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ABSTRACT

In this paper it is shown that for highly nonuniformly refined triangulations the condition number of the BPX preconditioner for elliptic finite element problems grows at most linearly in the depth of refinement. This is achieved by viewing the computational available version of the BPX preconditioner as an abstract additive Schwarz method with exact solvers.

AMS CLASSIFICATION: 65F10, 65F35, 65N20, 65N30.

Contents

INTRODUCTION

The recently introduced BPX preconditioner of BRAMBLE/PASCIAK/XU [5] for elliptic finite element problems has drawn a lot of attention to it due to several interesting features:

- its condition number estimate is independent of the space dimension [5, 15]
- it is fully parallel [5]
- it can be easily extended to preconditioners of other problems like problems arising from semidiscretization in time of a parabolic problem [3, 4, 15]

The theory presented in [5, 15] yielded as condition number estimate $\kappa = \mathcal{O}(j^2)$ for a large class of problems on *quasi-uniform* triangulations of refinement depth j. This result was recently improved in several ways:

- For *highly nonuniform* triangulations YSERENTANT [18] showed that $\kappa = \mathcal{O}(j^2)$.
- By means of a strengthened Cauchy–Schwarz inequality ZHANG (Courant Institute, New York) obtained $\kappa = \mathcal{O}(j)$ for quasi-uniform triangulations, [14].
- By means of best approximation arguments in Besov–Sobolev spaces OSWALD [11] obtained that $\kappa = \mathcal{O}(1)$ on quasi-uniform triangulations.

This paper is devoted to the proof of $\kappa = \mathcal{O}(j)$ for *highly nonuniform* triangulations, thus combining the results of YSERENTANT and ZHANG.

This can be achieved by viewing the *computational available* version of the BPX preconditioner in a new way: directly as an abstract additive Schwarz method with *exact* solvers. This interpretation seems to be known in the domain decomposition community but was not consequently used for purposes of a proof. DRYA/WIDLUND viewed instead in [7] the orthogonal projection version of the BPX preconditioner as an additive Schwarz method with inexact solvers, XU considered in [16] the computational version as a multilevel domain decomposition method.

Our interpretation as an additive Schwarz method with exact solvers is conceptually simple, allows to apply the well developed proof machinery of the additive Schwarz methods and helps to clarify the understanding of the computational version of the BPX preconditioner. A special feature of our proof is — in contrast to all other known proofs — that we do *not* use the weighted L^2–projections, which constitute the orthogonal projection version of the BPX preconditioner.

In Section 1 we provide some notation used in the following sections. Section 2 briefly discusses abstract additive Schwarz methods as a general construction principle for preconditioners and provides the proof machinery.

Section 3 gives the interpretation of the computational BPX version as an additive Schwarz method.

Section 4 derives a bound for the smallest eigenvalue of the preconditioned system. Here the stability and approximation results of YSERENTANT [18] for certain restriction or quasi–interpolation operators come into play.

In Section 5 a constant bound for the largest eigenvalue is derived by using orthogonalities introduced by colorings of the triangulations and a strengthened Cauchy–Schwarz inequality between subspaces generated by those colorings.

In Section 6 we discuss the implication for the orthogonal projection version of the BPX preconditioner and the connection to the hierarchical basis preconditioner.

1. TRIANGULATIONS AND FINITE ELEMENT SPACES

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected polygonal domain and Γ_D a boundary piece of Ω which is assumed to be composed of straight lines. Introducing the space of weak solutions

$$H_D^1(\Omega) = \{u \in H^1(\Omega) \mid u|_{\Gamma_D} = 0\}$$

we consider the variational problem for some $f^* \in (H_D^1(\Omega))^*$: Find $u \in H_D^1(\Omega)$ such that

$$a(u,v) = f^*(v)$$

for all $v \in H_D^1(\Omega)$. The bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij}\partial_i u \partial_j v dx.$$

We assume that $a_{ij} \in L^\infty(\Omega)$ and $a_{ij} = a_{ji}$.

A *triangulation* $\mathcal{T}$ of the polygonal domain Ω is given as the set of triangles resulting from a simplicial partition of Ω.

We start with a coarse triangulation $\mathcal{T}_0$ of Ω with the property that the Dirichlet boundary piece Γ_D is composed of edges of triangles $T \in \mathcal{T}_0$.

We assume that there are positive constants $0 < \delta \leq 1 \leq \Delta$ and $\omega(T)$ such that

$$\delta\omega(T)\sum_{i=1}^{2}\xi_i^2 \leq \sum_{i,k=1}^{2} a_{ik}(x)\xi_i\xi_k \leq \Delta\omega(T)\sum_{i=1}^{2}\xi_i^2 \tag{1.1}$$

for all $T \in \mathcal{T}_0$, almost all $x \in T$ and all $\xi \in \mathbb{R}^2$. Thus $a(\cdot,\cdot)$ is a symmetric, bounded and coercive bilinear form on $H_D^1(\Omega)$.

In addition to the usual (semi-)norms $\|\cdot\|_0$ and $|\cdot|_1$ of $L^2(\Omega)$ and $H^1(\Omega)$, we introduce the semi-inner product

$$(u,v)_{1|\Omega_0} = \sum_{i=1}^{2}\int_{\Omega_0} \partial_i u \partial_i v dx$$

for $\Omega_0 \subset \Omega$ measurable with induced norm $|u|_{1|\Omega_0}^2 = (u,u)_{1|\Omega_0}$. Furthermore we introduce the weighted $H^1(\Omega)$ semi-inner product

$$(u,v)_{1;\mathcal{T}_0} = \sum_{T\in\mathcal{T}_0} \omega(T)\sum_{i=1}^{2}\int_T \partial_i u \partial_i v dx,$$

which induces the seminorm $|u|^2_{1;\mathcal{T}_0} = (u,u)_{1;\mathcal{T}_0}$, and the weighted $L^2(\Omega)$ inner product

$$(u,v)_{0;\mathcal{T}_0} = \sum_{T\in\mathcal{T}_0} \frac{\omega(T)}{|T|} \int_T u\, v dx,$$

which induces the norm $\|u\|^2_{0;\mathcal{T}_0} = (u,u)_{0;\mathcal{T}_0}$.

Relation (1.1) may now be written as

$$\delta |u|^2_{1;\mathcal{T}_0} \le a(u,u) \le \Delta |u|^2_{1;\mathcal{T}_0}. \tag{1.2}$$

The triangulation $\mathcal{T}_0$ is refined several times, giving a family of *nested* triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_j$. A triangle of $\mathcal{T}_{k+1}$ is either a triangle of $\mathcal{T}_k$ or is generated by subdividing a triangle of $\mathcal{T}_k$ into four congruent triangles or into two triangles by connecting one of its vertices with the midpoint of the opposite side. The first case is called a regular or *red* refinement and the resulting triangles as well as the triangles of the initial triangulation are called regular triangles. The second case is an irregular or *green* refinement and results in two so–called irregular triangles.

However, the irregular refinement has the character of a *closure* which we force by the following *rule*:

(T1) Each new vertex of $\mathcal{T}_k$, i.e., a vertex which does not belong to $\mathcal{T}_{k-1}$, is a vertex of a triangle which was generated by regular refinement.

The irregular refinement is potentially dangerous because interior angles are reduced. Therefore, we add the following rule:

(T2) Irregular triangles may not be further refined.

This rule insures that every triangle of any triangulation $\mathcal{T}_k$ is geometrically similar to a triangle of the initial triangulation $\mathcal{T}_0$ or to a green refinement of a triangle in $\mathcal{T}_0$. These triangulations are meanwhile standard and have been introduced by BANK et al. in [1].

The index of the final triangulation will always be denoted by j and will be fixed in most of the following considerations.

By the *depth* of a triangle

$$T \in \bigcup_{k=0}^{j} \mathcal{T}_k$$

we mean the number of successive ancestors in the family of triangulations. If we add the rule

(T3) Only triangles of depth $k-1$ are refined for the construction of $\mathcal{T}_k$, we get the following expression for the depth of a triangle $T \in \bigcup_{k=0}^{j} \mathcal{T}_k$

$$\mathrm{depth}(T) = \min\{0 \le k \le j \mid T \in \mathcal{T}_k\}.$$

Equipped with rule (T3) we can uniquely reconstruct the sequence $\mathcal{T}_1, \ldots, \mathcal{T}_{j-1}$ from the knowledge of the initial triangulation $\mathcal{T}_0$ and the final triangulation $\mathcal{T}_j$ *alone*, without knowing the actual dynamic refinement process leading to $\mathcal{T}_j$ in an adaptive algorithm, see [6]. However, if we choose the data-structures representing the triangulation cleverly, the sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_j$ is *implicitly* given. For example this is the case in the adaptive FEM code KASKADE, cf. ROITZSCH [12] or LEINEN [10].

The refinement structure (T1)–(T3) implies the property of *local quasi-uniformity*, i.e., the existence of a positive constant K depending only on the local geometry of the initial triangulation $\mathcal{T}_0$ such that

$$\frac{h(T)}{h(T')} \le K \tag{1.3}$$

for all $T, T' \in \mathcal{T}_k$ such that $T \cap T' \neq \emptyset$. Here $h(T)$ denotes the diameter of a triangle.

Corresponding to the triangulations $\mathcal{T}_k$ we have *finite element* spaces $\mathcal{S}_k$. $\mathcal{S}_k$ consists of all functions which are linear on each triangle $T \in \mathcal{T}_k$ and continuous on Ω. Furthermore they vanish on the Dirichlet boundary piece Γ_D. Because the triangulations are nested we have

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_j \subset H_D^1(\Omega).$$

Let $\mathcal{N}_k = \{x_1^{(k)}, \ldots, x_{n_k}^{(k)}\}$ be the set of vertices of triangles in $\mathcal{T}_k$, which do not lie on the Dirichlet boundary piece Γ_D.

The nodal basis. The set $\Gamma_k = \{\psi_1^{(k)}, \ldots, \psi_{n_k}^{(k)}\}$ of nodal basis functions, where

$$\psi_i^{(k)}(x_l^{(k)}) = \delta_{il} \quad \text{for } 1 \le i, l \le n_k,$$

forms a basis of $\mathcal{S}_k$. For $\psi \in \Gamma_k$ we denote by $x_\psi \in \mathcal{N}_k$ the *supporting point* of ψ, i.e.

$$\psi(x_\psi) = 1.$$

Structuring of the nodal bases of varying index k. We set

$$\begin{aligned}
\text{i)}\quad \Psi &= \bigcup_{k=0}^{j} \Gamma_k, \\
\text{ii)}\quad \Psi_0 &= \Gamma_0, \\
\text{iii)}\quad \Psi_k &= \Gamma_k \setminus \Gamma_{k-1}, \qquad \text{whenever } 1 \le k \le j.
\end{aligned}$$

[3, Lemma 4.1] states the easily proven fact that the set Ψ is the *disjoint* union of the sets Ψ_k, $k = 0, \ldots, j$. For $\psi \in \Psi$ we denote the set of indices, for which a nodal basis function ψ occurs, by

$$K_\psi = \{k \mid \psi \in \Gamma_k\}.$$

Here we abbreviate the first resp. the last occurrence of ψ in a set Γ_k by

$$\begin{aligned} \text{i)}\quad k_\psi^0 &= \min K_\psi, \\ \text{ii)}\quad k_\psi^1 &= \max K_\psi. \end{aligned}$$

2. THE ABSTRACT SETTING OF ADDITIVE SCHWARZ METHODS

The abstract framework of *additive Schwarz methods* as introduced by DRYJA/WIDLUND [7, 8] can be viewed as a general construction principle for preconditioners of elliptic finite element equations. Moreover it provides methods for the analysis of such preconditioners.

Let V be a finite dimensional vector space with the two inner products $a(\cdot,\cdot)$ and $(\cdot,\cdot)$. Let $\{V_i\}_{i=1}^N$ be a family of subspaces of V such that

$$V = \sum_{i=1}^{N} V_i. \tag{2.1}$$

Define the projections $\pi_i, \sigma_i : V \to V_i$ as

$$(\pi_i u, v_i) = (u, v_i) \text{ and } a(\sigma_i u, v_i) = a(u, v_i)$$

for all $v_i \in V_i$. Furthermore define the representation operators $A : V \to V$ and $A_i : V_i \to V_i$ through

$$(Au, v) = a(u, v) \text{ and } (A_i u_i, v_i) = a(u_i, v_i)$$

for all $u, v \in V$ resp. $u_i, v_i \in V_i$. Observe that

$$\pi_i A = A_i \sigma_i. \tag{2.2}$$

The corresponding additive Schwarz method transforms the equation $Au = f$ into an equivalent one

$$\sigma u = f',$$

where σ denotes the sum of the Ritz projections σ_i

$$\sigma = \sum_{i=1}^{N} \sigma_i. \tag{2.3}$$

This transformed equation will be solved iteratively. Thus the aim of additive Schwarz methods is to find subspace decompositions (2.1) such that $\kappa(\sigma)$ is small.

It can be easily seen with the help of (2.2) that

$$\sigma = BA$$

with

$$B = \sum_{i=1}^{N} A_i^{-1}\pi_i. \tag{2.4}$$

Hence a good additive Schwarz method yields a good *preconditioner* B for A since

$$\kappa(\sigma) = \kappa(BA).$$

The following meanwhile well-known Lemma provides a technique of bounding the smallest and largest eigenvalue of σ.

LEMMA 1.

i) *Let there exists a positive constant c_0 such that for all $u \in V$ we get a decomposition $u = \sum_{i=1}^{N} u_i$, $u_i \in V_i$, such that*

$$\sum_{i=1}^{N} a(u_i, u_i) \le c_0 a(u, u). \tag{2.5}$$

Then we get the estimate for $u \in V$

$$c_0^{-1} a(u, u) \le a(\sigma u, u).$$

ii) *Let there exist constants ϵ_{ij} for $i, j = 1, \ldots, N$ such that the following Cauchy–Schwarz like inequality*

$$a(u_i, u_j) \le \epsilon_{ij} a(u_i, u_i)^{1/2} a(u_j, u_j)^{1/2}$$

holds for $u_i \in V_i$, $u_j \in V_j$. Then we get the estimate

$$a(\sigma u, u) \le \rho(E) a(u, u)$$

for $u \in V$ where $\rho(E)$ denotes the spectral radius of the matrix $E = \{\epsilon_{ij}\}_{i,j=1}^{N}$.

An elementary proof of i) may be found in [13], of ii) in [9].

3. BPX AS AN ADDITIVE SCHWARZ METHOD

Now we turn to the finite element equations. The following considerations strongly rely on the notation introduced in Section 1.

We specify the subspace decomposition (2.1) to be

$$\mathcal{S}_j = \mathcal{S}_0 + \sum_{\psi \in \Psi} V_\psi$$

with $V_\psi = \text{span}\{\psi\}$.

Corresponding to the considerations of the last section we introduce the projections $\pi_0, \sigma_0 : \mathcal{S}_j \to \mathcal{S}_0$ and $\pi_\psi, \sigma_\psi : \mathcal{S}_j \to V_\psi$ defined as

$$(\pi_0 u, v_0)_{0;\mathcal{T}_0} = (u, v_0)_{0;\mathcal{T}_0} \text{ and } a(\sigma_0 u, v_0) = a(u, v_0)$$

for all $v_0 \in \mathcal{S}_0$ and

$$(\pi_\psi u, \psi)_{0;\mathcal{T}_0} = (u, \psi)_{0;\mathcal{T}_0} \text{ and } a(\sigma_\psi u, \psi) = a(u, \psi)$$

for all $\psi \in \Psi$.

Furthermore we introduce the representation operators $A_k : \mathcal{S}_k \to \mathcal{S}_k$ and $A_\psi : V_\psi \to V_\psi$ by

$$(A_k u_k, v_k)_{0;\mathcal{T}_0} = a(u_k, v_k)$$

for all $u_k, v_k \in \mathcal{S}_k$, $k = 0, 1, \ldots, j$, and

$$(A_\psi \psi, \psi)_{0;\mathcal{T}_0} = a(\psi, \psi)$$

for all $\psi \in \Psi$.

The preconditioner of A_j which corresponds to the abstract version (2.4) is given by

$$B_j = A_0^{-1} \pi_0 + \sum_{\psi \in \Psi} A_\psi^{-1} \pi_\psi.$$

A straightforward computation shows that for $u \in \mathcal{S}_j$ and $\psi \in \Psi$

$$A_\psi^{-1} \pi_\psi u = \frac{(u, \psi)_{0;\mathcal{T}_0}}{a(\psi, \psi)} \psi.$$

Thus

$$B_j u = A_0^{-1} \pi_0 u + \sum_{\psi \in \Psi} \frac{(u, \psi)_{0;\mathcal{T}_0}}{a(\psi, \psi)} \psi \tag{3.1}$$

for $u \in \mathcal{S}_j$. This is just the *computational* available version of the BPX preconditioner as considered by YSERENTANT[18], cf. especially the discussion around formulas [18, (5.46)–(5.48)]. Note that for the application of B_j in a conjugate gradient method the values $(u,\psi)_{0;\mathcal{T}_0}$ are already known and need not to be computed, as explained in [4, 18].

An estimate of the condition number of

$$B_jA_j = \sigma = \sigma_0 + \sum_{\psi\in\Psi} \sigma_\psi$$

will be obtained with the help of Lemma 1.

4. A Bound for the Smallest Eigenvalue

In order to bound the smallest eigenvalue of σ by means of Lemma 1.i we have to specify the splitting for which the estimate (2.5) can be shown. An effective estimate of this kind relies strongly on the existence of linear *restriction operators*

$$R_k : \mathcal{S}_j \to \mathcal{S}_k$$

for $k = 0, 1, \ldots, j$ such that certain approximation and stability results hold. If we specify $R_k = \bar{M}_k$, where the $\bar{M}_k$ are the *quasi-interpolation* operators introduced by YSERENTANT in [18, (4.11)], then [18, Lemma 4.2] gives the *approximation estimate*

$$\|(I - R_k)u\|^2_{0;\mathcal{T}_0} \leq c_2^* \bar{\eta} 4^{-k} |u|^2_{1;\mathcal{T}_0}, \tag{4.1}$$

for $k = 0, 1, \ldots, j$, and [18, Lemma 4.4] gives the *stability estimate*

$$|R_k u|^2_{1;\mathcal{T}_0} \leq c_1^* \bar{\eta} |u|^2_{1;\mathcal{T}_0}, \tag{4.2}$$

for $k = 0, 1, \ldots, j$. Here the constants c_1^*, c_2^* depend only on the local geometry of the initial triangulation $\mathcal{T}_0$ and the constant $\bar{\eta}$ measures the jumps of ω as follows:

$$\bar{\eta} = \max_{T\in\mathcal{T}_0} \eta(T)$$

with

$$\eta(T) = \frac{\max\{\omega(T') \mid T' \in \mathcal{T}_0, T' \cap T \neq \emptyset\}}{\min\{\omega(T') \mid T' \in \mathcal{T}_0, T' \cap T \neq \emptyset\}}.$$

REMARK 1. Note that we also could take the R_k as the weighted L^2–projections $Q_k : L^2(\Omega) \to \mathcal{S}_k$ defined by

$$(Q_k u, v_k)_{0;\mathcal{T}_0} = (u, v_k)_{0;\mathcal{T}_0}$$

for all $v_k \in \mathcal{S}_k$. For these projections the same estimates as (4.1) and (4.2) hold by [18, Theorem 4.3/4.5].

These restriction operators induce the splitting

$$u = u_0 + \sum_{k=1}^{j} u_k$$

with

$$u_j = u - R_{j-1}u, \qquad u_k = (R_k - R_{k-1})u$$

for $k = 1, \ldots, j-1$ and $u_0 = R_0 u$. Put $\bar{u} = \sum_{k=1}^{j} u_k = u - u_0$, $\bar{u}_k = u_k$ for $k = 1, \ldots, j$ and $\bar{u}_0 = 0$. Since $\bar{u}_k \in \mathcal{S}_k$ for $k = 0, 1, \ldots, j$ we can decompose as follows

$$\begin{aligned} \bar{u} &= \sum_{k=0}^{j} \bar{u}_k = \sum_{k=0}^{j} \sum_{\psi \in \Gamma_k} \bar{u}_k(x_\psi)\psi \\ &= \sum_{\psi \in \Psi} \left[\sum_{k \in K_\psi} \bar{u}_k(x_\psi) \right] \psi = \sum_{\psi \in \Psi} u_\psi \end{aligned}$$

where we have put $u_\psi = \beta(u, \psi)\psi$ with

$$\beta(u, \psi) = \sum_{k \in K_\psi} \bar{u}_k(x_\psi).$$

We thus end up with the decomposition

$$u = u_0 + \sum_{\psi \in \Psi} u_\psi \tag{4.3}$$

where $u_0 \in \mathcal{S}_0$ and $u_\psi \in V_\psi$.

First we estimate $\sum_{\psi \in \Psi} a(u_\psi, u_\psi)$ from above by $a(u, u)$. Usage of the *inverse inequality* [18, Lemma 3.3] together with (1.2) yields

$$\sum_{\psi \in \Psi} a(u_\psi, u_\psi) \le \Delta K_0 \sum_{\psi \in \Psi} \beta^2(u, \psi)\, 4^{k_\psi^0} (\psi, \psi)_{0;\mathcal{T}_0}, \tag{4.4}$$

where the constant K_0 depends only on the shape regularity of the triangles of the initial triangulation $\mathcal{T}_0$. With the help of the Cauchy–Schwarz inequality we estimate

$$\begin{aligned} \beta^2(u, \psi) &= \left[\sum_{k \in K_\psi} 2^{-k} \left(2^k \bar{u}_k(x_\psi) \right) \right]^2 \\ &\le \left(\sum_{k \in K_\psi} 4^{-k} \right) \left(\sum_{k \in K_\psi} 4^k |\bar{u}_k(x_\psi)|^2 \right) \\ &\le \frac{4}{3} 4^{-k_\psi^0} \sum_{k \in K_\psi} 4^k |\bar{u}_k(x_\psi)|^2. \end{aligned}$$

Insertion in (4.4) yields

$$\sum_{\psi\in\Psi} a(u_\psi,u_\psi) \le \frac{4}{3}\Delta K_0 \sum_{k=0}^{j} 4^k \sum_{\psi\in\Gamma_k} |\bar u_k(x_\psi)|^2(\psi,\psi)_{0;\mathcal{T}_0}. \tag{4.5}$$

The last expression can be further estimated by means of the following Lemma.

LEMMA 2. *For $u \in \mathcal{S}_k$ the estimates*

$$\frac{1}{2}(u,u)_{0;\mathcal{T}_0} \le \sum_{\psi\in\Gamma_k} |u(x_\psi)|^2(\psi,\psi)_{0;\mathcal{T}_0} \le 2(u,u)_{0;\mathcal{T}_0}$$

holds.

Proof. We have

$$\sum_{\psi\in\Gamma_k} |u(x_\psi)|^2(\psi,\psi)_{0;\mathcal{T}_0} = \sum_{\hat T\in\mathcal{T}_0} \frac{\omega(\hat T)}{|\hat T|}\left(\frac{1}{6}\sum_{T\in\mathcal{T}_k, T\subset\hat T} |T| \sum_{x\in\mathcal{N}_k\cap T} |u(x)|^2\right).$$

The estimates of [4, Lemma 2.3] show that

$$\frac{1}{2}\int_{\hat T} u^2dx \le \frac{1}{6}\sum_{T\in\mathcal{T}_k, T\subset\hat T} |T| \sum_{x\in\mathcal{N}_k\cap T} |u(x)|^2 \le 2\int_{\hat T} u^2dx$$

for any $\hat T \in \mathcal{T}_0$. Thus the Lemma follows by the definition of $(\cdot,\cdot)_{0;\mathcal{T}_0}$. ■

Replacing the discrete term in (4.5) in this way gives

$$\sum_{\psi\in\Psi} a(u_\psi,u_\psi) \le \frac{8}{3}\Delta K_0 \sum_{k=1}^{j} 4^k \|\bar u_k\|^2_{0;\mathcal{T}_0}. \tag{4.6}$$

By the approximation estimate (4.1) we get

$$\|\bar u_j\|^2_{0;\mathcal{T}_0} = \|(I-R_{j-1})u\|^2_{0;\mathcal{T}_0} \le c_2^*\bar\eta 4^{-(j-1)}|u|^2_{1;\mathcal{T}_0}$$

and

$$\begin{aligned}\|\bar u_k\|^2_{0;\mathcal{T}_0} &\le 2\left(\|(I-R_k)u\|^2_{0;\mathcal{T}_0} + \|(I-R_{k-1})u\|^2_{0;\mathcal{T}_0}\right)\\ &\le \frac{5}{2}c_2^*\bar\eta 4^{-(k-1)}|u|^2_{1;\mathcal{T}_0}\end{aligned}$$

for $k = 1,\ldots,j-1$.

The missing term $a(u_0, u_0)$ can be easily bounded by $a(u,u)$ with the help of the stability estimate (4.2). Thus we obtain by (1.2) the following estimate of type (2.5):

$$a(u_0, u_0) + \sum_{\psi \in \Psi} a(u_\psi, u_\psi) \leq c_0 \bar{\eta} \frac{\Delta}{\delta} (j+1)\, a(u,u)$$

for the decomposition (4.3), where c_0 depends only on the local geometry of the initial triangulation $\mathcal{T}_0$.

Lemma 1.i shows that we have proven the following

THEOREM 1. *There exists a positive constant c_0, depending only on the local geometry of the initial triangulation $\mathcal{T}_0$, such that*

$$\frac{\delta}{\Delta \bar{\eta}} \frac{1}{c_0 (j+1)} a(u,u) \leq a(B_j A_j u, u)$$

for all $u \in \mathcal{S}_j$.

5. A Bound for the Largest Eigenvalue

Here we have to establish orthogonalities and some strengthened Cauchy–Schwarz inequalities in order to estimate $a(\sigma u, u)$ effectively from above.

First we consider *colorings* of $\mathcal{T}_k$: A coloring of $\mathcal{T}_k$ is a mapping

$$\chi_k : \mathcal{N}_k \rightarrow \{1, 2, \ldots, \chi(\mathcal{T}_k)\},$$

such that $\chi(x_1) \neq \chi(x_2)$ for $x_1, x_2 \in \mathcal{N}_k$ connected by an edge of $\mathcal{T}_k$. A well known, easily proven theorem ([2, Chap. 12, Theorem 12]) states that a minimal $\chi(\mathcal{T}_k) \leq 5$ can always be found. The famous four color theorem of Appel/Haken shows that even $\chi(\mathcal{T}_k) \leq 4$ for the minimal $\chi(\mathcal{T}_k)$. We take for the following a four coloring of the $\mathcal{T}_k$.

We put for $i = 1, \ldots, 4$

$$\Gamma_k^i = \{\psi \in \Gamma_k \mid \chi_k(x_\psi) = i\}.$$

Obviously the $\psi \in \Gamma_k^i$ are mutually orthogonal for fixed i, k with respect to $a(\cdot, \cdot)$ and any other inner products involving integrals of functions and their derivatives. Thus the operators

$$\sigma_{k,i} = \sum_{\psi \in \Psi_k \cap \Gamma_k^i} \sigma_\psi$$

are in fact *orthogonal projections*

$$\sigma_{k,i} : \mathcal{S}_j \to V_k^i = \text{span}(\Psi_k \cap \Gamma_k^i).$$

Hence the new decomposition (remember that Ψ is the disjoint union of the sets Ψ_k)

$$\sigma = \sigma_0 + \sum_{k=0}^{j} \sum_{i=1}^{4} \sigma_{k,i}$$

implies that for $u \in \mathcal{S}_j$

$$\begin{aligned} a(\sigma u, u) &= a(\sigma_0 u, \sigma_0 u) + \sum_{k=0}^{j} \sum_{i=1}^{4} a(\sigma_{k,i} u, \sigma_{k,i} u) \\ &\leq (1 + 4(j+1))\, a(u,u). \end{aligned} \tag{5.1}$$

However, we intend to improve that upper bound for the largest eigenvalue of σ. This improvement relies on a strengthened Cauchy–Schwarz inequality for the spaces V_k^i, which follows from the next two Lemmas.

LEMMA 3. *There exist constants* $0 < \gamma_0 \leq \gamma_1$ *depending only on the shape regularity of triangles of the initial triangulation* $\mathcal{T}_0$, *such that for* $u \in \text{span}\,\Gamma_k^i$

$$\gamma_0 \sum_{\psi \in \Gamma_k^i} |u(x_\psi)|^2 \leq |u|_1^2 \leq \gamma_1 \sum_{\psi \in \Gamma_k^i} |u(x_\psi)|^2$$

holds.

Proof. Since $u = \sum_{\psi \in \Gamma_k^i} u(x_\psi)\psi$ and those ψ are mutually orthogonal with respect to the $H^1(\Omega)$–inner product we have

$$|u|_1^2 = \sum_{\psi \in \Gamma_k^i} |u(x_\psi)|^2 |\psi|_1^2.$$

Now by the usual affine-transformation technique we see that there exist constants γ_0, γ_1 with the asserted properties such that

$$\gamma_0 \leq |\psi|_1^2 \leq \gamma_1$$

for all $\psi \in \Psi$. This proves the Lemma. ∎

LEMMA 4. *For* $u \in \mathcal{S}_k$, $v \in V_l^i$ *with* $l > k$ *the inequality*

$$(u,v)_{1;\mathcal{T}_0} \leq c \left(\frac{1}{\sqrt{2}}\right)^{l-k} |u|_{1;\mathcal{T}_0} |v|_{1;\mathcal{T}_0}$$

holds, where the positive constant c depends only on the local geometry of the initial triangulation $\mathcal{T}_0$.

Proof. The proof is inspired by the proof of YSERENTANT's [17, Lemma 2.7].

Consider a fixed triangle $\hat{T} \in \mathcal{T}_k$. Decompose $v = v_0 + v_1$ with

$$v_0 = \sum_{\psi \in \Psi_l \cap \Gamma_l^i, x_\psi \in \partial \hat{T}} v(x_\psi)\psi$$

and

$$v_1 = \sum_{\psi \in \Psi_l \cap \Gamma_l^i, x_\psi \in \hat{T} \setminus \partial \hat{T}} v(x_\psi)\psi.$$

As we have $v_1 = 0$ on the boundary of $\hat{T}$ and u is linear on $\hat{T}$ we get by partial integration

$$(u, v_1)_{1|\hat{T}} = 0$$

hence

$$(u, v)_{1|\hat{T}} = (u, v_0)_{1|\hat{T}}.$$

Define Γ as the boundary strip of $\hat{T}$ consisting of all those triangles $T \in \mathcal{T}_l$ such that

(*) there exists a $\psi \in \Psi_l \cap \Gamma_l^i$ such that $T \subset \operatorname{supp} \psi \cap \hat{T}$ and $x_\psi \in \partial \hat{T}$.

Since v_0 is identically zero in $\hat{T}$ outside Γ we get

$$(u, v)_{1|\hat{T}} = (u, v_0)_{1|\Gamma} \leq |u|_{1|\Gamma} \, |v_0|_{1|\Gamma}.$$

By Lemma 3, more precisely its proof, we estimate

$$\begin{aligned} |v_0|_{1|\Gamma}^2 &\leq \gamma_1 \sum_{\psi \in \Psi_l \cap \Gamma_l^i, x_\psi \in \partial \hat{T}} |v_0(x_\psi)|^2 = \gamma_1 \sum_{\psi \in \Psi_l \cap \Gamma_l^i, x_\psi \in \partial \hat{T}} |v(x_\psi)|^2 \\ &\leq \gamma_1 \sum_{\psi \in \Psi_l \cap \Gamma_l^i, x_\psi \in \hat{T}} |v(x_\psi)|^2 \leq \frac{\gamma_1}{\gamma_0} |v|_{1|\hat{T}}^2. \end{aligned}$$

On the other hand the derivatives of u are constant on $\hat{T}$. This gives

$$|u|_{1|\Gamma}^2 = \frac{|\Gamma|}{|\hat{T}|} |u|_{1|\hat{T}}^2.$$

Thus we have to estimate the ratio $|\Gamma|/|\hat{T}|$.

Take any $T \in \mathcal{T}_l$ which belongs to Γ and take $\psi \in \Psi_l$ as the corresponding nodal basis function with property (*). Now [3, Lemma 6.10] states the existence of a triangle $T' \in \mathcal{T}_l$, $T' \subset \operatorname{supp} \psi$ with $\operatorname{depth} T' = l$ such that T'

is regular. Because of $x_\psi \in T' \cap T$ the the local quasi-uniformity property (1.3) states

$$\frac{1}{K}h(T) \leq h(T') \leq Kh(T).$$

Since T' is regular there is due to (T2) and (T3) a regular $T'' \in \mathcal{T}_k$ such that $T' \subset T''$ and therefore due to the regular refinement over $l-k$ steps

$$h(T') = 2^{-(l-k)}h(T'').$$

Due to $x_\psi \in \hat{T} \cap T''$ the same argument as above yields

$$\frac{1}{K}h(\hat{T}) \leq h(T'') \leq Kh(\hat{T}).$$

Thus we get

$$\frac{1}{K^2}2^{-(l-k)}h(\hat{T}) \leq h(T) \leq K^2 2^{-(l-k)}h(\hat{T}).$$

We can restate these inequalities as follows : There exist positive constants α_0, α_1 such that for any $T \in \mathcal{T}_l$ which belongs to Γ the estimate

$$|T| \leq \alpha_1 \left(\frac{1}{4}\right)^{l-k} |\hat{T}|$$

holds and there are at most N such triangles where N is bounded by

$$N \leq \frac{1}{\alpha_0}2^{l-k}.$$

Hence we end up with the estimate

$$\frac{|\Gamma|}{|\hat{T}|} \leq \frac{\alpha_1}{\alpha_0}\left(\frac{1}{2}\right)^{l-k}.$$

Summarizing we get

$$(u,v)_{1|\hat{T}} \leq c\left(\frac{1}{\sqrt{2}}\right)^{l-k} |u|_{1|\hat{T}}|v|_{1|\hat{T}},$$

where c has the asserted property. Summing over all triangles $\hat{T} \in \mathcal{T}_k$ with $\hat{T} \subset T$ for a fixed $T \in \mathcal{T}_0$ gives with an application of the Cauchy–Schwarz inequality

$$(u,v)_{1|T} \leq c\left(\frac{1}{\sqrt{2}}\right)^{l-k} |u|_{1|T}|v|_{1|T}.$$

Thus we can estimate

$$\begin{aligned}
(u,v)_{1;\mathcal{T}_0} &= \sum_{T\in\mathcal{T}_0} \omega(T)(u,v)_{1|T} \le c\left(\frac{1}{\sqrt{2}}\right)^{l-k} \sum_{T\in\mathcal{T}_0} \omega(T)|u|_{1|T}|v|_{1|T} \\
&\le c\left(\frac{1}{\sqrt{2}}\right)^{l-k} \left(\sum_{T\in\mathcal{T}_0} \omega(T)|u|^2_{1|T}\right)^{1/2} \left(\sum_{T\in\mathcal{T}_0} \omega(T)|v|^2_{1|T}\right)^{1/2} \\
&= c\left(\frac{1}{\sqrt{2}}\right)^{l-k} |u|_{1;\mathcal{T}_0}|v|_{1;\mathcal{T}_0}.
\end{aligned}$$

■

Application of (1.2) gives the strengthened Cauchy–Schwarz inequality.

COROLLARY 1. *There exists a positive constant c depending only on the local geometry of the initial triangulation $\mathcal{T}_0$ such that*

$$a(u,v) \le c\frac{\Delta}{\delta}\left(\frac{1}{\sqrt{2}}\right)^{|l-k|} a(u,u)^{1/2}a(v,v)^{1/2}$$

for any $u \in V_k^i$, $v \in V_l^i$.

Usage of this inequality yields for $\sigma_i = \sum_{k=0}^{j} \sigma_{k,i}$, $i = 1,\dots,4$, by Lemma 1.ii that

$$a(\sigma_i u, u) \le c\frac{\Delta}{\delta}\rho(E)a(u,u)$$

with $E = \{2^{-|l-k|/2}\}_{l,k=0}^{j}$. The spectral radius of this matrix can be estimated as

$$\rho(E) \le \|E\|_\infty \le 1 + 2\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k = \frac{\sqrt{2}+1}{\sqrt{2}-1}.$$

Thus

$$a(\sigma u, u) = \sum_{i=0}^{4} a(\sigma_i u, u) \le c_1\frac{\Delta}{\delta}a(u,u)$$

where c_1 depends only on the local geometry of $\mathcal{T}_0$.

We have therefore proven the following

THEOREM 2. *There exists a positive constant c_1 depending only on the local geometry of the initial triangulation $\mathcal{T}_0$ such that*

$$a(B_jA_ju, u) \le c_1\frac{\Delta}{\delta}a(u,u)$$

for all $u \in \mathcal{S}_j$.

REMARK 2. We proved in fact the somewhat sharper result

$$a(B_jA_ju,u) \le \min\left(c_1\frac{\Delta}{\delta}, 1+4(j+1)\right) a(u,u).$$

This estimate helps to explain why for small j one can observe sometimes a linear dependence of the largest eigenvalue on j.

6. Consequences and Remarks

1. Tracing the arguments of [18] from formulas (5.46)–(5.48), i.e., the discussion of our B_j, *backward* to [18, Theorem 4.6] we get for the orthogonal projection version

$$\hat{B}_j^{-1} = A_0Q_0 + \sum_{k=1}^{j} 4^k(Q_k - Q_{k-1})$$

(the operators Q_k have been introduced in Remark 1) of the BPX preconditioner the following

THEOREM 3. *There exist positive constants $\hat{c}_0, \hat{c}_1$ depending only on the local geometry of the initial triangulation $\mathcal{T}_0$ such that*

$$\frac{\delta}{\Delta\bar{\eta}}\frac{1}{\hat{c}_0(j+1)}(\hat{B}_j^{-1}u,u) \le (A_ju,u) \le \hat{c}_1\frac{\Delta}{\delta}(\hat{B}_j^{-1}u,u)$$

for all $u \in \mathcal{S}_j$.

For the dependency on δ and Δ one has to consider first the bilinear form $(\cdot,\cdot)_{1;\mathcal{T}_0}$ instead of $a(\cdot,\cdot)$, i.e., $\delta = \Delta = 1$, and thereafter transform back to the final result by means of (1.2).

2. Theorem 3 may be extended to the case $\mathrm{mes}(\Gamma_D) = 0$. One considers the space $V_{\mathrm{const}} = \mathrm{span}\{1\}$ and the quotient space

$$\mathcal{S}_j/V_{\mathrm{const}} = \mathcal{S}_0/V_{\mathrm{const}} + \sum_{\psi\in\Psi} V_\psi/V_{\mathrm{const}}.$$

Observe that $V_\psi/V_{\mathrm{const}} \cong V_\psi$ and that $a(\cdot,\cdot)$ can be defined canonically on $\mathcal{S}_j/V_{\mathrm{const}}$. Since $\mathcal{S}_j/V_{\mathrm{const}} \cong \mathcal{S}_j \ominus V_{\mathrm{const}}$, where the last term denotes the orthogonal complement of V_{const} in $\mathcal{S}_j$ with respect to $(\cdot,\cdot)_{0;\mathcal{T}_0}$, one can also define $(\cdot,\cdot)_{0;\mathcal{T}_0}$ on $\mathcal{S}_j/V_{\mathrm{const}}$. The Poincaré inequality states the coercivity of $a(\cdot,\cdot)$ on $\mathcal{S}_j/V_{\mathrm{const}}$ and all the considerations of the last sections can be done on that space in a canonical way. Reinterpretation of Theorem 3 in terms of the space $\mathcal{S}_j$ is possible since this theorem does not involve an inversion of any A_k. This yields the validity of the theorem even in the case $\mathrm{mes}(\Gamma_D) = 0$, a fact which is used in [4].

3. The *hierarchical basis preconditioner* of YSERENTANT [17] is given in our setting as the additive Schwarz method with exact solvers corresponding to the *direct* subspace decomposition

$$\mathcal{S}_j = \mathcal{S}_0 \oplus \bigoplus_{\psi \in \Psi_H} V_\psi,$$

where $\Psi_H \subset \Psi$ denotes the set of all hierarchical basis functions of a depth greater or equal than one. Thus the hierarchical basis preconditioner

$$B_H u = A_0^{-1} \pi_0 u + \sum_{\psi \in \Psi_H} \frac{(u, \psi)_{0;\mathcal{T}_0}}{a(\psi, \psi)} \psi$$

consists simply of fewer terms in the last sum compared to the computational BPX preconditioner B_j of (3.1).

Since we have less freedom in decomposing $u \in \mathcal{S}_j$ for an estimate like (2.5), Lemma 1 shows that the thus obtained bound for the smallest eigenvalue of $B_H A_j$ must be smaller than the one for $B_j A_j$.

This helps to understand why *more* terms in the sum improve the relation $\lambda_{\min}(B_H A_j) = \mathcal{O}(1/j^2)$, which is known to be optimal, to $\lambda_{\min}(B_j A_j) = \mathcal{O}(1/j)$ or even better.

Note that the usual estimate for the hierarchical basis preconditioner B_H can be obtained along the same lines of Section 4 and 5 using the stability property [18, Theorem 3.1] and approximation property [18, Theorem 3.2] of the interpolation operators and the strengthened Cauchy-Schwarz inequality [17, Lemma 2.7].

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