7.4. The first 5 cases are analogous, so we restrict ourselves to the last case. Hence, $\delta>0$, $\gamma^q=0$, for all $q\in T$ and $\gamma^q<0$ for all $q\in Q\setminus T$. Let $(\bar{x},\bar{f})$ be an arbitrary optimal solution in $M(\delta,\gamma)$. Note that $\bar{x}\leq 1$, since the set is $|Q|-1$-strong and $\delta>0$. Let $\hat{q}=\arg\min_{q\in Q}\gamma^q$ (hence $\gamma^{\hat{q}}<0$). If $\bar{x}=1$ holds in an optimal solution, then surely $\bar{f}^{\hat{q}}=1$, and if $\bar{x}=0$ in an optimal solution, then $\bar{f}^{\hat{q}}=0$. Hence, $M(\delta,\gamma)\subseteq\{(x,f): x\geq D^{\hat{q}} f^{\hat{q}}\}$. ■

**Corollary 7.1** *If $|Q|\leq 2$ then the polytope $conv(X)$ is completely described by the trivial inequalities and c-strong inequalities.*

**Proof.** If $|Q|\leq 2$, then the set $Q$ is either 0-strong or $|Q|-1$-strong, hence, the previous propositions prove our claim. ■

# 8 The Directed Edge Capacity Polytope

This section reports on the directed edge capacity polytopes $Y_{ij}^{DF}$ and $Y_{ij}^{DP}$, as described in Section 2. We derive a class of valid inequalities and identify conditions under which the valid inequalities are facet defining. Next, we show that sequential lifting for the directed edge capacity polytope is $\mathcal{NP}$-hard in general, even for lifting orders which first lift all flow variables in the same direction.

Define the directed edge capacity polytope $Y$, where $f^q$ is a binary variable that denotes whether commodity $q$ is routed on the forward arc and $h^q$ represents whether the commodity $q$ is routed on the backward arc, as follows:

$$Y = \{(x, f, h) \in \mathbb{Z}_0^+ \times \{0,1\}^{2|Q|} : x \geq \textstyle\sum_{q \in Q} d^q f^q, x \geq \sum_{q \in Q} d^q h^q\}$$

**Proposition 8.1** *Let $\hat{q} \in Q$ and let $\alpha \in \mathbb{Z}_0^+$ such that $1 \leq \alpha \leq D^{\hat{q}}$. Then*

$$x \geq \alpha f^{\hat{q}} + \textstyle\sum_{q \in Q}(D^q - \alpha) h^q \tag{29}$$

*is a valid inequality for $Y$.*

**Proof.** Consider an arbitrary feasible solution $(\bar{x}, \bar{f}, \bar{h}) \in Y$ and let $\bar{Q} = \{q \in Q : \bar{h}^q = 1\}$. If $|\bar{Q}| = 0$, then $\bar{x} \geq D^{\hat{q}} \bar{f}^{\hat{q}} \geq \alpha \bar{f}^{\hat{q}} = \alpha \bar{f}^{\hat{q}} + \sum_{q \in Q}(D^q - \alpha)\bar{h}^q$. If $|\bar{Q}| \geq 1$, then

$$\begin{aligned} \bar{x} &\geq \lceil \max\{\textstyle\sum_{q \in Q} d^q \bar{f}^q, \sum_{q \in Q} d^q \bar{h}^q\} \rceil \geq D(\bar{Q}) \\ &= \textstyle\sum_{q \in \bar{Q}}(D^q - \alpha) + \sum_{q \in \bar{Q}}(\alpha - 1) + \lceil \sum_{q \in \bar{Q}} r^q \rceil \\ &\geq \textstyle\sum_{q \in \bar{Q}}(D^q - \alpha) + (\alpha - 1) + 1 \geq \alpha \bar{f}^{\hat{q}} + \sum_{q \in Q}(D^q - \alpha)\bar{h}^q \end{aligned}$$

which proves our claim. ■

**Proposition 8.2** *Let $\hat{q} \in Q$. If*

*(i).* $\alpha = 1$

*(ii).* $\forall q \in Q : D^q \geq D^{\hat{q}}$

*(iii).* $\forall q \in Q \setminus \{\hat{q}\} : \exists \tilde{Q} \subseteq Q$, *with* $|\tilde{Q}| = 2$, $\left\lceil \sum_{q \in \tilde{Q}} r^q \right\rceil = 1$, *and* $D(\tilde{Q}) \geq D(\{q, \hat{q}\})$

*then (29) is facet defining for $conv(Y)$.*

**Proof.** The dimension of $conv(Y)$ is $2|Q| + 1$. We give $2|Q|$ affinely independent vectors in $Y$ satisfying the inequality at equality. These vectors are

- $(x, f, h) = (0, 0, 0)$
- $(x, f, h) = (D^q, e^{\hat{q}}, e^q)$, for all $q \in Q$

- $(x, f, h) = (D(\tilde{Q}), e^{\hat{q} \cup q}, e^{\tilde{Q}})$, for all $q \in Q \setminus \{\hat{q}\}$

■

**Example 8.1** *Consider an instance of the set $Y$ with $|Q| = 4$ and the demands are $d^1 = 1.8, d^2 = 2.1, d^3 = 2.6, d^4 = 3.2$. If $\hat{q} = 1$, then the inequalities*

$$x \geq f^1 + h^1 + 2h^2 + 2h^3 + 3h^4$$
$$x \geq h^1 + f^1 + 2f^2 + 2f^3 + 3f^4$$

*are valid and facet defining for $conv(Y)$.*

As follows from Section 7, 0-strong inequalities which are facet defining for $conv(X)$ can be obtained by starting with the valid inequality $x \geq 0$ for $conv(X(Q, \emptyset))$ (i.e. the polytope in which all $f$ variables are set equal to zero), and then applying sequential lifting to the set of commodities $Q$. This lifting can be done in polynomial time as a result of Theorem 7.1. A similar approach could be employed to obtain facet defining inequalities for the directed polytope $conv(Y)$. First one could fix all variables (both $f$ and $h$ variables) to zero, and next one could apply sequential lifting on the valid inequality $x \geq 0$. However, we show that lifting under an arbitrary lifting order is an $\mathcal{NP}$-hard problem in general.

SUBSET SUM (cf. Garey and Johnson [9])
INSTANCE: A set of items $A$, a size $s^q \in \mathbb{Z}_0^+$ for all $q \in A$, and a positive integer $B$.
QUESTION: Does there exist a subset $\bar{A} \subseteq A$ such that $\sum_{q \in \bar{A}} s^q = B$ ?

LIFTING FOR DIRECTED EDGE CAPACITY MODEL
INSTANCE: A set of commodities $Q$, a demand size $d^q \in \mathbb{Z}_0^+$ for all $q \in Q$, and a capacity $\lambda \in \mathbb{Z}_0^+$ (this defines an instance of $Y$, using the inequalities $x \geq \sum_{q \in Q} \tilde{d}^q f^q$ and $x \geq \sum_{q \in Q} \tilde{d}^q h^q$, where $\tilde{d}^q = d^q / \lambda$). A complete order $\pi$ on a set of variables $T = \cup_{q \in Q} (f^q \cup h^q)$, a specific variable $z \in T$ and an integer $K \in \mathbb{Z}_0^+$.
QUESTION: If maximal lifting is applied to the inequality $x \geq 0$ in the lifting order $\pi$ to obtain a facet defining inequality for $Y$, is the lifting coefficient of variable $z$ less than or equal to $K$ ?

**Proposition 8.3** LIFTING FOR DIRECTED EDGE CAPACITY MODEL *is $\mathcal{NP}$-complete.*

**Proof.** We show that SUBSET SUM polynomially reduces to LIFTING FOR DIRECTED EDGE CAPACITY MODEL. Given an instance of SUBSET SUM, construct an instance of LIFTING FOR DIRECTED EDGE CAPACITY MODEL as follows. Let $Q = \{1, 2, 3\} \cup A$, $\lambda = B$, and define $d^q = s^q(\lambda + 1)$, for all $q \in A$. Define an integer $m = \left\lceil \frac{\sum_{q \in A} d^q}{\lambda} \right\rceil$, and let $d^1 = (m+1)\lambda + 1, d^2 = (m+1)\lambda + \lambda - 1, d^3 = (2m+4)\lambda + \lambda^2$. Define the order $\pi$ on $T$ as $f^1, f^2, f^q, \forall q \in A, h^3, h^2, h^1, h^q, \forall q \in A, f^3$. Finally, let $z = h^3$ and $K = 1$.

Next we show that an instance for SUBSET SUM yields an affirmative answer if and only if the corresponding instance for LIFTING FOR DIRECTED EDGE CAPACITY MODEL yields an affirmative answer. For convenience, we will use $\tilde{d}^q = d^q/\lambda$ throughout the proof. First, note that, starting with $x \geq 0$, after lifting the variables $f^1, f^2, f^q, \forall q \in A$ the valid inequality reads

$$x \geq b^T f = \lceil \tilde{d}^1 \rceil f^1 + (\lceil \tilde{d}^2 \rceil - 1) f^2 + \sum_{q \in A} (\lceil \tilde{d}^q \rceil - 1) f^q$$

Next, if we apply maximal lifting to variable $h^3$ and if $\bar{A} \subseteq A$ is a subset with $\sum_{q \in \bar{A}} s^q = B$, then the lifting coefficient $b^3$ for the variable $h^3$ satisfies

$$\begin{aligned} b^3 &= \min_{(x,f,h) \in Y : h^3 = 1, f^3 = 0, h^q = 0, \forall q \in Q \setminus \{3\}} \{x - (b^T f)\} \\ &\leq \left\lceil \max\{\tilde{d}^{\hat{q}}, \tilde{d}(\{1,2\} \cup \bar{A})\} \right\rceil - \sum_{q \in \{1,2\} \cup \bar{A}} b^q \\ &= 2m + 4 + \lambda - (2m + 3 + \lambda) = 1 = K \end{aligned}$$

Conversely, let $(\bar{x}, \bar{f}, \bar{h})$ be the vector for which the minimum value less than or equal to $K = 1$ in the lifting problem is attained, and let $\bar{Q} = \{q \in Q : \bar{f}^q = 1\}$. If $1 \notin \bar{Q}$, then $\sum_{q \in \bar{Q}} \tilde{d}^q \leq \tilde{d}^{\hat{q}}$ and hence,

$$\begin{aligned} b^3 &= \min_{(x,f,h) \in Y : h^3 = 1, f^3 = 0, h^q = 0, \forall q \in Q \setminus \{3\}} \{x - (b^T f)\} \\ &\geq \tilde{d}^{\hat{q}} - (b^T f) = 2m + 4 + \lambda - (m + 1 + \sum_{q \in A} s^q) \geq 3 + \lambda > K \end{aligned}$$

since $m \geq \sum_{q \in A} s^q$. Hence, $\bar{f}^1 = 1$. Similarly, one can prove that $\bar{f}^2 = 1$. Next define $\tilde{A} = \{q \in A : \bar{f}^q = 1\}$, and let $p = \sum_{q \in \tilde{A}} s^q$. If $p < B$, then $\tilde{d}(\{1,2\} \cup \tilde{A}) < \tilde{d}^{\hat{q}}$, thus,

$$b^3 = \bar{x} - \sum_{q \in \bar{Q}} b^q = 2m + 4 + \lambda - (2m + 3 + p) = \lambda - p + 1 > K$$

so, $p < B$ cannot be the case. If $p > B$, then $\tilde{d}(\{1,2\} \cup \tilde{A}) > \tilde{d}^{\hat{q}}$, hence,

$$b^3 = \bar{x} - \sum_{q \in \bar{Q}} b^q = 2m + 3 + p + \left\lceil \tfrac{p}{\lambda} \right\rceil - (2m + 3 + p) = \left\lceil \tfrac{p}{\lambda} \right\rceil \geq 2 > K$$

hence, neither $p > B$ can be the case. But this yields that $\sum_{q \in \tilde{A}} s^q = p = B$, hence $\tilde{A}$ is the required subset. ■

# 9 Computational Issues

To test the effect of the developed theory on the solvability of network loading problems we implemented a Branch-and-Cut algorithm, using A Branch-And-CUt System (ABACUS), version 2.2 [19], in combination with CPLEX 6.5 [14]. The algorithm was executed on a Sun Ultra-1 m140 workstation with 128MB internal memory. The program was tested on two sets of instances. First, we tested the quality of the valid inequalities for the edge capacity polytope with a set of randomly generated edge capacity instances. Second, we compare the results of the Branch-and-Cut algorithm with/without these inequalities for instances of the $DNFM$ (or

$DNPM$) model. These latter instances were made available to us by KPN Research, Leidschendam, The Netherlands.

This section is divided in three parts. In Section 9.1, we discuss the computational aspects of the separation of the different inequalities. Next, in Section 9.2, we compare the different separation techniques for the randomly generated edge capacity instances. Finally, in Section 9.3 the results for the real-life instances of KPN Research are presented.

## 9.1 Separation of Valid inequalities

In the Sections 5, 6, and 7.1, several (overlapping) classes of valid inequalities for the undirected edge capacity polytope are described. The separation problem for the already known $c$-strong inequalities is described by Brockmüller et al. [7]. They show that for a given value of $c$, finding the most violated $c$-strong inequality requires solving a knapsack problem. They propose heuristic methods to find the most violated inequality for values $c = 0, 1, 2$, and our computational experiments support their findings that this method yields good results.

Besides $c$-strong inequalities, the more general class of lower convex envelop inequalities have been introduced in this paper. Although lifting of lower convex envelope inequalities can be performed in polynomial time (as described in Section 5), there remain several unsolved issues regarding the separation of these inequalities. In order to obtain the most violated inequality in this class, it is yet unclear what choice of starting polytope $X(Q^0, Q^1)$ should be employed, and which lifting order to use. We therefore propose to search for violated lower convex envelope inequalities as follows. For a given LP-solution $(\bar{x}, \bar{f})$ and an arbitrary arc $(i, j) \in A$, let $S$ be the set of commodities with fractional routing variables on the arc ($S = \{q \in Q : 0 < \bar{f}^q_{ij} <= 1\}$). Moreover, let $Q^0 = \{q \in Q : \bar{f}^q_{ij} = 0\}$, and $Q^1 = \emptyset$. Then one can easily construct a two-dimensional picture similar to Figure 2 containing all feasible solutions on the subset of variables $(\cup_{q \in S} f^q_{ij}) \cup x$, and define all corresponding lower convex envelop inequalities for this subset of variables. After we found a violated lower envelope for $X(Q^0, Q^1)$ we lift the variables of $Q^0$ in order of non-increasing reduced cost.

A third way to obtain violated inequalities for the edge capacity is through the relation with the knapsack polytope (cf. Section 6). For a given LP-solution $(\bar{x}, \bar{f})$, we consider for every arc $(i, j) \in A$ the knapsack polytope $X(Q^0, Q^1, \tilde{x})$, with $Q^0 = \{q \in Q : \bar{f}^q_{ij} = 0\}$, $Q^1 = \{q \in Q : \bar{f}^q_{ij} = 1\}$, and $\tilde{x} = \lceil \bar{x} \rceil$. The search for a violated cover inequality is done in the way proposed by Gu et al. [12]. After a violated cover is found, first the $x$ variable is lifted, and afterwards the remaining flow variables (including the sets $Q^0$ and $Q^1$) according to the order defined in [12].

Besides the inequalities for the undirected edge capacity polytope, in Section 8, the two-side inequalities are proposed. For a given arc $(i, j) \in A$ and a fixed value of $\alpha$, finding the most violated two-side inequality (29) is an easy task. For a given LP-solution $(\bar{x}, \bar{f})$ a violated two-side inequality exists if and only if there exists an element $\hat{q} \in Q$ such that $\alpha \bar{f}^{\hat{q}}_{ij} > \bar{x} - \sum_{q \in Q} (D^q - \alpha) \bar{f}^q_{ji}$. Since the right hand side of the latter inequality is a constant for the given LP-solution, finding the most violated two-side inequality on arc $(i, j)$ for the specific value of $\alpha$ (if one exists) is equivalent to finding the maximal value $\bar{f}^q_{ij}$ over all commodities $q \in Q$. This can be done by any sorting algorithm. Likewise as for $c$-strong inequalities, computational experiments indicate that two-side inequalities should only be considered for small values of $\alpha$, for instance $\alpha \in \{1, 2\}$.

Apart from the inequalities for the edge capacity polytope as described in this paper, we also incorporated some other classes of well-known valid inequalities for network loading problems. Cut-set inequalities are used quite extensively for network loading problems (see for instance Barahona [2], Magnanti et al. [16, 17], Bienstock and Günlük [6], among others). Given a partition of the node set $V$ into two sets $S$ and $T$, let $d[S,T]$ denote the accumulated demand of all commodities with source node in $S$ and sink node in $T$. Then it is clear that the total capacity on the edges in the cut $\delta[S,T]$ should exceed this accumulated demand since all of these commodities must cross the cut. Since, capacity can only be installed in integer amounts, the cut-set inequalities read

$$\sum_{\{i,j\}\in\delta[S,T]} x_{ij} \geq \max\{\lceil d[S,T]\rceil, \lceil d[T,S]\rceil\} \tag{30}$$

Likewise, three partition inequalities (based on a partition of the node set into three sets) have been considered (see [6]), as well as the general $K$-cuts (see Barahona [3]). For small to medium sized graphs as considered in our experiments an exact separation that considers all possible partitions of the graph can be performed reasonably fast, and is therefore used.

### 9.2 Computational results Edge Capacity instances

To compare the performance of the different ways of separating inequalities for the edge capacity polytope, we have generated 20 instances of this single edge problem. For $|Q| = 10, 25, 50, 100$ commodities, we constructed 5 instances each with $\lambda = 155$ and $d^q$ uniformly distributed in the domain $\{10, \ldots, 155\}$. As optimization criterion we minimize $\lambda x - \sum_{q\in Q} c^q f^q$ with $c^q$ a randomly generated integer in the range $[0.8d^q, 1.2d^q]$ for all $q \in Q$.

Four different strategies are compared: branch-and-bound (B&B), $c$-strong separation ($c$-STR), lower envelope separation (LCE), and separation of lifted knapsack covers (COVER). As enumeration strategy we selected Best-First-Search with an initial upper bound of 0.

In Table 1 the results are compared on the number of nodes in the Branch-and-Cut tree, the number of added violated inequalities, and the CPU time. The table shows that with separation of lower envelope inequalities not only the smallest branch-and-cut trees are obtained, but also the number of added cuts is minimized (compared with the other strategies). This indicates that with this strategy very effective inequalities are added. The CPU time of LCE, however, is less attractive. This is due to the more complex lifting procedure. The order of variable lifting for lifted knapsack covers is even more complex which explains the CPU times in this case. In most cases the largest number of violated inequalities has been found with the separation of $c$-strong inequalities. It should be mentioned, however, that the results also show that the performance of the different strategies strongly depends on the individual instances. For example, the $c$-strong separation performs badly compared with the other ones on instance `ecp050b`, whereas it outperforms LCE and COVER on the instance `ecp025b` in both the number of nodes and cuts. A more detailed investigation of the computations learns that (almost) all lower envelope inequalities with slope 1 are also generated with the $c$-strong procedure. Therefore, we propose a combination of $c$-strong separation with separation of the lower envelope inequalities for the network loading instances.

| instance | $\|Q\|$ | # nodes | | | | # cuts | | | CPU time (sec) | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | B&B | $c$-STR | LCE | COVER | $c$-STR | LCE | COVER | B&B | $c$-STR | LCE | COVER |
| ecp010a | 10 | 77 | 11 | 19 | 11 | 16 | 15 | 15 | 0.43 | 0.08 | 0.15 | 0.20 |
| ecp010b | 10 | 43 | 31 | 13 | 19 | 35 | 20 | 31 | 0.36 | 0.30 | 0.17 | 0.36 |
| ecp010c | 10 | 53 | 19 | 15 | 17 | 21 | 17 | 20 | 0.40 | 0.22 | 0.12 | 0.26 |
| ecp010d | 10 | 33 | 13 | 13 | 13 | 10 | 13 | 9 | 0.14 | 0.16 | 0.13 | 0.16 |
| ecp010e | 10 | 103 | 23 | 21 | 21 | 41 | 23 | 24 | 0.63 | 0.25 | 0.22 | 0.28 |
| average | 10 | 61.8 | 19.4 | 16.2 | 16.2 | 24.6 | 17.6 | 19.8 | 0.39 | 0.20 | 0.16 | 0.25 |
| ecp025a | 25 | 149 | 55 | 49 | 55 | 71 | 53 | 74 | 0.80 | 0.68 | 1.44 | 2.26 |
| ecp025b | 25 | 107 | 31 | 47 | 39 | 46 | 61 | 59 | 0.46 | 0.31 | 1.66 | 1.76 |
| ecp025c | 25 | 235 | 87 | 73 | 99 | 127 | 88 | 144 | 1.30 | 1.05 | 2.36 | 4.22 |
| ecp025d | 25 | 131 | 63 | 55 | 49 | 92 | 45 | 42 | 0.79 | 0.80 | 1.19 | 1.43 |
| ecp025e | 25 | 501 | 203 | 135 | 193 | 256 | 126 | 247 | 2.97 | 2.40 | 3.45 | 7.81 |
| average | 25 | 224.6 | 87.8 | 71.8 | 87.0 | 118.4 | 74.6 | 113.2 | 1.26 | 1.05 | 2.02 | 3.50 |
| ecp050a | 50 | 455 | 155 | 157 | 127 | 411 | 124 | 228 | 2.72 | 5.04 | 11.50 | 21.00 |
| ecp050b | 50 | 13 | 41 | 9 | 3 | 93 | 1 | 2 | 0.13 | 1.22 | 0.17 | 0.23 |
| ecp050c | 50 | 353 | 53 | 59 | 53 | 377 | 69 | 75 | 1.97 | 3.23 | 6.28 | 7.25 |
| ecp050d | 50 | 723 | 431 | 99 | 167 | 661 | 115 | 204 | 4.73 | 11.67 | 10.53 | 20.38 |
| ecp050e | 50 | 353 | 147 | 61 | 167 | 271 | 90 | 332 | 2.54 | 4.28 | 7.92 | 30.93 |
| average | 50 | 379.4 | 165.4 | 77.0 | 103.4 | 362.6 | 79.8 | 168.2 | 2.42 | 5.09 | 7.28 | 15.96 |
| ecp100a | 100 | 109 | 69 | 57 | 151 | 125 | 65 | 345 | 0.99 | 5.05 | 22.04 | 116.41 |
| ecp100b | 100 | 79 | 13 | 23 | 13 | 9 | 27 | 9 | 0.62 | 0.58 | 9.10 | 3.51 |
| ecp100c | 100 | 693 | 223 | 273 | 251 | 263 | 279 | 531 | 6.93 | 11.78 | 95.67 | 182.41 |
| ecp100d | 100 | 371 | 197 | 57 | 85 | 241 | 81 | 114 | 3.08 | 10.60 | 27.21 | 42.27 |
| ecp100e | 100 | 1155 | 349 | 271 | 295 | 642 | 425 | 491 | 11.97 | 27.00 | 143.42 | 170.92 |
| average | 100 | 481.4 | 170.2 | 136.2 | 159.0 | 256.0 | 175.4 | 298.0 | 4.72 | 11.00 | 59.49 | 103.10 |

Table 1: Computational results Edge Capacity instances.

### 9.3 Computational results Network Loading instances

For the network loading problem we can either implement a Branch-and-Cut algorithm based on the flow formulation $DNFM$ as described by (9)-(13) or based on the path formulation $DNPM$ as described by (14)-(18). For a suitable choice of the set of paths for the commodities, and the exclusion of cycles from the flow formulation, it holds that every feasible solution to the former formulation corresponds to a feasible solution to the latter formulation, and vice versa. Consequently, the optimal values are the same. More importantly, this property also holds for the LP-relaxations of the formulations, even after the addition of valid inequalities as discussed in the sequel. Still, we implemented both formulations to test whether either of the two formulations would yield better results due to the difference in number of variables. Moreover, standard branching strategies use fixing of variables. If a path variable related to a certain commodity is fixed to its upper bound, then the routing of that specific commodity is completely known. Fixing a flow variable for a certain commodity to its upper bound only gives limited information on the routing for that specific commodity. Hence, this might also lead to different running times for both formulations. Although it is hard to draw general conclusions from the limited set of instances available to us, our computational experiments indicated that for larger graphs the exponential growth of the number of path variables is a serious problem. The computational results stated in the sequel are therefore obtained using the flow formulation.

Besides the class of lower envelope inequalities, and the two-side inequalities of Section 8, for network loading problems, also the already mentioned cut-set inequalities and three partition inequalities are available. To show the importance of both the edge-related inequalities, as well as the cut-related inequalities, we compare 4 different strategies: branch-and-bound (B&B), separation of edge-related inequalities (EDGE), separation of non-edge-related inequalities (NONEDGE), and separation of all available inequalities (ALL). For the last strategy we consider two variants; with and without an initial upper bound obtained with the heuristics described in [15] (ALL+UB). In all other cases the algorithm starts without an upper bound. To compare the through quality of the inequalities no primal heuristics are implemented at the nodes of the branch-and-cut tree. Computational experiments show that the DiveAndBest enumeration strategy (Depth-First-Search until a first integer solution is found, Best-First-Search afterwards) performs best for all strategies (in case of an initial upper bound this strategy results in a Best-First-Search).

In Table 2 we compare the quality of the lower bounds derived by the different strategies in the root node of the branch-and-cut tree. The comparison is done for fifteen real-life instances of KPN Research. These instances are defined on complete graphs in the range of 4 to 8 nodes, and for each graph size three different instances with fully dense non-symmetric demand matrices were available. The name of each instance, stated in the first column, refers to the number of nodes in the graph (first digit), whereas the second number in the name defines the demand matrix. Besides the lower bounds obtained with the different strategies, the table states the optimal value (if available), and an upper bound obtained with the heuristics described in [15] (used in the second variant of the strategy ALL). In case the branch-and-cut algorithms could not produce an optimal solution within acceptable time and memory requirements, the best global lower bound is given.

In Table 3 statistics concerning the number of branch-and-cut nodes, the number of added valid inequalities, and the computation times are reported. The results in both tables show that

| Instance | lower bound root node | | | | optimal value | upper bound |
|---|---|---|---|---|---|---|
| | B&B (LP value) | EDGE | NONEDGE | ALL | | |
| kpn_4_3 | 1.74 | 2.68 | 3.00 | 3.00 | 3 | 3 |
| kpn_4_10 | 5.81 | 7.17 | 6.75 | 7.19 | 8 | 8 |
| kpn_4_20 | 11.61 | 13.00 | 12.67 | 13.00 | 13 | 13 |
| kpn_5_3 | 1.89 | 3.02 | 3.75 | 3.75 | 4 | 4 |
| kpn_5_10 | 6.32 | 7.98 | 7.67 | 8.50 | 9 | 9 |
| kpn_5_20 | 12.64 | 15.04 | 13.77 | 15.06 | 16 | 16 |
| kpn_6_3 | 1.94 | 3.39 | 4.11 | 4.11 | 5 | 5 |
| kpn_6_10 | 6.45 | 8.07 | 8.11 | 8.36 | 9 | 10 |
| kpn_6_20 | 12.90 | 15.30 | 14.08 | 15.47 | 16 | 17 |
| kpn_7_3 | 1.95 | 3.79 | 4.42 | 4.42 | 6 | 6 |
| kpn_7_10 | 6.52 | 8.35 | 8.54 | 8.81 | 10 | 10 |
| kpn_7_20 | 13.03 | 15.38 | 14.59 | 15.73 | 17 | 18 |
| kpn_8_3 | 2.17 | 4.29 | 4.89 | 4.89 | 7 | 7 |
| kpn_8_10 | 7.22 | 9.44 | 9.74 | 9.83 | 11 | 12 |
| kpn_8_20 | 14.45 | 17.15 | 16.44 | 17.33 | 19 | 21 |

Table 2: Computational results for KPN network loading instances: values

the network loading problems become very difficult in case the difference between the optimal value and lower bound of the root node is larger than one. This observation holds for all strategies: kpn_5_20 and kpn_6_20 are already too difficult for the NONEDGE strategy, whereas the instance kpn_6_10 cannot be solved with the EDGE strategy. The results of the NONEDGE strategy also indicate that the cut and three-partition inequalities are less powerful in case of larger demands. For the EDGE strategy the relation between the size of the demands and the difficulty of the instance is less clear. In fact, the percentage of the gap closed in the root node by the edge-related inequalities increases as the demand increases, whereas the percentage decreases for the NONEDGE strategy. This observation agrees with our theoretical results that only the fractional part of the demand is important. Another, more surprising, observation that can be made is that not in all cases the separation of more inequalities results in a better overall performance of the algorithm. The instances kpn_5_10, kpn_5_20, and kpn_6_20 can be solved more efficiently with separation of only the edge-related inequalities than with separation of all inequalities. Also the addition of an initial upper bound does not improve the performance of the algorithm in all cases (see for instance kpn_6_10).

Finally, we have to conclude that for graphs as small as 8 nodes the inequalities are not strong enough to solve the instances to optimality. For the instance kpn_8_3 the optimal value can be determined by the combinatorial argument that at least 7 links are needed to have a connection between every pair vertices. Since, an upper bound of 7 is also available we can conclude that this is the optimal value. The branch-and-cut algorithms, however, cannot produce a lower bound better than 6 within acceptable time and memory requirements. Also for the other two instances with 8 nodes the branch-and-cut algorithm is not able to solve them. The optimal values for these instances are obtained by solving a preprocessed version of the linear programming relaxation

| Instance | # nodes | | | | | # cuts | | | | CPU time (sec) | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | B&B | EDGE | NONEDGE | ALL | ALL+UB | EDGE | NONEDGE | ALL | ALL+UB | B&B | EDGE | NONEDGE | ALL | ALL+UB |
| kpn_4_3 | 31 | 5 | 1 | 1 | 1 | 185 | 10 | 34 | 34 | 0.47 | 0.67 | 0.10 | 0.11 | 0.13 |
| kpn_4_10 | 77 | 7 | 47 | 7 | 1 | 99 | 8 | 129 | 35 | 1.82 | 0.47 | 1.28 | 0.49 | 0.15 |
| kpn_4_20 | 51 | 1 | 125 | 1 | 1 | 40 | 20 | 54 | 26 | 0.72 | 0.15 | 1.87 | 0.16 | 0.13 |
| kpn_5_3 | 183 | 7 | 5 | 5 | 1 | 478 | 37 | 202 | 74 | 8.95 | 3.76 | 0.48 | 1.16 | 0.34 |
| kpn_5_10 | - | 15 | 3,745 | 5 | 1 | 447 | 1,164 | 262 | 65 | - | 4.52 | 209.36 | 1.90 | 0.36 |
| kpn_5_20 | - | 65 | - | 73 | 1 | 1,097 | - | 1,532 | 156 | - | 16.62 | - | 17.18 | 0.90 |
| kpn_6_3 | - | 43 | 9 | 9 | 1 | 2,879 | 117 | 572 | 147 | - | 90.24 | 2.19 | 7.48 | 1.14 |
| kpn_6_10 | - | - | 1,431 | 11 | 33 | - | 2,043 | 625 | 1,891 | - | - | 240.36 | 10.04 | 33.71 |
| kpn_6_20 | - | 297 | - | 319 | 125 | 6,271 | - | 9,238 | 3,831 | - | 226.01 | - | 238.72 | 89.45 |
| kpn_7_3 | - | - | 27 | 27 | 19 | - | 522 | 1,831 | 1,628 | - | - | 28.56 | 122.83 | 65.90 |
| kpn_7_10 | - | - | - | 693 | 19 | - | - | 19,313 | 1,568 | - | - | - | 1,129.28 | 64.71 |
| kpn_7_20 | - | - | - | - | 255 | - | - | - | 7,093 | - | - | - | - | 415.06 |
| kpn_8_3 | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| kpn_8_10 | - | - | - | - | - | - | - | - | - | - | - | - | - | - |
| kpn_8_20 | - | - | - | - | - | - | - | - | - | - | - | - | - | - |

Table 3: Computational results for KPN network loading instances: statistics

(with the violated inequalities added in the root node) with the CPLEX integer programming solver. The computation of these optimal values took respectively 3,606 and 163,586 seconds.

## 10 Concluding Remarks

The computational results of the previous section show that the edge capacity polytope is an important substructure of the network loading problem in the sense that its strong valid inequalities increase the value of the LP-relaxation quite substantially. Similar results were obtained by Bienstock and Bley [4], who applied the results of this paper to a multicast network design problem. We did, however, not take into account the inequalities that concentrate on larger structures than a single edge within the network loading problem. Therefore, a potential gain on our results may lie in finding valid inequalities for more general structures of the integer programming formulation, such as cuts in the graph. Although cut-set inequalities (or the more general partition inequalities) are facet defining for bifurcated versions of network loading problems (see [6, 17]), for non-bifurcated network loading problems this is in general not the case, and several possibilities for strengthening arise. We have performed some initial computational experiments to test the effect of a strengthening of a cut-set inequality, but so far the gain has been limited.

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