



FABIO D'ANDREAGIOVANNI<sup>1,2</sup>, & ANNIE RAYMOND<sup>2</sup>

## **Multiband Robust Optimization and its Adoption in Harvest Scheduling**

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<sup>1</sup>DFG Research Center MATHEON, Technical University Berlin, Straße des 17. Juni 135,  
10623 Berlin, Germany

<sup>2</sup>Dept. of Optimization, Zuse-Institut Berlin (ZIB), Takustr. 7, 14195 Berlin, Germany,  
e-mail: {d.andreagiovanni, raymond}@zib.de

Herausgegeben vom  
Konrad-Zuse-Zentrum für Informationstechnik Berlin  
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Telefon: 030-84185-0  
Telefax: 030-84185-125

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# Multiband Robust Optimization and its Adoption in Harvest Scheduling

Fabio D’Andreagiovanni and Annie Raymond

**Abstract** A central assumption in classical optimization is that all the input data of a problem are exact. However, in many real-world problems, the input data are subject to uncertainty. In such situations, neglecting uncertainty may lead to nominally optimal solutions that are actually suboptimal or even infeasible. Robust optimization offers a remedy for optimization under uncertainty by considering only the subset of solutions protected against the data deviations. In this paper, we provide an overview of the main theoretical results of multiband robustness, a new robust optimization model that extends and refines the classical theory introduced by Bertsimas and Sim. After introducing some new results for the special case of pure binary programs, we focus on the harvest scheduling problem and show how multiband robustness can be adopted to tackle the uncertainty affecting the volume of produced timber and grant a reduction in the price of robustness.

## 1 Introduction

Tackling uncertain data in optimization problems has attracted academic attention since the seminal study by Dantzig [13] and, over the years, many models and approaches such as stochastic programming [20] and robust optimization [3] have been proposed. In recent years, practitioners have shown particular interest in robust optimization, especially because of its computational tractability and the possibility of having a strong control over the characterization of the uncertainty. Many collaborations between practitioners and academic as well as research institutions were

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Fabio D’Andreagiovanni  
DFG Research Center MATHEON, Technical University Berlin, Straße des 17. Juni 135, 10623 Berlin, Germany  
Dept. of Optimization, Zuse-Institut Berlin (ZIB), Takustr. 7, 14195 Berlin, Germany,  
e-mail: d.andreagiovanni@zib.de

Annie Raymond  
Dept. of Optimization, Zuse-Institut Berlin (ZIB), Takustr. 7, 14195 Berlin, Germany,  
e-mail: raymond@zib.de

formed to adapt robust optimization to real-world optimization problems (see e.g., [7]).

In this work, we provide an overview of the main theoretical results of multiband robustness ([8, 9]), a model recently proposed by Büsing and D'Andreagiovanni to overcome some limitations of the classical Bertsimas-Sim model, while maintaining its accessibility and simplicity of use. The idea at the basis of the new model is to partition the single deviation band that characterizes the Bertsimas-Sim model into multiple bands to allow a better representation of arbitrary-shaped uncertainty distributions. After outlining the main features and results of the Bertsimas-Sim model, we discuss three of the main results for multiband robustness: 1) the robust counterpart of a mixed-integer linear program can be formulated as a compact mixed-integer linear program; 2) the separation of robustness cuts can be operated efficiently by solving a min-cost flow problem; 3) in the case of a pure binary program, the tractability and approximability of the original nominal problem are maintained if a constant number of constraints are uncertain. Finally, we define a new family of strong valid inequalities for binary programs.

After this overview, we show how multiband robustness can be applied to tackle uncertainty in a forest management problem, namely the harvest scheduling problem. The goal of this problem is to select areas of a forest to be harvested in each period of a finite time horizon in order to maximize revenues. Though the harvest scheduling problem has been thoroughly studied in a deterministic fashion, limited research focused on its uncertain counterpart (we refer the reader to the recent work [19] for an overview of related works in forest management). We apply multiband robustness to the cluster packing formulation with volume constraints to address the issue of uncertain timber production volumes, and we conclude by presenting preliminary computational results.

## 2 An Introduction to Robust Optimization

Consider the following generic mixed integer linear program (MILP):

$$\begin{aligned}
 Z^* &:= \max \sum_{j \in J} c_j x_j && (MILP) \\
 \sum_{j \in J} a_{ij} x_j &\leq b_i && i \in I \\
 x_j &\geq 0 && j \in J \\
 x_j &\in \mathbb{Z}_+ && j \in J_{\mathbb{Z}} \subseteq J.
 \end{aligned}$$

A typical assumption in classical optimization is that all the data of the problem are known exactly. However, for many problems, the value of the coefficients  $a_{ij}$ ,  $b_i$ ,  $c_j$  may be uncertain. For example, they might represent historical averages of what happened in the past, but the future might not behave as the past has on average. There might be some erroneous or unprecise measures as well as typos. Some of

these values might also be irrational numbers that will be cut off by the computer for numerical reasons. These are all sources of uncertainties.

These uncertainties might seem unimportant, especially if they are small. However, neglecting uncertainty can lead to disastrous consequences. Indeed, the presence of uncertainties actually corresponds to the presence of additional deviation terms in the problem:

$$\begin{aligned}
 Z^{DEV} &:= \max \sum_{j \in J} (c_j + \delta_j^c) x_j \\
 \sum_{j \in J} (a_{ij} + \delta_{ij}^a) x_j &\leq (b_i + \delta_i^b) & i \in I \\
 x_j &\geq 0 & j \in J \\
 x_j &\in \mathbb{Z}_+ & j \in J_{\mathbb{Z}} \subseteq J,
 \end{aligned}$$

where  $\delta_j^c, \delta_{ij}^a, \delta_i^b$  represent the deviation that the coefficients  $c_j, a_{ij}, b_i$  respectively experience.

The optimal solution  $Z^*$  of the first program might be unfeasible for the second program. If it is still feasible, it might be suboptimal under the objective function  $\max \sum_{j \in J} (c_j + \delta_j^c) x_j$ , i.e.  $Z^* < Z^{DEV}$ . Losing feasibility or optimality can have very bad effects in practice, and so different techniques have been developed to deal with uncertainty over the years (we refer the reader to [3, 4, 20] for an introduction of optimization under uncertainty).

From now on, we assume without loss of generality that uncertainty in *MILP* only affects the coefficients  $a_{ij}$ . Indeed, if the coefficients  $b_i$  and/or  $c_j$  are uncertain, it is easy to derive an equivalent program where the uncertainty only affects the coefficient matrix. We now illustrate the essential features of a robust optimization approach by referring to the classical model by Bertsimas and Sim [5], undoubtedly the most famous and successful robust model which has been applied in many different contexts. The Bertsimas-Sim model is based on the following assumptions:

1. for each uncertain coefficient, the decision maker identifies a *nominal value*  $\bar{a}_{ij}$  (for example, this could be the expected value derived from historical data) and the *maximum deviation possible*  $d_{ij}$  from the nominal value;
2. the *actual value*  $a_{ij}$  of an uncertain coefficient belongs to the *symmetric deviation band*  $[\bar{a}_{ij} - d_{ij}, \bar{a}_{ij} + d_{ij}]$ ;
3. the uncertain coefficients  $a_{ij}$  are stochastically independent random variables defined over their own deviation ranges according to an unknown symmetric distribution;
4. for each constraint  $i \in I$ , the decision maker specifies a value  $0 \leq \Gamma_i \leq |J|$  representing the maximum number of coefficients deviating simultaneously from their nominal value.

The set of deviations that respect these assumptions is called the *uncertainty set*.

The parameter  $\Gamma_i$  controls the conservativeness of the model: increasing its value increases also the protection against deviation, but at the same time, we incur the

so-called *price of robustness*, that is, the deterioration of the optimal value caused by the exclusion of non-robust solutions from the feasible set.

Under these assumptions, a bilevel robust counterpart of MILP can be written as follows:

$$\begin{aligned}
 & \max \sum_{j \in J} c_j x_j && (NL-ROB-MILP) \\
 & \sum_{j \in J} \bar{a}_{ij} x_j + DEV(x, \Gamma_i) \leq b_i && i \in I \\
 & x_j \geq 0 && j \in J \\
 & x_j \in \mathbb{Z}_+ && j \in J_{\mathbb{Z}} \subseteq J,
 \end{aligned}$$

where each feasibility constraint includes the additional term  $DEV(x, \Gamma_i)$ , that represents the worst deviation for constraint  $i$  allowed by the uncertainty set for a given solution  $x$  when at most  $\Gamma_i$  coefficients deviate. Specifically, the term  $DEV(x, \Gamma_i)$  corresponds to the optimal value of the following knapsack problem:

$$DEV(x, \Gamma_i) := \max \sum_{j \in J} d_{ij} x_j y_{ij} \quad (1)$$

$$\begin{aligned}
 & \sum_{j \in J} y_{ij} \leq \Gamma_i && (2) \\
 & y_{ij} \in \{0, 1\} && j \in J,
 \end{aligned}$$

in which the binary variable  $y_{ij}$  is 1 if coefficient  $a_{ij}$  deviates to its worst value, which is its most positive feasible value since we're maximizing (i.e.,  $a_{ij} = \bar{a}_{ij} + d_{ij}$ ) and 0 if the coefficient experiences no deviation (i.e.,  $a_{ij} = \bar{a}_{ij}$ ). The objective function (1) aims to maximize the total deviation in the constraint and the inequality (2) imposes that at most  $\Gamma_i$  coefficients deviate from their nominal value.

By pointing out the integrality of the knapsack polyhedron and by using duality results, Bertsimas and Sim removed the non-linearity of the previous robust counterpart and reached the following nice result:

**Theorem 1 (Bertsimas & Sim, 2004).** *The non-linear mixed-integer program (NL-ROB-MILP) is equivalent to the following compact and linear mixed integer program:*

$$\begin{aligned}
& \max \sum_{j \in J} c_j x_j \\
& \sum_{j \in J} \bar{a}_{ij} x_j + \Gamma_i w_i + \sum_{j \in J} z_{ij} \leq b_i & i \in I \\
& w_i + z_{ij} \geq d_{ij} x_j & i \in I, j \in J \\
& w_i \geq 0 & i \in I \\
& z_{ij} \geq 0 & i \in I, j \in J \\
& x_j \geq 0 & j \in J \\
& x_j \in \mathbb{Z}_+ & j \in J_{\mathbb{Z}} \subseteq J.
\end{aligned}$$

Notwithstanding its elegant simplicity, which has contributed to its wide success even beyond the mathematical programming community, the Bertsimas-Sim model presents some limitations. In particular, the hypothesis that each coefficient is a random variable symmetrically distributed over a symmetric range can be very limitative in practice and makes robust solutions more conservative than necessary, as Sim himself pointed out in [11]. Indeed, this model focuses on the extreme deviations of each coefficients and completely neglects the specific behaviour of the uncertainty within the deviation range. The drawbacks of this modeling approach are particularly evident in real-world problems, where it is very common to have coefficients whose value is asymmetrically distributed over asymmetric ranges. In order to overcome the limitations of the Bertsimas-Sim model, several refinements have been proposed over the years (see e.g., [3, 4, 11]). In particular, a recent model, multiband robustness, has tried to combine practical and theoretical considerations by taking into account the specific needs of practitioners who were asking for a refined representation of the uncertainty. Moreover, this new model maintains the accessibility and simplicity of use of the Bertsimas-Sim model. We provide an overview of multiband robustness and of its main theoretical results in the following section.

### 3 The Multiband Robustness Model

As we pointed out in the previous section, the use of a single and symmetric deviation band affects negatively the conservativeness of robust solutions, as it becomes evident when the probability of deviation changes within the range. In order to increase the modeling resolution of the uncertainty, we do a very simple operation: breaking the single deviation band into multiple narrower bands, each associated with its own parameter  $\Gamma_i$ . An uncertainty model with these features is able to effectively approximate an arbitrary-shaped uncertainty distribution, thus guaranteeing a much higher modeling power. The multiband idea was first exploited by Bienstock for the special case of robust portfolio optimization [6] and was then extended to wireless network design [12]. The first general theoretical study of the multiband model was done by Büsing and D'Andreagiovanni in 2012 (see [8], [9] and [10]).

The multiband uncertainty model assumes that:

1. for each coefficient  $a_{ij}$ , the decision maker knows its nominal value  $\bar{a}_{ij}$  as well as the maximum negative and positive deviations possible  $d_{ij}^{K-}, d_{ij}^{K+}$  from  $\bar{a}_{ij}$  (so  $a_{ij} \in [\bar{a}_{ij} + d_{ij}^{K-}, \bar{a}_{ij} + d_{ij}^{K+}]$ );
2. the overall single deviation band  $[d_{ij}^{K-}, d_{ij}^{K+}]$  of each coefficient  $a_{ij}$  is partitioned into  $K$  bands, defined on the basis of  $K$  deviation values:  
 $-\infty < d_{ij}^{K-} < \dots < d_{ij}^{K-2} < d_{ij}^{K-1} < d_{ij}^0 = 0 < d_{ij}^1 < d_{ij}^2 < \dots < d_{ij}^{K+} < +\infty$ ;
3. through these deviation values,  $K$  deviation bands are defined, namely: a set of positive deviation bands  $k \in \{1, \dots, K^+\}$  and a set of negative deviation bands  $k \in \{K^- + 1, \dots, -1, 0\}$ , such that a band  $k \in \{K^- + 1, \dots, K^+\}$  corresponds to the range  $(d_{ij}^{k-1}, d_{ij}^k]$ , and band  $k = K^-$  corresponds to the single value  $d_{ij}^{K-}$ ;
4. for each constraint  $i \in I$  and each band  $k \in K$ , a lower bound  $l_{ik}$  and an upper bound  $u_{ik}$  on the number of deviations that may fall in  $k$  are defined, so  $0 \leq l_{ik} \leq u_{ik} \leq n$ ;
5. the number of coefficients that take their nominal value is not limited, i.e.  $u_{i0} = n$  for all  $i \in I$ ;
6.  $\sum_{k \in K} l_{ik} \leq n$  for all  $i \in I$ , so that there always exists a feasible realization of the coefficient matrix.

The set of deviations satisfying these assumptions is called the *multiband uncertainty set*.

The multiband uncertainty model (MB) thus generalizes the uncertainty definition of the Bertsimas-Sim model: the single deviation band is partitioned into multiple bands and each band  $k \in K$  is associated not only with an upper bound  $u_{ik}$ , but also with a lower bound  $l_{ik}$  on the number of coefficients deviating in that band. The lower bound improves the modeling power of the decision maker and, more important, allows to take into account the presence of negative value deviations, that are neglected by the Bertsimas-Sim; the negative deviations reduce the value of the overall worst deviation.

As for the non-linear robust counterpart (NL-ROB-MILP) defined for the Bertsimas-Sim model, an additional term  $DEV(x, MB_i)$  is introduced in every feasibility constraint in the MB model, to represent the maximum total deviation that could be incurred by constraint  $i$  under the multiband uncertainty set for a solution  $x$ .

$$\begin{aligned}
 & \max \sum_{j \in J} c_j x_j \\
 & \sum_{j \in J} \bar{a}_{ij} x_j + DEV(x, MB_i) \leq b_i & i \in I \\
 & x_j \geq 0 & j \in J \\
 & x_j \in \mathbb{Z} & j \in J_{\mathbb{Z}} \subseteq J.
 \end{aligned}$$

Here, the term  $DEV(x, MB_i)$  is associated to the following problem:



$$DEV(x, MB_i) := \max \sum_{j \in J} \sum_{k \in K} d_{ij}^k x_j y_{ij}^k \quad (3)$$

$$l_{ik} \leq \sum_{j \in J} y_{ij}^k \leq u_{ik} \quad k \in K \quad (4)$$

$$\sum_{k \in K} y_{ij}^k \leq 1 \quad j \in J \quad (5)$$

$$y_{ij}^k \in \{0, 1\} \quad j \in J.$$

The decision variables  $y_{ij}^k$  determine whether or not coefficient  $a_{ij}$  deviates into its  $k$ th band. The objective function maximizes the total deviation incurred by constraint  $i$  under the multiband uncertainty set. Constraints (4) impose that the number of coefficients that deviate in their  $k$ th band is between  $l_{ki}$  and  $u_{ki}$ , whereas (5) impose that each coefficient deviates in at most one band (note that if  $\sum_{k \in K} y_{ij}^k = 0$  then the coefficient is experiencing no deviation and thus is implicitly falling in band  $k = 0$ ).

Again, this is a non-linear robust counterpart, since it is a bilevel max-max problem. However, it has been proven that it can be reformulated as a compact and linear problem. We refer the reader to [8,9] for the complete proofs of the results presented in this section.

**Theorem 2 (Büsing & D'Andreagiovanni, 2012).** *The robust counterpart of problem MILP under the multiband uncertainty set MB is equivalent to the following compact and linear mixed integer program:*

$$\begin{aligned} \max \sum_{j \in J} c_j x_j & \quad (\text{Rob-MILP}) \\ \sum_{j \in J} \bar{a}_{ij} x_j - \sum_{k \in K} l_{ik} v_{ik} + \sum_{k \in K} u_{ik} w_{ik} + \sum_{j \in J} z_{ij} & \leq b_i \quad i \in I \\ -v_{ik} + w_{ik} + z_{ij} & \geq d_{ij}^k x_j \quad i \in I, j \in J, k \in K \\ v_{ik}, w_{ik} & \geq 0 \quad i \in I, k \in K \\ z_{ij} & \geq 0 \quad i \in I, j \in J \\ x_j & \geq 0 \quad j \in J \\ x_j & \in \mathbb{Z}_+ \quad j \in J_{\mathbb{Z}} \subseteq J. \end{aligned}$$

As an alternative to solving (Rob-LP), a cutting-plane approach [18] has been investigated. Given a solution  $\bar{x}$ , we want to test if it is robust feasible, i.e. if  $\sum_{j \in J} \bar{a}_{ij} \bar{x}_j + DEV(x, MB_i) \leq b_i$  for all the feasible deviations included in the multiband uncertainty set, and that, for all  $i \in I$ . In the case of the Bertsimas-Sim model, the problem of separating a robustness cut for a given constraint is very simple and essentially consists in sorting the deviations in increasing order and choosing the worst  $I_i > 0$ . In the case of multiband uncertainty, this simple approach does not guarantee the robustness of a computed solution. However, it has been proven that the separation can be done in polynomial time by solving a min-cost flow problem as follows.

**Theorem 3.** *Let  $x$  be a solution to MILP and let  $MB$  be a multiband uncertainty set. Moreover, let  $(G, c)_x^i$  be the min-cost flow instance corresponding to a solution  $x$  and a constraint  $i \in I$  of MILP. The solution  $x$  is robust for constraint  $i$  with respect to  $MB$  if and only if  $\bar{a}_i'x - c_i^*(x) \leq b_i$ , where  $c_i^*(x)$  is the minimum cost of a flow of the instance  $(G, c)_x^i$ .*

The proof is based on showing the existence of a one-to-one correspondence between the integral flows of the min-cost flow problem and the non-dominated feasible solutions of the pure 0 – 1 program that computes the maximum deviation allowed by the multiband uncertainty set.

Let us now say a few more words about how to choose the band partition. Obviously, having more bands allows us to represent the uncertainty distribution more accurately and thus reduce conservativeness. On the other hand, doing so increases the size of the problem and thus the computing time. Given a fixed number of bands, we would like to find the partition that yields the least conservative robust solution. Two factors affect the conservativeness of the solution: the width of the bands, and the size of the gap between the lower and upper bounds  $l_{ki}$  and  $u_{ki}$  for that band. Indeed, looking at the bilevel program, it is clear that, the bigger the gap between  $d_{ij}^{k-1}$  and  $d_{ij}^k$ , then the more conservative the solution will be since the program goes for the worst deviation possible and picks deviation  $d_{ij}^k$ . So the further apart  $d_{ij}^{k-1}$  and  $d_{ij}^k$  are, the less likely it is that the actual deviation is very close to  $d_{ij}^k$ . Thus, one way to make solutions less conservative for a constant number of bands is to make them all the same width.

However, by doing that, one might get big gaps between some  $l_{ki}$ 's and  $u_{ki}$ 's since the  $k$ th band of one coefficient might contain a very different percentage of deviations of its distribution than for another coefficient. The bigger the gap, the more wildly the deviations can behave: more coefficients will deviate into bands that are worse than necessary. So another idea to make the solution less conservative for a constant number of bands is to make sure that the  $k$ th band in the multiband setup of each coefficient contains more or less the same percentage of deviations of its distribution, so that the gap between  $l_{ki}$  and  $u_{ki}$  is very small. However, by doing that, we might have some very wide bands.

Depending on the problem, one of these two ideas might yield a better solution; it might be worthwhile to try both strategies when applying multiband robustness.

**Exploiting domination to simplify the robust counterpart.** In [9], Büsing and D'Andreagiovanni propose a refinement of their multiband results, based on the exploitation of domination among deviation scenarios. In particular, they define what they call the *profile*  $(p, \theta)$  of a multiband uncertainty set for each constraint  $i \in I$  as follows:

$$p_i = \min\{k \in K : \sum_{\ell=K^-}^k l_{i\ell} + \sum_{\ell=k+1}^{K^+} u_{i\ell} \leq |J|\},$$

$$\theta_{ik} = \begin{cases} l_{ik} & k \leq p_i - 1 \\ u_{ik} & k \geq p_i + 1 \\ |J| - \sum_{k \in K \setminus \{p\}} \theta_{ik} & k = p_i. \end{cases}$$

Note that  $p_i \geq 0$  since  $u_{i0} = n$  and  $\sum_{k \in K} \theta_{ik} = |J|$ , for all  $i \in I$ . For each constraint  $i \in I$  and deviation band  $k \in K$ , the profile identifies the number of coefficients  $\theta_{ik}$  that must fall in the band  $k$  to maximize the deviation. Intuitively, since the aim is to maximize the total deviation,  $\theta_{ik}$  is set equal to the lower bound  $l_{ik}$  for the negative bands (a deviation occurring in them indeed reduce the total deviation), while as many coefficients as possible are requested to fall in the most positive deviation bands (while respecting the corresponding bounds).

Thanks to the definition of the profile concept, the term  $DEV(x, MB_i)$  can be associated to the following pure binary program with a reduced number of constraints:

$$\begin{aligned} DEV(x, MB_i) = \max \quad & \sum_{j \in J} \sum_{k \in K} d_{ij}^k x_j y_{ij}^k \\ & \sum_{j \in J} y_{ij}^k = \theta_{ik} & k \in K \\ & \sum_{k \in K} y_{ij}^k \leq 1 & j \in J \\ & y_{ij}^k \in \{0, 1\} & j \in J. \end{aligned}$$

By duality, the reduced number of constraints corresponds to a reduced number of variables in the compact robust counterpart, as follows:

$$\begin{aligned} \max \quad & \sum_{j \in J} c_j x_j & (\theta\text{-Rob-LP}) \\ & \sum_{j \in J} \bar{a}_{ij} x_j + \sum_{k \in K} \theta_{ik} w_{ik} + \sum_{j \in J} z_{ij} \leq b_i & i \in I \\ & w_{ik} + z_{ij} \geq d_{ij}^k x_j & i \in I, j \in J, k \in K \\ & w_{ik} \geq 0 & i \in I, k \in K \\ & z_{ij} \geq 0 & i \in I, j \in J \\ & x_j \geq 0 & j \in J \\ & x_j \in \mathbb{Z}_+ & j \in J_{\mathbb{Z}} \subseteq J. \end{aligned}$$

**Specializing results for pure 0 – 1 linear programs.** Another nice result that Büsing and D'Andreagiovanni achieved is that if the original problem is a purely binary problem that can be solved in polynomial time, then there exists a polynomial-

time algorithm to solve the robust counterpart if a constant number of constraints are uncertain. See [8, 9] for more details.

In addition to these results, we investigate here a new family of valid inequalities for  $DEV(x, MB_i)$  by adopting a proof strategy similar to that of Atamtürk for the Bertsimas-Sim model [1].

**Proposition 1.** *Consider the multiband model for an uncertain pure 0–1-program with only positive deviations. For any  $T := \{j_1, j_2, \dots, j_t\} \subseteq J$  with  $0 := d_{j_0}^k \leq d_{j_1}^k \leq \dots \leq d_{j_t}^k$  and for any  $k \in K$  and  $i \in I$ , the following inequality is valid for  $\theta$ –Rob–LP:*

$$\sum_{j_l \in T} (d_{ij_l}^k - d_{ij_{l-1}}^k) x_{j_l} \leq w_{ik} + \sum_{j \in T} z_{ij}.$$

*Proof.* Since we're considering a single  $i \in I$ , we drop its index in the proof to make the notation lighter.

Let

$$M = \max\{l \in [0, t] : d_{j_l}^k \leq w_k\}$$

and let  $d_{j_M}^k = w_k - \varepsilon$  with  $\varepsilon \geq 0$ . Consider the sum  $\sum_{j_l \in T} (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l}$ .

First, note how  $d_{j_l}^k - d_{j_{l-1}}^k \geq 0$  for all  $l \in [1, t]$ . Thus the sum will be maximized if  $x_{j_l} = 1$  for all  $l \in [1, t]$ . So by splitting the sum into three parts and using the fact that  $d_{j_M}^k = w_k - \varepsilon$ , we get:

$$\begin{aligned} & \sum_{j_l \in T} (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l} \\ &= \sum_{l=1}^M (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l} + (d_{j_{M+1}}^k - d_{j_M}^k) x_{j_{M+1}} + \sum_{l=M+2}^t (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l} \\ &\leq d_{j_M}^k + (d_{j_{M+1}}^k - d_{j_M}^k) x_{j_{M+1}} + \sum_{l=M+2}^t (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l} \\ &= w_k - \varepsilon + (d_{j_{M+1}}^k - (w_k - \varepsilon)) x_{j_{M+1}} + \sum_{l=M+2}^t (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l}. \end{aligned}$$

Then two scenarios arise: either  $x_{j_{M+1}} = 1$  or  $x_{j_{M+1}} = 0$ . In the first case, we get that the two  $\varepsilon$ -s cancel and the previous line becomes:

$$w_k + (d_{j_{M+1}}^k - w_k) x_{j_{M+1}} + \sum_{l=M+2}^t (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l},$$

with  $x_{j_{M+1}} = 1$ . In the second case, the second  $\varepsilon$  disappears and, since  $w_k - \varepsilon \leq w_k$ , the previous line can be seen as being at most:

$$w_k + (d_{j_{M+1}}^k - w_k) x_{j_{M+1}} + \sum_{l=M+2}^t (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l},$$

with  $x_{j_{M+1}} = 0$ . Thus, whether  $x_{j_{M+1}} = 1$  or  $x_{j_{M+1}} = 0$ , we get that:

$$\begin{aligned} \sum_{j_l \in T} (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l} &\leq w_k + (d_{j_{M+1}} - w_k) x_{j_{M+1}} + \sum_{l=M+2}^t (d_{j_l}^k - d_{j_{l-1}}^k) x_{j_l} \\ &\leq w_k + (d_{j_{M+1}} - w_k) x_{j_{M+1}} + \sum_{l=M+2}^t (d_{j_l}^k - w_k) x_{j_l} \\ &= w_k + \sum_{l=M+1}^t (d_{j_l}^k - w_k) x_{j_l}. \end{aligned}$$

The second line is obtained by noticing that  $d_{j_{l-1}}^k \geq w_k$  for  $M+2 \leq l \leq t$ . Now, recall that  $w_k + z_j \geq d_j^k x_j$  in  $\theta - \text{Rob} - LP$ . So, if  $x_{j_l} = 0$ , then  $z_{j_l} \geq (d_{j_l}^k - w_k) \cdot 0 = 0$ , and if  $x_{j_l} = 1$ , then  $z_{j_l} \geq (d_{j_l}^k - w_k) \cdot 1 = d_{j_l}^k x_{j_l} - w_k$ . Thus, we have that

$$\begin{aligned} w_k + \sum_{l=M+1}^t (d_{j_l}^k - w_k) x_{j_l} &\leq w_k + \sum_{l=M+1}^t z_{j_l} \\ &\leq w_k + \sum_{j \in T} z_j. \end{aligned}$$

□

**Proposition 2.** *If  $0 = d_{j_0} < d_{j_1} < \dots < d_{j_t}$ , then the preceding inequalities are facet-defining for  $DEV(x, MB_i)$ .*

*Proof.* Once again, we drop index  $i$  to lighten the notation since it is unnecessary.

We exhibit  $2|J| + |K|$  affinely independent points of the polytope which are tight with

$$\sum_{j_l \in T} (d_{j_l}^{k_1} - d_{j_{l-1}}^{k_1}) x_{j_l} \leq w_{k_1} + \sum_{j \in T} z_j,$$

thus implying that it is a facet.

$$\begin{array}{lll} x & z & w \\ \left( \begin{array}{ccc} 0, & 0, & 0 \end{array} \right) & & \\ \left( \begin{array}{ccc} e_j, & D_j e_j, & 0 \end{array} \right) & \forall j \in [n] \setminus T & \\ \left( \begin{array}{ccc} e_j, & (D_j + \varepsilon) e_j, & 0 \end{array} \right) & \forall j \in [n] \setminus T & \\ \left( \begin{array}{ccc} \sum_{l=1}^m e_{j_l}, & 0, & d_{j_m}^{k_1} e_{k_1} + \sum_{k \neq k_1} D^k e_k \end{array} \right) & \forall m \in \{1, \dots, t\} & \\ \left( \begin{array}{ccc} \sum_{l=1}^m e_{j_l}, & \varepsilon e_{j_m}, & (d_{j_m}^{k_1} - \varepsilon) e_{k_1} + \sum_{k \neq k_1} D^k e_k \end{array} \right) & \forall m \in \{1, \dots, t\} & \\ \left( \begin{array}{ccc} 0, & 0, & e_k \end{array} \right) & \forall k \neq k_1 & \end{array}$$

where  $D_j := \max_{k \in K} d_j^k$ ,  $D^k := \max_{j \in J} d_j^k$  and  $0 < \varepsilon$ .

Conversely, if  $d_{j_l}^{k_1} = d_{j_{l-1}}^{k_1}$  for some  $l \in [1, t]$ , then the inequality defined by  $T$  is implied by the one defined by  $T \setminus \{l\}$  and  $z_{j_l} \geq 0$   $\square$

## 4 The Harvest Scheduling Problem

The Harvest Scheduling Problem (HSP) is a central problem in forestry management that has received a lot of attention over the last two decades. The essential goal of the HSP is to select the areas of a forest that will be harvested in each period of a time horizon with the aim of maximizing a revenue function. Over the years, many optimization models have been developed and several variants have been presented in the literature to find schedules that take into account additional side constraints (e.g., so-called volume constraints to provide an even flow of timber over the planning horizon). We refer the reader to [17] for a good overview on the main variants of the HSP and related works.

One possible way to model HSP is to partition a forest into *stands* and plan harvesting at the stand-level: in each *period* either a stand is harvested or it is not. There is also a *maximum contiguous area*  $A_M$  that can be harvested in a single time period. It is assumed that the area of each stand is smaller or equal to  $A_M$ . A set of contiguous stands for which the total area is less than  $A_M$  is called a *feasible cluster*. Two clusters are said to be *incompatible* if they are adjacent or if they share a stand. Finally, it is also assumed that the considered time horizon is shorter than the time needed for a *green-up*: a stand can be harvested at most once during the time horizon. The following input is assumed to be given:

- $T$  the set of time periods;
- $S$  the set of stands;
- $\Omega$  the set of feasible clusters;
- $\Omega(U)$  the set of feasible clusters containing at least one stand in  $U \subseteq S$ ;
- $p_{C,t}$  the revenue obtained from harvesting cluster  $C$  in period  $t$ ;
- $\alpha_{C,t}$  the volume of timber obtained from harvesting cluster  $C$  in period  $t$ .

As said before, the objective is to find the harvesting schedule for  $T$  that maximizes the total revenue. It is also required that the volume of harvested timber is more or less constant from period to period for economical and ecological reasons. We use the *cluster packing formulation*, which has been proven to be particularly effective from a computational point of view [15]:

$$\begin{aligned} \max \quad & \sum_{t \in T} \sum_{C \in \Omega} p_{Ct} x_{Ct} \\ \sum_{t \in T} \sum_{C \in \Omega(s)} x_{Ct} & \leq 1 \quad s \in S \end{aligned} \quad (6)$$

$$\begin{aligned} x_{Ct} + x_{Bt} & \leq 1 \quad t \in T, \\ & C, B \in \Omega : C, B \text{ incompatible} \end{aligned} \quad (7)$$

$$\sum_{C \in \Omega} \alpha_{Ct+1} x_{Ct+1} \geq L_t \sum_{C \in \Omega} \alpha_{Ct} x_{Ct} \quad t \in T \setminus \{|T|\} \quad (8)$$

$$\sum_{C \in \Omega} \alpha_{Ct+1} x_{Ct+1} \leq U_t \sum_{C \in \Omega} \alpha_{Ct} x_{Ct} \quad t \in T \setminus \{|T|\} \quad (9)$$

$$x_{Ct} \in \{0, 1\} \quad t \in T, C \in \Omega.$$

The binary variables  $x_{Ct}$  determine whether or not cluster  $C$  is harvested in time period  $t$ . The objective function maximizes the total revenue granted by a harvesting schedule. Constraints (6) impose that a stand can be harvested at most once throughout the whole time horizon. Constraints (7) state that, given two incompatible clusters, at most one can be harvested in any given time period. Indeed, if we harvest both, either their total area is greater than the permissible area  $A_M$  or it is smaller and their union is already in the set of feasible clusters  $\Omega$ . Constraints (8)-(9) ensure that the volume of timber harvested in period  $t+1$  is between  $L_t$  and  $U_t$  times the volume of timber harvested in period  $t$ , thus enforcing a somewhat constant flow of timber volume.

#### 4.1 Including Uncertainty in the Harvest Scheduling Problem

We now observe that certain values of the program are uncertain, namely the timber volumes  $\alpha_{Ct}$ . Assuming that we know a nominal value  $\bar{\alpha}_{C,t}$  and set a system of deviations  $d_{Ct}^k$ ,  $k \in K$  for each coefficient, we know from the previous section that we can reformulate the uncertain model as the following multiband compact robust counterpart:

$$\begin{aligned}
& \max \sum_{t \in T} \sum_{C \in \Omega} p_{Ct} x_{Ct} \\
& \sum_{t \in T} \sum_{C \in \Omega(s)} x_{Ct} \leq 1 & s \in S \\
& x_{Ct} + x_{Bt} \leq 1 & t \in T \\
& & C, B \in \Omega : C, B \text{ incompatible} \\
& \sum_{C \in \Omega} \bar{\alpha}_{Ct+1} x_{Ct+1} - \sum_{k \in K} \theta_{ik}^L w_{ik}^L - \sum_{C \in \Omega} z_{Ct+1}^L \\
& \geq L_t \sum_{C \in \Omega} \bar{\alpha}_{Ct} x_{Ct} + \sum_{C \in \Omega} z_{Ct}^L & t \in T \setminus \{|T|\} \\
& \sum_{C \in \Omega} \bar{\alpha}_{Ct+1} x_{Ct+1} + \sum_{k \in K} \theta_{ik}^U w_{ik}^U + \sum_{C \in \Omega} z_{Ct+1}^U \\
& \leq U_t \sum_{C \in \Omega} \bar{\alpha}_{Ct} x_{Ct} - \sum_{C \in \Omega} z_{Ct}^U & t \in T \setminus \{|T|\} \\
& w_{ik}^L + z_{Ct}^L \geq L_t d_{Ct}^k x_{Ct} & t \in T, C \in \Omega, k \in K \\
& w_{ik}^L + z_{Ct+1}^L \geq -d_{Ct+1}^{k-1} x_{Ct+1} & t \in T \setminus \{|T|\}, C \in \Omega, k \in K \\
& w_{ik}^U + z_{Ct}^U \geq -U_t d_{Ct}^{k-1} x_{Ct} & t \in T, C \in \Omega, k \in K \\
& w_{ik}^U + z_{Ct+1}^U \geq d_{Ct+1}^k x_{Ct+1} & t \in T \setminus \{|T|\}, C \in \Omega, k \in K \\
& w_{ik}^U, w_{ik}^L, z_{Ct}^U, z_{Ct}^L \geq 0 & t \in T, C \in \Omega, k \in K \\
& x_{Ct} \in \{0, 1\} & t \in T, C \in \Omega.
\end{aligned}$$

## 4.2 Preliminary Computational Results

We conducted a preliminary computational experience over a set of ten instances derived from the data sets available on the FMOS website [14]. Our aim was to compare the price of robustness (i.e., the decrease in the value of the objective function that we must face to guarantee protection against deviations) of the multiband model versus the one of the Bertsimas-Sim model. To this end, we considered the two instances “PhyllisLeeper” and “Bear Town” which both include five time periods and respectively 80 and 71 elementary forest stands. All experiments were made on a 2.70 GHz Intel Core i7 with 8 GB. The programs were written in the C/C++ programming language and the optimization problems were solved by IBM ILOG CPLEX 12.1 with the support of Concert Technology. All runs were made using the default settings of Cplex with a time limit of 7200 seconds.

Concerning the volume constraints, we considered five different settings: denoting by  $V_t$  the total volume of timber harvested in period  $t$ , we adopted constraints  $(1 - \beta)V_t \leq V_{t+1} \leq (1 + \beta)V_t$  with  $\beta$  in  $\{0.10, 0.15, 0.20, 0.25, 0.30\}$ . Concerning the nature of timber volume uncertainty, we assumed that the deviation of each volume coefficient is a random variable following a standard normal distribution and we allowed deviations up to 15% of the nominal value. The nominal value  $\bar{\alpha}_{Ct}$  of a



volume coefficient is computed by summing up the volumes of the stands that form it (derived from the FMOS instances). The deviation values of each coefficient are defined considering the standard normal distribution in correspondence with integer multiples of the standard deviation. For the experiments, we considered an increasing number of bands: 3, 5, 7 (respectively covering the well-known 68-95-99.7 areas of the standard distribution), each including the zero deviation band and an equal number of positive and negative deviation bands. The bounds  $l_{ik}, u_{ik}$  were defined taking into account the fraction of coefficients deviating in each band according to the considered standard normal distribution and then decreasing/increasing and rounding the resulting value. For a fair comparison between single and multiband uncertainty, we set the Bertsimas-Sim parameter  $\Gamma_i$  of each constraint  $i$  equal to a weighted summation of the profile values  $\theta_{ik}$ , taking into account the proportion between the worst deviation of the considered band and the worst deviation of the worst band.

**Table 1** Comparisons between the single and the multiband price of robustness

ID	$\beta$	PoR-BS (%)			$\Delta$ PoR-MB (%)		
		K=3	K=5	K=7	K=3	K=5	K=7
PL1	0.10	13.0	23.5	24.1	7.8	9.3	8.8
PL2	0.15	9.7	21.4	22.3	9.5	12.1	7.5
PL3	0.20	8.2	15.4	18.7	11.4	11.6	9.4
PL4	0.25	13.8	24.4	27.1	7.5	7.7	8.5
PL5	0.30	7.9	18.5	20.1	12.4	13.2	9.7
BT1	0.10	6.3	14.3	15.7	6.5	6.8	9.1
BT2	0.15	7.9	18.3	23.0	9.2	9.9	9.5
BT3	0.20	8.1	17.0	18.2	11.9	12.4	12.0
BT4	0.25	11.4	18.8	20.2	7.4	9.3	9.1
BT5	0.30	10.2	27.2	27.9	8.2	8.6	7.6

The results are shown in Table 4.2. Each row reports the ID and volume constraint coefficient  $\beta$ . Furthermore, the three columns of *PoR-BS* report the price of robustness in the case of the Bertsimas-Sim model, as a percentage increase with respect to the optimal value of the nominal problem (the three columns corresponds with the three distinct  $\Gamma$ -settings adopted to fairly compare with the three band settings considered for the multiband model). Finally, the three columns of  $\Delta$ *PoR-MB* report the percentage reduction in the price of robustness that the multiband model grants with respect to the robust optimal value of the Bertsimas-Sim model. Two main observations can be made about the results: 1) imposing protection sensibly decreases the value of the optimal solutions and the decrease is naturally bigger when higher protection is demanded, as shown by columns *PoR-BS*; 2) the refined representation of the uncertainty set allowed by the multiband model grants a very remarkable reduction in the price of robustness, as shown by columns  $\Delta$ *PoR-MB*. This encourage further and deeper investigations about a refined definition of the multiband uncertainty set for the HSP.

### 4.3 Final remarks

We presented an overview of the main theoretical results of multiband robustness, a model for robust optimization that Büsing and D'Andreagiovanni recently proposed to extend and refine the classical Bertsimas-Sim model. Furthermore, we introduced a new family of strong valid inequalities for uncertain binary programs. We showed how these theoretical results can be used in forest management to tackle the uncertainty affecting the volume of produced timber in a harvest scheduling problem. Preliminary computational results highlight the improved performance of multiband robustness in terms of price of robustness. Our current work focuses on evaluating the computational efficiency and efficacy of the new family of valid inequalities and developing ad-hoc algorithms for the solution of larger robust harvest scheduling problems.

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### References

1. Atamtürk, A.: Strong Formulations of Robust Mixed 01 Programming. *Math. Program. B*, **108**, 235–250 (2006)
2. Bauschert, T., Büsing, C., D'Andreagiovanni, F., Koster, A.M.C.A., Kutschka, M., Steglich, U.: Network Planning under Demand Uncertainty with Robust Optimization. To appear in *IEEE Communications Magazine* (2013)
3. Ben-Tal, A., El Ghaoui, L., Nemirovski, A.: *Robust Optimization*. Springer, Heidelberg (2009)
4. Bertsimas, D., Brown, D., Caramanis, C.: Theory and Applications of Robust Optimization. *SIAM Review* **53** (3), 464–501 (2011)
5. Bertsimas, D., Sim, M.: The Price of Robustness. *Oper. Res.* **52** (1), 35–53 (2004)
6. Bienstock, D.: Histogram Models for Robust Portfolio Optimization. *J. Comp. Finance*, **11** (1), 1–64 (2007)
7. Bley, A., D'Andreagiovanni, F., Hanemann, A.: Robustness in Communication Networks: Scenarios and Mathematical Approaches. In: *Proc. of the ITG Symposium on Photonic Networks 2011*, pp. 10–13. VDE Verlag, Berlin (2011)
8. Büsing, C., D'Andreagiovanni, F.: New Results about Multi-band Uncertainty in Robust Optimization. In: Klasing, R. (ed.) *Experimental Algorithms - SEA 2012, LNCS*, vol. 7276, pp. 63–74. Springer, Heidelberg (2012)
9. Büsing, C., D'Andreagiovanni, F.: Robust Optimization under Multi-band Uncertainty. Preprint: Optimization Online 2013-01-3748. Submitted for publication (2012)
10. Büsing, C., D'Andreagiovanni, F.: A new theoretical framework for Robust Optimization under multi-band uncertainty. In: Helber, S. et al. (eds.) *Operations Research Proceedings 2012*, Springer, Heidelberg
11. Chen, X., Sim, M., Peng, S.: A Robust Optimization Perspective on Stochastic Programming. *Oper. Res.* **55** (6), 1058–1071 (2007)
12. D'Andreagiovanni, F.: Pure 0 – 1 Programming approaches to Wireless Network Design. *4OR-Q. J. Oper. Res.* **10** (2), 211–212 (2012)
13. Dantzig, G.: Linear programming under uncertainty. *Management Sci.* **1**, 197–206 (1955)

14. Forest Management Optimization Site (FMOS), <http://ifmlab.for.unb.ca/fmos/>
15. Goycoolea, M., Murray, A. T., Barahona, F., Epstein, R., Weintraub, A.: Harvest Scheduling Subject to Maximum Area Restrictions: Exploring Exact Approaches. *Oper. Res.* **53** (3), 490–500 (2005)
16. Goycoolea, M., Vielma, J. P., Murray, A. T., Weintraub, A.: Evaluating Approaches for Solving the Area Restrictions Model in Harvest Scheduling. *Forest Science* **55** (2), 149–165 (2009)
17. Goycoolea, M., Murray, A. T., Barahona, F., Epstein, R., Weintraub, A.: Harvest Scheduling Subject to Maximum Area Restrictions: Exploring Exact Approaches. *Forest Science* **55** (2), 149–165 (2009)
18. Nemhauser, G., Wolsey, L.: *Integer and Combinatorial Optimization*. John Wiley & Sons, Hoboken (1988)
19. Palma, C.D., Nelson, J.D.: A robust optimization approach protected harvest scheduling decisions against uncertainty. *Can. J. For. Res.* **39** 342–355 (2009)
20. Shapiro, A., Dentcheva, D., Ruszczyński, A.: *Lectures on Stochastic Programming: modeling and theory*. MPS/SIAM Series on Optimization 9. SIAM, Philadelphia (2009)