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**Adaptive Incompressible Flow Computations  
with Linearly Implicit Time Discretization  
and Stabilized Finite Elements**

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# Adaptive Incompressible Flow Computations with Linearly Implicit Time Discretization and Stabilized Finite Elements

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## Abstract

Fully adaptive solutions of incompressible flow problems employing the discretization sequence first in time then in space are presented. The time discretization is done by linearly implicit one-step methods possibly of high order with automatic step size control. A posteriori error estimates for the stabilized finite element discretization in space are obtained by solving local Dirichlet problems with higher accuracy. Once those estimates have been computed, we are able to control time and space grids with respect to required tolerances and necessary computational work. The devised method is applied to two benchmark problems in 2D.

## 1 Introduction

Stabilization techniques for the finite element computation of incompressible flows are well established in the literature. They are widely accepted to prevent numerical instabilities forced by advection-dominated terms (i.e., high Reynolds number) and by inappropriate combinations of interpolation functions for the velocity and pressure fields. Among others with interest in stabilization methods, Hughes, Franca, Tezduyar, Johnson, Lube, and their groups have been involved most.

One more general stabilization approach is the Galerkin/least-squares method (GLS) where weighted residuals of the basic differential equations are added locally to the Galerkin finite element discretization. This approach has been successfully applied to Stokes flows, compressible and incompressible flows as well, see [1] for an overview.

To extend the GLS stabilization technique to time–dependent equations space–time elements and semidiscretizations based on standard time integration methods have been considered in several papers (eg. [2, 3]). Additionally, these approaches can often be improved using adaptive techniques based on a posteriori error estimates.

The aim of this paper is to present a general adaptive approach to solve incompressible flow problems efficiently. It includes thermally coupled flows satisfying the thermodynamic assumptions for the Boussinesq approximation. The equations governing this flow are

$$\begin{aligned}
 \rho_0[\partial_t v + (v \cdot \nabla)v] - \mu \nabla^2 v + \nabla p &= \rho_0 g[1 - \beta(T - T_0)] + F_v, \\
 \nabla \cdot v &= 0, \\
 \rho_0 c_p[\partial_t T + (v \cdot \nabla)T] - \kappa \nabla^2 T &= F_T,
 \end{aligned}
 \tag{1}$$

where  $v$  describes the velocity field,  $p$  is the pressure,  $\rho_0$  is the (constant) density of the fluid,  $\mu$  is the dynamical viscosity,  $T$  is the temperature,  $\kappa$  is the thermal conductivity,  $g$  is the gravitational acceleration,  $c_p$  is the specific heat at constant pressure,  $F_v$  and  $F_T$  are force terms. The parameter  $\beta$  is the volume expansion coefficient and  $T_0$  refers to a reference temperature state.

We discretize first in time applying linearly implicit one–step methods possibly of high order with automatic step size control. These methods have excellent stability properties and require only the solution of systems of linear equations in each time step. In the spirit of full adaptivity the arising spatial problems are approximated by a self–adaptive multilevel finite element method based on a GLS stabilization. To determine during the course of time integration where spatial refinement is necessary and where coarse meshes are sufficient, we compute a posteriori error estimates. Therefore, we capture local discretization errors involving all variables of the model equations to solve Dirichlet problems on small subdomains with higher accuracy. The mesh adaptation is done until a global prescribed tolerance is reached. So, by repeated application of solving linear equations, error estimation, and local refinement the final grid is well adapted to the required solution at each time. Furthermore, computing the solution on grids of different scales speeds up the iterations on each level due to the related multilevel decomposition and gives optimal multigrid complexity. The implementation of those complex algorithms requires modern software design and programming languages as C or C++. Our code KARDOS is based on the finite element programming environment KASKADE [8].

The method has been successfully applied to reaction–diffusion problems

[4, 5, 6] and now has been extended to instationary incompressible flow problems. The method and its implementation have shown to be satisfactory in terms of flexibility with respect to the equations to be solved, the boundary conditions, and the geometry. Steep gradients and moving fronts are handled optimally.

## 2 Discretization in Time and Space

### 2.1 Stabilized Rosenbrock–Galerkin Finite Element Methods

Equations (1) are a special case of a more general system of nonlinear equations

$$\begin{aligned} H\partial_t u &= \nabla \cdot (D(x, t, u)\nabla u) + B(x, t, u) \cdot \nabla u + F(x, t, u) \\ x &\in \Omega \subset \mathbb{R}^n, \quad t > 0, \end{aligned} \quad (2)$$

equipped with suitable initial and boundary conditions for the unknown vector-valued function  $u$ . Setting  $u = (p, v, T)$  we are able to describe the Boussinesq approximation of an incompressible thermally coupled flow with an appropriate choice of the matrix  $H$ , the tensors  $D$  and  $B$ , and the vector  $F$ . For convenience, we restrict ourselves to the autonomous case and rewrite (2) as an abstract Cauchy problem

$$H\partial_t u = f(u), \quad u(0) = u_0, \quad (3)$$

where  $f$  represents the right-hand side in (2) and the function  $u_0$  is taken as initial value. We discretize (3) in time employing linearly implicit one-step methods proposed to achieve higher-order temporal discretizations for stiff problems by working the Jacobian matrix directly into the integration formula [7]. With variable time step  $\tau_n > 0$  a so-called s-stage Rosenbrock method reads as follows

$$\begin{aligned} \left( \frac{1}{\gamma\tau_n} H - \partial_u f(u_{n-1}) \right) l_j &= f(u_{n-1} + \sum_{i=1}^{j-1} a_{ji} l_i) + \frac{1}{\tau_n} H \sum_{i=1}^{j-1} c_{ji} l_i \\ u_n &= u_{n-1} + \sum_{i=1}^s b_i l_i, \quad j = 1, \dots, s. \end{aligned} \quad (4)$$

Here,  $u_n$  stands for an approximation of  $u(t)$  at  $t = t_n$ . The coefficients  $b_i$ ,  $a_{ji}$ ,  $c_{ji}$ , and  $\gamma$  are suitable chosen to obtain a desired order of consistency and stability for stiff problems. Note, only linear systems with one and the same operator have to be solved successively.

In the spirit of full adaptivity we solve the equations (4) by a multilevel finite element method. According to finite element meshes  $\mathcal{T}_n^k$  at  $t = t_n$  with refinement level  $k$ , we choose finite dimensional subspaces  $S_k^q$  consisting of all continuous functions which are polynomials of order  $q$  on each finite element. The standard Galerkin finite element solutions  $l_j^k \in S_k^q$  satisfy the equations

$$(A_k l_j^k, w) = (r_j^k, w), \quad \forall w \in S_k^q, j = 1, \dots, s, \quad (5)$$

with

$$A_k := \frac{1}{\gamma \tau_n} H - \partial_u f(u_{n-1}^k), \quad (6)$$

$$r_j^k := f(u_{n-1}^k + \sum_{i=1}^{j-1} a_{ji} l_i^k) + \frac{1}{\tau_n} H \sum_{i=1}^{j-1} c_{ji} l_i^k. \quad (7)$$

It is a well-known inconvenience that the solutions  $l_j^k$  may suffer from numerical oscillations caused by dominant convective and reactive terms as well. We locally add weighted residuals to stabilize the discretization:

$$(A_k l_j^k, w) + \sum_{\mathcal{T} \in \mathcal{T}_n^k} (A_k l_j^k, \phi(w))_{\mathcal{T}} = (r_j^k, w) + \sum_{\mathcal{T} \in \mathcal{T}_n^k} (r_j^k, \phi(w))_{\mathcal{T}}. \quad (8)$$

To define the function  $\phi(w)$ , we first have to describe more precisely the differential form of the operator  $A_k$ , namely

$$A_k = -\nabla \cdot (d(x, u_{n-1}^k) \nabla w) + b(x, u_{n-1}^k) \cdot \nabla w + c(x, u_{n-1}^k) w, \quad w \in S_k^q. \quad (9)$$

Using definition (6), the functions  $d$ ,  $b$ , and  $c$  can be derived easily from the given data. We consider now the general class of Galerkin/least-squares methods with

$$\phi(w)|_{\mathcal{T}} := \delta_1^{\mathcal{T}} (-\nabla \cdot (d \nabla w) + cw) + \delta_2^{\mathcal{T}} b \cdot \nabla w, \quad \forall \mathcal{T} \in \mathcal{T}_n^k, \quad (10)$$

where we allow, in addition, that the parameters  $\delta_1^{\mathcal{T}}$  and  $\delta_2^{\mathcal{T}}$  can be different for each component of  $u$ . We set for the pressure

$$\delta_{2,p}^{\mathcal{T}} = \delta_2^* \frac{h^\sharp}{2V} \frac{\text{Re}}{\sqrt{1 + (\text{Re})^2}}, \quad \text{Re} := \frac{\rho_0 h^\sharp V}{\mu}. \quad (11)$$

Here,  $V$  denotes a global reference velocity and  $h^\sharp$  is the diameter of the  $n$ -dimensional ball which is area-equivalent to the element  $\mathcal{T}$ . For the velocity components and the temperature we choose

$$\delta_{2,v}^{\mathcal{T}} = \delta_2^* \frac{h}{2|v|} \frac{\text{Re}}{\sqrt{1 + (\text{Re})^2}}, \quad \text{Re} := \frac{\rho_0 h |v|}{\mu}, \quad (12)$$

and

$$\delta_{2,T}^{\mathcal{T}} = \delta_2^* \frac{h}{2|v|} \frac{\text{Pe}}{\sqrt{1 + (\text{Pe})^2}}, \quad \text{Pe} := \frac{\rho_0 c_p h |v|}{\kappa}. \quad (13)$$

The definitions of the local Reynolds and Peclet numbers are based on the element length  $h$  taken in the direction of the local velocity  $v$ . In this paper we choose  $\delta_1^{\mathcal{T}} = \delta_2^{\mathcal{T}}$  for simplicity. Obviously, we recover the standard Galerkin method for  $\delta_1^{\mathcal{T}} = \delta_2^{\mathcal{T}} = 0$ . In the case  $\delta_1^{\mathcal{T}} = 0$  and  $\delta_2^{\mathcal{T}} > 0$ , we get the well-known streamline upwind finite element method.

From the computational point of view it is attractive to use the same finite element functions for all components of  $u$ . Unfortunately, as a consequence of the Babuška–Brezzi condition this simplification is not allowed in general. If we do not satisfy this compatibility condition the discrete solution may be affected by spurious pressure modes. Yet, there is a way to get stable discretizations even in this case relaxing the incompressibility constraint at the differential level as follows

$$\nabla \cdot u = \delta \nabla \cdot (\rho_0 [\partial_t v + (v \cdot \nabla) v] - \mu \nabla^2 v + \nabla p - \rho_0 g [1 - \beta(T - T_0)] - F_v), \quad (14)$$

where the parameter  $\delta$  is chosen in such a way that the parasitic modes will be eliminated. The term on the right-hand side is identically zero from (1). Since this procedure aims at a stabilization of the pressure, we set  $\delta = \delta_{2,p}^{\mathcal{T}}$  after spatial discretization and summation over all finite elements  $\mathcal{T} \in \mathcal{T}_n^k$ .

### 3 Adaptive Approach

In this section we briefly describe the basis of our fully adaptive approach. For more details we refer to [4].

To estimate the error in time we apply an embedding strategy. Replacing the coefficients  $b_i$  in (4) by different coefficients  $\hat{b}_i$  a second solution  $\hat{u}_n$  of inferior order is derived. The difference

$$\varepsilon_n := \|u_n - \hat{u}_n\| \quad (15)$$

can be used for an efficient step size control. A new step size with respect to a desired tolerance  $TOL_t$  is selected by

$$\tau_n = \frac{\tau_{n-1}}{\tau_{n-2}} \left( \frac{TOL_t \varepsilon_{n-2}}{\varepsilon_{n-1} \varepsilon_{n-1}} \right)^{1/(p+1)} \tau_{n-1}. \quad (16)$$

Here,  $p$  is the local order of the embedded solution  $\hat{u}_n$ .

After computation of each intermediate stage value  $l_j^k$  in (8), an a posteriori spatial error estimation has to be given for an assessment of the new solution

$$u_n^k = u_{n-1}^k + \sum_{i=1}^s b_i l_i^k. \quad (17)$$

One common method is to compute local solutions with higher order on small subdomains  $\Theta_i \subset \Omega$ ,  $i = 1, \dots, M$ ,  $\cup \overline{\Theta_i} = \Omega$ , which in general are unions of several finite elements. Let us consider a local hierarchical decomposition

$$S_k^{q+1}(\Theta_i) = S_k^q(\Theta_i) \oplus W_k^{q+1}(\Theta_i) \quad (18)$$

where  $W_k^{q+1}(\Theta_i)$  contains all additional basis functions needed to enrich the space  $S_k^q(\Theta_i)$  to order  $q+1$ . Defining the spatial stage errors  $e_j := l_j - l_j^k$ , we approximate them locally in  $W_k^{q+1}(\Theta_i)$  by  $e_j^k$  satisfying the equations

$$\begin{aligned} (A_k e_j^k, w) + \sum_{\mathcal{T} \in \mathcal{T}_n^k} (A_k e_j^k - \hat{r}_j^k, \phi(w))_{\mathcal{T}} &= (\hat{r}_j^k, w), \quad \forall w \in W_k^{q+1}(\Theta_i), \\ e_j^k &= 0, \quad x \in \partial\Theta_i, \\ \hat{r}_j^k &:= r_j^k(e_1^k + l_1^k, \dots, e_{j-1}^k + l_{j-1}^k) - A_k l_j^k, \quad j = 1, \dots, s. \end{aligned} \quad (19)$$

The local quantities  $e_j^k$  can be utilized to compute local error estimates for each subdomain  $\Theta_i$ . These estimates are then the basis for grid refinement and coarsening until a prescribed tolerance  $TOL_x$  in space is reached [4].

## 4 Numerical Examples

To illustrate the performance of the described method we solve two different two-dimensional problems. We choose a Rosenbrock method of order 3 with embedded order 2 ( $s=3$ ) and use linear finite elements on unstructured triangular grids ( $q=1$ ). The small subdomains for spatial error estimation are all unions of two neighboring triangles having one common edge. We set  $\delta_2^*=1$  except for the stabilization of the incompressibility condition where  $\delta_2^*=0.4$  is used.

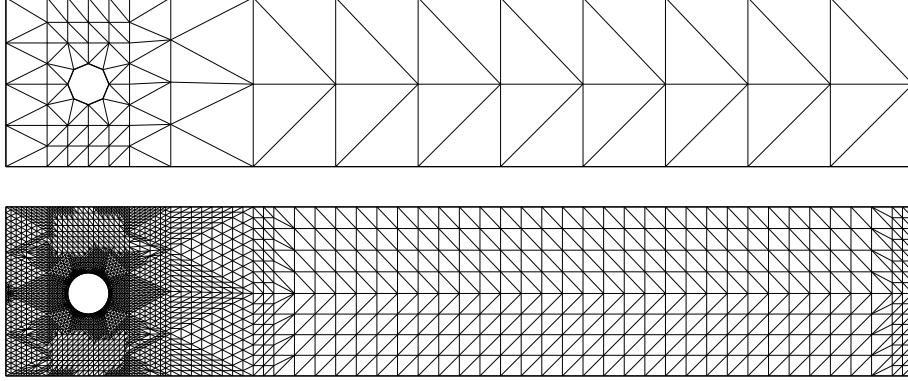
### 4.1 Laminar Flow Around a Cylinder

The flow past a cylinder is a widely solved problem. To make our computations comparable with the results of a benchmark [9], we skip the temperature and solve the conservation equations of mass and momentum. The fluid density is defined as  $\rho_0 = 1.0kg/m^3$ , and the dynamic viscosity is  $\mu = 0.001m^2/s$ . No force term  $F_v$  is considered. The computational domain has length  $L = 2.2m$  and height  $H = 0.41m$ . The midpoint of the cylinder with diameter  $D = 0.1m$  is placed at  $(0.2m, 0.2m)$ . The inflow condition at the left boundary is

$$v_x(0, y, t) = 4Vy(H - y)/H^2, \quad v_y = 0,$$

with a mean velocity  $V = 0.3m/s$  yielding a Reynolds number  $Re = 20$ . We further use non-flux conditions at the right outflow boundary, and  $v_x = v_y = 0$  otherwise. The flow becomes steady and two unsymmetric eddies develop behind the cylinder. We start with a very coarse approximation of the given geometry (81 points) to test our automatic mesh controlling. The resulting fine grid at the steady state contains 2785 points. The drag and lift coefficients as well as the pressure difference  $\Delta p = p(0.15m, 0.2m) - p(0.25m, 0.2m)$  are in good agreement with the results given in [9].





	KARDOS	Benchmark
Drag coefficient $c_w$	5.510	5.500
Lift coefficient $c_a$	0.003	0.010
Pressure difference $\Delta p$	0.119	0.117

Fig. 1: Flow around a cylinder with Reynolds number  $Re = 20$ ; coarse (81 points) and fine (2785 points) grid, comparison of drag and lift coefficient, pressure difference

## 4.2 Thermoconvective Instability of Plane Poiseuille Flow

Introducing suitable reference values, the system (1) may be written for the so-called forced convection problem in dimensionless form as follows

$$\begin{aligned}
 \partial_t v + (v \cdot \nabla)v - \frac{1}{Re} \nabla^2 v + \nabla p &= -\frac{1}{Fr} T \hat{g}, \\
 \nabla \cdot v &= 0, \\
 \partial_t T + (v \cdot \nabla)T - \frac{1}{Pe} \nabla^2 T &= 0,
 \end{aligned}$$

where source terms have been omitted.  $Fr$  is the Froude number and the vector  $\hat{g}$  in the momentum equation denotes now the normalized gravity acceleration vector. We consider a two-dimensional laminar flow in a horizontal channel  $\Omega = [0, 10] \times [0, 1]$  suddenly heated from below with constant temperature  $T = 1.0$ . At the opposite wall we choose  $T = 0.0$  and non-flux conditions for the temperature at the inlet and outlet. The boundary conditions for the velocity field are taken from the previous problem, but a parabolic inlet profile is prescribed now by

$$v_x(0, y, t) = 6y(1 - y), \quad v_y = 0.$$

The dimensionless parameters have been taken as  $Re = 10$ ,  $Fr = 1/150$  and  $Pe = 40/9$ . The same setting was studied in [10]. Travelling transverse waves can be observed (see Fig. 2). We plot also the transient evolution of the temperature at the central point  $(5.0, 0.5)$ . Comparing our curve with that given in [10], we observe a smoother function due to the higher accuracy provided by the devised adaptive approach.

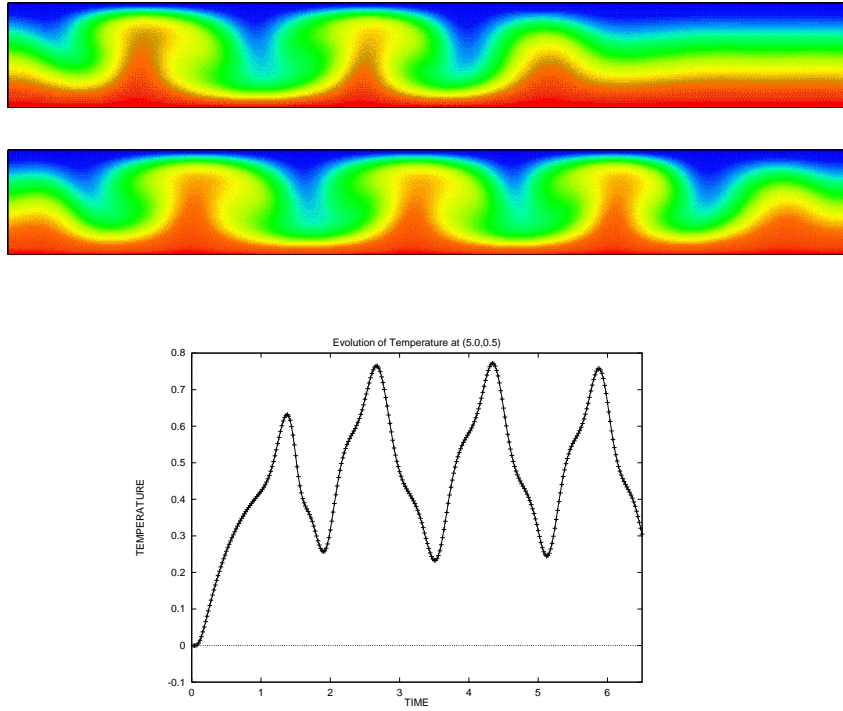


Fig. 2: Poiseuille flow; temperature field at  $t=1.4$  and  $t=1.8$ , evolution of temperature at central point  $(5.0,0.5)$

## 5 Conclusion

In this paper we have presented a mesh-controlling algorithm based on linearly-implicit time schemes and stabilized Galerkin/least-squares finite

element methods in space to solve incompressible flows of viscous fluids. The proposed method permits an efficient solution of the Navier–Stokes equations and may also be a valuable tool for problems with more realistic physics. The ability to handle the above prototype equations allows for generalization to much broader sets of equations such as low Mach number flows with complicated chemistry.

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