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Associated Classical Discrete Orthogonal
Polynomials**

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Abstract

We derive the fourth order difference equation satisfied by the associated of order r of the classical orthogonal polynomials of a discrete variable.

The coefficients of this equation are given in terms of the polynomials σ and τ which appear in the discrete Pearson equation $\Delta(\sigma \rho) = \tau \rho$ defining the weight $\rho(x)$ of the classical discrete orthogonal polynomials.

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1 Introduction

The fourth order difference equation for the associated Meixner and Charlier polynomials were given for all r (order of association) in [6], using an explicit solution of the recurrence relation built from the symmetry properties of this recurrence. On the other hand, the equation for the first associated ($r = 1$) of all classical discrete polynomials was obtained in [10] using a useful relation proved in [1].

In this work, we give a single fourth order difference equation which is valid for all integers r and for all classical discrete orthogonal polynomials. This equation is important in birth and death processes [6] and also for some connection coefficient problems [7].

Let $(P_n)_n$ be a sequence of monic orthogonal polynomials of degree n with respect to the regular linear functional \mathcal{L} . $(P_n)_n$ satisfies the following second order recurrence relation:

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$$\begin{cases} P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x), & n \geq 1, \gamma_n \neq 0 \\ P_0(x) = 1, P_1(x) = x - \beta_0. \end{cases}$$

The associated orthogonal polynomials of order r , $P_n^{(r)}$, are defined by the shifted recurrence relation ($n \rightarrow n+r$ in β_n and γ_n)

$$\begin{cases} P_{n+1}^{(r)}(x) = (x - \beta_{n+r}) P_n^{(r)}(x) - \gamma_{n+r} P_{n-1}^{(r)}(x), & n \geq 1, \\ P_0^{(r)}(x) = 1, P_1^{(r)}(x) = x - \beta_r, & r \geq 0. \end{cases}$$

When the family $(P_n)_n$ is classical (continuous), the polynomials P_n are solutions of the hypergeometric equation

$$L_{2,0}[y] \equiv \sigma(x) y''(x) + \tau(x) y'(x) + \lambda_n y(x) = 0,$$

where σ is a polynomial of degree at most two, τ is a polynomial of degree one, and λ_n is a constant [8].

From the following coupled second order relations [2,9],

$$L_{2,r}[P_n^{(r)}] = K_r [P_{n-1}^{(r+1)}]', \quad L_{2,r}^*[P_{n-1}^{(r+1)}] = K_r^* [P_n^{(r)}]' \quad (1)$$

with

$$\begin{aligned} L_{2,r} &= \sigma \frac{d^2}{dx^2} + (x - \beta_r) (r\sigma'' + \tau') \frac{d}{dx} + (\lambda_n - nr\sigma'') \\ L_{2,r}^* &= \sigma \frac{d^2}{dx^2} - [(x - \beta_r) (r\sigma'' + \tau') - 2\sigma'] \frac{d}{dx} + (\lambda_n^* - (n+1)r\sigma'') \\ K_r^* &= -[(2r-1)\sigma'' + 2\tau'], \quad K_r = \begin{cases} \gamma_r[(2r-3)\sigma'' + 2\tau'] & \text{if } r \geq 1 \\ 0 & \text{if } r = 0 \end{cases}. \end{aligned} \quad (2)$$

in [11], using the representation of γ_r and β_r in terms of σ and τ the generic fourth order differential equation $M_4^{(r)}(P_n^{(r)}(x)) = 0$ satisfied by the associated of any integer order r of the classical class was derived, where

$$\begin{aligned} M_4^{(r)} &= \sigma^2 \frac{d^4}{dx^4} + 5\sigma\sigma' \frac{d^3}{dx^3} + [-\tau^2 + 2\tau\sigma' + 3\sigma'^2 - (2n+4r)\sigma\tau' \\ &\quad + (4+n-n^2+4r-2nr-2r^2)\sigma\sigma''] \frac{d^2}{dx^2} \\ &\quad - \frac{3}{2} [2\tau\tau' + (2n-2+4r)\sigma'\tau' - 2\tau\sigma'' + (n^2-n-4r+2nr+2r^2)\sigma'\sigma''] \frac{d}{dx} \\ &\quad + \frac{1}{4} \{n(2+n)[2\tau' + (n+2r-3)\sigma''] [2\tau' + (n+2r-1)\sigma'']\} \end{aligned} \quad (3)$$

In this letter we want to extend these results to the classical discrete class, i.e., the polynomials P_n of Hahn, Hahn-Eberlein, Krawtchouk, Meixner, and Charlier, which are solutions of the second order difference equation

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0$$

with $\Delta y(x) = y(x+1) - y(x)$ and $\nabla y(x) = y(x) - y(x-1)$. It turns out that the coefficient λ_n is given by $2\lambda_n = -n[(n-1)\sigma'' + 2\tau']$, see [8]. From the known relations between the recurrence coefficients β_n , γ_n and the polynomials σ and τ [5,4],

$$\begin{aligned}\beta_n &= \frac{-\tau(0)(\tau' - \sigma'') - n(\tau' + 2\sigma'(0))(\tau' + (n-1)\frac{\sigma''}{2})}{((n-1)\sigma'' + \tau')(n\sigma'' + \tau')}, \quad n \geq 0 \\ \gamma_n &= -\frac{n(\tau' + (n-2)\frac{\sigma''}{2})}{(\tau' + (2n-3)\frac{\sigma''}{2})(\tau' + (2n-1)\frac{\sigma''}{2})}[\sigma(\eta_{n-1}) + \tau(\eta_{n-1})], \quad n \geq 1 \\ \eta_n &= -\frac{\tau(0) + n\sigma'(0) - n^2\frac{\sigma''}{2}}{\tau' + n\sigma''}, \quad n \geq 0\end{aligned}\tag{4}$$

it is possible to write the corresponding equations (1), (2) and (3) again in terms of σ and τ , for the generic classical discrete polynomials.

Proposition 1 [3] *The associated polynomials satisfy*

$$\mathcal{D}_{r,n} \left[P_n^{(r)} \right] = \mathcal{N}_{r+1,n-1} \left[P_{n-1}^{(r+1)} \right], \tag{5}$$

$$\bar{\mathcal{D}}_{r+1,n-1} \left[P_{n-1}^{(r+1)} \right] = \bar{\mathcal{N}}_{r,n} \left[P_n^{(r)} \right], \tag{6}$$

where

$$\mathcal{D}_{r,n} = a_2 \mathcal{T}^2 + a_1 \mathcal{T} + a_0 \mathcal{T}^0, \quad \mathcal{N}_{r+1,n-1} = \tilde{a}_1 \mathcal{T} + \tilde{a}_0 \mathcal{T}^0, \tag{7}$$

$$\bar{\mathcal{D}}_{r+1,n-1} = b_2 \mathcal{T}^2 + b_1 \mathcal{T} + b_0 \mathcal{T}^0, \quad \bar{\mathcal{N}}_{r,n} = \tilde{b}_1 \mathcal{T} + \tilde{b}_0 \mathcal{T}^0, \tag{8}$$

$$a_2 = k_{9,0}, \quad a_1 = -k_{2,1}k_{10,0}, \quad a_0 = k_{11,0}, \quad \tilde{a}_1 = k_{4,0}k_{10,0}, \quad \tilde{a}_0 = -k_{4,0}k_{12,0} \tag{9}$$

$$b_2 = k_{9,0}, \quad b_1 = -k_{5,1}k_{10,0}, \quad b_0 = k_{13,0}, \quad \tilde{b}_1 = k_{6,0}k_{10,0}, \quad \tilde{b}_0 = -k_{6,0}k_{14,0}, \tag{10}$$

\mathcal{T} is the shift operator: $\mathcal{T}P(x) = P(x+1)$, and the coefficients $k_{i,j}$ are polynomials given by

$$k_{i,0}(x) \equiv k_i(x), \quad \text{and} \quad k_{i,j}(x) = k_i(x+j) \tag{11}$$

and (16).

2 Fourth Order Difference Equation for Associated Polynomials

Replacing $\mathcal{T}^2 P_{n-1}^{(r+1)}$ given by (6) in the shifted equation (5), we obtain

$$[c_3 \mathcal{T}^3 + c_2 \mathcal{T}^2 + c_1 \mathcal{T} + c_0] \left(P_n^{(r)} \right) = [\tilde{c}_1 \mathcal{T} + \tilde{c}_0] \left(P_{n-1}^{(r+1)} \right). \tag{12}$$

By the same process, using again $\mathcal{T}^2 P_{n-1}^{(r+1)}$ given by (6) in the shifted equation (12), we obtain

$$[d_4 \mathcal{T}^4 + d_3 \mathcal{T}^3 + d_2 \mathcal{T}^2 + d_1 \mathcal{T} + d_0] \left(P_n^{(r)} \right) = [\tilde{d}_1 \mathcal{T} + \tilde{d}_0] \left(P_{n-1}^{(r+1)} \right), \quad (13)$$

where the polynomial coefficients c_i , \tilde{c}_i , d_i and \tilde{d}_i are easily computed from the coefficients a_i , \tilde{a}_i , b_i and \tilde{b}_i .

Now, use of equations (5), (12) and (13) gives the expected fourth order difference equation satisfied by each $P_n^{(r)}$

$$\begin{vmatrix} a_2 \mathcal{T}^2 P_n^{(r)} + a_1 \mathcal{T} P_n^{(r)} + a_0 P_n^{(r)} & \tilde{a}_1 \tilde{a}_0 \\ c_3 \mathcal{T}^3 P_n^{(r)} + c_2 \mathcal{T}^2 P_n^{(r)} + c_1 \mathcal{T} P_n^{(r)} + c_0 P_n^{(r)} & \tilde{c}_1 \tilde{c}_0 \\ d_4 \mathcal{T}^4 P_n^{(r)} + d_3 \mathcal{T}^3 P_n^{(r)} + d_2 \mathcal{T}^2 P_n^{(r)} + d_1 \mathcal{T} P_n^{(r)} + d_0 P_n^{(r)} & \tilde{d}_1 \tilde{d}_0 \end{vmatrix} = 0, \quad (14)$$

which can be written in the form $\left[\sum_{j=0}^4 I_j(r, n, x) \mathcal{T}^j \right] P_n^{(r)}(x) = 0$.

We have used Maple V Release 4 to compute the coefficients I_j depending on r, n and x , and after canceling common factors, we obtain

$$\begin{aligned} I_4 &= k_{9,2} (k_{10,0} k_{10,1} - k_{12,0} k_{12,1}), \\ I_3 &= k_{10,2} (k_{12,0} (k_{2,3} k_{12,1} + k_{13,1}) - k_{10,0} k_{10,1} (k_{2,3} + k_{5,2})) + k_{9,1} k_{10,0} k_{12,2}, \\ I_2 &= k_{10,1} (k_{10,2} (k_{10,0} k_{10,1} + k_{13,0} - k_{5,1} k_{12,0}) \\ &\quad - k_{9,1} k_{10,0}) - k_{12,1} (k_{12,2} k_{13,0} + k_{11,2} k_{12,0}), \\ I_1 &= k_{10,0} k_{12,2} (k_{2,2} k_{12,0} + k_{13,0}) + k_{10,2} k_{12,0} (k_{9,0} - k_{10,0} k_{10,1}), \\ I_0 &= k_{9,-1} (k_{10,1} k_{10,2} - k_{12,1} k_{12,2}) \end{aligned} \quad (15)$$

where the polynomials $k_{i,j}$ are given by (11) and

$$\begin{aligned} E_r(x) &= \tau(x) - \frac{\tau(\beta_r)}{2} + r \frac{\tau'}{2} + (r^2 - r(1 + 2\beta_r) - 2) \frac{\sigma''}{4} \\ &\quad + (r - 2) \frac{\sigma'(x)}{2} - \sigma'(0) \frac{r}{2} \\ F_r(x) &= -\frac{\tau(\beta_r)}{2} - r \frac{\tau'}{2} - (r^2 - r(3 - 2\beta_r)) \frac{\sigma''}{4} + (\sigma'(x) - \sigma'(0)) \frac{r}{2} \\ \zeta_n &= (2n - 1) \frac{\sigma''}{2} + \tau', \quad k_1(x) = \sigma(x + 1) + E_{n+r+1}(x), \end{aligned} \quad (16)$$

$$\begin{aligned} k_2(x) &= \sigma(x + 1) - F_r(x), \quad k_3(x) = \zeta_{n+r}, \quad k_4(x) = \begin{cases} \gamma_r \zeta_{r-1} & \text{if } r \geq 1 \\ 0 & \text{if } r = 0 \end{cases} \\ k_5(x) &= \sigma(x + 1) + E_r(x), \quad k_6(x) = -\zeta_r, \quad k_7(x) = \sigma(x + 1) - F_{n+r+1}(x), \\ k_8(x) &= -\gamma_{n+r+1} \zeta_{n+r+1}, \quad k_9(x) = k_7(x + 1) k_1(x + 1) - k_3(x) k_8(x), \\ k_{10}(x) &= k_7(x + 1) + k_1(x), \quad k_{11}(x) = k_2(x + 1) k_2(x) + k_4(x) k_6(x), \\ k_{12}(x) &= k_2(x + 1) + k_5(x), \quad k_{13}(x) = k_5(x + 1) k_5(x) + k_4(x) k_6(x), \\ k_{14}(x) &= k_5(x + 1) + k_2(x). \end{aligned}$$

The polynomials k_3 , k_4 , k_6 , k_8 are constant with respect to the variable x and β_r , γ_r are given by (4).

If $r = 0$, from (2), and (16) we have $k_4 = K_0 = 0$. Then, $\mathcal{N}_{1,n-1}$ is equal to zero, thus the fourth order difference equation for the first associated $P_n^{(1)}$, factorizes in the form [1,3,10] $(\bar{A}_1 \mathcal{T}^2 + \bar{B}_1 \mathcal{T} + \bar{C}_1 \mathcal{T}^0)(A_1 \mathcal{T}^2 + B_1 \mathcal{T} + C_1 \mathcal{T}^0)[P_n^{(1)}] = 0$.

For $r = 0$, if we are inside the Hahn class with $\alpha + \beta + 1 = 0$ (discrete Grosjean polynomials), from (2), (16), and [2,9] we have $K_0^* = 2k_6 = 0$. Then $\bar{\mathcal{N}}_{1,n}$ is equal to zero and the difference equation in this case reduces to the second order difference equation $\bar{\mathcal{D}}_{1,n} [P_n^{(1)}] = 0$.

Using the result of this letter, we have computed the coefficients I_j for all classical polynomials of a discrete variable, generalizing the results given in [6,10].

For the Krawtchouk case for example, $(\sigma(x) = x, \tau(x) = \frac{1}{q}((1-q)N - x))$, the r th Krawtchouk associated $P_n^{(r)}$ with $n + r \leq N$ is annihilated by the following difference operator, where t is given by $t = r + x - 2xq + qN - 5q - N + 2$.

$$\begin{aligned}
& q(4+x)(x+3-N)(q-1)(n-2+4q+2t)\mathcal{T}^4 - (10xq+nq-6Nq \\
& - 42q^2 - 4xq^3N - 2xNq + 2x^2q - 12q^3N + 20xq^3 + 4x^2q^3 + 28q^3 \\
& - 3nq^2 + 14q - 2t - n^2q + 18q^2N + 6xq^2N - 30xq^2 - 6x^2q^2 + 3nt^2 \\
& + 2t^3 - 6tq^2 + n^2t - nt + 6tq)\mathcal{T}^3 - (10xq - 8nq - 6Nq - 42q^2 \\
& - 5nNq^2 + 8xq^2n - 4xq^3N + 2x^2nq^2 + 5nNq + 2xNqn + n^2 \\
& - 2xNq + 2x^2q - 12q^3N + 20xq^3 + 4x^2q^3 - 2xNq^2n + 28q^3 + 6nq^2 \\
& + 14q - 2t - 8xqn - n^3 - 4n^2q + 18q^2N + 6xq^2N - 30xq^2 - 2x^2nq \\
& - 6x^2q^2 - 12nqt - 4x^2qt - 6nt^2 - 4t^3 + 12tq^2 - 12qt^2 - 4n^2t \\
& + 4nt - 12tq + 6t^2 - 10q^2Nt - 16xqt + 16xq^2t - 4xq^2Nt \\
& + 4xNqt + 10Nqt + 4x^2q^2t)\mathcal{T}^2 + (6xq + 9nq - 4Nq - 12q^2 \\
& - 4xq^3N + 2n^2 - 2xNq + 2x^2q - 8q^3N + 12xq^3 + 4x^2q^3 + 8q^3 \\
& - 9nq^2 + 4q - 2n - 4t - 3n^2q + 12q^2N + 6xq^2N - 18xq^2 - 6x^2q^2 \\
& - 12nqt - 3nt^2 - 2t^3 - 18tq^2 - 12qt^2 - n^2t + 7nt + 18tq + 6t^2)\mathcal{T} \\
& + q(1+x)(x-N)(q-1)(2t+n).
\end{aligned}$$

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