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**Asymptotic Mesh Independence
of Newton–Galerkin Methods
via a Refined Mysovskii Theorem**

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0. Introduction

Mesh independence of Newton's method means that Newton's method applied to a family of finite dimensional discretizations of a Banach space nonlinear operator equation behaves essentially the same for all sufficiently fine discretizations. This fact has been frequently observed and even used in designing efficient mesh refinement strategies for certain classes of nonlinear equations (e.g. [1], [9]).

In order to explain these observations, a *mesh independence principle* for Newton's method has been proved in [2] for general families of discretizations that are bounded, stable, consistent and *Lipschitz uniform*. The last property means that the Fréchet derivatives of the finite dimensional operators are Lipschitz continuous with a Lipschitz constant that is independent of the mesh size. Such a uniform Lipschitz condition may be hard to verify for a number of operator equations (such as the biharmonic equation). In the present paper we do not require this property as an axiom, but obtain it asymptotically in a rather natural way by assuming certain simple convergence conditions that implicitly contain a stability assumption. In our framework, the proof of mesh independence is much simpler and gives more insight. Another advantage of our approach is that we are only using norms corresponding to the domain of definition of the operators and no norms in the domain of values or corresponding operator norms as in [2]. On one hand this leads to better theoretical bounds, and on the other hand this opens the possibility that all quantities needed for applications can be numerically estimated.

In proving our results we use a presumably new refined version of the Mysovskii theorem which guarantees, under some affine-invariant assumptions, the existence and the uniqueness of the solution, as well as the quadratic convergence of the Newton iterates towards this solution. This theorem may be of interest in a much wider context, especially its uniqueness result.

In the first section of our paper, we state and prove the refined version of the Mysovskii theorem and show that, for operators that are twice Fréchet differentiable, its hypothesis is equivalent to the hypothesis of the affine invariant version of the Mysovskii theorem due to DEUFLHARD/HEINDL [7], which has been slightly refined by BOCK [5]. The second section contains different aspects of our asymptotic mesh independence principle for Newton's method when applied in the context of Galerkin approximations.

1. A Refined Newton–Mysovskii Theorem

Consider a nonlinear operator equation of the form

$$F(x) = 0 \tag{1.1}$$

where $F : D \subset X \rightarrow Y$ is a nonlinear mapping defined on a convex domain D of a Banach space X with values in a Banach space Y . Under certain conditions Newton's method

$$x^{k+1} = x^k - F'(x^k)^{-1}F(x^k) \quad k = 0, 1, \dots \tag{1.2}$$

produces a sequence $\{x^k\}_{k=0}^{\infty}$ which converges to a solution x^* of the equation (1.1). In (1.2) $F'(x^k)$ denotes the Fréchet derivative of the nonlinear operator F at the point x^k . At each step of Newton's method a linear equation of the form

$$F'(x^k)\Delta x^k = -F(x^k) \tag{1.3}$$

must be solved yielding the correction

$$x^{k+1} = x^k + \Delta x^k. \tag{1.4}$$

The first convergence theorem for Newton's method in Banach spaces is due to L.V. KANTOROVITCH [8] — originally even in an affine-invariant form. The Kantorovitch theorem gives sufficient conditions under which equation (1.1) has a unique solution x^* in a certain neighborhood of x^0 , and proves that under those conditions the iterates given by (1.2) converge quadratically to x^* . There is a huge literature concerning different aspects of the Kantorovitch theorem. For an account on some of the results, we refer the reader to the monograph of POTRA/PTÁK [11] and the paper of YAMAMOTO [13].

Another approach in studying the convergence of Newton's method was considered by MYSOVSKII [10]. We start from an affine invariant version of the Mysovskii theorem that has been first considered by DEUFLHARD and HEINDL [7] and later refined by BOCK [5].

Theorem 1.1 *Let $F : D \subseteq X \rightarrow Y$ be continuously Fréchet differentiable, with $F'(x)$ invertible for all $x \in D$, D open and convex. Assume that one can find a starting point $x^0 \in D$ and constants $\alpha, \omega \geq 0$ such that*

$$\|F'(x^0)^{-1}F(x^0)\| \leq \alpha < \infty \tag{1.5}$$

$$\| F'(y)^{-1} \left(F'(x + s \cdot (y - x)) - F'(x) \right) (y - x) \| \leq s \cdot \omega \|y - x\|^2, \quad (1.6)$$

$$\forall x, y \in D, s \in [0, 1]$$

$$h := \alpha\omega/2 < 1 \quad (1.7)$$

$$\bar{S}(x^0, \rho) := \{x \in X \mid \|x - x^0\| \leq \rho\} \subset D \quad (1.8)$$

$$\text{with } \rho = \alpha \sum_{j=0}^{\infty} h^{2^j-1} \leq \alpha/(1-h).$$

Then the following results hold:

(A) The sequence $\{x^k\}$ of Newton iterates generated by (1.2) remains in $\bar{S}(x^0, \rho)$ and converges to a solution $x^* \in \bar{S}(x^0, \rho)$ of (1.1).

(B) The convergence rate can be estimated by

$$\|x^{k+1} - x^k\| \leq \omega/2 \|x^k - x^{k-1}\|^2. \quad (1.9)$$

(C) In addition, one obtains the error estimates

$$\|x^k - x^*\| \leq \varepsilon_k \|x^k - x^{k-1}\|^2 \quad (1.10)$$

where

$$\varepsilon_k := \omega/2 \sum_{j=0}^{\infty} (h^{2^k})^{2^j-1} \leq \frac{\omega/2}{(1-h^{2^k})}. \quad (1.11)$$

While the hypothesis of the above theorem implies the *existence* of a solution $x^* \in \bar{S}(x^0, \rho)$ of equation (1.1), nothing is said about the *uniqueness* of this solution. In what follows we will show that if the Lipschitz assumption (1.6) is exchanged by

$$\| F'(x)^{-1} \left(F'(x + s(y - x)) - F'(x) \right) (y - x) \| \leq s\omega \|y - x\|^2, \quad (1.12)$$

$$\forall x, y \in D, s \in [0, 1]$$

then x^* can be proved to be the unique solution of (1.1) in the region

$$D^* := \bigcup_{k=0}^{\infty} S(x^k, 2/\omega) \cap D \quad (1.13)$$

where

$$S(x^k, 2/\omega) = \{x \in X \mid \|x - x^k\| < 2/\omega\}.$$

We note that under the hypothesis of Theorem 1.1 we have $x^{k+1} \in S(x^k, 2/\omega)$ because

$$\|x^{k+1} - x^k\| \leq \|x^1 - x^0\| \leq \alpha < 2/\omega \quad (1.14)$$

so that D^* is a connected set. We have clearly $S(x^*, 2/\omega) \cap D \subset D^*$. Moreover, it is easily seen from (1.8) that if $h < 1/2$ then $x^* \in S(x^0, 2/\omega)$.

An immediate straightforward consequence of assumption (1.12) is — in the just introduced notation — that

$$\|F'(x)^{-1}(F(y) - F(x) - F'(x)(y - x))\| \leq \omega/2 \|y - x\|^2, \quad \forall x, y \in D. \quad (1.15)$$

This auxiliary result permits now a more detailed study of the attraction ball of Newton's method.

Theorem 1.2 *Let $F : D \subset X \rightarrow Y$ be a nonlinear operator defined on a convex domain D of a Banach space X with values in a Banach space Y . Suppose that F is Fréchet differentiable on D , that $F'(x)$ is invertible for each $x \in D$, and that (1.12) is satisfied. Moreover, assume that equation (1.1) has a solution $y^* \in D$. Let $y^0 \in D$ be such that*

$$\overline{S}(y^*, \|y^0 - y^*\|) \subset D \quad (1.16)$$

and

$$y^0 \in S(y^*, 2/\omega). \quad (1.17)$$

Then the iterates given by Newton's method

$$y^{k+1} = y^k - F'(y^k)^{-1}F(y^k) \quad k = 0, 1, \dots \quad (1.18)$$

remain in the open ball $S(y^*, \|y^0 - y^*\|)$, converge to y^* and satisfy the following inequality

$$\|y^{k+1} - y^*\| \leq \omega/2 \|y^k - y^*\|^2, \quad k = 0, 1, \dots \quad (1.19)$$

Moreover, y^* is the unique solution in $S(y^*, 2/\omega)$.

Proof. Suppose that $y^k \in \overline{S}(y^*, \|y^0 - y^*\|)$. Using (1.18) and the fact that $F(y^*) = 0$, we have

$$\begin{aligned} y^{k+1} - y^* &= y^k - y^* - F'(y^k)^{-1}(F(y^k) - F(y^*)) \\ &= F'(y^k)^{-1}(F(y^*) - F(y^k) - F'(y^k)(y^* - y^k)). \end{aligned}$$

From (1.15) and (1.17), it follows that

$$\|y^{k+1} - y^*\| \leq \omega/2 \|y^k - y^*\|^2 \leq \omega/2 \|y^0 - y^*\|^2 < \|y^0 - y^*\|,$$

and the first part of the theorem is proved by a simple inductive argument.

In order to prove uniqueness in $S(y^*, 2/\omega)$ let $y^0 := y^{**}$ for some $y^{**} \neq y^*$ with $F(y^{**}) = 0$, which implies $y^1 = y^{**}$ as well. Upon insertion into (1.19), this yields

$$\|y^{**} - y^*\| \leq \omega/2 \|y^{**} - y^*\|^2 < \|y^{**} - y^*\| ,$$

which is a contradiction. This completes the proof. ■

The result contained in the above theorem is certainly interesting in itself and — though being so simple — appears to be new. We will use it in proving our uniqueness theorem. In order to do that, we first observe that assumption (1.16) is required only to ensure the fact that the iterates (1.18) stay in D . If we can ensure by other means that

$$y^j \in D , \quad j = 0, 1, \dots \quad (1.20)$$

then assumption (1.16) may be dropped from the hypothesis of Theorem 1.2.

Theorem 1.3 *Given the hypothesis of Theorem 1.1 replace the Lipschitz condition (1.6) by (1.12). Then x^* is the unique solution of (1.1) in the region D^* defined by (1.13).*

Proof. Suppose there is an $y^* \in D$ such that $F(y^*) = 0$. First, take some x^0 such that

$$x^0 \in S(y^*, 2/\omega) ,$$

which is equivalent to

$$y^* \in S(x^0, 2/\omega) .$$

Theorem 1.2 guarantees that y^* is unique in $S(y^*, 2/\omega)$. In order to show that y^* is also unique in $S(x^0, 2/\omega)$, assume that there exists another solution

$$y^{**} \in S(x^0, 2/\omega) \setminus S(y^*, 2/\omega)$$

with $F(y^{**}) = 0$. Then y^{**} must be the unique solution in $S(y^{**}, 2/\omega)$ and the sequence of Newton iterates starting at x^0 must converge to y^{**} — which is a contradiction to the uniqueness of the sequence of Newton iterates. Hence, y^* is unique in $S(x^0, 2/\omega)$. Upon extending the same argument over all Newton iterates x^k , one arrives at the natural definition of D^* in (1.13). This completes the proof. ■

In what follows we will discuss the relationship between assumptions (1.6) and (1.12). First, observe that both are particular cases of the stronger assumption

$$\|F'(z)^{-1}(F'(u) - F'(x))(u - x)\| \leq \omega \|u - x\|^2 \quad \forall u, x, z \in D \text{ collinear} . \quad (1.21)$$

Indeed (1.6) follows from (1.21) by taking $z = y$ and $u = x + s(y - x)$, while (1.12) follows from (1.21) for $z = x$ and $u = x + s(y - x)$. Let $u, x \in D$ and denote by L the line containing u and x , and by L_1 the half line of origin u , contained in L but not containing x . Assumption (1.6) says that (1.21) is satisfied only for $z \in L_1$ and not necessarily for all $z \in L$. However, we will prove that in case F is twice Fréchet differentiable assumptions (1.6) and (1.21) are equivalent. First note that if F is twice Fréchet differentiable then, by dividing both sides of (1.6) by s , and then setting $s \rightarrow 0$, we obtain

$$\|F'(y)^{-1}F''(x)(y-x)^2\| \leq \omega\|y-x\|^2 \quad \forall x, y \in D. \quad (1.22)$$

Lemma 1.4 *Assume that the operator $F : D \subseteq X \rightarrow Y$ is twice continuously Fréchet differentiable on D , where D is open and convex, and X, Y are Banach spaces. Then properties (1.6), (1.21) and (1.22) are equivalent.*

Proof. We have already seen that (1.21) implies (1.6) and (1.6) implies (1.22). We are left to prove that (1.22) implies (1.21). Because F is twice continuously differentiable on D , we may apply the integral mean value theorem to obtain

$$F'(z)^{-1}(F'(u) - F'(x))(u-x) = \int_0^1 F'(z)^{-1}F''(x+t(u-x))(u-x)^2 dt$$

for any $u, x, z \in D$. If u, x, z are collinear, then by writing

$$u = x + v, \quad z = x + \mu v, \quad x(t) = x + tv$$

it follows that

$$F'(z)^{-1}(F'(u) - F'(x))(u-x) = \int_0^1 \frac{1}{(\mu-t)^2} F'(z)^{-1}F''(x(t))(z-x(t))^2 dt.$$

By majorizing under the integral sign and using (1.22), we have

$$\begin{aligned} \|F'(z)^{-1}(F'(u) - F'(x))(u-x)\| &\leq \omega \int_0^1 \frac{\|z-x(t)\|^2}{(\mu-t)^2} dt \\ &= \omega \int_0^1 \|u-x\|^2 dt = \omega\|u-x\|^2, \end{aligned}$$

which proves that (1.21) is satisfied. ■

The above lemma motivates us to replace condition (1.6) in the hypothesis of Theorem 1.1 by the stronger, but practically equivalent condition (1.21). By doing this we obtain both existence and uniqueness of the solution of (1.1). For convenience, we repeat all assumptions explicitly.

Theorem 1.5 (refined Newton–Mysovskii theorem)

Let $F : D \subseteq X \rightarrow Y$ be continuously Fréchet differentiable, with $F'(x)$ invertible for all $x \in D$, D open and convex. Assume that one can find a starting point $x \in D$ and constants $\alpha, \omega \geq 0$ such that

$$\|F'(x^0)^{-1}F(x^0)\| \leq \alpha < \infty ,$$

$$\|F'(z)^{-1}(F'(u) - F'(x))(u - x)\| \leq \omega \|u - x\|^2 \quad \forall u, x, z \in D \text{ collinear} ,$$

$$h := \alpha\omega/2 < 1 ,$$

$$\bar{S}(x^0, \rho) := \{x \in X \mid \|x - x^0\| \leq \rho\} \subset D$$

$$\text{with } \rho = \alpha \sum_{j=0}^{\infty} h^{2^j-1} \leq \alpha/(1 - h) .$$

Then:

(A) The sequence $\{x^k\}$ of Newton iterates (1.2) remains in $S(x^0, \rho)$ and converges to a solution $x^* \in \bar{S}(x^0, \rho)$ of equation (1.1).

(B) x^* is the unique solution of (1.1) in the region

$$D^* := \bigcup_{k=0}^{\infty} S(x^k, 2/\omega) \cap D$$

(C) The following error estimates hold

$$\|x^k - x^{k+1}\| \leq \omega/2 \|x^{k-1} - x^k\|^2$$

$$\|x^k - x^*\| \leq \omega/2 \|x^{k-1} - x^*\|^2$$

$$\|x^k - x^*\| \leq \frac{\|x^k - x^{k+1}\|}{1 - \omega/2 \|x^k - x^{k+1}\|}$$

The proof has been given in the above careful stepwise derivation.

2. Asymptotic Mesh Independence of Newton–Galerkin Methods

In the previous section we have given sufficient conditions under which the Newton iterates (1.2) converge toward a unique solution x^* of the nonlinear equation (1.1). However, if X and Y are infinite dimensional spaces then the solution of the linear equation (1.3) can rarely be found in applications, so that in practice equation (1.1) is replaced by a family of discretized equations

$$F_j(x_j) = 0 \quad j = 0, 1, \dots \quad (2.1)$$

where $F_j : D_j \subset X_j \rightarrow Y_j$ is a nonlinear operator defined on a convex domain D_j of a finite-dimensional subspace $X_j \subset X$ with values in a finite dimensional subspace $Y_j \subset Y$. This means that we restrict our attention to Galerkin methods. For finite difference methods, similar results should hold — but the proofs would certainly be much more technical.

One would like to choose the “discretized operators” F_j in such a way that a solution x_j^* of (2.1) can be computed for each j so that

$$\lim_{j \rightarrow \infty} x_j^* = x^* . \quad (2.2)$$

Eventually x_j^* should be obtainable via Newton’s method applied to the finite dimensional equation (2.1):

$$x_j^{k+1} = x_j^k - F_j'(x_j^k)^{-1} F_j(x_j^k), \quad k = 0, 1, \dots \quad (2.3)$$

Corresponding to (1.3) and (1.4) one Newton step for (2.1) is performed by first solving the finite dimensional linear equation

$$F_j'(x_j^k) \Delta x_j^k = -F_j(x_j^k) \quad (2.4)$$

and then by setting

$$x_j^{k+1} = x_j^k + \Delta x_j^k . \quad (2.5)$$

In order for (2.2) to hold, one has to assume that the elements of the infinite dimensional space X can be well approximated by elements of the finite dimensional subspaces X_j . In fact we may well restrict ourselves to a subset $W^* \subset X$ consisting of elements that have “better smoothness properties” than the generic elements of X . We assume that

$$\{x^*, x^k, x^k - x^*, \Delta x^k\} \subset W^*, \quad k = 0, 1, \dots \quad (2.6)$$

This assumption is quite natural because x^* is the solution of the nonlinear equation (1.1) and Δx^k is the solution of the linear equation (1.3), and therefore under reasonable conditions they have better smoothness properties.

Let $\pi_j : X \rightarrow X_j$, $j = 0, 1, \dots$ be some family of linear projection operators. Assume that these operators are uniformly bounded on W^* , which means that they satisfy a *stability condition* of the type

$$\|\pi_j x\| \leq q_j \|x\|, \quad x \in W^*, \quad q_j \leq \bar{q} < \infty. \quad (2.7)$$

The projection property $\pi_j^2 = \pi_j$ then immediately implies that

$$q_j \geq 1. \quad (2.8)$$

Subsequently, the most interesting case will turn out to be that the constants q_j satisfy

$$\lim_{j \rightarrow \infty} q_j = q. \quad (2.9)$$

The fact that the subspaces X_j can be used to approximate the elements of W^* well, at least asymptotically, is expressed by the *approximation condition*

$$\|x - \pi_j x\| \leq \delta_j \|x\|, \quad x \in W^*, \quad j = 0, 1, \dots \quad (2.10)$$

$$\lim_{j \rightarrow \infty} \delta_j = 0. \quad (2.11)$$

For most of the subsequent results we will need monotonicity of the kind

$$\delta_{j+1} \leq \delta_j, \quad j = 0, 1, \dots \quad (2.12)$$

Thus the discretization method is characterized here by the family of triplets

$$\{F_j, \pi_j, \delta_j\}, \quad j = 0, 1, \dots \quad (2.13)$$

We assume that the domains D_j of the discretized operators are such that

$$S_j(\pi_j x^*, 2/\omega) \subset D_j \subset D, \quad j = 0, 1, \dots \quad (2.14)$$

where $S_j(z_j, r) := \{x \in X_j \mid \|x - z_j\| \leq r\}$. We also assume that the discretization is *consistent* in the sense that there are positive constants c_0, c_1 such that the following properties hold:

\mathcal{C}_0 : if $u_j \in X_j$ and $u \in X$ are solutions of the linear equations

$$\begin{aligned} F'_j(\pi_j x) u_j &= F'_j(\pi_j x) \\ F'(x) u &= F'(x) \end{aligned}$$

where $x \in W^* \cap D$, $\pi_j x \in D_j$, then

$$\|u_j - \pi_j u\| \leq c_0 \delta_j, \quad j = 0, 1, \dots \quad (2.15)$$

\mathcal{C}_1 : if $w_j \in X_j$ and $w \in X$ are solutions of the linear equations

$$\begin{aligned} F'_j(z_j)w_j &= F''_j(x_j)v_j^2 \\ F'(z_j)w &= F''(x_j)v_j^2 \end{aligned}$$

where $x_j, z_j \in D_j$, $v_j \in \text{span}(z_j - x_j)$, then

$$\|w_j - \pi_j w\| \leq c_1 \delta_j \|v_j\|^2 \quad j = 0, 1, \dots \quad (2.16)$$

We note that the consistency condition \mathcal{C}_1 may be rephrased in first Fréchet derivative terms, if we replace

$$F''(x_j)v_j^2$$

by

$$\left(F'(x_j + v_j) - F'(x_j)\right)v_j$$

(see (1.21) and (1.22)). For the simplicity of the subsequent presentation, however, we prefer to use the second derivative formulation. The connection between the Lipschitz constant ω for the operator equation and the associated ω_j for the finite-dimensional approximating nonlinear systems is studied in the following lemma.

Lemma 2.1 *Let $F : D \subset X \rightarrow Y$ satisfy the hypothesis of Lemma 1.4 and suppose that the discretization method (2.13) satisfies (2.7), (2.10), (2.12), (2.14) as well as \mathcal{C}_1 . Then*

$$\begin{aligned} \left\| F'_j(z_j)^{-1} \left(F'_j(u_j) - F'_j(x_j) \right) (u_j - x_j) \right\| &\leq \omega_j \|u_j - x_j\|^2 \\ \forall u_j, x_j, z_j \in D_j \quad &\text{collinear,} \end{aligned} \quad (2.17)$$

where

$$\omega_j = \omega q_j + c_1 \delta_j \quad (2.18)$$

Proof. With the choice of assumption \mathcal{C}_1 , the proof is elementary. Just apply, in the above notation, the definitions

$$\begin{aligned} \|w_j\| &\leq \omega_j \|v_j\|^2, \\ \|w\| &\leq \omega \|v_j\|^2, \end{aligned}$$

and the inequalities

$$\|w_j\| \leq \|w_j - \pi_j w\| + \|\pi_j w\| \leq (c_1 \delta_j + q_j \omega) \|v_j\|^2 .$$

■

Recall that the Lipschitz constant ω_j (chosen best possible) governs the convergence speed of Newton's iteration in subspace X_j . Note that the assumptions (2.9) and (2.11) imply that

$$\lim_{j \rightarrow \infty} \omega_j = \omega q , \quad (2.19)$$

which means that the *convergence speed is asymptotically constant*.

For a more precise statement, the initial guesses x_j^0 need to be additionally related. In what follows, two selections will be made: first, setting $x_j^0 := \pi_j x^*$ will lead to a local existence and uniqueness result (Theorem 2.2) and second, setting x_j^0 "sufficiently close" to x_j^* , will lead to a formulation of the asymptotic mesh independence property (Theorem 2.3).

Theorem 2.2 *Under the hypothesis of Theorem 1.5 suppose that F is twice continuously Fréchet differentiable on D . Consider a discretization method given by the family of triplets (2.13) that satisfy (2.7), (2.10), (2.12) and (2.14). Assume also that the operators F_j are twice continuously differentiable on D_j , that $F_j'(x_j)$ is invertible for each $x_j \in D_j$ and that conditions \mathcal{C}_0 and \mathcal{C}_1 are satisfied.*

Then there is a refinement index $j^ \in \mathbb{N}$ such that for any $j \geq j^*$ equation (2.1) has a solution x_j^* that satisfies the inequality*

$$\|x_j^* - \pi_j x^*\| \leq 2c_0 \delta_j . \quad (2.20)$$

Moreover x_j^ is the unique solution of (2.1) in the region*

$$S_j(\pi_j x^*, 2/\omega_j) \cap D_j \quad (2.21)$$

where ω_j is given by (2.18).

Proof. The basic idea is to set a formal starting point $x_j^0 := \pi_j x^*$ and to apply Theorem 1.5 to the associated Newton iteration. For ease of writing denote

$$u_j^* := F_j'(\pi_j x^*)^{-1} F_j(\pi_j x^*), \quad \alpha_j^* := \|u_j^*\| .$$

Because x^* is a solution of (1.1), we have

$$u^* := F'(x^*)^{-1} F(x^*) = 0$$

and by using property \mathcal{C}_0 we obtain

$$\alpha_j^* = \|u_j^*\| = \|u_j^* - \pi_j u^*\| \leq c_0 \delta_j . \quad (2.22)$$

Denote the Kantorovitch quantity by

$$h_j^* := \alpha_j^* \omega_j / 2 .$$

From (2.18) and (2.22), it follows that

$$h_j^* \leq (\omega q_j + c_1 \delta_j) c_0 \delta_j / 2 ,$$

so that by virtue of (2.7) and (2.12) there is a refinement index $j^* > 0$ such that $h_j^* < 1/2$ for $j \geq j^*$. It follows that

$$\rho_j^* := \alpha_j^* \sum_{i=0}^{\infty} (h_j^*)^{2^i - 1} \leq \frac{\alpha_j^*}{1 - h_j^*} \leq 2\alpha_j^* \leq 2c_0 \delta_j .$$

Now, applying Theorem 1.5 to Newton's process (2.3) with starting point x_j^0 , we deduce the existence of a solution $x_j^* \in S_j(\pi_j x^*, \rho_j^*)$ which is the unique solution of (2.1) in the region

$$D_j^* = \bigcup_{k=0}^{\infty} S_j(x_j^k, 2/\omega_j) \cap D .$$

In particular x_j^* is the unique solution of (2.1) in the region (2.21). \blacksquare

Next consider Newton's process (2.3) with some starting point $x_j^0 \in D_j$ and let $\alpha_j(x_j^0)$ be given by

$$\alpha_j(x_j^0) := \|F_j'(x_j^0)^{-1} F_j(x_j^0)\| . \quad (2.23)$$

If

$$h_j(x_j^0) = \alpha_j(x_j^0) \omega_j / 2 < 1 \quad (2.24)$$

then we may define

$$\rho_j(x_j^0) := 2/\omega_j \sum_{i=0}^{\infty} [h_j(x_j^0)]^{2^i} . \quad (2.25)$$

If besides (2.24) we also have

$$\overline{S}_j(x_j^0, \rho_j(x_j^0)) \subset D_j \quad (2.26)$$

then according to Theorem 1.5 and Lemma 2.1 the sequence $\{x_j^k\}_{k=0}^{\infty}$ given by (2.3) will converge to a solution of the equation (2.1). In general it may converge to a solution that is different from the solution x_j^* whose existence

and uniqueness is guaranteed by Theorem 2.2. However this is not the case for properly chosen starting points. For example, if

$$\|x_j^0 - x_j^*\| < 2/\omega_j \quad (2.27)$$

then according to Theorem 1.2 the sequence $\{x_j^k\}_{k=0}^\infty$ will converge to x_j^* . Whenever the Newton iterates (2.3) stay in D_j and converge to x_j^* , they satisfy the error estimates

$$\|x_j^k - x_j^{k+1}\| \leq \omega_j/2 \|x_j^{k-1} - x_j^k\|^2 \quad (2.28)$$

$$\|x_j^k - x_j^*\| \leq \omega_j/2 \|x_j^{k-1} - x_j^*\|^2. \quad (2.29)$$

Under the asymptotic assumption (2.19), the estimates (2.28) and (2.29) show that for j sufficiently large all sequences of Newton iterates associated with the finite-dimensional equations (2.1) behave the same. This is the essence of our *asymptotic mesh independence property*. If, in addition, the relation (2.9) is satisfied with

$$\lim q_j = q = 1, \quad (2.30)$$

then the number of steps required to reduce an initial error by a given factor is essentially the same for the continuous Newton process (1.2) and the discrete Newton process (2.3), provided j is large enough. Indeed, suppose that the initial error satisfies (2.27) and let $\varepsilon > 0$ be a given (small) positive number. We want to find an integer $k_j := k_j(x_j^0, \varepsilon)$ such that

$$\|x_j^k - x_j^*\| \leq \varepsilon, \quad k \geq k_j. \quad (2.31)$$

By multiplying both sides of (2.29) by $\omega_j/2$ we deduce that

$$\omega_j/2 \|x_j^k - x_j^*\| \leq [\omega_j/2 \|x_j^0 - x_j^*\|]^{2^k}. \quad (2.32)$$

Hence (2.31) is satisfied, whenever

$$[\omega_j/2 \|x_j^0 - x_j^*\|]^{2^k} \leq \omega_j \varepsilon / 2. \quad (2.33)$$

It follows that we may take

$$k_j(x_j^0, \varepsilon) = \left\lceil \log_2 \left(\frac{\ln(\omega_j \varepsilon / 2)}{\ln(\omega_j \|x_j^0 - x_j^*\| / 2)} \right) \right\rceil \quad (2.34)$$

where $\lceil x \rceil := \min\{p \mid p \text{ integer}, p \geq x\}$. Similarly, if

$$\|x^0 - x^*\| < 2/\omega \quad (2.35)$$

then

$$\|x^k - x^*\| \leq \varepsilon \quad (2.36)$$

for $k \geq k(x^0, \varepsilon)$, where

$$k(x^0, \varepsilon) = \left\lceil \log_2 \left(\frac{\ln(\omega\varepsilon/2)}{\ln(\omega\|x^0 - x^*\|/2)} \right) \right\rceil. \quad (2.37)$$

Because $k_j(x_j^0, \varepsilon)$ and $k(x^0, \varepsilon)$ are integer valued, they will differ by at most one whenever $\omega_j\|x_j^0 - x_j^*\|$ is close enough to $\omega\|x^0 - x^*\|$. This is certainly the case when condition (2.30) holds and when we relate the starting points for (2.3) by

$$x_j^0 = \pi_j x^0. \quad (2.38)$$

This leads to our main asymptotic mesh independence result.

Theorem 2.3 *Under the hypothesis of Theorem 2.2 assume that (2.30) holds and that*

$$h = \alpha\omega/2 < 1/2. \quad (2.39)$$

Then there is $j_1 \geq j^$ such that for each $j \geq j_1$ the sequence of Newton iterates (2.3) with starting point (2.38) converges to x_j^* . Moreover, inequalities (2.35) and (2.27) are satisfied for $j \geq j_1$ and*

$$|k_j(\pi_j x^0, \varepsilon) - k(x^0, \varepsilon)| \leq 1, \quad j \geq j_1. \quad (2.40)$$

If, instead of (2.30) only (2.9) holds for $q > 1$, then the above results are valid with ω to be replaced by ωq .

Proof. Let $q = 1$ w.l.o.g. (2.39) implies $\alpha < 1/\omega$ so that by using Theorem 1.1 we obtain

$$\|x^0 - x^*\| \leq \rho \leq \alpha/(1-h) < 2\alpha < 2/\omega \quad (2.41)$$

which proves (2.35). Moreover, with (2.38)

$$| \|\pi_j(x^0 - x^*)\| - \|\pi_j x^* - x_j^*\| | \leq \|x_j^0 - x_j^*\| \leq \|\pi_j(x^0 - x^*)\| + \|\pi_j x^* - x_j^*\|$$

and by using Theorem 2.2 and (2.30) we deduce that

$$\lim_{j \rightarrow \infty} \|x_j^0 - x_j^*\| = \|x^0 - x^*\|. \quad (2.42)$$

Then from (2.19) and (2.35) it follows that

$$\lim_{j \rightarrow \infty} \omega_j \|x_j^0 - x_j^*\| = \omega \|x^0 - x^*\| < 2. \quad (2.43)$$

This shows that both (2.27) and (2.40) are satisfied for all j sufficiently large. \blacksquare

The integer (2.37) cannot be exactly computed in practice because the quantity $\|x^0 - x^*\|$ is in general unknown. In this case we may use the last inequality from the statement of Theorem 1.5 to conclude that (2.36) is satisfied whenever

$$\|x^k - x^{k+1}\| \leq \frac{\varepsilon}{1 + \omega\varepsilon/2}. \quad (2.44)$$

By using (1.9) and proceeding the same as in the deduction of (2.34) it follows that (2.44) is satisfied for all $k \geq \tilde{k}(x^0, \varepsilon)$, where

$$\tilde{k}(x^0, \varepsilon) = \left\lceil \log_2 \left(\ln \left(\frac{\omega\varepsilon}{2 + \omega\varepsilon} \right) / \ln h \right) \right\rceil. \quad (2.45)$$

Once more, ω may be replaced by ωq whenever $q \neq 1$. In particular (2.36) is also satisfied for all $k \geq \tilde{k}(x^0, \varepsilon)$. Similarly, if (2.24) holds, and if the sequence (2.3) converges to x_j^* , then inequality (2.31) is satisfied for all $k \geq \tilde{k}_j(x_j^0, \varepsilon)$, where

$$\tilde{k}_j(x_j^0, \varepsilon) = \left\lceil \log_2 \left(\ln \left(\frac{\omega_j \varepsilon}{2 + \omega_j \varepsilon} \right) / \ln h_j(x_j^0) \right) \right\rceil. \quad (2.46)$$

Suppose that the constants α and h in Theorem 1.1 are determined exactly, i.e.:

$$\alpha = \alpha(x^0) := \|F'(x^0)^{-1}F(x^0)\|, \quad h = h(x^0) = \alpha(x^0)\omega/2. \quad (2.47)$$

Then from assumption \mathcal{C}_0 and (2.11) it follows that

$$\lim_{j \rightarrow \infty} \alpha_j(\pi_j x^0) = \alpha. \quad (2.48)$$

By applying (2.19), we obtain

$$\lim_{j \rightarrow \infty} h_j(\pi_j x^0) = h, \quad (2.49)$$

which leads to the following corollary:

Corollary 2.4 *Under the hypothesis of Theorem 2.3, there is a $j_2 \geq j_1$, such that (2.24) holds for $x_j^0 = \pi_j x^0$, $j \geq j_2$ and*

$$|\tilde{k}_j(\pi_j x^0, \varepsilon) - \tilde{k}(x^0, \varepsilon)| \leq 1, \quad j \geq j_2. \quad (2.50)$$

Condition (2.30) is generally not satisfied for important discretization methods. Nevertheless, under the hypothesis of Theorem 2.2 it always follows that

$$\lim_{j \rightarrow \infty} \omega_j \|\pi_j x^0 - x^*\| \leq \omega \bar{q}^2 \|x^0 - x^*\|. \quad (2.51)$$

On the other hand from Theorem 1.1 we have

$$\omega \|x^0 - x^*\|/2 \leq \omega \rho/2 \leq h/(1-h). \quad (2.52)$$

From (2.51) and (2.52) we deduce that if the constant h satisfies the inequality

$$h < 1/(1 + \bar{q}^2) \quad (2.53)$$

then

$$\lim_{j \rightarrow \infty} \omega_j \|\pi_j x^0 - x_j^*\| < 2. \quad (2.54)$$

Hence (2.27) is satisfied for j sufficiently large and $x_j^0 = \pi_j x^0$. We also have

$$\lim_{j \rightarrow \infty} h_j(\pi_j x^0) \leq \bar{q}^2 h < \bar{q}^2/(1 + \bar{q}^2) \quad (2.55)$$

so that (2.24) is also satisfied for j sufficiently large. Hence we obtain the following result.

Corollary 2.5 *Under the hypothesis of Theorem 2.2 assume that inequality (2.53) is satisfied. Then there is $j_3 \geq j^*$ such that for any $j \geq j_3$ the sequence $\{x_j^k\}_{k=0}^\infty$ given by (2.3) with starting point (2.38) converges to x_j^* and the inequalities (2.24) and (2.27) are satisfied.*

In order to illustrate the above theory, the key result (2.18) will now be discussed for a special example.

Example: Spline collocation in 1-D. This is a rather popular technique of solving boundary value problems (BVP's) for ordinary differential equations (ODE's) — see e.g. the recent textbook by ASCHER/MATTHEIJ/RUSSELL [3]. Following the basic theoretical paper by DE BOOR/SWARTZ [6], the subspaces X_j are defined on a (strict) partition

$$\Delta_j := \{a = t_0 < t_1 < \dots < t_{n_j} = b\}$$

with $(n_j + 1)$ nodes over the finite interval $[a, b]$. The projection π_j maps, say, the space $C[a, b]$ to some space

$$C_{\Delta_j} := C[t_0, t_1] \times \dots \times C[t_{n_j-1}, t_{n_j}]$$

of piecewise continuous functions with certain additional specifications at the nodes of Δ_j (to be ignored here). Upon equipping this space with the canonical C^0 -norm — piecewise, of course — it is a Banach space. Assume that polynomial splines of fixed order k are chosen throughout Δ_j . Now, we maximize over each single sub-interval $I_i \subset \Delta_j$ separately, use the Lagrangian representation and recall that the fundamental Lagrange polynomials are invariant under stretching of each sub-interval. Thus we end up with the stability result

$$q_j = q = \Lambda_k := \max_{s \in [0,1]} \sum_{l=0}^k |L_{lk}(s)|, \quad (2.56)$$

in terms of the Lebesgue constant Λ_k . It is well-known that independent of the choice of the internal nodes (within each sub-interval)

$$\Lambda_k \geq 1, \quad k = 1, 2, \dots, \quad (2.57)$$

in agreement with (2.8). Furthermore, let

$$|\Delta_j| := \max_i |t_{i+1} - t_i| \quad (2.58)$$

and denote the modulus of continuity of some $f \in C[a, b]$ by σ_f , then the following *approximation* result from [6] can be used:

$$\|f - \pi_j f\| \leq 6(\Lambda_k + 1)\sigma_f(|\Delta_j|/(2k - 2)). \quad (2.59)$$

At this point, in order to obtain a bounded δ_j , we introduce for fixed $K > 0$, $0 < \gamma \leq 1$ the subspace

$$W^* := \left\{ f \in C[a, b] \mid \begin{array}{l} |f(s) - f(t)| \leq K|s - t|^\gamma, \\ s, t \in [a, b] \end{array} \right\}. \quad (2.60)$$

In view of the underlying BVP, Hölder continuity is certainly a reasonable concept. With these preparations, we obtain

$$\sup_{f \in W^*, \|f\|=1} \|f - \pi_j f\| \leq \delta_j := 6(\Lambda_k + 1)K \cdot \left(\frac{|\Delta_j|}{2k - 2} \right)^\gamma. \quad (2.61)$$

For $\gamma > 0$, the behavior of $|\Delta_j|$ directly carries over to the behavior of δ_j . Hence

$$\lim_{j \rightarrow \infty} |\Delta_j| = 0 \quad (2.62)$$

implies (2.11) and the monotonicity property

$$|\Delta_{j+1}| \leq |\Delta_j| \quad (2.63)$$

implies (2.12) — without any quasi-uniformity assumption. This means that the above results cover the *adaptive* implementations of spline collocation as realized in codes like COLSYS or COLNEW.

At this point, the key result (2.18) can already be discussed — assuming that bounded constants c_0, c_1 can be found for the collocation approach, but without specifying these constants. As a first aspect, 1-D collocation for fixed order k can be seen to have the asymptotic mesh independence property, since

$$\lim_{j \rightarrow \infty} \omega_j = \omega \Lambda_k . \quad (2.64)$$

Moreover, with Gaussian nodes as internal collocation points (in view of the desirable superconvergence property at the nodes of Δ_j), the Lebesgue constant Λ_k remains decent — growing roughly like $k^{1/2}$ for the small values of k , which are of practical interest. As a second aspect, Theorem 2.2 applied to the present context guarantees — under mild assumptions (boundedness of c_0, c_1) — that the discretized nonlinear systems have locally unique discrete solutions x_j^* for sufficiently fine meshes. In particular, a rather constructive characterization of the term “sufficiently fine mesh” is given, which at the same time gives an error bound in comparison with $\pi_j x^*$. In this sense, the above theory improves the comparable, but non-constructive results of DE BOOR/SWARTZ [6] and the constructive, but less transparent results of BADER’s thesis [4].

Remark. In [4], BADER had already used the basic idea behind our Theorem 2.2. Instead of our refined Mysovskii theorem, however, he applied an affine invariant modification of the convergence theorem of RALL [12]. Moreover he used a different norm of the projection π_j , namely

$$p_j := \frac{\|x^* - \pi_j x^*\|}{\text{dist}(x^*, X_j)} \geq 1 , \quad (2.65)$$

which had been suggested in [6].

As illustrated above, the asymptotic mesh independence property for 1-D collocation does not really rely on essentials of the collocation approach. In fact, the same arguments hold for an adaptive finite element method in 1-D as well. In particular, $\Lambda_1 = 1$ covers the case of *linear* elements. Even more interesting consequences for the construction of adaptive Newton-multilevel FEM in 2-D and 3-D follow from the above theory — a topic, which is, however, beyond the scope of the present paper.

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