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An Adaptive Finite Element Method for Convection–Diffusion Problems by Interpolation Techniques

Technical Report TR 91–4 (March 1991)

Herausgegeben vom
Konrad-Zuse-Zentrum für Informationstechnik Berlin
Heilbronner Str. 10
1000 Berlin 31
Verantwortlich: Dr. Klaus André
Umschlagsatz und Druck: Rabe KG Buch-und Offsetdruck Berlin

ISSN 0933-789X

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Abstract

For adaptive solution of convection–diffusion problems with the streamline–diffusion finite element method, an error estimator based on interpolation techniques is developed. It can be shown that for correctness of this error estimator a restriction of the maximum angle is to be sufficient. Compared to usual methods, the adaptive process leads to more accurate solutions at much less computational cost. Numerical tests are enclosed.

Keywords: Adaptive finite elements, convection–diffusion equation, internal and boundary layers, streamline–diffusion

Subject Classifications: AMS(MOS): 65N15, 65N30

The author would like to thank Erlinda Cadano-Körnig for her excellent TEX–typing of the manuscript and my colleagues at the Konrad–Zuse–Zentrum for stimulating discussions.

0. Introduction

In the last years adaptive strategies in solving partial differential equations more and more enter into applications. Still more, efficient adaptivity is often the only way to compute accurate solutions to complex "real life" problems, such as problems in fluid mechanics with boundary and internal layers. Following the first step of applications to complicated flows, one can observe that adaptive methods open fascinating new possibilities using locally refined meshes.

For quantitative error control a posteriori error estimates of the form

$$\|u - u_h\| \sim \mathrm{ERROR}(h, u_h) \tag{0.1}$$

in a suitable error norm are needed. Here, u is the exact solution and u_h is the finite element solution on the mesh T described through the mesh parameter h. Now the goal is to reduce the error constructing an algorithm for the mesh T, such that the number of degrees of freedom is nearly optimal. It means that a sequence of meshes should be generated which realize an approximation solution of the complex nonlinear problem

$$\|u - u_h\| \Longrightarrow \min_T \tag{0.2}$$

for all T with a fixed number of degrees of freedom. Note that the minimization process based on seeking the equidistribution of all element errors to the global quantity in (0.1).

In the recent paper of ERIKSSON/JOHNSON [3], adaptive streamline–diffusion finite element methods for convection–diffusion problems are discussed. Therein, the error estimates are based on a representation of the error in terms of the solution of a certain dual problem. For their discretization, they use triangulations which satisfy the minimal angle condition. In some sense this condition is very restrictive, especially in the case where functions change more rapidly in one direction than in another direction. Therefore, it is of interest to develop methods for computational fluid problems where the mesh can be stretched, more preciously, the mesh orientation should be adjusted to the flux direction. That mean· in (0.1)

$$\mathrm{ERROR}(h, u_h) \leq C \tag{0.3}$$

if one angle tends to zero with C independent of that angle.

An anisotropic refinement strategy which can be used if some information about the appearance of layers is known in advance is introduced in KORNHUBER/ROITZSCH [8]. Therein, the mesh control is based on the gradient of the numerical solution and therefore mesh refinement takes place only in regions where the numerical solution rapidly changes.

Here, an error estimator is obtained by using the standard local finite element interpolation estimates. Together with the main result of BABUŠKA/AZIS [1] it can be shown that for correctness of the error estimator a restriction of the maximum angle is to be sufficient. For a one–dimensional convection-diffusion problem with dominated convection, a theoretical foundation of the equidistribution of such an error estimator over all elements was developed in LANG [9].

This paper consists of six sections. In Section 1 the model convection–diffusion type problem is considered. The transformation from the standard triangle to a general triangle is studied in Section 2. The choice of the local parameter of the streamline–diffusion method is motivated in Section 3. A priori interpolation error estimates are developed in Section 4 and the adaptive process is described in Section 5 . Finally, in Section 6, the results of some numerical experiments are given.

1. The Model Problem

In this paper we consider the model problem

$$\begin{aligned} -\varepsilon\Delta u + \beta\nabla u &= f \text{ in } \Omega \\ u &= u_0 \text{ on } \Gamma_0 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_1 . \end{aligned} \tag{1.1}$$

Here, Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, $\beta : \Omega \longrightarrow \mathbb{R}^2$ is a given smooth vector field with

$$-\operatorname{div} \beta \geq 0 \text{ in } \Omega , \tag{1.2}$$

$0 < \varepsilon \ll 1$ is a given small diffusion coefficient. The part Γ_0 of Γ includes all inflow boundary, that means

$$\beta \cdot n \geq 0 \text{ on } \Gamma_1 . \tag{1.3}$$

It is well known that internal or boundary layer may occur in the solution of (1.1).
Let $T_h = \{K\}$ be a partition of the domain Ω into triangular elements K, such that

$$\zeta(K) \leq \zeta_0 < \pi , \quad \forall K \in T_h \tag{1.4}$$

where $\zeta(K)$ denotes the maximum angle of K.
Given $D \subset \mathbb{R}^2$ we define the usual norm and seminorm in the Sobolev spaces $H^l(D)$, $l \geq 1$, with

$$|v|_{l,D}^2 = \|D^\nu v\|_{0,D}^2 , \quad |\nu| = l$$

and

$$\|v\|_{l,D}^2 = \|v\|_{0,D}^2 + \sum_{1\leq m\leq l} |v|_{m,D}^2 ,$$

$$D^\nu = \frac{\partial^{\nu_1+\nu_2}}{\partial_{x_1}^{\nu_1}\partial_{x_2}^{\nu_2}} , \; \nu = (\nu_1,\nu_2) , \; |\nu| = \nu_1 + \nu_2$$

where $\|\cdot\|_{0,D}$ stand for the usual norm in $L_2(D)$.
Denoting by $S(T_h)$ the set of all continuous functions on Ω, which are linear on each $K \in T_h$, we define the finite element solution space by

$$S_h := \{v_h \in S(T_h) \mid v_h = u_0 \text{ on } \Gamma_0\}$$

and the special spaces

$$\begin{aligned} H_0^1 &:= \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0\} , \\ V_h &:= \{v_h \in S(T_h) \mid v_h = 0 \text{ on } \Gamma_0\} . \end{aligned}$$

For the solution of (1.1) we use the following streamline–diffusion method:
Find $u_h \in S_h$ such that

$$\varepsilon(\nabla u_h, \nabla v_h) + (\beta\nabla u_h, v_h + \delta\beta\nabla v_h) = (f, v_h + \delta\beta v_h), \; \forall\, v_h \in V_h . \tag{1.5}$$

Here $\delta : \Omega \longrightarrow \mathbb{R}$ is a positive, piecewise constant function which will be specified later, and $(\cdot,\cdot)$ denotes the L_2–inner product.
For the method (1.5), we introduce the bilinear form

$$B_\varepsilon(w,v) := \varepsilon(\nabla w, \nabla v) + (\beta\nabla w, v + \delta\beta\nabla v), \; v,w \in H^1(\Omega) \tag{1.6}$$

such that we can rewrite (1.5) as

$$B_\varepsilon(u_h, v_h) = (f, v_h) , \quad \forall\, v_h \in V_h . \tag{1.7}$$

Further, for an error estimate in Section 4 we introduce the special norm

$$\|v\|_\varepsilon^2 := \varepsilon\|\nabla v\|_{0,\Omega}^2 + \delta\|\beta\nabla v\|_{0,\Omega}^2 + \tfrac{1}{2}\|\,|\operatorname{div}\beta|^{1/2} v\|_{0,\Gamma}^2 + \tfrac{1}{2}|v|_{0,\Gamma_1}^2 \tag{1.8}$$

for all $v \in H_0^1$ where

$$|v|_{0,\Gamma_1}^2 := \int_{\Gamma_1} v^2 \beta \cdot n \; ds . \tag{1.9}$$

2. Local Interpolation Error Bounds

Let K denote a typical triangle of T_h. Since the norms of $H^l(D)$ are invariant with respect to the rotation of the coordinates, we can restrict ourselves to the case when one side of the triangle is along a coordinate axis (see Fig. 2.1).

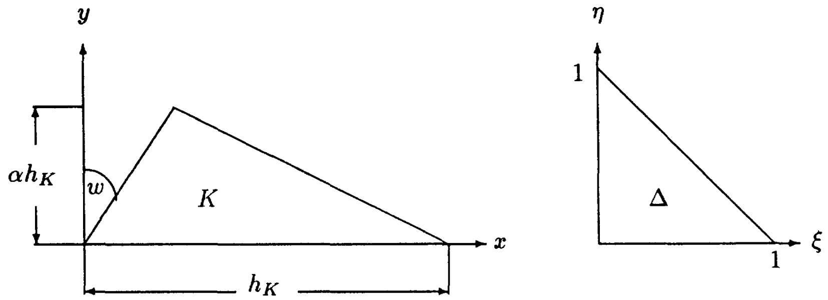


Figure 2.1: Triangles

Without any restriction of the general validity we have $0 < \alpha \leq 1$ and $|w| < \pi/2$. Further let $\gamma := \tan w$. Denoting by Δ the unit triangle with the vertices $P_1 = (0,0)$, $P_2 = (1,0)$ and $P_3 = (1,0)$ the affine–linear mapping $F : \Delta \to K$ is provided by

$$\begin{aligned} x &= h_K \xi + \alpha\gamma h_K \eta \\ y &= \alpha h_K \eta \end{aligned} \tag{2.1}$$

where $(\xi, \eta) \in \Delta$.
For the Jacobian matrix J of this mapping we have $|J| = \alpha h_K^2$. The inverse mapping F^{-1} : $K \to \Delta$ is given by

$$\begin{aligned} \xi &= \frac{1}{h_K} x - \frac{\gamma}{h_K} y \\ \eta &= \frac{\alpha}{h_K} y \end{aligned} \tag{2.2}$$

for $(x, y) \in K$.
We define for $D = K$ and $D = \Delta$ the special spaces

$$\mathcal{E}_D := \{ v \in H^2(D) \mid v(x,y) = 0 \text{ in the vertices of } D \} .$$

For our further investigation it is necessary to use the following

Lemma 2.1 (Babuška/Aziz (1976) [1])
Let for $v \in \mathcal{E}_\Delta$

$$I_1(\alpha) := \int_\Delta \left[\left(\frac{\partial v}{\partial \xi} \right)^2 + \alpha^{-2} \left(\frac{\partial v}{\partial \eta} \right)^2 \right] d\xi d\eta$$

and

$$I_2(\alpha) := \int_\Delta \left[\left(\frac{\partial^2 v}{\partial \xi^2} \right)^2 + 2\alpha^{-2} \left(\frac{\partial v}{\partial \xi \partial \eta} \right)^2 + \alpha^{-4} \left(\frac{\partial^2 v}{\partial \eta^2} \right)^2 \right] d\xi d\eta .$$

Then there exist $A^2 > 0$, such that for all α with $0 < \alpha \leq 1$

$$A^2 \leq \inf_{v \in \mathcal{E}_\Delta} \frac{I_2(\alpha)}{I_1(\alpha)} .$$

Proof. See [1]. ∎

Now we can formulate the main result of this section.

Lemma 2.2 *Let $v \in \mathcal{E}_K$. Then*

$$\|v\|_{0,K}^2 \le 0.2587(1+|\gamma|+\gamma^2)^2 h_K^4 |v|_{2,K}^2 \tag{2.3}$$

and

$$|v|_{1,K}^2 \le A^{-2}(1+|\gamma|+\gamma^2)^3 h_K^2 |v|_{2,K}^2| \,. \tag{2.4}$$

Proof. To prove this lemma we shall employ the standard technique using transformations of $K \in T_h$ into Δ. However, all estimates are to be done more finely than usually. Transforming the integrals over K to Δ by means of (2.1) we obtain with

$$\hat{v}(\xi,\eta) := v\Big(x(\xi,\eta), y(\xi,\eta)\Big) \in \mathcal{E}_K$$

for all α with $0 < \alpha \le 1$

$$\|v\|_{0,K}^2 = \int_\Delta |\hat{v}(\xi,\eta)|^2 \, |J| d\xi d\eta = \alpha h_K^2 |\hat{v}|_{0,\Delta}^2 \tag{2.5}$$

and

$$\begin{aligned} |v|_{1,K}^2 &= \int_\Delta |J^{-T}\nabla\hat{v}(\xi,\eta)|^2 \, |J| d\xi d\eta \\ &= \alpha \int_\Delta \left[(1+|\gamma|+\gamma^2)\left(\frac{\partial\hat{v}}{\partial\xi}\right)^2 + \alpha^{-2}(1+|\gamma|)\left(\frac{\partial\hat{v}}{\partial\eta}\right)^2\right] d\xi d\eta \\ &\le \alpha(1+|\gamma|+\gamma^2)\int_\Delta \left[\left(\frac{\partial\hat{v}}{\partial\xi}\right)^2 + \alpha^{-2}\left(\frac{\partial\hat{v}}{\partial\eta}\right)^2\right] d\xi d\eta \,. \end{aligned} \tag{2.6}$$

Due to Lemma 2.1 there exist $A^2 > 0$ such that in (2.6)

$$|v|_{1,K}^2 \le \alpha(1+|\gamma|+\gamma^2)A^{-2} I_2(\alpha) \,. \tag{2.7}$$

In (2.5) we get

$$\|v\|_{0,K}^2 \le 0.2587\alpha h_K^2 |\hat{v}|_{2,K}^2 \le 0.2587\alpha h_K^2 I_2(\alpha) \,. \tag{2.8}$$

The first inequality is a result of WEISS [11] obtained by solving an eigenvalue problem. Remaining the back transformation of $I_2(\alpha)$ to the triangle K, we have by (2.2)

$$\begin{aligned} I_2(\alpha) &= \int_\Delta \left[\left(\frac{\partial^2\hat{v}}{\partial\xi^2}\right)^2 + 2\alpha^{-2}\left(\frac{\partial^2\hat{v}}{\partial\xi\partial\eta}\right)^2 + \alpha^{-4}\left(\frac{\partial^2\hat{v}}{\partial\eta^2}\right)^2\right] d\xi d\eta \\ &= h_K^4 \int_K \left[\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + 2\alpha^{-2}\left(\alpha\gamma\frac{\partial^2 v}{\partial x^2} + \alpha\frac{\partial^2 v}{\partial x\partial y}\right)^2 \right. \\ &\qquad \left. + \alpha^{-4}\left(\alpha^2\gamma^2\frac{\partial^2 v}{\partial x^2} + 2\alpha^2\gamma\frac{\partial^2 v}{\partial x\partial y} + \alpha^2\frac{\partial^2 v}{\partial y^2}\right)^2\right] |J^{-1}| dx dy \,. \end{aligned}$$

Squaring the three expressions and applying the inequality $2ab \le a^2 + b^2$ to the mixed terms, we easily found that

$$I_2(\alpha) \le (1+|\gamma|+\gamma)^2 \alpha^{-1} h_K^2 |v|_{2,K}^2 \,. \tag{2.9}$$

Combining the inequalities (2.7), (2.8) and (2.3) we arrive at the lemma. ∎

Remark 2.1: An immediate consequence of Lemma 2.2 is the inequality

$$||v||_{0,K}^2 + h_K^2|v|_{1,K}^2 \le C(\zeta(K))h_K^4|v|_{2,K}^2 \tag{2.10}$$

for all $K \in T_h$ where $C(\zeta(K))$ is an increasing finite function of the maximum angle ζ of triangle K. Note that there is an essential difference to the analogous theorems in the literature, where the minimum angle must be bounded below. Here, to obtain a uniform estimate only the inequality (1.4) is needed. This fact is useful for adaptive mesh refinement.

For the interpolation function, which belongs to $\mathcal{E}_K$, SYNGE [10] was probably the first who suggested that it is better to pay attention to angles that tend to π rather than those that tend to zero. After that there are several approaches to this problem, for instance see BABUŠKA/AZIZ [1] and JAMET [5].

Remark 2.2: The unknown constant A^2 in Lemma 2.1 was specified in BABUŠKA/AZIZ [1] as

$$A^2 = \inf_{v\in\Theta} \frac{|v|_{1,\Delta}^2}{||v||_{0,\Delta}^2}$$

where $\Theta := \{v \in H^1(\Delta) | \int_0^1 v(0,\eta)d\eta = 0\}/\{0\}$. For practical computation one can estimate A^2 in the following way

$$A^2 \ge \inf_{v\in\widetilde{\Theta}} \frac{|v|_{1,\Delta}^2}{||v||_{0,\Delta}^2} = \pi^2 , \tag{2.11}$$

$\widetilde{\Theta} = \{v \in H^1(\Delta) | v \ne \text{const.}\}$. The inequality is a direct consequence of $\Theta \subset \widetilde{\Theta}$. The minimization of the new Rayleigh quotient is equivalent to seeking for the minimal positive eigenvalue of the Neumannn problem for the Laplacian over the standard triangle Δ (cf. IWANOW/KORNEEW/-LANG [4]).

3. A Geometric Motivation for δ

In the following we use the well-known inverse estimate

$$||\nabla v||_{0,\Omega}^2 \le C_I h^{-2} ||v||_{0,\Omega}^2 \tag{3.1}$$

for all $v \in H^1(\Omega)$ which are piecewise linear over each $K \in T_h$. To prove a standard stability estimate the inequality

$$\delta \le C C_I^{-1} \frac{h^2}{\varepsilon} \tag{3.2}$$

is to be satisfied. Therefore, in the streamline–diffusion method by $\varepsilon < h$ one has to choose $\delta \sim ch$ with arbitrary small c. For more details see JOHNSON [6].

Now we would give a geometric motivation for the choice of δ in the context of adaptive techniques. With the help of the results from section 2 we are able to specified the constant C_I in (3.1) more precisely. Let us again consider the triangle from Fig. 2.1.

Lemma 3.1 *Let $v \in P_1(K)$. Then*

$$|v|_{1,K}^2 \le (1 + |\gamma| + \gamma^2) \frac{C_I(\Delta)}{(\alpha h_K)^2} ||v||_{0,K}^2 \tag{3.3}$$

where $C_I(\Delta)$ is the inverse constant for the standard triangle.

Proof. The proof directly follows from the relations (2.5) and (2.6) and the fact that

$$|v|_{1,\Delta}^2 \le C_I(\Delta)\|v\|_{0,\Delta}^2$$

holds for all $v \in P_1(\Delta)$. ∎

Lemma 3.1 shows that according to (3.1) locally holds

$$\delta_K \le C(\alpha h_K)^2 \frac{1}{\varepsilon} . \tag{3.4}$$

From the view-point that the adaptive algorithm should lead to resolution of the layers, one has $\varepsilon \sim \alpha h_K$ for boundary layers and $\sqrt{\varepsilon} \sim \alpha h_K$ for internal layers. That means in (3.4) $\delta_K \sim c(\alpha h_K)$ and $\delta_K \sim c$ respectively. So we can identify the paramater δ_K with the desired mesh size in flux direction. Denoting by $hflux(K)$ the maximal diameter of $K \in T_h$ in flux direction, we get

$$\delta_K \sim C\ hflux(K)\ , \quad K \in T_h\ . \tag{3.5}$$

Remark 3.1: The characterization of δ_K in (3.2) according to $\varepsilon < h$ is typical for nonadaptive techniques. All sides of the triangle have the same rights. In contrast to this we associate with any triangle two real numbers (α, h_K) and get (3.5) for adaptive strategies.

4. A Priori Error Estimates

In this section we will give a priori error estimates based on local interpolation error estimates. For this we assume the regularity $u \in H^2(\Omega)$.

Let us denote by $\Pi_h u$ the usual Lagrange interpolation of the exact solution u over the nodal points of T_h. Then $\Pi_h u$ satisfies the well-known approximation property

$$\|u - \Pi_h u\|_{0,K} + h_K|u - \Pi_h u|_{1,K} \le C h_K^2 |u|_{2,K} \tag{4.1}$$

for all $K \in T_h$. Note that according to $(u - \Pi_h u)|_K \in \mathcal{E}_K$ for the more detailed description of the constant C in the right-hand side we can use the results of Section 2.

Now we would give for the used streamline-diffusion method (1.5) the natural connection between the global discretization error and the global interpolation error.

Lemma 4.1 *Let u and u_h be the solutions of* (1.1) *and* (1.5) *repectively. Then*

$$\|u - u_h\|_\varepsilon^2 \le \varepsilon\|\nabla(u - \Pi_h u)\|_{0,\Omega}^2 + 2\delta\|\beta\nabla(u - \Pi_h u)\|_{0,\Omega}^2 + 2\delta^{-1}\|u - \Pi_h u\|_{0,\Omega}^2 . \tag{4.2}$$

Proof. At first we note that for $v \in H_0^1(\Omega)$ Greens-formula supplies in (1.6)

$$(\beta\nabla v, v) = \tfrac{1}{2}\int\limits_{\Gamma_1} v^2 \beta\cdot n\, ds\ - \tfrac{1}{2}(\operatorname{div}\beta\cdot v, v)\ .$$

We let the discretization error be $e := u - u_h$ and define $\eta := u - \Pi_h u$. This implies $e \in H_0^1(\Omega)$ and the equality

$$\|e\|_\varepsilon^2 = B_\varepsilon(e,e) = B_\varepsilon(e,\eta) - B_\varepsilon(e, u_h - \Pi_h u)\ . \tag{4.3}$$

Following $(u_h - \Pi_h u) \in V_h$ the second term of the right-hand side vanishes. Now applying the Cauchy-Schwarz-inequality and the inequality $2ab \le \nu a^2 + \frac{1}{\nu}b^2$ for any $\nu > 0$ we get

$$\begin{aligned}
B_\varepsilon(e,\eta) &= \varepsilon(\nabla e, \nabla\eta) + \delta(\beta\nabla e, \beta\nabla\eta) + (\beta\nabla e, \eta) \\
&\le \tfrac{\varepsilon}{2}\left(\|\nabla e\|_{0,\Omega}^2 + \|\nabla\eta\|_{0,\Omega}^2\right) + \tfrac{\delta}{2}\left(\tfrac{1}{2}\|\beta\nabla e\|_{0,\Omega}^2 + 2\|\beta\nabla\eta\|_{0,\Omega}^2\right) \\
&\quad + \tfrac{1}{2}\left(\tfrac{\delta}{2}\|\beta\nabla e\|_{0,\Omega}^2 + \tfrac{2}{\delta}\|\eta\|_{0,\Omega}^2\right) \\
&\le \tfrac{1}{2}\|e\|_\varepsilon^2 + \tfrac{\varepsilon}{2}\|\nabla\eta\|_{0,\Omega}^2 + \delta\|\beta\nabla\eta\|_{0,\Omega}^2 + \delta^{-1}\|\eta\|_{0,\Omega}^2
\end{aligned}$$

and finally

$$||e||_\varepsilon^2 \leq \varepsilon||\nabla\eta||_{0,\Omega}^2 + 2\delta||\beta\nabla\eta||_{0,\Omega}^2 + 2\delta^{-1}||\eta||_{0,\Omega}^2$$

which proves the desired estimate. ∎

According to $(u - \Pi_h u)|_K \in \mathcal{E}_K$ for a further estimation of (4.2) we can use Lemma 2.2.

Theorem 4.2 *Let u and u_h be the solutions of* (1.1) *and* (1.5) *respectively. Then holds*

$$||u - u_h||_\varepsilon^2 \leq \sum_{K\in T_h} \left(C_1(\varepsilon,\delta,\beta,\gamma) + C_2(\delta,\gamma)h_K^2\right) h_K^2 |u|_{2,K}^2 \tag{4.4}$$

where

$$C_1(\varepsilon,\delta,\beta,\gamma) = \pi^{-2}\left(\varepsilon + 2\delta_K \max_K(|\beta_1|,|\beta_2|)\right)(1+|\gamma|+\gamma^2)^3$$

and

$$C_2(\delta,\gamma) = 0.5174\,\delta_K^{-1}(1+|\gamma|+\gamma^2)^2 \, .$$

Proof. Splitting up the norm in the right–hand side of (4.2) and using directly Lemma 2.2 to $v = (u - \Pi_h u)|_K \in \mathcal{E}_K$ we obtain the inequality (4.4). ∎

Now the error control will be based on this optimal a priori estimate.

5. The Adaptive Algorithm

Given now a tolerance TOL > 0 our aim is to find a finite element solution u_h satisfying

$$||u - u_h||_\varepsilon \leq \mathrm{TOL}\, . \tag{5.1}$$

The obvious idea is to improve u_h in an adaptive process through equidistribution of all element errors. Thus, Theorem 4.2 leads us to the following choice of the local mesh size h_K:

$$\left((C_1 + C_2 h_K^2)h_K^2|u|_{2,K}^2\right)^{1/2} \sim \frac{\mathrm{TOL}}{\sqrt{\mathcal{N}_e}} =: \mathrm{ETOL} \tag{5.2}$$

where $\mathcal{N}_e$ denotes the number of elements. To give this guideline a practical meaning, the approximation of the second derivatives of the exact solution u is needed. It can be done by using certain local difference quotients of computed gradients of the numerical solution u_h. Thus, our adaptive algorithm is based on a optimal a posteriori error estimate of the form (4.4) replacing $|u|_{2,K}^2$ by a function of u_h.

To compute approximations of the seminorm $|u|_{2,K}^2$ locally we shall apply the operator

$$D_K^2 u_h := \mathrm{meas}(K)\cdot \tfrac{1}{2} \sum_{\tau\in\partial K} \left(|[\nabla u_h]_\tau|/h_\tau\right)^2\, , \quad K \in T_h \tag{5.3}$$

where $[\cdot]_\tau$ denotes the jump across the edge τ of K which is of length h_τ. The operator was also propagated in ERIKSSON/JOHNSON [3] and may be viewed as a discrete counterpart of the seminorm $|u|_{2,K}^2$.

The algorithm can now be formulated as follows:

Step 0: Choose an initial mesh T_h^0 satisfying the maximum angle condition.

Step 1: Given a mesh T_h, compute the corresponding finite element solution $u_h \in S_h$.

Step 2: Compute the error indicator for each $K \in T_h$

$$Z_K := \Big((C_1 + C_2 h_K^2) h_K^2 D_K^2 u_h\Big)^{1/2}.$$

Step 3: If $Z_k \leq$ ETOL for all $K \in T_h$ then stop else construct a new mesh and go to step 1.

Our refinement strategy consists of the well-known local refinement dividing certain triangles into four similar triangles by connecting the midpoints of the sides (red refinement). To remedy irregular nodes an additional irregular refinement is taken (green refinement). In a special case we will apply an adjusted refinement giving preference to the flow direction which has to be derived from a posteriori information (blue refinement). For more details we refer to BANK ET AL. [2] and KORNHUBER/ROITZSCH [8].

Finally, a triangle $K \in T_h$ is marked for refinement, if

$$Z_K \geq \frac{C}{\mathcal{N}_e} \sum_{K \in T_h} Z_K \tag{5.4}$$

with some constant C specified later.

6. Numerical Results

Example 6.1: Interior and boundary layers. At first, for comparison, we consider a similar problem to that in JOHNSON/ERIKSSON [7]. For actual computation we choose $\Omega = (0,1) \times (0,1)$, $\Gamma_0 = \Gamma_{\text{in}} \cup \{(x,y)|x = 1, y \leq 0.8\}$ with $\Gamma_{\text{in}} := \{(x,y) \in \partial\Omega | \max(x,y) < 1\}$, $\Gamma_1 = \partial\Omega \backslash \Gamma_0$, $\varepsilon = 10^{-4}$, $\beta = (1.0, 0.5)$, $f \equiv 0$ and

$$u_0(x,y) = \begin{cases} 0, \; y > 0.3, \; (x,y) \in \Gamma_{\text{in}} \\ 1, \; y \leq 0.3, \; (x,y) \in \Gamma_{\text{in}} \\ 0, \; x = 1, \; y \leq 0.8 \; . \end{cases}$$

It is easily seen that the exact solution shows a linear internal layer proceeding from the discontinuity transport in the flux direction β and an outflow layer.

The constant in (5.4) was chosen such that the nodes on each level are comparable. In Fig. 6.1 – 6.2 we give some results with our algorithm for ETOL = 0.1.

One immediately sees a concentration of mesh points in the layers. This leads to a reduction of the error. In a direct comparison with the results in JOHNSON/ERIKSSON [7] we can emphasize that our adaptive method detects and resolves the layers in a similar good way. Note, here the streamline diffusion without shock capturing is used and therefore oscillations of the numerical solution may still occur.

Example 6.2: Boundary layer. In many problems of practical interest boundary layers play an important role. As a simple example we choose $\Omega = (0,1) \times (0,1)$, $\Gamma_0 = \partial\Omega$, $\varepsilon = 10^{-5}$, $\beta = (0,1)$, $f \equiv 1$ and $u_0(x,y) = 0$. The exact solution exhibits an ordinary boundary layer at the outflow boundary $\Gamma_{\text{out}} = \{(y,y) \in \partial\Omega | y = 1\}$ and the birth of a parabolic layer at the characteristic boundary $\Gamma_{\text{char}} = \{(x,y) \in \partial\Omega | 0 < y < 1\}$.

Here the constant in (5.4) was chosen such that the coefficient $\mathcal{N}_e^{\text{new}}/\mathcal{N}_e^{\text{old}}$ lies in the interval $[2,3]$.

The results on the final level 6 for ETOL $= 0.1$ are shown in Fig. 6.3.

We notice that the adaptive method results in a good resolution of the solution in the layer without oscillations.

Example 6.3: Interior curved layer. Let us change in Example 6.1 the flux direction to $\beta = (y, -x)$, the diffusion parameter to $\varepsilon = 10^{-5}$ and the boundary condition to

$$u_0(x,y) = \begin{cases} 0\,, & y > 0.7 \\ 1\,, & y \leq 0.7 \end{cases}\,, \quad (x,y) \in \Gamma_0$$

with $\Gamma_0 = \Gamma_{\text{in}} = \{(x,y) \in \partial\Omega | x = 0 \text{ or } y = 1\}$ and $\Gamma_1 = \partial\Omega \backslash \Gamma_{\text{in}}$. At hand this problem we would demonstrate how the local estimator works if anisotropic refinement is applied. For details on realization of so-called blue refinement we refer to KORNHUBER/ROITZSCH [8]. Here we realize $\mathcal{N}_e^{\text{new}}/\mathcal{N}_e^{\text{old}} \in [1,2]$.

Figure 6.4 shows that only with 184 nodes the solution is approximated with an acceptable accuracy.

Note that for this problem the usual minimal angle condition no longer holds.

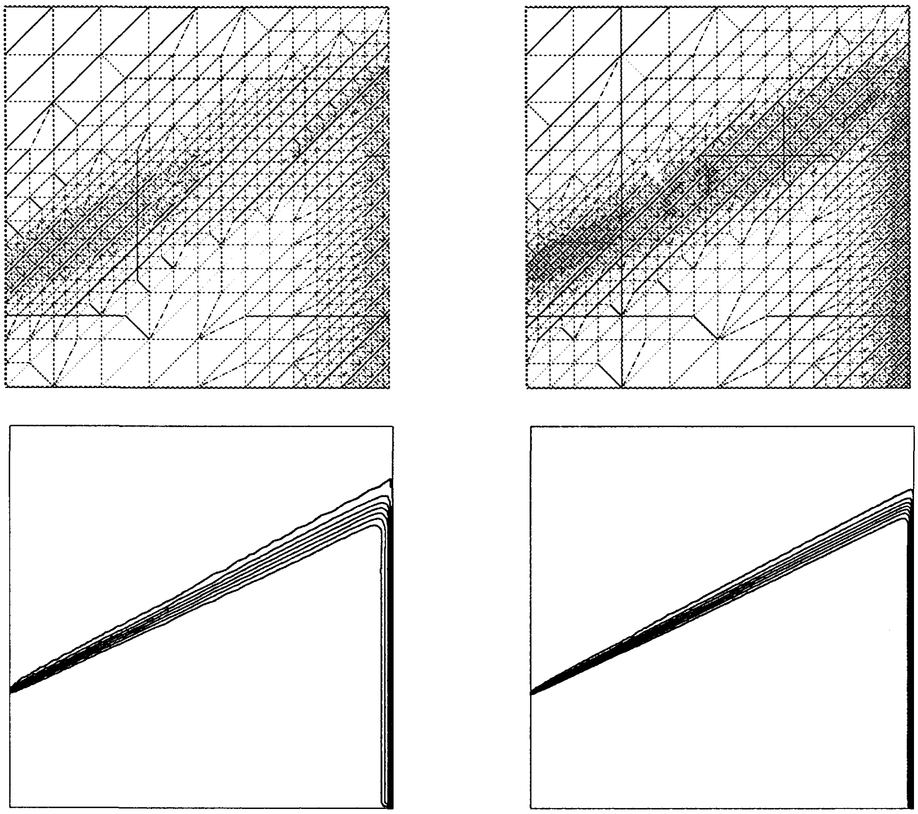
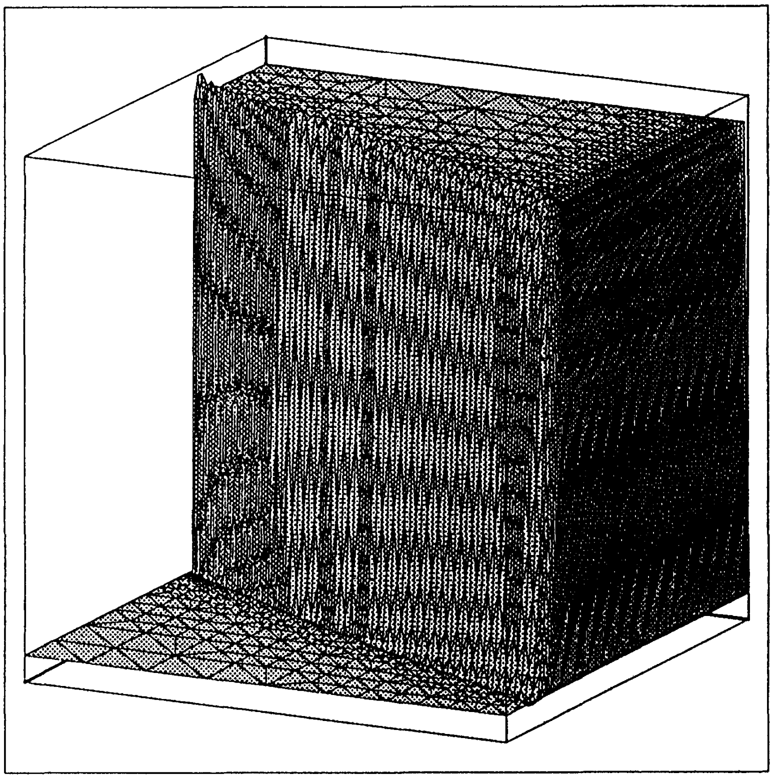
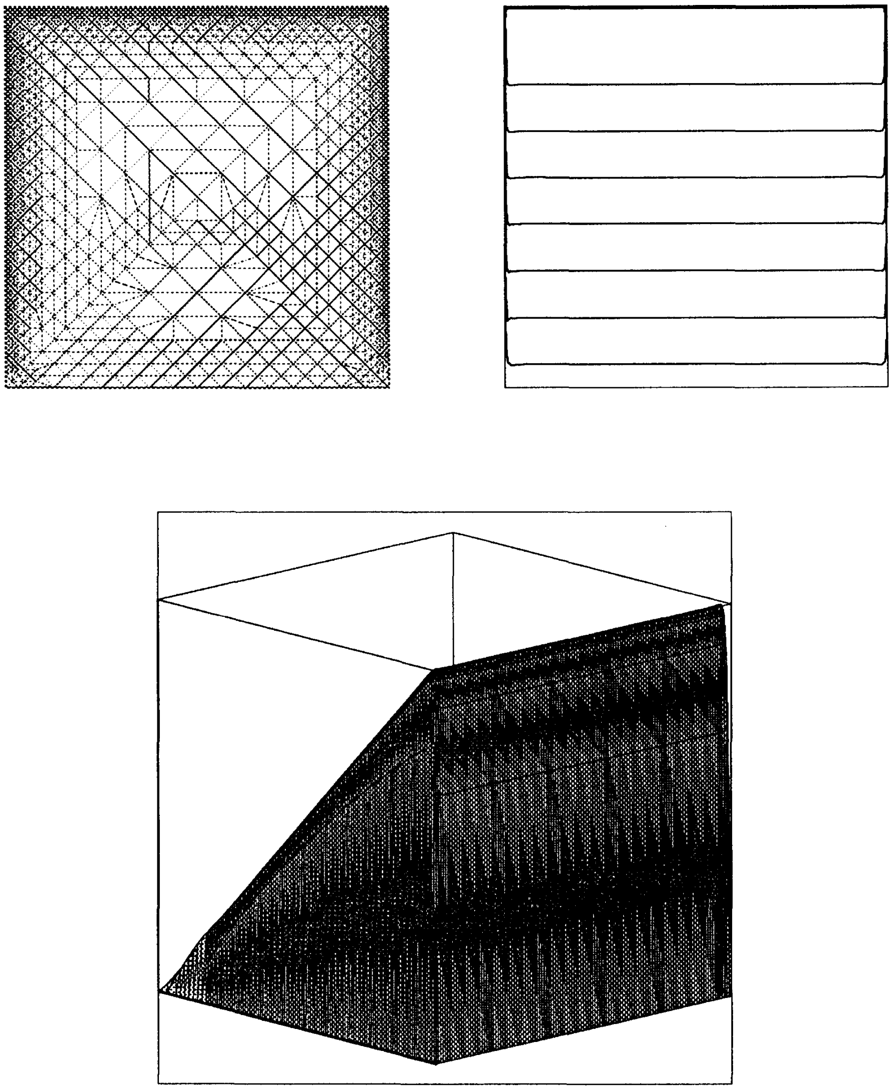


Figure 6.1: Display of mesh and elevation of approximate solution for computation level 5 and level 6.



Level	Nodes	Maximum of approximate local error	Approximate global error
1	25	0.4945	1.4223
2	79	0.3530	1.1259
3	190	0.2496	1.0709
4	451	0.1765	0.9296
5	1021	0.1248	0.8461
6	2304	0.0883	0.7701

Figure 6.2: Curve of final solution. Estimated error in the $\|\cdot\|_\varepsilon$–norm and nodes on sequence of meshes.



Level	Nodes	Maximum of approximate local error	Approximate global error
1	41	0.4876	1.3460
2	123	0.3844	1.4359
3	293	0.2883	1.5705
4	641	0.2020	1.5162
5	1400	0.1345	1.3376
6	3073	0.0798	1.0874

Figure 6.3: Display of mesh, elevation and curve of final solution on level 6, history of errors in the $||\cdot||_\varepsilon$–norm.

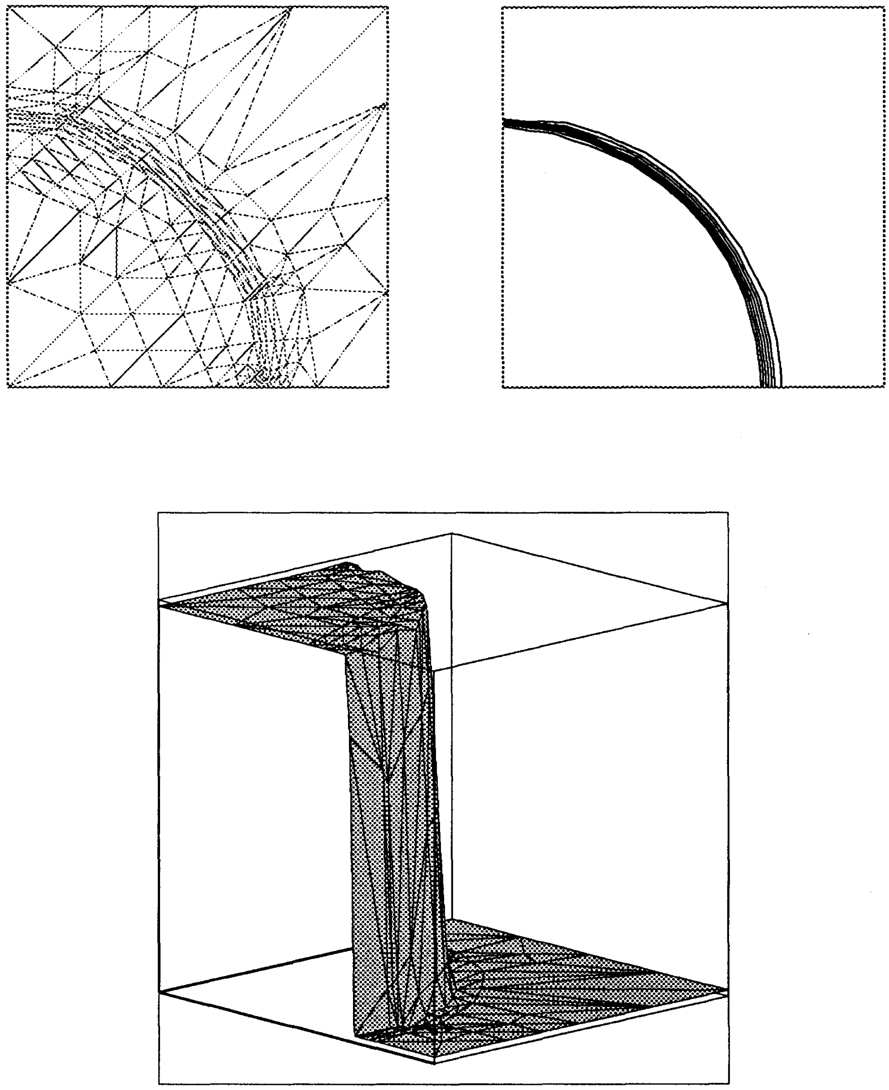


Figure 6.4: Display of mesh, elevation and curve of approximate solution with 184 points.

References

[1] I. Babuška, A. K. Aziz: *On the Angle Condition in the Finite Element Method.* SIAM J. Numer. Anal. **13**, pp. 214–226 (1976).

[2] R. E. Bank, A. H. Shermann, A. Weiser: *Refinement Algorithms and Data Structures for Regular Local Mesh Refinement.* In: R. Stepleman, et al. (eds.), Scientific Computing, Amsterdam: IMACS North-Holland, pp. 3–17 (1983).

[3] K. Eriksson, C. Johnson: *Adaptive Streamline Diffusion Finite Element Methods for Convection–Diffusion Problems.* Technical Report, Dept. of Mathematics, Chalmers Univ. of Technology (1990).

[4] S. A. Iwanow, W. G. Korneew, J. Lang: *Estimates for Constants in Inequality of Basic Elements.* J. Numer. Math. & Math. Phys., Vol. **28**, 9, Moscow (1988).

[5] P. Jamet: *Estimation d' erreur pour des éléments finis droits presque dégénérés.* R.A.I.R.O., Série Analyse Numérique **10**, no. 3, pp. 43–61 (1976).

[6] C. Johnson: *Numerical Solutions of Partial Differential Equations by the Finite Element Method.* Cambridge University Press, Cambridge (1987).

[7] C. Johnson: *Adaptive Finite Element Methods for Diffusion and Convection Problems.* Comp. Meth. Appl. Mech. & Engrg., **82**, pp 301–322 (1990).

[8] R. Kornhuber, R. Roitzsch: *On Adaptive Grid Refinement in the Presence of Internal or Boundary Layers.* Preprint SC 89–5, Konrad–Zuse–Zentrum Berlin (1989).

[9] J. Lang: *Ein Beitrag zur adaptiven Netzsteuerung bei der Methode der finiten Elemente auf der Grundlage von Interpolationsfehler–Abschätzungen.* Diss. A, Martin–Luther–Universität Halle–Wittenberg (1988).

[10] J. L. Synge: *The Hypercircle in Mathematical Physics.* Cambridge University Press, New York (1957).

[11] R. Weiss: *Diploma Thesis.* Sektion Mathematik, Martin–Luther–Universität Halle–Wittenberg (1985).