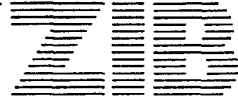


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Multilevel Preconditioned CG-Iterations for Variational Inequalities

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Abstract. We consider such variational inequalities which either describe obstacle problems or result from an implicit time discretization of moving boundary problems of two phase Stefan type. Based on a discretization in space by means of continuous, piecewise linear finite elements with respect to a nested hierarchy of triangulations, in both cases we use iterative processes consisting of inner and outer iterations. The outer iterations are either active set strategies or generalized Newton methods while the inner iterations are preconditioned cg-iterations with multilevel preconditioners.

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1. Introduction

We consider elliptic variational inequalities of the following form: Given a closed subspace V of the Sobolev space $H^1(\Omega)$, Ω being a bounded polygonal domain in Euclidean space \mathbb{R}^2 , a bounded V -elliptic and symmetric bilinear form $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$, a lower semicontinuous convex functional $\phi : V \rightarrow \mathbb{R}$ and a continuous linear functional $l : V \rightarrow \mathbb{R}$, a function $u \in V$ is sought satisfying

$$a(u, v - u) + \phi(v) - \phi(u) \geq l(v - u), \quad v \in V. \quad (1.1)$$

Note that (1.1) constitutes the necessary and sufficient optimality condition for the minimization problem

$$J(u) = \inf_{v \in V} J(v), \quad (1.2)$$

where

$$J(v) = \frac{1}{2}a(v, v) + \phi(v) - l(v), \quad v \in V. \quad (1.3)$$

It is well known (e.g. [1]) that the preceding assumptions guarantee both the existence and the uniqueness of a solution $u \in V$ to (1.1), (1.2), respectively.

In the following we will specialize on two particular cases: The first one is an obstacle problem where the functional ϕ represents the indicator function of a closed convex subset $\mathcal{K} \subset V$ given by

$$\mathcal{K} = \{v \in V | v \leq \psi\} \quad (1.4)$$

with $\psi \in L^\infty(\Omega)$ and \leq denoting the canonical ordering in $L^2(\Omega)$. In the second case we assume the functional ϕ to be of the form

$$\phi(v) = \int_{\Omega} \Phi(v(x)) dx, \quad v \in V, \quad (1.5)$$

where $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is the piecewise quadratic function

$$\Phi(\lambda) = \frac{1}{2}a_1\lambda_-^2 + \frac{1}{2}a_2\lambda_+^2 + s\lambda_+, \quad \lambda \in \mathbb{R} \quad (1.6)$$

with $a_i > 0$, $1 \leq i \leq 2$, and $s > 0$. The variational inequality (1.1) with ϕ given by (1.5) can be interpreted as the weak formulation of an implicitly in time discretized two phase Stefan problem with nominal change of phase temperature at zero (see [11, 18]).

Discretizing (1.1) in space by continuous, piecewise linear finite elements with respect to a triangulation of $\bar{\Omega}$, standard numerical schemes for the solution of the resulting finite dimensional variational inequality are projected relaxation methods in case of an obstacle problem (e.g. [12]) and a modified SOR technique in case of the fully discretized Stefan problem [11]. These iterative methods suffer from rapidly deteriorating convergence rates when proceeding

to more and more refined triangulations which renders them inefficient from a numerical point of view. However, this drawback can be overcome by using multilevel techniques such as multigrid methods with respect to a hierarchy of triangulations. Multigrid approaches to obstacle problems have been developed by various authors ([5, 13, 16, 23, 24]) while a multigrid solution of the two phase Stefan problem has been considered in [18] (see also [17, 19] for treatment of coupled systems of Stefan type equations).

For obstacle type problems an alternative to projected relaxation is to use some sort of linearization techniques based on active set strategies (e.g. [13, 14, 15]). This is an iterative scheme where in each iteration step a set of active constraints is prespecified and then an auxiliary linear system has to be solved for the computation of the new iterate. Note that the multigrid techniques used in [13, 14, 15] consist of outer and inner iterations where the outer iteration is an active set strategy and the inner iterations are multigrid iterations for the approximate solution of the auxiliary problems.

Since for the obstacle problems under consideration the coefficient matrices of the auxiliary systems are symmetric positive definite, an alternative choice for the inner iterations are preconditioned conjugate gradient (pcg) methods, especially those based on multilevel preconditioners such as Yserentant's hierarchical basis preconditioner [27] and the BPX-preconditioner [4] (see also [29] for a comparison). In case of the two phase Stefan problem the use of pcg can also be facilitated by using some linearization technique as an outer iteration. Such approaches have been recently applied in [26] using SSOR as a preconditioner and in [6] where different kinds of standard preconditioners have been implemented depending on different formulations of the basic governing equation (source based technique with liquid fraction and enthalpy method). While a thorough analysis of the pcg techniques has neither been performed in [6] nor in [26], the displayed numerical results did not show an improvement compared to Elliott's modified SOR and to the corresponding cg methods without preconditioning, respectively. A plausible reason for the bad performance of the preconditioners used in [6] and [26] seems to us that these preconditioners are not adapted to the appearance of a discrete free boundary and thus result in an inadequate coupling between the discrete liquid and solid region.

In the following sections we will focus on multilevel preconditioned cg methods for both types of variational inequalities. In particular, we will show that these preconditioners can be derived by only slight modifications of the standard multilevel preconditioners as developed for the cg solution of finite element discretized elliptic boundary value problems.

2. Variational Inequalities of Obstacle Type

In this section we shall deal with variational inequalities arising from obstacle problems where ϕ in (1.1) is the indicator function of a closed convex set $\mathcal{K} \subset V$ of type (1.4) and the bilinear form $a(\cdot, \cdot)$ is given by

$$a(u, v) = \int_{\Omega} \sum_{i,j=1}^2 a_{ij} \partial_i u \partial_j v dx \quad (2.1)$$

with $a_{ij} \in L^{\infty}(\Omega)$ satisfying for almost all $x \in \Omega$

$$\begin{aligned} \text{a) } & a_{ij}(x) = a_{ji}(x), \quad 1 \leq i, j \leq 2, \\ \text{b) } & \alpha_0 |\xi|^2 \leq \sum_{i,j=1}^2 a_{ij}(x) \xi_i \xi_j \leq \alpha_1 |\xi|^2, \quad \xi \in R^2, 0 < \alpha_0 \leq \alpha_1. \end{aligned} \quad (2.2)$$

Note that in this case (1.1) is equivalent to the computation of a function $u \in K$ such that

$$a(u, v - u) \geq l(v - u), \quad v \in \mathcal{K}. \quad (2.3)$$

For the finite element discretization of (2.3) we assume $(\mathcal{T}_k)_{k=0}^l$ to be a regular family of nested triangulations of $\bar{\Omega}$ generated by the refinement process as described in [2, 3]: Given an initial coarse triangulation \mathcal{T}_0 , for $1 \leq k \leq l$ the triangulation \mathcal{T}_k is a direct refinement of \mathcal{T}_{k-1} in the sense that a triangle in \mathcal{T}_k either corresponds to a triangle in \mathcal{T}_{k-1} or is obtained by regular ("red") or irregular ("green") refinement of a triangle in \mathcal{T}_{k-1} . In particular, a refinement of a triangle $T \in \mathcal{T}_{k-1}$ is called regular, if T is subdivided into four congruent triangles by joining the midpoints of the sides while an irregular refinement of T means a bisection by connecting one of the vertices with the midpoint of the opposite side. Triangles in \mathcal{T}_0 are said to be level 0 elements while for $1 \leq k \leq l$ triangles generated by refinement of a level $k-1$ element are referred to as level k elements. In order to keep the interior angles of the triangles bounded away from zero, irregular triangles may not be further refined and to guarantee uniqueness of the decomposition only level $k-1$ elements are permitted for refinement in the construction of \mathcal{T}_k . We refer to $(\mathcal{S}_k)_{k=0}^l$ as the nested sequence of finite element spaces $\mathcal{S}_k \subset V$, $0 \leq k \leq l$, with respect to continuous, piecewise linear finite elements associated with the triangulation \mathcal{T}_k . We denote by $\mathcal{N}_k = \{x_1, \dots, x_{n_k}\}$ the set of nodal points of \mathcal{T}_k and by $\{w_i^{(k)}\}_{i=1}^{n_k}$ the associated nodal basis of \mathcal{S}_k , i.e. $w_i^{(k)}(x_j) = \delta_{ij}$, $x \in \mathcal{N}_k$. Further, we assume that $(\psi^{(k)})_{k=0}^l$ is a sequence of discrete obstacles $\psi^{(k)} \in \mathcal{S}_k$, $0 \leq k \leq l$, approximating the given obstacle ψ in an appropriate sense. For example, the $\psi^{(k)}$'s may be chosen as the L^2 -projections of ψ onto \mathcal{S}_k or if $\psi \in C(\bar{\Omega})$, as the \mathcal{S}_k -interpolates. Correspondingly, we denote by $\mathcal{K}_k = \{v \in \mathcal{S}_k | v \leq \psi^{(k)}\}$ the sets of discrete

constraints. Then, on levels $0 \leq k \leq l$ the finite element approximation of (2.3) amounts to the computation of an element $u \in \mathcal{K}_k$ satisfying

$$a(u, v - u) \geq l(v - u), \quad v \in \mathcal{K}_k. \quad (2.4)$$

It is easy to see that the finite dimensional variational inequality (2.4) is equivalent to a linear complementarity problem.

Lemma 2.1 *An element $u \in \mathcal{K}_k$ is a solution to (2.4) if and only if the vector $u_k \in \mathbb{R}^{n_k}$ with components $u_{k,i} = u(x_i)$, $x_i \in \mathcal{N}_k$, $1 \leq i \leq n_k$, satisfies*

$$\max(A_k u_k - b_k, u_k - \psi_k) = 0, \quad (2.5)$$

where A_k is the (n_k, n_k) stiffness matrix with entries $a_{ij} = a(w_i^{(k)}, w_j^{(k)})$, $1 \leq i, j \leq n_k$, and $b_k \in \mathbb{R}^{n_k}$ and $\psi_k \in \mathbb{R}^{n_k}$ are the vectors with components $b_{k,i} = l(w_i^{(k)})$, $\psi_{k,i} = \psi^{(k)}(x_i)$, $x_i \in \mathcal{N}_k$, $1 \leq i \leq n_k$. Note that (2.5) has to be understood componentwise.

Proof. Since $u \in \mathcal{K}_k$ is equivalent to $u_k \leq \psi_k$, and since the choice $v = u - z$, $z \in \mathcal{S}_k$, $z \geq 0$, arbitrarily given, in (2.4) leads to $A_k u_k \leq b_k$, we have $(u_k - \psi_k)^T (A_k u_k - b_k) \geq 0$. On the other hand, $v = \psi^{(k)}$ in (2.4) yields $(u_k - \psi_k)^T (A_k u_k - b_k) \leq 0$ and thus $(u_k - \psi_k)^T (A_k u_k - b_k) = 0$ which proves (2.5). The converse statement is obvious. ■

In the sequel we will concentrate on an outer-inner iteration technique for the numerical solution of the level l complementarity problem (2.5). The outer iterations are governed by the iterative scheme as presented in [14, 15]:

Outer iteration:

Step 1. Choose a startvector $u_l^{(0)} \in \mathbb{R}^{n_l}$.

Step 2. Given $u_l^{(\nu)} \in \mathbb{R}^{n_l}$, $\nu \geq 0$, compute $u_l^{(\nu+1)} \in \mathbb{R}^{n_l}$ as the solution of the linear algebraic system

$$A_l^{(\nu)} u_l^{(\nu+1)} = b_l^{(\nu)}, \quad (2.6)$$

where for $i = 1, \dots, n_l$ the entries $a_{ij}^{(\nu)}$, $1 \leq j \leq n_l$, of the (n_l, n_l) matrix $A_l^{(\nu)}$ are given by

$$a_{ij}^{(\nu)} = \begin{cases} a_{ij}, & \text{if } (A_l u_l^{(\nu)} - b_l)_i > (u_l^{(\nu)} - \psi_l)_i \\ \delta_{ij}, & \text{otherwise} \end{cases} \quad (2.7.a)$$

while the components b_i^ν , $1 \leq i \leq n_l$, of the vector $b_i^\nu \in R$ are as follows

$$b_i^{(\nu)} = \begin{cases} b_{l,i}, & \text{if } (A_l u_i^{(\nu)} - b_l)_i > (u_i^{(\nu)} - \psi_l)_i \\ \psi_{l,i}, & \text{otherwise.} \end{cases} \quad (2.7.b)$$

Under the assumption (2.2) monotone convergence can be easily deduced [14, 15]:

Theorem 2.2 *For arbitrarily given $u_i^{(0)} \in \mathbb{R}^{n_l}$ let $(u_i^{(\nu)})_{\nu \geq 1}$ be the sequence of iterates obtained by the successive solution of (2.6). Then $(u_i^{(\nu)})_{\nu \geq 1}$ is a monotonically decreasing sequence converging to the unique solution of (2.5).*

In view of (2.7) the computation of the iterate $u_i^{(\nu+1)}$ actually requires the solution of a lower dimensional linear algebraic system: We decompose the set \mathcal{N}_l of nodal points according to $\mathcal{N}_l = \mathcal{N}_l^{(1)} \cup \mathcal{N}_l^{(2)}$ where

$$\begin{aligned} \mathcal{N}_l^{(1)} &= \{x_i \in \mathcal{N}_l \mid a_{ij}^{(\nu)} = a_{ij}, 1 \leq j \leq n_l\}, \\ \mathcal{N}_l^{(2)} &= \mathcal{N}_l \setminus \mathcal{N}_l^{(1)}. \end{aligned}$$

Note that on levels $0 \leq k \leq l$ we have an analogous decomposition $\mathcal{N}_k^{(1)} \cup \mathcal{N}_k^{(2)}$ where

$$\mathcal{N}_k^{(\mu)} = \mathcal{N}_k \cap \mathcal{N}_l^{(\mu)}, \quad 1 \leq \mu \leq 2.$$

Moreover, denoting by N_k , $0 \leq k \leq l$, the index set $N_k = \{1, \dots, n_k\}$ we have $N_k = N_k^{(1)} \cup N_k^{(2)}$ with $N_k^{(\mu)} = \{i \in N_k \mid x_i \in \mathcal{N}_k^{(\mu)}\}$, $1 \leq \mu \leq 2$. We refer to $N_k^{(2)}$ as the set of active nodal points, since with regard to $u_{l,i}^{(\nu+1)} = \psi_{l,i}$, $x_i \in \mathcal{N}_l^{(2)}$, the constraints are active in such points. Correspondingly, the set $\mathcal{N}_l^{(1)}$ is said to be the set of inactive nodal points. Obviously, the computation of $u_i^{(\nu+1)}$ reduces to the determination of an n_l^1 -vector $v_l \in \mathbb{R}^{n_l^1}$, $n_l^1 = \#N_l^{(1)}$, with components $v_{l,i} = u_{l,i}^{(\nu+1)}$, $i \in N_l^{(1)}$, as the solution of the "reduced" algebraic system

$$B_l v_l = c_l, \quad (2.8)$$

where B_l denotes the symmetric, positive definite (n_l^1, n_l^1) matrix with entries $b_{ij} = a_{ij}$, $i, j \in N_l^{(1)}$, and c_l is the n_l^1 -vector with components

$$c_{l,i} = b_{l,i} - \sum_{j \in N_l^{(2)}} a_{ij} \psi_{l,j}, \quad i \in N_l^{(1)}.$$

Given the linear system (2.8) prescribed by the outer iterate $u_i^\nu \in \mathbb{R}^{n_l}$, the inner iterations will be of conjugate gradient type with an appropriate multilevel preconditioner H_l :

Inner iteration:

Step 1. Set $v_l^{(0)} = (u_{l,i}^{(\nu)})_{i \in \mathcal{N}_l^{(1)}}$ and arbitrarily define $p_l^{(-1)} = (p_{l,i}^{(-1)})_{i \in \mathcal{N}_l^{(1)}}$.

Step 2. Given $v_l^{(\mu)}$, $\mu \geq 0$, determine $v_l^{(\mu+1)}$ as follows:

- Solve $H_l z_l^{(\mu)} = r_l^{(\mu)}$, $r_l^{(\mu)} = c_l - B_l v_l^{(\mu)}$.
- Compute

$$\begin{aligned}\beta_\mu &= \begin{cases} (z_l^{(\mu)})^T r_l^{(\mu)} / (z_l^{(\mu-1)})^T r_l^{(\mu-1)}, & \mu \geq 1, \\ 0, & \mu = 0, \end{cases} \\ p_l^{(\mu)} &= z_l^{(\mu)} + \beta_\mu p_l^{(\mu-1)}, \\ \alpha_\mu &= (z_l^{(\mu)})^T r_l^{(\mu)} / (p_l^{(\mu)})^T B_l p_l^{(\mu)}, \\ v_l^{(\mu+1)} &= v_l^{(\mu)} + \alpha_\mu p_l^{(\mu)}.\end{aligned}$$

In the following we will exclusively deal with the construction of the preconditioner H_l . For this purpose we define subspaces $\mathcal{S}_k^{(1)}$ of \mathcal{S}_k , $0 \leq k \leq l$, as the linear spaces of all function in \mathcal{S}_k vanishing in active nodal points, i.e.

$$\mathcal{S}_k^{(1)} = \{v \in \mathcal{S}_k \mid v(x_i) = 0, x_i \in \mathcal{N}_k^{(2)}\}.$$

Obviously, $\dim \mathcal{S}_k^{(1)} = n_k^1$ and

$$\mathcal{S}_k^{(1)} = \text{span}\{w_i^{(k)} \mid i \in N_k^{(1)}\}, \quad (2.9)$$

but in contrast to $(\mathcal{S}_k)_{k=0}^l$, in general the sequence $(\mathcal{S}_k^{(1)})_{k=0}^l$ is nonnested, since the function $v \in \mathcal{S}_{k-1}^{(1)}$ does not necessarily vanish in active nodal points $x_i \in \mathcal{N}_k^{(2)}$. However, if we consider the hierarchical basis functions $\hat{w}_i^{(1)}$, $i \in N_k^{(1)}$, given by

$$\hat{w}_i^{(1)} = \begin{cases} w_i^{(0)}, & \text{if } i \in N_0^{(1)} \\ w_i^{(j)}, & \text{if } i \in N_j^{(1)} \setminus N_{j-1}^{(1)}, 1 \leq j \leq k \end{cases} \quad (2.10)$$

the subspaces

$$\hat{\mathcal{S}}_k^{(1)} = \text{span}\{\hat{w}_i^{(1)} \mid i \in N_k^{(1)}\} \quad (2.11)$$

of dimension $\dim \hat{\mathcal{S}}_k^{(1)} = n_k^1$ form a nested sequence $(\hat{\mathcal{S}}_k^{(1)})_{k=0}^l$. We refer to $I_k : \mathcal{S}_l \rightarrow \mathcal{S}_k$, $0 \leq k \leq l$, as the interpolation operators as defined by $(I_k u)(x_i) = u(x_i)$, $x_i \in \mathcal{N}_k$, and introduce a bilinear form $\hat{a}_H(\cdot, \cdot) : \mathcal{S}_l^{(1)} \times \hat{\mathcal{S}}_l^{(1)} \rightarrow \mathbb{R}$ according to

$$\begin{aligned}\hat{a}_H(u, v) &= a(I_0 u, I_0 v) + \\ &+ \sum_{k=1}^l \sum_{x_i \in \mathcal{N}_k^{(1)} \setminus \mathcal{N}_{k-1}^{(1)}} d_i^{(k)} (I_k u - I_{k-1} u)(x_i) (I_k v - I_{k-1} v)(x_i)\end{aligned} \quad (2.12)$$

where $d_i^{(k)} = a(\hat{w}_i^{(k)}, \hat{w}_i^{(k)})$. We further denote by \hat{H}_l the matrix representation of $\hat{a}_H(\cdot, \cdot)$ with respect to the hierarchical basis (2.10). Note that $\hat{H}_l = (\hat{h}_{ij})_{i,j \in N_l^{(1)}}$ is explicitly given by

$$\hat{h}_{ij} = \begin{cases} a(w_i^{(0)}, w_j^{(0)}) & , \text{ if } i, j \in N_0^{(1)} \\ \delta_{ij} d_i^{(k)} & , \text{ if } i, j \in N_k^{(1)} \setminus N_{k-1}^{(1)}, 1 \leq k \leq l. \end{cases} \quad (2.13)$$

Then, if \hat{B}_l stand for the matrix representation of $a(\cdot, \cdot)$ restricted to $\mathcal{S}_l^{(1)} \times \mathcal{S}_l^{(1)}$, the following result is crucial for the performance of the preconditioned cg inner iteration:

Theorem 2.3 *There exist positive constants c_μ , $1 \leq \mu \leq 2$, depending only on the shape regularity of the triangles in \mathcal{T}_0 and on α_μ , $1 \leq \mu \leq 2$, from (2.2) such that for all $u_l = (u_{l,i})_{i \in N_l^{(1)}} \in \mathbb{R}^{n_l^{(1)}}$*

$$c_1(l+1)^{-2} u_l^T \hat{H}_l u_l \leq u_l^T \hat{B}_l u_l \leq c_2 u_l^T \hat{H}_l u_l. \quad (2.14)$$

Proof. The assertion is a direct consequence of the inequalities

$$c_1(l+1)^{-2} \hat{a}_H(u, u) \leq a(u, u) \leq c_2 \hat{a}_H(u, u), \quad u \in \hat{\mathcal{S}}_l^{(1)},$$

which can be shown following the same lines of proof as in [27] for the finite element discretization of standard second order elliptic boundary value problems. \blacksquare

Since the inner iteration is in terms of the matrix B_l which is the nodal basis representation of $a(\cdot, \cdot)$ restricted to $\mathcal{S}_l^{(1)} \times \mathcal{S}_l^{(1)}$, we need a suitable map $S_l : \hat{\mathcal{S}}_l^{(1)} \rightarrow \mathcal{S}_l^{(1)}$ such that $(S_l u)(x_i) = u(x_i)$, $x_i \in \mathcal{N}_l^{(1)}$, $u \in \hat{\mathcal{S}}_l^{(1)}$. For this purpose we assume $(\hat{u}_{l,i})_{i \in N_l^{(1)}}$ and $(u_{l,i})_{i \in N_l^{(1)}}$ to be the vector representation of $u \in \hat{\mathcal{S}}_l^{(1)}$ and $S_l u \in \mathcal{S}_l^{(1)}$, respectively. Further, if $x_i \in \mathcal{N}_k^{(1)} \setminus \mathcal{N}_{k-1}^{(1)}$, $0 \leq k \leq l$, we denote by $i_\mu(i) \in N_{k-1}$, $1 \leq \mu \leq 2$, the indices of the neighboring nodal points in \mathcal{N}_{k-1} . Then the components $u_{l,i}$, $i \in N_l^{(1)}$, can be recursively computed according to

$$\begin{aligned} u_{l,i} &= \hat{u}_{l,i}, \quad i \in N_0^{(1)}, \\ u_{l,i} &= \hat{u}_{l,i} + \frac{1}{2}(\bar{u}_{l,i_1(i)} + \bar{u}_{l,i_2(i)}), \quad i \in N_k^{(1)} \setminus N_{k-1}^{(1)}, \quad 1 \leq k \leq l, \end{aligned} \quad (2.15)$$

where

$$\bar{u}_{l,i_\mu(i)} = \begin{cases} u_{l,i_\mu(i)} & , \text{ if } i_\mu(i) \in N_{k-1}^{(1)} \\ 0 & , \text{ otherwise.} \end{cases}$$

The matrix representation of the map S_l given by (2.15) results in a lower triangular matrix $S_l = S^{(l)} \cdot S^{(l-1)} \cdot S^{(1)}$ with $S^{(k)} = (S_{ij}^{(k)})_{i,j \in N_l^{(1)}}$ and

$$S_{ij}^{(k)} = \begin{cases} 1 & \text{if } i = j \\ \frac{1}{2} & \text{if } i \in N_k^{(1)}, j = i_\mu(i) \in N_{k-1}^{(1)} \\ 0 & \text{otherwise} \end{cases} \quad (2.16)$$

Theorem 2.4 *Assume that \mathcal{T}_0 is fine enough and the coefficients $a_{ij}(x)$ are sufficiently regular. Then*

$$C_1 u_l^T \hat{B}_l u_l \leq (S_l u_l)^T B_l S_l u_l \leq C_2 u_l^T \hat{B}_l u_l$$

holds with constants C_1, C_2 independent of l .

Proof. The proof is based on the fact that the area of a triangle with diameter h is of order h^2 and for the reason is restricted to two space dimensions. For details we refer to [20]. ■

Hence the matrix

$$H_l = S_l^{-T} \hat{H}_l S_l^{-1} \quad (2.17)$$

is a good preconditioner for B_l and we obtain:

Theorem 2.5 *Let H_l be the (n_l^1, n_l^1) matrix defined by (2.17). Then the condition number estimate holds true*

$$\kappa(H_l^{-1/2} B_l H_l^{-1/2}) \leq C(l+1)^2 \quad (2.18)$$

with a constant C independent l .

Proof. The assertion is a direct consequence of the Theorems 2.3 and 2.4. ■

Remark. Recently, a result similar to the preceding one has been reported in the slightly different context of the iterative solution of elliptic boundary value problems where after some iterations certain components of the iterates practically do not change any more and thus are kept fixed during the subsequent iterations [30].

In the unrestricted case $\mathcal{K}_l = \mathcal{S}_l$ the preconditioner H_l exactly corresponds to that developed in [27] for the hierarchically preconditioned cg iterative solution of second order elliptic boundary value problems. Hence, it is not surprising that (2.18) displays the same moderate growth in the number of refinement levels. Moreover, in view of (2.13) and (2.16) the implementation of the preconditioner can be done inexpensively requiring approximately $7n_l^1$

floating point operations plus the computational work for the solution of a linear system with the (n_0^1, n_0^1) matrix \hat{H}_0 .

Instead of the preconditioner H_l based on the splitting of S_l by means of the interpolation operators I_k , $0 \leq k \leq l$, we can alternatively use the so-called BPX-preconditioner H_X (see [4, 29]) which relies on a splitting involving the L^2 -projections $Q_k : S_l \rightarrow S_k$ given by

$$(Q_k u, v) = (u, v), \quad v \in S_k, \quad 0 \leq k \leq l,$$

where (\cdot, \cdot) stands for the standard L^2 inner product. For the obstacle problems under consideration, denoting by $r^{(\mu)} \in \hat{S}_l^{(1)}$ the defect with respect to the μ -th iterate $u^{(\mu)} \in \hat{S}_l^{(1)}$, the action of H_X^{-1} on $r^{(\mu)}$ is given by

$$H_X^{-1} r^{(\mu)} = (A_0^{(1)})^{-1} Q_0 r^{(\mu)} + \sum_{k=1}^l \sum_{i \in N_k^{(1)}} (d_i^{(k)})^{-1} (r^{(\mu)}, w_i^{(k)}) w_i^{(k)} \quad (2.19)$$

where $A_0^{(1)}$ is the operator associated with the bilinear form $a(\cdot, \cdot)$ restricted to $S_0^{(1)} \times S_0^{(1)}$. Without going into details, using the BPX-preconditioner H_X results in the same moderate growth $\mathcal{O}(l+1)^2$ of the preconditioned stiffness matrix B_l with respect to the number l of refinement levels. However, in contrast to the hierarchical basis preconditioner which deteriorates in 3-D the condition number estimate for the BPX-preconditioner does not depend on the dimension of the underlying problem. Moreover, the computation of $H_X^{-1} r^{(\mu)}$ can be done almost as cheaply as in the case of the hierarchical basis preconditioner by evaluating the double sum in (2.19) in a recursive manner.

Finally, we remark that without major difficulties the presented outer-inner iteration technique can be built into existing adaptive finite element codes like PLTMG [2] or KASKADE [9] provided a reliable local error estimator for obstacle problems is at hand. An edge oriented local error estimator for such problems has recently been developed in [22] by heuristic arguments and successfully applied within a self-adaptive scheme for the solution of a special obstacle problem arising in semi-conductor device simulation. In a forthcoming paper a theoretical justification of this error estimator will be given along with a detailed discussion of the algorithmic aspects of the complete code for both unilateral and bilateral obstacle type problems.

3. Variational Inequalities Arising from two Phase Stefan Problems

The two phase Stefan problem is a moving boundary problem describing the temporal and spatial temperature distribution $\Theta(x, t)$, $(x, t) \in Q := \Omega \times (0, T)$, of a heat conducting substance undergoing a change of phase at a certain change of phase temperature Θ_c which we assume to be zero. In its fixed domain formulation the problem can be described by the degenerate parabolic pde

$$\frac{\partial}{\partial t} H(u) - \Delta u = f \text{ in } Q \quad (3.1)$$

where $u = K(\Theta) = \int_0^\Theta \kappa(\tau) d\tau$ is the generalized temperature, $H(u) = E(K^{-1}u)$ the generalized enthalpy with $E = \alpha + s$, $\alpha = \int_0^\Theta c(\tau) d\tau$, denoting the standard enthalpy, and f stands for source/sink terms (c, κ and s are the volumetric heat capacity, thermal conductivity and latent energy content, respectively). Of course (3.1) has to be completed by prescribing initial and boundary values, and a solution has to be understood in an appropriate weak sense (for details including existence and uniqueness results we refer to [21]). The fixed domain formulation has the advantage that the free boundary $\Sigma = \{(x, t) \in Q | u(x, t) = 0\}$ can be computed a posteriori and that it allows the occurrence of so-called mushy regions, i.e. subdomains of constant change of phase temperature having positive measure. In the following the physical data c, κ and s are supposed to be piecewise constant, i.e.

$$c(\Theta) = c_\mu, \kappa(\Theta) = \kappa_\mu, s(\Theta) = s_\mu \text{ for } (-1)^\mu \Theta < 0, 1 \leq \mu \leq 2,$$

with $0 < c_1 \leq c_2, 0 < \kappa_1 \leq \kappa_2$ and $0 = s_1 < s_2 = s$.

Given a partition $\{t_0 = 0 < t_1 < \dots < t_M = T\}$ of the time interval $[0, T]$, we assume $H^{m-1} \in H(u(\cdot, t_{m-1}))$ to be known and then discretize (3.1) implicitly in time by the backward Euler scheme with respect to the subinterval $[t_{m-1}, t_m]$. In its weak formulation the semidiscretized problem can then be written as a variational inequality of the form (1.1) where $u \in V \subset H^1(\Omega)$ is an approximation to $u(\cdot, t_m)$, $a(\cdot, \cdot)$ and l are given by

$$a(u, v) = \tau_m \int_\Omega \nabla u \cdot \nabla v dx, u, v \in V, \tau_m = t_m - t_{m-1}, \quad (3.2)$$

$$l(v) = \int_\Omega (\tau_m f^m + H^{m-1}) v dx, v \in V, f^m = f(\cdot, t_m) \quad (3.3)$$

and the functional ϕ is defined by (1.5), (1.6) with $a_\mu = c_\mu/\kappa_\mu, 1 \leq \mu \leq 2$ (for details see e.g. [11, 18]).

As in the preceding section we consider a family $(\mathcal{T}_k)_{k=0}^l$ of triangulations of $\bar{\Omega}$ but this time we assume the family to be quasiuniform, i.e.

$$\eta_1 2^{-l} H \leq \text{diam}(T) \leq \eta_2 2^{-l} H, \quad T \in \mathcal{T}_l \quad (3.4)$$

with constants $0 < \eta_1 \leq \eta_2$ and $H > 0$. We further assume the time step size τ_m to be chosen such that the temporal and spatial discretizations are coupled by

$$4^{-l} h^2 \leq \tau_m \leq 4^{-l} H^2. \quad (3.5)$$

As in Section 2 we denote by $(\mathcal{S}_k)_{k=0}^l$ the nested sequence of finite element spaces $\mathcal{S}_k \subset V$, $0 \leq k \leq l$, generated by continuous, piecewise linear finite elements with respect to \mathcal{T}_k , by $\mathcal{N}_k = \{x_1, \dots, x_{n_k}\}$ and $N_k = \{1, \dots, n_k\}$ the set of nodal points and its related index set, respectively, and by $w_i^{(k)}$, $1 \leq i \leq n_k$, the standard nodal basis functions. The usual finite element approach is marred by the fact that the computation of $\phi(v)$, $v \in \mathcal{S}_k$, would require to determine in each triangle $T \in \mathcal{T}_k$ where v changes sign. Since this is impracticable from a computational point of view, we replace the convex integrand in (1.5) by its \mathcal{S}_k -interpolate, i.e., we approximate ϕ by the functional $\phi_k : \mathcal{S}_k \rightarrow R$ given by

$$\phi_k(v) = \sum_{T \in \mathcal{T}_k} \frac{1}{3} \text{area}(T) \sum_{i=1}^3 \Phi(v(x_i^T)), \quad v \in \mathcal{S}_k \quad (3.6)$$

where x_i^T , $1 \leq i \leq 3$, are the vertices of the triangle $T \in \mathcal{T}_k$. Then the level k finite element discretization of (1.1) amounts to the computation of a function $u \in \mathcal{S}_k$ satisfying

$$\tau_m a(u, v - u) + \phi_k(v) - \phi_k(u) \geq l(v - u), \quad v \in \mathcal{S}_k. \quad (3.7)$$

We denote by A_k the (n_k, n_k) stiffness matrix associated with $a(\cdot, \cdot)$ on $\mathcal{S}_k \times \mathcal{S}_k$, by Γ_k the (n_k, n_k) diagonal matrix with entries

$$\gamma_{ii}^{(k)} = \frac{1}{3} \sum_{T \subset \text{supp } w_i^{(k)}} \text{area}(T), \quad 1 \leq i \leq n_k,$$

and by $b_k \in R^{n_k}$ the vector with components $b_{k,i} = l(w_i^{(k)})$, $1 \leq i \leq n_k$. Then, if $\partial\Phi$ represents the subgradient of the convex function Φ , the variational inequality (3.7) is equivalent to the algebraic inclusion

$$\Gamma_k^{-1}(b_k - \tau_m A_k u_k) \in \partial\Phi(u_k) \quad (3.8)$$

which has to be understood componentwise. By a well known argument from convex analysis [10] the inclusion (3.8) is equivalent to

$$u_k \in \partial\Phi^*(\Gamma_k^{-1}(b_k - \tau_m A_k u_k)) \quad (3.9)$$

where Φ^* stands for the Fenchel conjugate of Φ . Since in the present case the subgradient $\partial\Phi^*$ of the Fenchel conjugate is the continuous, piecewise linear function

$$\partial\Phi^*(\lambda) = \begin{cases} a_2^{-1}(\lambda - s) & , \lambda > s \\ 0 & , \lambda \in [0, s] \\ a_1^{-1}\lambda & , \lambda < 0, \end{cases} \quad (3.10)$$

the inclusion (3.9) reduces to a nonlinear algebraic system which can be written as

$$F_k(u_k) := u_k - \partial\Phi^*(\Gamma_k^{-1}(b_k - \tau_m A_k u_k)) = 0. \quad (3.11)$$

It has been shown in [18] that F_k is a continuous, surjective M-function which implies that (3.11) possesses a unique solution $u_k^* \in \mathbb{R}^{n_k}$.

In the sequel we will solve (3.11) on level l by an outer-inner iteration where the outer iteration is of generalized Newton type

$$J_l(u_i^{(\nu)})(u_i^{(\nu+1)} - u_i^{(\nu)}) = F_l(u_i^{(\nu)}), \quad \nu \geq 0, \quad (3.12)$$

with $J_l(u_i^{(\nu)}) \in \partial F_l(u_i^{(\nu)})$, $\partial F_l(u_i^{(\nu)})$ being the generalized Jacobian of the locally Lipschitzian function F_l in the sense of Clark [7] (see [18] for details).

Outer iteration:

Step 1 Choose an appropriate startvector $u_i^{(0)}$.

Step 2 Given $u_i^{(\nu)}$, $\nu \geq 0$, choose $J_l(u_i^{(\nu)}) \in \partial F_l(u_i^{(\nu)})$ and compute $u_i^{(\nu+1)}$ as the solution to (3.12).

Only for the purpose to simplify notations, in the sequel we suppose that the solution $u_i^* \in \mathcal{S}_l$ of (3.11) satisfies the discrete nondegeneracy condition

$$u_{i,i}^* = 0 \Leftrightarrow \partial\Phi(u_{i,i}^*) \in (0, s), \quad 1 \leq i \leq n_l. \quad (3.13)$$

Then there exists a neighborhood $\mathcal{U}(u_i^*)$ such that for all $v_l \in \mathcal{U}(u_i^*)$ the Jacobian $\partial F_l(v_l)$ is single-valued with elements $(\partial F_l(v_l))_{ij}$, $1 \leq j \leq n_l$, given by

$$(\partial F_l(v_l))_{ij} = \begin{cases} \delta_{ij} + a_2^{-1} \tau_m a_{ij} / \gamma_{ii}^{(l)}, & (b_l - \tau_m A_l v_l)_i / \gamma_{ii}^{(l)} \geq s \\ \delta_{ij}, & 0 < (b_l - \tau_m A_l v_l)_i / \gamma_{ii}^{(l)} < s \\ \delta_{ij} + a_1^{-1} \tau_m a_{ij} / \gamma_{ii}^{(l)}, & (b_l - \tau_m A_l v_l)_i / \gamma_{ii}^{(l)} \geq 0 \end{cases} \quad (3.14)$$

We assume $u_i^{(\nu)} \in \mathcal{U}(u_i^*)$ and decompose the set \mathcal{N}_l of nodal points by $\mathcal{N}_l = \bigcup_{\mu=1}^3 \mathcal{N}_l^{(\mu)}$ where

$$\begin{aligned} \mathcal{N}_l^{(1)} &= \{x_i \in \mathcal{N}_l \mid (b_l - \tau_m A_l u_i^{(\nu)})_i / \gamma_{ii}^{(l)} \leq 0\}, \\ \mathcal{N}_l^{(2)} &= \{x_i \in \mathcal{N}_l \mid (b_l - \tau_m A_l u_i^{(\nu)})_i / \gamma_{ii}^{(l)} \geq s\}, \\ \mathcal{N}_l^{(3)} &= \{x_i \in \mathcal{N}_l \mid 0 < (b_l - \tau_m A_l u_i^{(\nu)})_i / \gamma_{ii}^{(l)} < s\}, \end{aligned} \quad (3.15)$$

and we set

$$\mathcal{N}_k^{(\mu)} = \mathcal{N}_k \cap \mathcal{N}_l^{(\mu)}, \quad 1 \leq \mu \leq 3, \quad 0 \leq k \leq l.$$

Further, for $0 \leq k \leq l$ we define $n_k^{(\mu)} = \#\mathcal{N}_k^{(\mu)}$, $1 \leq \mu \leq 3$, and we denote by $N_k^{(\mu)} = \{i \in \mathcal{N}_k \mid x_i \in \mathcal{N}_k^{(\mu)}\}$, $1 \leq \mu \leq 3$, the corresponding index sets. Then, if we order the nodal points in a blockwise manner according to the decomposition (3.15), in view of (3.14) the linear system (3.12) can be written as the following 3×3 block system

$$\begin{bmatrix} \Gamma_{11} + a_1^{-1}\tau_m A_{11} & a_1^{-1}\tau_m A_{12} & a_1^{-1}\tau_m A_{13} \\ a_2^{-1}\tau_m A_{12} & \Gamma_{22} + a_2^{-1}\tau_m A_{22} & a_2^{-1}\tau_m A_{23} \\ 0 & 0 & \Gamma_{33} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad (3.16)$$

where $z_l = (z_1, z_2, z_3)^T = u_l^{(\nu+1)} - u_l^{(\nu)}$ and $c_l = (c_1, c_2, c_3)^T$ with

$$c_\mu = -(\Gamma_l F_l(u_l^{(\nu)}))_\mu, \quad 1 \leq \mu \leq 2, \quad c_3 = -(\Gamma_l u^{(\nu)}). \quad (3.17)$$

Since $z_3 = -(u_l^{(\nu)})_3$, we may eliminate the unknowns corresponding to nodal points $x_i \in \mathcal{N}_l^{(3)}$ thus reducing (3.16) to a 2×2 block system. Although the associated 2×2 block coefficient matrix is not symmetric, a simple multiplication with the block diagonal matrix $\text{diag}(a_\mu I_{\mu\mu})_{\mu=1}^2$ results in the following 2×2 block system with symmetric, positive definite block coefficient matrix

$$B_l z_l = \begin{bmatrix} a_1 \Gamma_{11} + \tau_m A_{11} & \tau_m A_{12} \\ \tau_m A_{12}^T & a_2 \Gamma_{22} + \tau_m A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{bmatrix} = \tilde{c}_l \quad (3.18)$$

where

$$\tilde{c}_\mu = a_\mu c_\mu - \tau_m A_{\mu 3} (u_l^{(\nu)})_3, \quad 1 \leq \mu \leq 2. \quad (3.19)$$

The inner iterations for the solution of (3.12) will be of preconditioned conjugate gradient type applied to (3.18) using $H_l = \text{diag}(H_{\mu\mu})_{\mu=1}^2$ as a preconditioner where $H_{\mu\mu}$ are appropriate preconditioners for the diagonal blocks $B_{\mu\mu} = a_\mu \Gamma_{\mu\mu} + \tau_m A_{\mu\mu}$, $1 \leq \mu \leq 2$.

Inner iterations:

Step 1 Set $z_l^{(0)} = 0$ and arbitrarily define $p_l^{(-1)}$.

Step 2 Given $z_l^{(\mu)}$, $\mu \geq 0$, determine $z_l^{(\mu+1)}$ as follows:

- Solve $H_l v_l^{(\mu)} = r_l^{(\mu)}$, $r_l^{(\mu)} = \tilde{c}_l - B_l z_l^{(\mu)}$.

• Compute

$$\begin{aligned}\beta_\mu &= \begin{cases} (v_i^{(\mu)})^T r_i^{(\mu)} / (v_i^{(\mu-1)})^T r_i^{(\mu-1)}, & \mu \geq 1 \\ 0, & \mu = 0 \end{cases} \\ p_i^{(\mu)} &= v_i^{(\mu)} + \beta_\mu p_i^{(\mu-1)}, \\ \alpha_\mu &= (v_i^{(\mu)})^T r_i^{(\mu)} / (p_i^{(\mu)})^T B_i p_i^{(\mu)}, \\ z_i^{(\mu+1)} &= z_i^{(\mu)} + \alpha_\mu p_i^{(\mu)}.\end{aligned}$$

Remarks. If $H_{\mu\mu} = B_{\mu\mu}$, $1 \leq \mu \leq 2$, the inner iteration corresponds to the generalized cg method in [8].

In case $A_{12} \neq 0$ the inner iteration is closely related to a domain decomposition technique, namely an additive Schwarz iteration with minimal overlap of the subregions $\Omega_\mu \subset \Omega$, $1 \leq \mu \leq 2$, given by $\bar{\Omega}_\mu = \bigcup_{i \in N(\mu)} \text{supp } w_i^{(l)}$, $1 \leq \mu \leq 2$.

The rest of this section is devoted to the construction of appropriate hierarchical preconditioners $H_{\mu\mu}$ for $B_{\mu\mu}$. Reminding the fact that (3.18) stems from the discretization of a parabolic problem with the time step size τ_m being related to the triangulations in space by (3.5), we follow Yserentant's approach in [28] and actually work with a hierarchy $(\mathcal{T}_k)_{k=k_{\min}}^l$ where k_{\min} is the smallest integer between 0 and $l-1$ such that

$$4^{-(k+1)_{\min}} h^2 \leq \tau_m \leq 4^{-k_{\min}} H^2. \quad (3.20)$$

For $\mu_1 \in \{1, 2\}$ we denote by μ_2 the complementary index $\mu_2 = \{1, 2\} \setminus \{\mu_1\}$. Then, for $k_{\min} \leq k \leq l$ we define finite element spaces $\mathcal{S}_k^{(1)}$ and $\mathcal{S}_k^{(2)}$ according to

$$\mathcal{S}_k^{(\mu)} = \{v \in \mathcal{S}_k \mid v(x_i) = 0, x_i \in \mathcal{N}_k^{(\mu)} \cup \mathcal{N}_k^{(3)}\}, \quad 1 \leq \mu_1 \leq 2. \quad (3.21)$$

Note that $\tau_m A_{\mu\mu}$ is the matrix representation of $\tau_m a(\cdot, \cdot)$ restricted to $\mathcal{S}_l^{(\mu)} \times \mathcal{S}_l^{(\mu)}$ and that $a_\mu \Gamma_{\mu\mu}$ is the matrix representation of $\partial \Phi_l(\cdot)$ on $\mathcal{S}_l^{(\mu)}$, $1 \leq \mu \leq 2$. We further define hierarchical basis functions $\hat{w}_i^{(\mu)}$, $i \in \mathcal{N}_k^{(\mu)}$, $k_{\min} \leq k \leq l$, $1 \leq \mu \leq 2$, by

$$\hat{w}_i^{(\mu)} = \begin{cases} w_i^{(k_{\min})} & , \quad \text{if } i \in \mathcal{N}_{k_{\min}}^{(\mu)} \\ w_i^{(j)} & , \quad \text{if } i \in \mathcal{N}_j^{(\mu)} \setminus \mathcal{N}_{j-1}^{(\mu)}, \quad k_{\min} + 1 \leq j \leq k, \end{cases}$$

we set

$$\hat{\mathcal{S}}_k^{(\mu)} = \text{span}\{\hat{w}_i^{(\mu)} \mid i \in \mathcal{N}_k^{(\mu)}\} \quad 1 \leq \mu \leq 2, \quad (3.22)$$

and we denote by $S_l^{(\mu)}$, $1 \leq \mu \leq 2$, the matrix representations of the maps $S_l^{(\mu)} : \hat{\mathcal{S}}_l^{(\mu)} \rightarrow \mathcal{S}_l^{(\mu)}$ being defined in the same way as in the previous section.

Following [28] we introduce bilinear forms $\hat{a}_{H_{\mu\mu}}(\cdot, \cdot)$ on $\hat{S}_l^{(\mu)} \times \hat{S}_l^{(\mu)}$, $1 \leq \mu \leq 2$, by

$$\begin{aligned} \hat{a}_{H_{\mu\mu}}(u, v) &= 4^{-k_{\min}} H^2 \sum_{x_i \in \mathcal{N}_{k_{\min}}^{(\mu)}} I_{k_{\min}} u(x_i) I_{k_{\min}} v(x_i) + \\ &+ \tau_m \sum_{k=k_{\min}+1}^l \sum_{x_i \in \mathcal{N}_k^{(\mu)} \setminus \mathcal{N}_{k-1}^{(\mu)}} (I_k u - I_{k-1} u)(x_i) (I_k v - I_{k-1} v)(x_i), \end{aligned}$$

and we refer to $\hat{H}_{\mu\mu}$, $1 \leq \mu \leq 2$, as the diagonal matrixes representing these bilinear forms with respect to the hierarchical basis functions. Setting

$$H_{\mu\mu} = (S_l^{(\mu)})^{-T} \hat{H}_{\mu\mu} (S_l^{(\mu)})^{-1} \quad 1 \leq \mu \leq 2, \quad (3.23)$$

we can show:

Theorem 3.1 *There exist positive constants $c_{\mu\nu}$, $1 \leq \mu, \nu \leq 2$, depending only on a lower bound for the interior angles of the triangles in \mathcal{T}_0 and on the constants η_μ in (3.4) and a_μ , $1 \leq \mu \leq 2$, such that*

$$(H_{\mu\mu}^{-1/2} B_{\mu\mu} H_{\mu\mu}^{-1/2}) \leq c_{\mu 1}^{-1} c_{\mu 2} (l - k_{\min} + 1)^2 \quad 1 \leq \mu \leq 2, \quad (3.24)$$

Proof. The condition number estimates (3.24) are direct consequences of the inequalities

$$(l - k_{\min} + 1)^2 c_1 \hat{a}_{H_{\mu\mu}}(v, v) \leq \partial \Phi_l(v) + \tau_m a(v, v) \leq c_2 \hat{a}_{H_{\mu\mu}}(v, v), \quad v \in \hat{S}_l^{(\mu)},$$

which can be verified in much the same way as in [28].

Although the discretization of the two phase Stefan problem (3.1) is based on a fixed domain approach, the splitting $B_l = D_l - C_l$ of the 2×2 block matrix B_l in its block diagonal part D_l and its block off-diagonal part $-C_l$ has the flavor of a front tracking technique, since the diagonal blocks $B_{\mu\mu}$, $1 \leq \mu \leq 2$, correspond to the discretization of the heat equation in the solid region Ω_1 and liquid region Ω_2 , respectively. The preceding result tells us that $H_l = \text{diag}(H_{\mu\mu})_{\mu=1}^2$ is a good preconditioner for $D_l = \text{diag}(B_{\mu\mu})_{\mu=1}^2$. On the other hand, the efficiency of the preconditioned conjugate gradient type inner iteration does not only depend on $\kappa(H_l^{-1/2} D_l H_l^{-1/2})$ but also on the algebraic properties of

$$H_l^{-1/2} C_l H_l^{-1/2} = \begin{bmatrix} 0 & \tau_m H_{11}^{-1/2} A_{12} H_{22}^{-1/2} \\ \tau_m H_{22}^{-1/2} A_{12}^T H_{11}^{-1/2} & 0 \end{bmatrix}.$$

As shown in [8] good performance of the pcg method can be expected if the off-diagonal blocks either have eigenvalues of small magnitude or are of small rank. In our situation the latter case does apply:

We denote by $\mathcal{N}_l^{(1,2)}$ respectively $\mathcal{N}_l^{(2,1)}$ the set of all nodal points x_i respectively $x_i \in \mathcal{N}_l^{(2)}$, having a level l neighboring nodal point in $\mathcal{N}_l^{(2)}$ respectively $\mathcal{N}_l^{(1)}$. Then, if we set $n_l^{(1,2)} = \#\mathcal{N}_l^{(1,2)}$, $n_l^{(2,1)} = \#\mathcal{N}_l^{(2,1)}$, both off-diagonal blocks are of rank $r_l = \min(n_l^{(1,2)}, n_l^{(2,1)})$. The actual size of r_l depends on the number $n_l^{(3)}$ of nodal points $x_i \in \mathcal{N}_l^{(3)}$ being located on the discrete free boundaries or within a discrete mushy region. The ideal situation would be when the nodal points in $\mathcal{N}_l^{(3)}$ completely separate the discrete liquid from the discrete solid region in which case $r_l = 0$. For example, if the continuous problem exhibits a mushy region separating both phases, we can expect this situation for sufficiently large l . On the other hand, the worst scenario would be $n_l^{(3)} = 0$ which implies $r_l = \mathcal{O}(h_l^{-1})$ and thus results in an exponential growth $\mathcal{O}(2^{-l})$ of r_l with respect to the refinement levels. However, this situation is unlikely to occur even if there is no mushy region and the continuous free boundaries are one-dimensional manifolds. In this case, under very moderate regularity assumptions the discrete free boundaries approximate the continuous ones, and we can expect decreasing ratios $r_l/n_l^{(3)}$ with increasing l . The quantitative behavior of r_l with respect to the number l of refinement levels is a function of the actual shape of the free boundaries and thus problem dependent.

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