Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

C. HELMBERG[1]

The m-Cost ATSP

[1]Konrad-Zuse-Zentrum für Informationstechnik Berlin, Takustraße 7, D-14195 Berlin, Germany, helmberg@zib.de, http://www.zib.de/helmberg

Preprint SC 99-48 (February 1999)

The m-Cost ATSP

Christoph Helmberg*

February 1999

Abstract

Although the m-ATSP (or multi traveling salesman problem) is well known for its importance in scheduling and vehicle routing, it has, to the best of our knowledge, never been studied polyhedraly, i.e., it has always been transformed to the standard ATSP. This transformation is valid only if the cost of an arc from node i to node j is the same for all machines. In many practical applications this is not the case, machines produce with different speeds and require different (usually sequence dependent) setup times. We present first results of a polyhedral analysis of the m-ATSP in full generality. For this we exploit the tight relation between the subproblem for one machine and the prize collecting traveling salesman problem. We show that, for $m \geq 3$ machines, all facets of the one machine subproblem also define facets of the m-ATSP polytope. In particular the inequalities corresponding to the subtour elimination constraints in the one machine subproblems are facet defining for m-ATSP for $m \geq 2$ and can be separated in polynomial time. Furthermore, they imply the subtour elimination constraints for the ATSP-problem obtained via the standard transformation for identical machines. In addition, we identify a new class of facet defining inequalities of the one machine subproblem, that are also facet defining for m-ATSP for $m \geq 2$. To illustrate the efficacy of the approach we present numerical results for a scheduling problem with non-identical machines, arising in the production of gift wrap at Herlitz PBS AG.

1 Introduction

For Herlitz PBS AG, Berlin, we are developing a software package for the following scheduling problem. Gift wrap has to be printed on two non-identical printing machines. The gift wrap is printed in up to six colors on various kinds of materials in various widths. The colors may differ considerably from one gift wrap to the next. The machines differ in speed and capabilities, not all jobs can be processed on both machines. Setup times for the machines are quite large in comparison to printing time. They depend strongly on whether

*Konrad-Zuse-Zentrum für Informationstechnik Berlin, Takustraße 7, 14195 Berlin, Germany, helmberg@zib.de, http://www.zib.de/helmberg

material, width, or colors have to be changed, so in particular on the sequence of the jobs. The task is to find an assignment of the jobs to the machines and an ordering on these machines such that the last job is finished as soon as possible, i.e., minimize the makespan for a scheduling problem on two (non-identical) machines with sequence dependent set-up times.

An obvious mathematical model for this problem is the asymmetric multi traveling salesman problem with separate arc costs for each salesman. Although there is considerable literature on the TSP/ATSP [10, 13, 14, 15, 3, 11, 7] as well as on the m-ATSP for vehicle routing (see [5] and references therein) it seems that the m-ATSP problem in full generality has never been studied from a polyhedral point of view. Existing work on the m-ATSP relies on the standard transformation to ATSP (see e.g. [16] or Section 6) which assumes that all salesmen need the same time for the same route. The general case of non-identical salesmen should also be of importance in vehicle routing where usually not all vehicles in a car park have the same capabilities or fuel consumption.

In this paper we give a precise definition of what we call the m-Cost ATSP and present a 'canonical' integer programming formulation. This formulation includes an exponential class of inequalities, which we call the *v0-cut inequalities*. They may be interpreted as a kind of conditional subtour elimination constraints and are, in a certain sense, equivalent to the inequalities introduced in [1], Theorem 2.5, for the Price Collecting TSP (PCTSP) (see also [8, 2, 9]). The PCTSP is tightly related to the one machine subproblem of the m-Cost ATSP and we will clarify this relation in Section 4. Another variant, the generalized traveling salesman problem [6], also allows for skipping certain jobs but differs in that the jobs are partitioned into sets and at least one job from each set has to be visited by the tour.

The main results of the paper are: Facets of the one machine subproblem define facets of the m-Cost ATSP polytope for $m \geq 3$; the non-negativity constraints and the v0-cut inequalities are facet defining for the m-Cost ATSP polytope for $m \geq 2$, they can be separated in polynomial time, and they imply the subtour elimination constraints if the standard transformation for identical machines from m-ATSP to ATSP is applied. In addition, we identify a new class of so called *nested conflict inequalities* that are facet defining for the polytope of the one machine subproblem and the m-Cost ATSP polytope for $m \geq 2$.

For our real-world instances the linear relaxation based on the v0-cut inequalities yields, in average, a relative gap of 0.2% with respect to the best solution known. In comparison, the relaxation using subtour elimination constraints on the transformed problem exhibits an average relative gap of 0.5%, so this approach reduces the gap by 60%.

The paper is organized as follows. Section 2 introduces the necessary notation and gives the problem definition. In Section 3 we present an integer programming formulation of the m-Cost ATSP and determine the dimension of the m-Cost ATSP polytope. Section 4 is devoted to the one machine subproblem, its relation to the PCTSP, and the separation algorithm. In Section 5 the facet defining inequalities of the subproblem are shown to be facet defining for the m-Cost ATSP polytope. In Section 6 we prove that the v0-cut inequalities imply the subtour elimination constraints for the transformed problem. Finally,

in Section 7 numerical results are presented for the scheduling problem at Herlitz PBS AG.

2 Notation and Problem Formulation

Let $D = (V, A)$ be a digraph on node set V and arc set A without multiple arcs but possibly with loops. An arc is an ordered pair (v, w) of nodes $v, w \in V$, v is called tail, w head of (v, w). Most of the time we will simply write vw instead of (v, w). For a subset $S \subseteq V$, $\delta^-(S) = \{vw \in A : v \in V \setminus S, w \in S\}$ is the set of arcs of A that have their head in S and their tail in $V \setminus S$. Similarly, $\delta^+(S) = \{vw \in A : v \in S, w \in V \setminus S\}$ is the set of arcs of A having tail in S and head in $V \setminus S$. By definition, loops are not contained in any set $\delta^-(S)$ or $\delta^+(S)$. For a node $v \in V$ we will write $\delta^-(v)$ instead of $\delta^-(\{v\})$. For an arc set $\hat{A} \subset A$, $V(\hat{A}) = \{v \in V : vw \in \hat{A} \text{ or } wv \in \hat{A}\}$ is the set of nodes that are tail or head of an arc in $\hat{A}$. For a node set $S \subset V$, $A(S) = \{vw \in A : v \in S \text{ and } w \in S\}$ is the set of arcs having both head and tail in S. For a vector $x \in \mathbb{Q}^{|A|}$ the sum of the weights over an arc set $\hat{A} \subset A$ is denoted by $x(\hat{A}) = \sum_{a \in \hat{A}} x_a$.

A dicycle is a set of $k > 0$ arcs $C = \{v_1v_2, v_2v_3, \ldots, v_kv_1\} \subseteq A$ with distinct vertices $v_1, \ldots, v_k \in V$. The number of arcs k is referred to as the length of the dicycle. A tour is a dicycle of length $|V|$. A loop is a dicycle of length 1.

Now consider the problem of scheduling n jobs on m distinct machines. Let $J = \{1, \ldots, n\}$ denote the set of jobs and $M = \{1, \ldots, m\}$ the set of machines. In order to model all possible schedules we introduce for each machine $k \in M$ a digraph $D_k^0 = (V_k, A_k)$ with $V_k = J \cup \{0\}$ and arc set $A_k = \{(0, 0)_k\} \cup \{(i, j)_k : i, j \in V_k, i \neq j\}$ (node 0 allows to model initial and final state of each machine as well as idle costs). We will write ijk instead of $(i, j)_k$. Usually we will regard the nodes in V_k and V_i as the same objects. Sometimes, however, we will have to discern between nodes in V_k and V_i, then we will do so by adding a subscript, e.g., 0_k for $0 \in V_k$. For $S \subseteq V_k$ the subscript in the symbols $\delta_k^-(S)$ and $\delta_k^+(S)$ will indicate on which arc set we are working. We will frequently need the union of all arc sets which we denote by $\mathcal{A} = \bigcup_{k=1}^m A_k$. With each arc $ijk \in \mathcal{A}$ we associate a cost or weight c_{ijk}.

A feasible solution $A_F = C_1 \cup \ldots \cup C_m$ is a set of $m + n$ arcs that is the union of m dicycles $C_k \subseteq A_k$ for $k \in M$, such that $0 \in V_k(C_k)$ for all $k \in M$ and for each $v \in J$ there is a $k \in M$ with $v \in V_k(C_k)$. The set of feasible solutions is

$$\mathcal{F}_n^m = \Big\{ C_1 \cup \ldots \cup C_m : \begin{array}{l} \sum_{k=1}^m |C_k| = n + m, \\ \forall k \in M : C_k \subseteq A_k \text{ is a dicycle with } 0 \in V_k(C_k), \\ \forall v \in J : \exists k \in M : v \in V_k(C_k) \Big\}. \end{array}$$

One proves easily that each job is assigned uniquely.

Proposition 2.1 *Let $A_F = C_1 \cup \ldots \cup C_m \in \mathcal{F}_n^m$, then for $v \in J$ there is a unique $k \in M$ with $v \in V(C_k)$.*

An $A_F \in \mathcal{F}_n^m$ may be interpreted in the following way: if $ijk \in A_F$ and $i, j \in J$ then job j has to be processed directly after job i on machine k. If $i = 0$ and $j \in J$, then j is the first job to be processed on machine k. If $i \in J$ and $j = 0$ then i is the last job to be processed on machine k. If $i = j = 0$ then machine k does not process any jobs.

The optimization problem we will deal with is to find

$$\text{(m-Cost ATSP)} \qquad \min\{c(A_F) : A_F \in \mathcal{F}_n^m\}.$$

It is not difficult to show that (m-Cost ATSP) is *NP*-complete.

3 The m-Cost ATSP Polytope

We will approach the problem from a polyhedral point of view. To this end define, for $A \subseteq \mathcal{A}$, the incidence vector $x^A \in \{0,1\}^{|\mathcal{A}|}$ by

$$x_{ijk}^A = \begin{cases} 1, & \text{if } ijk \in A \\ 0, & \text{otherwise.} \end{cases}$$

The m-Cost ATSP polytope is defined as the convex hull of the incidence vectors of all feasible solutions, i.e.,

$$P_n^m = \operatorname{conv}\left\{x^A : A \in \mathcal{F}_n^m\right\}.$$

In order to study this polytope consider the following set of constraints

$$\begin{aligned}
x_{00k} + x(\delta_k^+(0)) &= 1 && k \in M && (1)\\
\textstyle\sum_{k\in M} x(\delta_k^+(j)) &= 1 && j \in J && (2)\\
x(\delta_k^-(j)) &= x(\delta_k^+(j)) && j \in J, k \in M && (3)\\
x(\delta_k^+(S)) &\geq x(\delta_k^+(v)) && k \in M, S \subseteq J, |S| \geq 2, v \in S && (4)\\
x_{ijk} &\in \{0,1\} && ijk \in \mathcal{A}. && (5)
\end{aligned}$$

The constraints of (1) ensure that for each node 0_k, $k \in M$, one arc is selected. In (2) we require that for each job $j \in J$ one outgoing arc of j is selected among all such arcs on all machines. Constraints (3) are flow conservation constraints. Constraints (4) are a kind of conditional subtour elimination constraints that ensure that any flow through v_k in graph k reaches node 0_k in this graph (note, that $0 \notin J$), i.e., the flow through v_k has to cross the cut from S to $V_k \setminus S$. Therefore, we call constraints (4) *v0-cut inequalities*.

Lemma 3.1 *An incidence vector x^A of $A \subseteq \mathcal{A}$ is a feasible solution of (1)–(5) if and only if $A \in \mathcal{F}_n^m$.*

Proof. The *if* part is easy to check, so consider the *only if* part. Let x be a solution of (1)–(5). Then, by the flow conservation constraints (3), x is the incidence vector of a union of r dicycles $C_1, \ldots, C_r$ and each $C_i \subseteq A_{k_i}$ for some $k_i \in M$. Because of (1) there is exactly one dicycle $C_i \subseteq A_k$ for each $k \in M$ with $0 \in V_k(C_i)$. Likewise, by (2), for each

$v \in J$ there is exactly one dicycle C_i with $v \in V_{k_i}(C_i)$. Furthermore, for each C_i we must have $0 \in V_{k_i}(C_i)$, because otherwise $S = V_{k_i}(C_i)$ would lead to a violated $v0$-cut inequality (4) for $k = k_i$ and arbitrary $v \in S$. Thus, $r = m$ and the cycles $C_1, \ldots, C_m$ fulfill the requirements for $\mathcal{F}_n^m$. ■

So P_n^m is the convex hull of the feasible solutions of (1)–(5). In order to determine the dimension of P_n^m we investigate equations (1)–(3) which we collect, for convenience, in a matrix B and a right hand side vector b,

$$Bx = b.$$

Lemma 3.2 *The coefficient matrix $B \in \{-1, 0, 1\}^{(m+n+mn)\times|\mathcal{A}|}$ corresponding to (1)–(3) has full row rank $m + n + mn$.*

Proof. Consider the sets of column indices

$$\begin{aligned} F_1 &= \{0i1, i01 : i \in J\} \cup \{121\} \\ F_k &= \{00k\} \cup \{0ik : i \in J\} \qquad \text{for } k \in M \setminus \{1\} \\ F &= \bigcup_{k \in M} F_k. \end{aligned}$$

We claim that the columns of B corresponding to F are linearly independent.

The linear independence of F_h for $h \in M \setminus \{1\}$ with respect to $F \setminus F_h$ follows from the fact that the variables are the only arcs in F that are in the support of the equation with $k = h$ in (1) and of the n equations with $k = h$ in (3). A proper arrangement of the submatrix of B consisting of these rows and the columns corresponding to F_h yields an upper triangular matrix with all diagonal entries equal to one. Thus the columns corresponding to F_h are linearly independent.

In the case of F_1 variables $0i1$ and $i01$ are both in the support of (3) for $k = 1$ and $j = i$ but only $i01$ is in (2) for $j = i$, so they are linearly independent with respect to these rows and the rows do not contain further variables in $F \setminus \{121\}$. Now observe that column 001 is the linear combination (of columns) $021 + 101 - 121$. Variable 001 is only in the support of the equation with $k = 1$ in (1) and its column is therefore not in the span of the columns of $F \setminus \{121\}$. We conclude that the columns of F are linearly independent. ■

To complement this upper bound on the dimension of P_n^m with a lower bound we will need a standard construction of linearly independent tours for the ATSP.

Theorem 3.3 *(see [10]) Let D_V be the complete digraph on nodes $V = \{0, \ldots, n\}$ with $n \geq 2$. There exist $n(n-1)$ tours in D_V that are linearly independent with respect to the arcs not incident to 0.*

Proof. Follows directly from [10], proof 1 of Theorem 7 and Theorem 20. ■

Now we are ready to determine the dimension of P_n^m.

Theorem 3.4 $\dim(P_n^m) = mn^2 - n$ *for $n \geq 3$ and $m \geq 2$.*

Proof. If follows from Lemma 3.2 that $\dim(P_n^m) \leq mn^2 - n$. To proof equality let $a^T x = a_0$ be a valid equation for P_n^m. We will show that it is linearly dependent with respect to $Bx = b$. By the proof of Lemma 3.2 the columns of B corresponding to the arcs in $F = \bigcup_{k \in M} F_k$ are linearly independent, so we can compute $\lambda \in \mathbb{Q}^{n+m+nm}$ such that $a_{ijk}^T = (\lambda^T B)_{ijk}$ for $ijk \in F$. The equation $d^T x = (a^T - \lambda^T B)x = a_0 - \lambda^T b$ is again valid for P_n^m and $d_{ijk} = 0$ for $ijk \in F$. We will show that all other coefficients d_{ijk} have to be zero as well.

First observe that on the support $A_1 \setminus \{001\}$ d has exactly $n(n-1)-1$ 'unknowns'. Employing the $n(n-1)$ linearly independent tours of Lemma 3.3 construct linearly independent feasible solutions $A^h \in \mathcal{F}_n^m$ for $h = 1, \ldots, n(n-1)$ that use these tours on machine 1 such that the x^{A^h} are linearly independent on the index set $\{ij1 : i \neq j, i, j \in J\}$. There exists an $\hat{h}$, $1 \leq \hat{h} \leq n(n-1)$, such that the $n(n-1)-1$ vectors $x^{A^h} - x^{A^{\hat{h}}}$ $(h \neq \hat{h})$ are linearly independent on the index set $\{ij1 : i \neq j, i, j \in J\} \setminus \{121\}$. These difference vectors have no support outside $A_1 \setminus \{001\}$ and satisfy $d^T(x^{A^h} - x^{A^{\hat{h}}}) = 0$. Collect the $n(n-1)-1$ difference vectors as columns in a matrix X. Since $d^T X = 0$ and $d_{ij1} = 0$ for $ij1 \in F_1$ we conclude that $d_{ij1} = 0$ for all $ij1 \in A_1 \setminus \{001\}$ by the linear independence of the columns of X.

We proceed to show that $d_{ij2} = 0$ for all $ij2 \in A_2$. The following two arc sets $A^1, A^2 \in \mathcal{F}_n^m$ correspond to a tour on machine 1 and to the same tour shortened by 'moving' node n to machine 2,

$$\begin{aligned} A^1 &= \{i(i+1)1 : i = 0, \ldots, n-1\} \cup \{n01\} \cup \{00k : 2 \leq k \in M\} \\ A^2 &= \{i(i+1)1 : i = 0, \ldots, n-2\} \cup \{(n-1)01\} \cup \{0n2, n02\} \\ &\quad \cup \{00k : 2 < k \in M\}. \end{aligned}$$

Since $0 = d^T(x^{A^1} - x^{A^2}) = -d_{n02}$ (in the support of $x^{A^1} - x^{A^2}$ only d_{n02} has not yet be shown to be zero) we obtain $d_{n02} = 0$. Moving any other vertex $i \in J$ from the tour on machine 1 to machine 2 yields $d_{i02} = 0$. Now it is easy to obtain $d_{ij2} = 0$ for $i \neq j$ and $i, j \in J$ by the dicycles $\{0i2, ij2, j02\}$ (here we use $n \geq 3$ as otherwise d_{001} would have to enter). So $d_{ij2} = 0$ for $ij2 \in A_2$. The same steps for $k > 2$ lead to $d_{ijk} = 0$ for $ijk \in A_k$ for $2 \leq k \in M$.

It remains to show that $d_{001} = 0$. This is easily achieved by taking the difference vector between x^{A^1} and an incidence vector of a feasible arc set $A^3 \in \mathcal{F}_n^m$ with $001 \in A^3$ (e.g., let A^3 contain a tour on machine 2). ■

Remark 3.5 *It can be shown that for $n = 2$ and $m \geq 2$ the dimension is $mn^2 - n - 1$, the additional affine constraint being*

$$\sum_{ijk \in \{01k, 10k, 12k, 21k : k \in M\}} x_{ijk} = 2.$$

Before we turn to the facets of P_n^m we study a tightly related problem.

4 The One-Machine Subproblem

Consider the subproblem on one machine where this machine is not required to process all jobs, but may process any subset of jobs, or even choose the idle loop. We model this problem on a digraph $D_s = (V_s, A_s)$ with $V_s = \{0\} \cup J$ and $A_s = \{(0,0)\} \cup \{(i,j) : i,j \in V_s, i \neq j\}$. The feasible set is

$$\mathcal{F}_n^s = \{C : C \subseteq A_s \text{ is a dicycle with } 0 \in V_s(C)\}$$

The corresponding polytope defined by the incidence vectors of the arc sets in the feasible set is

$$P_n^s = \operatorname{conv}\left\{x^A : A \in \mathcal{F}_n^s\right\}$$

We can model the incidence vectors by the following constraints,

$$x_{00} + x(\delta^+(0)) = 1 \tag{6}$$
$$x(\delta^-(j)) = x(\delta^+(j)) \quad j \in J \tag{7}$$
$$x(\delta^+(S)) \geq x(\delta^+(v)) \quad S \subseteq J, |S| \geq 2, v \in S \tag{8}$$
$$x_{ij} \in \{0,1\}\, \forall ij \in A_s. \tag{9}$$

Lemma 4.1 *An incidence vector x^A of $A \subset A_s$ is a feasible solution of (6)–(9) if and only if $A \in \mathcal{F}_n^s$.*

Proof. The *if* part is easy to check, so consider the *only if* part. Let x be a solution of (6)–(9). Then, by the flow conservation constraints (7), x is the incidence vector of a union of r dicycles $C_1, \ldots, C_r$. Because of (6) exactly one dicycle of these, w.l.o.g. C_1, satisfies $0 \in V(C_1)$. Furthermore, for all C_i we must have $0 \in V(C_i)$, because otherwise $S = V(C_i)$ would lead to a violated $v0$-cut inequality (8) with arbitrary $v \in S$. Thus $r = 1$ and the cycles C_1 fulfill the requirements for $\mathcal{F}_n^s$. ■

P_n^s is therefore the convex hull of the feasible solutions of (6)–(9). We determine the dimension of P_n^s by the same steps as in Section 3. We collect equations (6) and (7) in a matrix B_s and a right hand side vector b_s, $B_s x = b_s$.

Lemma 4.2 *The coefficient matrix $B_s \in \{-1,0,1\}^{(n+1)\times|A_s|}$ corresponding to (6) and (7) has full row rank $n+1$.*

Proof. The set of column indices

$$F_s = \{00\} \cup \{0i : i \in J\}$$

corresponds to the sets F_k for $k > 1$ of the proof of Lemma 3.2. The proof can be completed analogous to the proof there. ■

Theorem 4.3 $\dim(P_n^s) = n^2$.

Proof. $\dim(P_n^s) \leq n^2$ follows from Lemma 4.2. We construct n^2+1 feasible dicycles,

$$\begin{array}{lcll} C_{00} & = & \{\mathbf{00}\} & \\ C_{i0} & = & \{0i, \mathbf{i0}\} & i \in J \\ C_{ij} & = & \{0i, \mathbf{ij}, j0\} & i,j \in J, i \neq j. \end{array}$$

To show the linear independence of the incidence vectors $x^{C_{ij}}$ arrange them as rows in a matrix B. Let F be the set of indices indicated in bold above, $F = \{00\} \cup \{i0 : i \in J\} \cup \{ij : i,j \in J, i \neq j\}$, and consider the submatrix $B_s \in \{0,1\}^{|F| \times |F|}$ consisting of the columns corresponding to F. Reorder rows and columns of B_s simultaneously arranging rows and columns corresponding to $\{ij : i,j \in J, i \neq j\}$ in front, followed by rows and columns $\{00\} \cup \{i0 : i \in J\}$. The reordered matrix is upper triangular with all diagonal elements equal to one. Thus the vectors are linearly independent. ■

Theorem 4.4 *For $ij \in A_s$, $x_{ij} \geq 0$ defines a facet of P_n^s if $n \geq 3$.*

Proof. For 00 or $ij \in A_s$ with $i,j \in J$, $i \neq j$ we have to drop dicycle C_{ij} from the dicycles defined in the proof of Theorem 4.3 to obtain n^2 linearly independent tours satisfying $x_{ij} = 0$. (Indeed, these are facets for $n \geq 1$.)

For $0v \in A_s$ with $v \in J$ we assume w.l.o.g. that $v = 1$. Then the n^2 feasible dicycles

$$\begin{array}{lcll} C_{00} & = & \{\mathbf{00}\} & \\ C_{0i} & = & \{0i, \mathbf{i0}\} & i \in J \setminus \{1\} \\ C_{ij} & = & \{0i, \mathbf{ij}, j0\} & i \in J \setminus \{1\}, j \in J, i \neq j \\ C_{1j} & = & \{0i, i1, \mathbf{1j}, j0\} & j \in J \setminus \{1\} \text{ with some } i \in J \setminus \{1,j\} \end{array}$$

are again easily seen to satisfy $x_{01} = 0$ and to be linearly independent (proceed as in the proof of Theorem 4.3). A similar construction shows that x_{v0} is facet defining for $v \in J$. ■

The following corollary will be especially useful in relating P_n^s with P_n^m.

Corollary 4.5 *Any facet defining inequality $a^T x \leq a_0$ of P_n^s not equivalent to $x_{00} \geq 0$ satisfies $a^T x^{C_{00}} = a_0$ where $C_{00} = \{00\}$.*

Proof. Let $a^T x \leq a_0$ be facet defining with $C_{00} \notin \mathcal{F}_a = \left\{A \in \mathcal{F}_n^s : a^T x^A = a_0\right\}$. Since $A = C_{00}$ is the only set $A \in \mathcal{F}_n^s$ satisfying $x_{00}^A = 1$ it follows that $\mathcal{F}_a \subset \left\{A \in \mathcal{F}_n^s : x_{00}^A = 0\right\}$, so $a^T x \leq a_0$ is equivalent to the facet $x_{00} \geq 0$. ■

Theorem 4.6 *For $S \subseteq J, |S| \geq 2, v \in S$ inequalities $x(\delta^+(S)) \geq x(\delta^+(v))$ are facet defining for P_n^s.*

Proof. W.l.o.g. let $S = \{1, \ldots, h\}$ with $1 < h \leq n$ and $v = 1$. For convenience, let $a^T x \geq 0$ denote the inequality $x(\delta^+(S)) \geq x(\delta^+(1))$, and $S_1 = S \setminus \{1\}$. We list n^2 feasible dicycles whose incidence vectors are linearly independent and satisfy $a^T x^{C_i} = 0$:

1. $C_{00} = \{\mathbf{00}\}$; the 'idle' loop.
2. $C_{i0} = \{0i, \mathbf{i0}\}$ for $i \in J \setminus S_1$; $n - h + 1$ dicycles.
3. $C_{ij} = \{0i, \mathbf{ij}, j0\}$ for $i, j \in J \setminus S_1$, $i \neq j$; $(n - h + 1)(n - h)$ dicycles.
4. $C_{i1} = \{0i, \mathbf{i1}, 10\}$ for $i \in S_1$; $h - 1$ dicycles.
5. $C_{ij} = \{0i, \mathbf{ij}, j1, 10\}$ for $i \in J \setminus \{1\}$, $j \in S \setminus \{1, i\}$; $(n - 2)(h - 1)$ dicycles.
6. $C_{ij} = \{01, 1i, \mathbf{ij}, j0\}$ for $i \in S_1$, $j \in J \setminus S$; $(n - h)(h - 1)$ dicycles.
7. $C_{i0} = \{01, 1i, \mathbf{i0}\}$ for $i \in S_1$; $h - 1$ dicycles.
8. $C_{1i} = \{01, \mathbf{1i}, i(i + 1), (i + 1)0\}$ for $i \in \{2, \ldots, h - 1\}$; $h - 2$ dicycles.

To see the linear independence, collect the incidence vectors of the dicycles as rows in a matrix. By arranging the rows corresponding to 1–6 in reverse order and the columns of the variables indicated in bold 'in front on the diagonal' then this submatrix is upper triangular with each diagonal element equal to 1.

On the other hand the submatrix corresponding to the rows of 7 and 8 is regular on the support indicated in bold (on this support the submatrix is the edge-node incidence matrix of an undirected path). The only other type of rows having these variables in their support is in 6, but the rows of 6 are already determined uniquely by their support in bold, which is not in the support of any other row. So 6–8 are linearly independent on their support in bold and none of the variables in this support appear in any other row, therefore the incidence vectors of the cycles 1–8 are linearly independent. ■

It is to be expected that the following theorem has already been proved for the PCTSP, but we could not find an explicit reference so far.

Theorem 4.7 *For $x \in \mathbb{Q}^{n(n+1)+1}$, $x \geq 0$ satisfying (7), inequalities (8) can be separated in polynomial time.*

Proof. For $v \in J$ construct a digraph $D_v = (V_s, A_v)$ with $A_v = \{ij : i \in J, j \in V_s \setminus \{v\}\}$ and define capacities $c^v_{ij} = x_{ij}$ for $ij \in A_v$. Solve, in polynomial time, a v0-max flow problem for these capacities to obtain a min-cut set $\delta_+(S_v)$ with $v \in S_v \subset J$ for these capacities. If $x(\delta^+(S_v)) < x(\delta^+(v))$ then a violated inequality has been found, otherwise (8) is satisfied for all $S \subset J$ with $v \in S$. ■

In order to show the relation between P^s_n and the polytope P_0 for the PCTSP as defined in [1, 2] we introduce loop variables x_{jj} (called y_j in [2]) and add to (6)–(9) the constraints

$$x_{jj} + x(\delta^-(j)) = 1 \qquad j \in J. \tag{10}$$

The polyhedron $P^s_n(n)$ corresponding to the convex hull of the points satisfying constraints (6)–(9) and (10) is not exactly P_0 of [2], but the only differences are that now the "long" cycle is required to pass through node 0 and that this long cycle may also be of length

1 as opposed to at least two. This close relationship allows to copy the lifting Theorem 3.1 of [2] more or less verbatim. Denote by $P_n^s(k)$ the polytope obtained from the ATSP polytope P (with respect to the complete digraph on nodes $J \cup \{0\}$) by introducing the first k loop variables x_{jj} for $j = 0, \ldots, k$.

Theorem 4.8 *Let $a^T x \leq a_0$ be any facet defining inequality for the ATSP polytope P with respect to the complete digraph on V_s. For $k = 0, \ldots, n$ define $b_{j_k} = a_0 - z(P_n^s(k))$, where*

$$z(P_n^s(k)) = \max \left\{ a^T x + \sum_{i=0}^{k-1} b_{j_i} x_{j_i j_i} : x \in P_n^s(k), x_{j_k j_k} = 1 \right\}.$$

Then the inequality $a^T x + \sum_{i=0}^{k-1} b_{j_i} x_{j_i j_i} \leq a_0$ is valid and facet defining for $P_n^s(k)$.

Proof. As in [2], but with $P_0(k)$ replaced with $P_n^s(k)$. ■

After having determined the lifting coefficients we eliminate variables x_{jj} for $j \in J$ by using (10) in order to obtain the lifted inequality for P_n^s.

For example the v0-cut inequalities can be seen to be liftings of the subtour elimination constraints for $0 \notin S \subset V_s$, $|S| \geq 2$, with respect to the sequence $j_1, \ldots, j_{|S|} = v \in S$ and afterwards for $j \in V_s \setminus S$. Here, we need the direct proof of Theorem 4.6 in order to arrive at Theorem 5.3.

It is not yet clear for which other classes of facet defining inequalities of the ATSP polytope it is possible to determine the lifting coefficients explicitly as in [2], because the special role of node 0 does not allow to use these results without some careful checking.

For the following class of facet defining inequalities we do not know whether there exists a corresponding class in the ATSP or PCTSP literature.

Theorem 4.9 *Let $S_0 \subset S_1 \subset \ldots \subset S_k = J$ be a nested sequence of proper subsets with $k \geq 2$ and let $t_0 \in S_0$, $t_i \in S_i \setminus S_{i-1}$ for $i = 1, \ldots, k$. For*

$$A_c = \bigcup_{i=1,\ldots,k} \left[\delta^+(S_{i-1}) \setminus \{vt_i : v \in S_{i-1}\} \right]$$

the nested conflict inequality

$$\sum_{i=1}^{k} x_{t_i t_0} - \sum_{ij \in A_c} x_{ij} \leq 0 \tag{11}$$

is facet defining for P_n^s if $n \geq 3$.

Proof. Validity: Any dicycle through 0 contains at most one of the arcs $t_i t_0$ with $1 \leq i \leq k$. If it does so for some i then it must include an arc from the cut set $\delta^+(S_{i-1}) \setminus \{vt_i : v \in S_{i-1}\}$ in order to return from the set S_{i-1} to 0 without visiting t_i once more.

Facet defining: We exhibit n^2 linearly independent dicycles through 0 that satisfy (11) with equality. In order to simplify notation we will only give the sequence of vertices of each dicycle, consecutive multiple appearances of the same vertex should be interpreted as a single appearance, a sequence $t_j, t_{j+1}, \ldots, t_k$ is regarded as empty if $j > k$. Furthermore, we will use $\Delta_i = S_i \setminus S_{i-1} \setminus \{t_i\}$ for $1 \leq i \leq k$ and $\Delta_0 = S_0 \setminus \{t_0\}$ (these sets may be empty).

1. The 'idle' loop: $0, 0$
2. $v \in \Delta_i$, $w \in \Delta_j$, $0 \le i < j \le k$: $0, t_j, t_0, t_1, \ldots, t_i, v, w, t_{j+1}, t_{j+2}, \ldots, t_k, 0$
3. $v \in \Delta_i$, $w \in \Delta_j$, $k \ge i \ge j \ge 0$, $v \ne w$: $0, v, w, t_{j+1}, \ldots, t_k, 0$
4. $v \in \Delta_i$, $w = t_j$, $0 \le i < j - 1 \le k - 1$: $0, t_{j-1}, t_0, t_1, \ldots, t_i, v, t_j, t_{j+1}, \ldots, t_k, 0$
5. $v = t_i$, $w \in \Delta_j$, $0 \le i < j \le k$: $0, t_j, t_0, t_1, \ldots, t_i, w, t_{j+1}, t_{j+2}, \ldots, t_k, 0$
6. $v = t_1$, $w \in \Delta_0$: $0, t_k, t_0, t_1, w, 0$
7. $v \in \Delta_i$, $w = 0$, $0 \le i \le k - 1$: $0, t_k, t_0, t_1, \ldots, t_i, v, 0$
8. $v = t_i$, $w \in \Delta_j$, $k \ge i \ge j \ge 0$, $i > 1$: $0, t_i, w, t_1, t_0, 0$
9. $v = t_i$, $w \in \Delta_i$, $0 \le i \le 1$: $0, t_i, w, t_{i+1}, t_{i+2}, \ldots, t_k, 0$
10. $v \in \Delta_i$, $w = t_j$, $k \ge i \ge 0$, $i + 1 \ge j$, $k \ge j \ge 0$: $0, v, t_j, t_{j+1}, \ldots, t_k, 0$
11. $v \in \Delta_k$, $w = 0$: $0, v, 0$
12. $v = t_i$, $w = 0$, $1 \le i \le k - 1$: $0, t_{i+1}, t_0, t_1, \ldots, t_i, 0$
13. $v = t_i$, $w = t_j$, $0 \le i < j - 1 \le k - 1$: $0, t_{i+1}, t_0, t_1, \ldots, t_i, t_j, t_{j+1}, \ldots, t_k, 0$
14. $v = t_i$, $w = t_{i+1}$, $0 \le i \le k - 1$: $0, t_i, t_{i+1}, \ldots, t_k, 0$
15. $v = t_i$, $w = t_j$, $k \ge i > j \ge 0$: $0, t_i, t_j, t_0, 0$
16. $v = t_k$, $w = 0$: $0, t_k, 0$

For each edge $vw \in A_s \setminus \{01, \ldots, 0n, t_0 0\}$ (in total n^2 edges) the list provides an associated dicycle via v and w as specified above. Observe that an identifying edge in this list does not appear in dicycles listed after the specification of its associated dicycle (an appropriate internal order can be defined for 14 and 15). Thus, by arranging the incidence vectors of the dicycles as rows and the edges as columns according to this ordering the resulting matrix is upper triangular with ones on the main diagonal and therefore the vectors are linearly independent. ■

5 Facets of the m-Cost ATSP Polytope

Theorem 5.1 *For $ijk \in \mathcal{A}$, $x_{ijk} \ge 0$ defines a facet of P_n^m if $n \ge 3$ and $m \ge 2$.*

Proof. Without loss of generality let $k = 2$.

For 002 or $ij2 \in \mathcal{A}$ with $i, j \in J$ we first construct $n(n-1)$ tours on machine 1 as in the proof of Theorem 3.4. Then we construct n^2 solutions corresponding to the dicycles given in the proof of Theorem 4.4, where these dicycles are used on machine 2, whereas the remaining nodes of J are covered by dicycles on machine 1. Finally we construct a tour on machine 2 without using the current arc in question, which is the first solution containing 001 in its support. If $m > 2$ further linear independent solutions can be constructed by using the dicycles of the proof of Theorem 4.3 on these machines and by completing them on machine 1.

For indices $0v2$ we show this, w.l.o.g., for $v = 1$. The proof is analogous, it uses the corresponding dicycles of Theorem 4.4. The final tour for exploiting 001 has the form $\{0n2\} \cup \{i(i-1)2 : i \in J\}$. Since for $n = 3$ this tour does not have common support with other tours of Theorem 4.4 with respect to their support indicated in bold, it is linearly dependent with respect to dicycles only, that do not have 001 in their support. ■

Theorem 5.2 *For $m \geq 3$ and $n \geq 3$ all facet defining inequalities $\hat{a}^T x \leq \hat{a}_0$ of P_n^s give rise to facet defining inequalities $a^T x \leq a_0$ for P_n^m by identifying A_s with A_h for some $h \in M$, i.e., by setting $a_{ijh} = \hat{a}_{ij}$ for $ij \in A_s$, $a_{ijk} = 0$ for $ijk \in \mathcal{A} \setminus A_h$, and $a_0 = \hat{a}_0$.*

Proof. W.l.o.g. let $h = m$. It follows from Theorem 5.1 that $x_{00m} \geq 0$ is facet defining. So let $\hat{a}^T y \leq \hat{a}_0$ be a facet defining inequality of P_n^s not equivalent to $x_{00} \geq 0$. Then Corollary 4.5 guarantees that any $x^{A^{m-1}}$ with $A^{m-1} \in \mathcal{F}_n^{m-1}$ is easily extended to a solution x^{A^m} with $A^m \in \mathcal{F}_n^m$ satisfying $a^T x^{A^m} = a^0$ by setting $A^m = A^{m-1} \cup \{00m\}$. By Theorem 3.4 this ensures the existence of $(m-1)n^2 - n + 1$ solutions that are linearly independent on the support outside A_m. Since $\hat{a}^T x \leq \hat{a}_0$ is a facet of P_n^s we can construct further $n^2 - 1$ linearly independent solutions with respect to the support in $A_m \setminus \{00m\}$ by completing the corresponding solutions of $\mathcal{F}_n^s$ on machine 1 to solutions of $\mathcal{F}_n^m$. ■

We do not know whether the theorem also holds for $m = 2$. The main difficulty is that one has to construct an additional solution that exploits the variable 001. We can do this for the $v0$-cut and the nested conflict inequalities.

Theorem 5.3 *The $v0$-cut inequalities (4) are facet defining for P_n^m for $m \geq 2$ and $n \geq 3$.*

Proof. By Theorem 5.2 we may concentrate on the case $m = 2$. W.l.o.g. consider in (4) the case $k = 2$, $S = \{1, \ldots, h\}$ with $1 < h \leq n$ and $v = 1$. For convenience, let $a^T x \geq 0$ denote the inequality $x(\delta_2^+(S)) \geq x(\delta_2^+(1))$, and $S_1 = S \setminus \{1\}$.

We use $n(n-1)$ tours on machine 1 as in the proof of Theorem 3.4 and $n^2 - 1$ additional solutions corresponding to the dicycles defined in the proof of Theorem 4.6 on machine 2, completed to full solutions on machine 1. These $2n^2 - n - 1$ solutions are linearly independent and satisfy $a^T x = 0$. If $n > 3$ then a tour on machine 2 suffices, since all other solutions have $x_{001} = 0$. If $n = 3$ then consider the tour $C = \{022, 212, 132, 302\}$. Eliminate in its incidence vector the support in 212 by subtracting an incidence vector of type 4 (in combination with the type 2 incidence vector $\{012, 102\}$) and eliminate the

support in 132 as well as 302 by subtracting an incidence vector of either type 3 or 7, then all relevant support on machine 2 has been eliminated while 001 is still in the support. So the tour is linearly independent with respect to all other solutions. ■

Theorem 5.4 *For $x \in \mathbb{Q}^{|\mathcal{A}|}$, $x \geq 0$ satisfying (3), the $v0$-cut inequalities (4) can be separated in polynomial time.*

Proof. Follows directly from Theorem 4.7. ■

Theorem 5.5 *The nested conflict inequalities (11) are facet defining for P_n^m for $m \geq 2$ and $n \geq 3$.*

Proof. By Theorem 5.2 we may concentrate on the case $m = 2$. Let k, S_i, t_i, Δ_i be defined as in Theorem 4.9 and its proof and identify the edges, w.l.o.g., with the edges of machine 2. Denote by $h_i = |\Delta_i|$ and let $\Delta_i = \left\{v_1^i, \ldots, v_{h_i}^i\right\}$, the sets may be empty. Consider the tour on machine 2 specified by the sequence of vertices

$$0, v_1^k, v_2^k, \ldots, v_{h_k}^k, v_1^{k-1}, \ldots, v_{h_{k-1}}^{k-1}, v_1^{k-2}, \ldots, v_{h_0}^0, t_1, t_2, \ldots, t_k, t_0, 0.$$

On the support $\mathcal{A} \setminus A_1 \setminus \{012, \ldots, 0n2, t_0 02\}$ the incidence vector (with respect to $\mathcal{A}$) of this tour is a linear combination of incidence vectors (with respect to $\mathcal{A}$) of dicycles from 3, 10, 14, and the dicycle $0, t_k, t_0, 0$ of 15 as specified in the proof of Theorem 4.9. None of these dicycles are tours on machine 2, so 001 is not in the support of any of these incidence vectors. However, 001 is in the support of the new tour. Thus, by the linear independence of the dicycles of the proof of Theorem 4.9, its incidence vector is linearly independent with respect to the $n(n-1)$ tours on machine 1 of the proof of Theorem 3.4 and the $n^2 - 1$ dicycles (the idle dicycle is covered by the tours on machine 1) of the proof of Theorem 4.9. ■

6 Subtour Elimination Constraints

We consider the standard transformation from m-ATSP (with identical machines) to standard ATSP (see e.g. [16]). To this end let $D = (V, \mathcal{B})$ be a digraph with $V = \{0_1, \ldots, 0_m\} \cup J$ and

$$\begin{array}{rll} \mathcal{B} = & \{0_k i, i0_k : k \in M, i \in J\} & \cup \\ & \{ij : i \neq j, i, j \in J\} & \cup \\ & \{0_k, 0_{k+1} : k \in M \setminus \{m\}\} & \cup \\ & \{0_m 0_1\}. & \end{array}$$

We map vectors $x \in \mathbb{Q}^{|\mathcal{A}|}$ to $Tx = y \in \mathbb{Q}^{|\mathcal{B}|}$ by

$$\begin{array}{lcll} y_{0_k i} & = & x_{0ik} & i \in J, k \in M \\ y_{i0_1} & = & x_{i0m} & i \in J \\ y_{i0_k} & = & x_{i0(k-1)} & i \in J, k \in \{2, \ldots M\} \\ y_{ij} & = & \sum_{k \in M} x_{ijk} & i \neq j, i, j \in J \\ y_{0_k 0_{k+1}} & = & x_{00k} & k \in \{1, \ldots m-1\} \\ y_{0_m 0_1} & = & x_{00m} & \end{array}$$

It is well known that any feasible solution x^A with $A \in \mathcal{F}_n^m$ maps to $y^B = Tx^A$ with B being a tour in D. Probably the most useful inequalities for the general ATSP are the subtour elimination constraints, usually stated in the form

$$y(\mathcal{B}(S)) \leq |S| - 1 \qquad \text{for } S \subset V, 2 \leq |S| \leq |V| - 2.$$

In order to show that Tx satisfies all subtour elimination constraints if x satisfies all v0-cut inequalities, we need the following observation.

Proposition 6.1 *Let $x \in \mathbb{Q}^{|\mathcal{A}|}$, $x \geq 0$, satisfy constraints (1) to (3). If for some $S \subset V$, $2 \leq |S| \leq |V| - 2$, there exist $h, k \in M$, $h \neq k$ such that $0_h \in S$ and $0_k \notin S$ then the subtour elimination constraint corresponding to S cannot be violated for $y = Tx$.*

Proof. W.l.o.g. assume $h = 1, k = 2$ and let $x \in \mathbb{Q}^{|\mathcal{A}|}$, $x \geq 0$. Decompose the flow through h into cycles C_1 to C_r. In the transformed graph each of these cycles correspond to a path from 0_1 to 0_2 and must therefore cross $\delta^+(S)$. Therefore, $1 = x_{001} + x(\delta_1^+(0)) \leq y(\delta^+(S))$, which implies because of $y(\delta^+(S)) + y(\mathcal{B}(S)) = \sum_{v \in S} y(\delta^+(v)) = |S|$ that $y(\mathcal{B}(S)) \leq |S| - 1$. ■

Theorem 6.2 *Let $x \in \mathbb{Q}^{|\mathcal{A}|}$, $x \geq 0$, satisfy constraints (1) to (4), let $y = Tx$, and let $S \subset V$ with $2 \leq |S| \leq |V| - 2$. Then $y(\mathcal{B}(S)) \leq |S| - 1$.*

Proof. Because of Proposition 6.1 we may assume $0_k \notin S$ for $k \in M$. First observe that

$$\sum_{v \in S} x(\delta_k^+(S)) - \sum_{v \in S} x(\delta_k^+(v)) = |S| x(\delta_k^+(S)) - x(\delta_k^+(S)) - x(A_k(S)).$$

By summing constraints (4) over all $k \in M$ and $v \in S$ we obtain

$$\begin{aligned}
0 \leq & \sum_{k \in M} \sum_{v \in S} \left[x(\delta_k^+(S)) - x(\delta_k^+(v)) \right] = \\
= & \sum_{k \in M} \left[|S| x(\delta_k^+(S)) - x(\delta_k^+(S)) - x(A_k(S)) \right] \\
= & (|S| - 1) \sum_{k \in M} x(\delta_k^+(S)) - \sum_{k \in M} x(A_k(S)) \\
= & (|S| - 1) y(\delta^+(S)) - y(\mathcal{B}(S)) \\
= & (|S| - 1) y(\delta^+(S)) + (|S| - 1) \left[y(\mathcal{B}(S)) - y(\mathcal{B}(S)) \right] - y(\mathcal{B}(S)) \\
= & (|S| - 1) \sum_{v \in S} y(\delta^+(v)) - |S| y(\mathcal{B}(S)) \\
= & (|S| - 1) \sum_{v \in S} 1 - |S| y(\mathcal{B}(S)) = (|S| - 1)|S| - |S| y(\mathcal{B}(S))
\end{aligned}$$

It follows that $y(\mathcal{B}(S)) \leq |S| - 1$ and the proof is complete. ■

7 Numerical Results

The scheduling problem at Herlitz PBS AG asks for a solution minimizing the makespan over two non-identical machines. Conflicts arising from jobs using the same printing cylinders are avoided by additional side constraints that require that these jobs be executed on the same machine. The problem is thus not a pure m-Cost ATSP, but the basic structure is the same. Our data instances are constructed by randomly choosing jobs (without repetitions) from a set of roughly 150 jobs provided by Herlitz PBS AG. The cost coefficients are computed according to data and rather involved rules determined in cooperation with experts at Herlitz PBS AG. The cost of one arc c_{ijk} includes the setup time from job i to job j and the printing time of job j on machine k.

Our computational experiments were performed on a Sun Ultra 10 with a 299 MHz SUNW,UltraSPARC-IIi CPU and with CPLEX 6.0 as LP-solver, computation times are given in seconds. Table 1 shows detailed results for 10 random instances of 60 jobs, the typical number of jobs processed in one week.

Table 1: 10 instances of 60 jobs each

n	feas	v0-cut (%)	secs	subtour (%)	subt+D_k (%)
60	9420	9406.8 (0.14)	19.6	9387.42 (0.35)	9387.42 (0.35)
60	9382	9366.93 (0.16)	10.5	9345.37 (0.39)	9345.48 (0.39)
60	9519	9481.32 (0.4)	6.8	9425.49 (0.98)	9425.49 (0.98)
60	9723	9707.24 (0.16)	12.2	9674.58 (0.5)	9674.58 (0.5)
60	9910	9900.59 (0.1)	31.4	9859.52 (0.51)	9860.24 (0.5)
60	9421	9411.46 (0.1)	8.5	9395.92 (0.27)	9395.92 (0.27)
60	9984	9967.58 (0.16)	14.4	9914.74 (0.69)	9915.24 (0.69)
60	9557	9542.84 (0.15)	17.9	9517.14 (0.42)	9517.14 (0.42)
60	9263	9261.73 (0.01)	7.4	9238.88 (0.26)	9238.88 (0.26)
60	9373	9343.13 (0.32)	23.9	9294.81 (0.83)	9294.81 (0.83)

Column n gives the size of the instance, *feas* displays the best integral solution we know, *v0-cut* shows the optimal value of the relaxation based on the v0-cut inequalities, in parenthesis we give the relative gap $1-$*(v0-cut/feas)*. *secs* is the number of seconds the code needed to solve this relaxation, including separation. We compare the value of relaxation *v0-cut* to the relaxations obtained by separating the subtour inequalities on the transformed problem and to the relaxation combining subtour and D_k^+/D_k^- inequalities (for the definition of D_k^+/D_K^- inequalities see [10], for a separation heuristic [7]). With respect to the latter relaxations the v0-cut inequalities close the gap by roughly 60%. Optimal solutions to the subtour relaxations without v0-cut inequalities would typically exhibit several subtours on the one machine subproblems. There is still much room for improvement in the separation of the v0-cut inequalities, so it should be possible to reduce the computation time in more sophisticated implementations. It is surprising that the

D_k^+/D_k^- inequalities are not very effective, quite contrary to the experience in usual ATSP-problems.

Table 2: Relative gap and computation time, averaged over 10 instances

n	#	v0-cut %	secs	subtour %	subt+D_k %
40	10	0.22	2.84	0.54	0.54
50	10	0.15	9.57	0.51	0.51
60	10	0.17	15.25	0.52	0.52
70	10	0.16	38.13	0.52	0.52
80	10	0.18	55.19	0.49	0.48
90	10	0.23	82.75	0.54	0.54
100	10	0.19	135.91	0.44	0.44

In Table 2 we summarize results on instances with 40 to 100 jobs. For each problem size we generated 10 instances, the table displays the average relative gap of the relaxations as well as the average computation time. In the application in question the error in the data is certainly significantly larger than 0.2%, so there is no immediate need to improve these solutions further. On the other hand, most fractional solutions we investigated so far exhibited violated nested conflict inequalities, so separation heuristics for this new class of inequalities might help to improve the performance.

I thank Norbert Ascheuer for many helpful discussions and pointers to the literature and software, Oleg Mänz for helping in the implementation of the v0-cut separator, and Thorsten Koch for making available his min-cut code.

References

[1] E. Balas. The prize collecting traveling salesman problem. *Networks*, 19:621–636, 1989.

[2] E. Balas. The prize collecting traveling salesman problem: II. polyhedral results. *Networks*, 25:199–216, 1995.

[3] E. Balas and M. Fischetti. A lifting procedure for the asymmetric traveling salesman polytope and a large new class of facets. *Mathematical Programming*, 58:325–352, 1993.

[4] M. O. Ball, T. L. Magnanti, C. L. Monma, and G. L. Nemhauser, editors. *Network Routing*, volume 8 of *Handbooks in Operations Research and Management Science*. Elsevier Sci. B.V., Amsterdam, 1995.

[5] J. Desrosiers, Y. Dumas, M. M. Solomon, and F. Soumis. *Time Constrained Routing and Scheduling*, chapter 2, pages 35–139. Volume 8 of Ball et al. [4], 1995.

[6] M. Fischetti, J. Salazar González, and P. Toth. A branch-and-cut algorithm for the generalized travelling salesman problem. Technical report, University of Padova, 1994.

[7] M. Fischetti and P. Toth. A polyhedral approach to the asymmetric traveling salesman problem. *Management Science*, 43:1520–1536, 1997.

[8] M. Fischetti and P. Toth. An additive approach for the optimal solution of the prize-collecting travelling salesman problem. In B. Golden and A. Assad, editors, *Vehicle Routing: Methods and Studies*, pages 319–343. Elsevier Science Publishers B.V. (North-Holland), 1998.

[9] M. Gendreau, G. Laporte, and F. Semet. A branch-and-cut algorithm for the undirected selective traveling salesman problem. *Networks*, 32:263–273, 1998.

[10] M. Grötschel and M. Padberg. Polyhedral theory. In Lawler et al. [12], chapter 8.

[11] M. Jünger, G. Reinelt, and G. Rinaldi. The traveling salesman problem. In M. Ball, T. Magnanti, C. Monma, and G. Nemhauser, editors, *Network Models*, volume 7 of *Handbooks in Operations Research and Management Science*, chapter 4, pages 225–330. North Holland, 1995.

[12] E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, and D. B. Shmoys, editors. *The Traveling Salesman Problem.* John Wiley & Sons Ltd, Chichester, 1985.

[13] M. Padberg and M. Grötschel. Polyhedral computations. In Lawler et al. [12], chapter 9.

[14] M. Padberg and G. Rinaldi. Facet identification for the symmetric traveling salesman polytope. *Mathematical Programming*, 47:219–257, 1990.

[15] M. Padberg and G. Rinaldi. A branch and cut algorithm for the resolution of large–scale symmetric traveling salesman problems. *SIAM Review*, 33:60–100, 1991.

[16] G. Reinelt. *The traveling salesman - Computational solutions for TSP applications.* Number 840 in Lecture Notes in Computer Science. Springer-Verlag, 1994.