

The Intersection of Knapsack Polyhedra and Extensions

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Abstract

This paper introduces a scheme of deriving strong cutting planes for a general integer programming problem. The scheme is related to Chvatal-Gomory cutting planes and important special cases such as odd hole and clique inequalities for the stable set polyhedron or families of inequalities for the knapsack polyhedron. We analyze how relations between covering and incomparability numbers associated with the matrix can be used to bound coefficients in these inequalities. For the intersection of several knapsack polyhedra, incomparabilities between the column vectors of the associated matrix will be shown to transfer into inequalities of the associated polyhedron. Our scheme has been incorporated into the mixed integer programming code SIP. About experimental results will be reported.

1 Introduction

LP-based branch-and-bound algorithms are currently the most important tool to deal with real-world general mixed integer programming problems computationally. Usually, the LP relaxations that occur during the execution of such methods are strengthened by cutting planes. Cutting planes for integer programs may be classified with regard to the question whether their derivation requires knowledge about the structure of the underlying constraint matrix. Examples of families of cutting planes that do not exploit the structure of the constraint matrix are Chvátal-Gomory cuts [G60] [C73] [S80] or lift-and-project cuts [BCC93]. An alternative approach to obtain cutting planes for an integer program follows essentially the scheme to derive relaxations associated with certain substructures of the underlying constraint matrix, and tries to find valid inequalities for these relaxations. Crowder, Johnson and Padberg [CJP83] applied this methodology by interpreting each single row of the constraint matrix as a knapsack relaxation and strengthened the integer program by adding violated knapsack inequalities. An analysis of other important relaxations of an integer program allows to incorporate odd hole and clique inequalities for the

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stable set polyhedron [P73] or flow cover inequalities for certain mixed integer models [PRW85]. Further recent examples of this second approach are given in [CCMW97], [MW97].

Our paper follows the methodology to obtain cutting planes for an integer program by investigating relaxations of it. We try to go one step further and investigate the intersection of two or more knapsack polyhedra. We describe a general family of valid inequalities for an integer program that are associated with its feasible solutions. Usually such inequalities must be lifted in order to induce high dimensional faces. We derive lower and upper bounds on the exact lifting coefficients of such an inequality and discuss special cases when these bounds can be computed in polynomial time. We also relate our family of inequalities to Chvátal-Gomory cuts and discuss in detail the special case where only two knapsacks are involved. The use of feasible set inequalities within an implementation for the solution of general mixed integer programming problems is investigated.

Consider some matrix $A \in \mathbf{R}^{m \times n}$, vectors $b \in \mathbf{R}^m$, $u \in \mathbf{R}^n$, and the polytope

$$P := \text{conv} \{x \in \mathbf{Z}^n : Ax \leq b, 0 \leq x \leq u\},$$

that is the convex hull of all integral vectors x satisfying $Ax \leq b$ and $0 \leq x \leq u$. Set $N := \{1, \dots, n\}$. For $S \subseteq N$, define $P_S := P \cap \{x \in \mathbf{R}^n : x_i = 0 \text{ for } i \in S\}$, where $P_\emptyset = P$, and, for $x \in \mathbf{R}^n$, denote by $x|_S := (x_i)_{i \in S}$ the vector restricted to the components in S .

Definition 1.1 *Let $T \subseteq N$ such that $\sum_{i \in T} A_i v_i \leq b$ for all $v \in \mathbf{R}^T$ with $v \leq u|_T$. T is called a feasible set. Let $w : T \mapsto \mathbf{Z}$ be some weighting of the elements of T . For $j \in N \setminus T$ with $\sum_{i \in T} A_i u_i + A_j u_j \not\leq b$, the inequality*

$$(1) \quad \sum_{i \in T} w_i x_i + w_j x_j \leq \sum_{i \in T} w_i u_i$$

is called a feasible set inequality associated with T (and $\{j\}$) if

$$(2) \quad \begin{aligned} w_j &\leq \min_{l=1, \dots, u_j} \frac{1}{l} \min_x \sum_{i \in T} w_i x_i \\ &\sum_{i \in T} A_i x_i \geq A_j - r(T) \\ &0 \leq x_i \leq u_i, x_i \in \mathbf{Z}, i \in T, \end{aligned}$$

where $r(T) := b - \sum_{i \in T} A_i u_i$.

Theorem 1.2 *Feasible set inequalities are valid for $P_{T \cup \{j\}}$.*

Examples of feasible set inequalities include $(1, k)$ -configuration and minimal-cover inequalities that are known for the knapsack polytope $K := \text{conv} \{x \in \{0, 1\}^n : a^T x \leq \alpha\}$ with $a \in \mathbf{R}_+^n, \alpha > 0$. Let $S \subseteq N$ be a minimal cover,

i.e., $a(S) > \alpha$ and $a(S \setminus \{i\}) \leq \alpha$ for all $i \in S$, and partition S into T and $\{j\}$ for some $j \in S$. Set $w_i := 1$ for all $i \in T$. The feasible set inequality reads $\sum_{i \in T} x_i + w_j x_j \leq |T| = |S| - 1$ with $w_j \leq \min\{|V| : V \subseteq T, \sum_{i \in V} a_i \geq a_j - r(T)\}$. Since $\sum_{i \in T} a_i + r(T) = \alpha$ and $\sum_{i \in S} a_i > \alpha$, this minimum is greater than or equal to one. Therefore, under the regularity condition imposed in [P80], the feasible set inequality is always a $(1, k)$ -configuration inequality. In case the coefficient happens to be one we get a minimal cover inequality, see, for instance [W75].

Theorem 1.2 states the validity of the feasible set inequality for $P_{T \cup \{j\}}$. To obtain a (strong) valid inequality for P we resort to *lifting*, see [P75]. Let $\pi_1, \dots, \pi_{n-|T|-1}$ be some permutation of the set $N \setminus (T \cup \{j\})$. For $k = 1, \dots, n - |T| - 1$ and $l = 1, \dots, u_{\pi_k}$ let

$$(3) \quad \begin{aligned} \gamma(k, l) = \max \quad & \sum_{i \in T \cup \{j\}} w_i x_i + \sum_{i \in \{\pi_1, \dots, \pi_{k-1}\}} w_i x_i \\ & \sum_{i \in T \cup \{j\}} A_i x_i + \sum_{i \in \{\pi_1, \dots, \pi_{k-1}\}} A_i x_i + A_{\pi_k} l \leq b \\ & 0 \leq x_i \leq u_i, x_i \in \mathbf{Z} \text{ for } i \in T \cup \{\pi_1, \dots, \pi_{k-1}\}. \end{aligned}$$

Let $\gamma = \sum_{i \in T} w_i u_i$, the lifting coefficients are

$$(4) \quad w_{\pi_k} := \min_{l=1, \dots, u_{\pi_k}} \frac{\gamma - \gamma(k, l)}{l}.$$

The following statement is immediate.

Theorem 1.3 *The (lifted) feasible set inequality $w^T x \leq \sum_{i \in T} w_i u_i$ is valid for P .*

Note that the right hand side of (2) coincides with (4) applied to variable j if we substitute in (3) the set $T \cup \{j\}$ by T . In other words, a lifted feasible set inequality associated with T and $\{j\}$, where the variables in $N \setminus (T \cup \{j\})$ are lifted according to the sequence $\pi_1, \dots, \pi_{n-|T|-1}$, coincides with the inequality associated with T , where j is lifted first, and the remaining variables $N \setminus (T \cup \{j\})$ are lifted in the same order $\pi_1, \dots, \pi_{n-|T|-1}$. Thus, instead of speaking of a feasible set inequality associated with T and $\{j\}$, we speak in the sequel of a feasible set inequality associated with T and view j as the variable that is lifted first.

Odd hole- and clique inequalities for the set packing polytope are examples of lifted feasible set inequalities. Consider the set packing polytope $P = \text{conv}\{x \in \{0, 1\}^n : Ax \leq \mathbf{1}\}$ for some 0/1 matrix $A \in \{0, 1\}^{m \times n}$. Let $G_A = (V, E)$ denote the associated column intersection graph whose nodes correspond to the columns of A and nodes i and j are adjacent if and only if the columns associated with i and j intersect in some row. Let $Q \subseteq V$ be a clique in G_A , then the clique inequality $\sum_{i \in Q} x_i \leq 1$ is valid for P . To see that this inequality is a lifted feasible set inequality, let $T = \{i\}$ for some $i \in Q$. The feasible set inequality $x_i \leq 1$ is valid for $P_{\{i\}}$. Lifting the remaining variables $k \in Q \setminus \{i\}$ by applying formula (4) yields $w_k = 1$, and the clique inequality follows.

2 Bounds on the Lifting Coefficients

For a feasible set inequality associated with T , the calculation of the lifting coefficients for the variables in $N \setminus T$ requires the solution of an integer program. In this section we study lower and upper bounds for these coefficients. It will turn out that these bounds are sometimes easier to compute. We assume throughout the section that $A \geq 0$ and $w_i \geq 0$ for $i \in T$.

Definition 2.1 Let $T \subseteq N$ be a feasible set and $w : T \mapsto \mathbf{R}_+^T$ a weighting of T . For $v \in \mathbf{R}^n$ we define the

Covering Number

$$\phi^\geq(v) := \min \left\{ \sum_{i \in T} w_i x_i : \sum_{i \in T} A_i x_i \geq v, 0 \leq x_i \leq u_i, x_i \in \mathbf{Z}, i \in T \right\},$$

\geq -Incomparability Number

$$\phi^\not\leq(v) := \min \left\{ \sum_{i \in T} w_i x_i : \sum_{i \in T} A_i x_i \not\leq v, 0 \leq x_i \leq u_i, x_i \in \mathbf{Z}, i \in T, \right. \\ \left. \wedge \exists j \in T, x_j < u_j : \sum_{i \in T} A_i x_i + A_j \geq v \right\},$$

\leq -Incomparability Number

$$\phi^\leq(v) := \min \left\{ \sum_{i \in T} w_i x_i : \sum_{i \in T} A_i x_i \leq v, 0 \leq x_i \leq u_i, x_i \in \mathbf{Z}, i \in T \right\},$$

where we set $\phi^\geq(v) := 0$ and $\phi^\leq(v) := 0$ for $v \leq 0$.

Consider a (lifted) feasible set inequality $w^T x \leq \sum_{i \in T} w_i u_i$ associated with T , where the variables in $N \setminus T$ are lifted in the sequence $\pi_1, \dots, \pi_{n-|T|}$. The following proposition gives upper bounds for the lifting coefficients derived from the covering number.

Proposition 2.2

- (a) $w_{\pi_1} = \min_{l=1, \dots, u_{\pi_1}} \frac{1}{l} \phi^\geq(A_{\pi_1} l - r(T))$.
- (b) $w_{\pi_k} \leq \min_{l=1, \dots, u_{\pi_k}} \frac{1}{l} \phi^\geq(A_{\pi_k} l - r(T))$, for $k = 2, \dots, n - |T|$.

To derive lower bounds on the lifting coefficients we need the following relations.

Lemma 2.3 For $v, w \in \mathbf{R}^n$ with $v, w \geq 0$ holds:

- (a) $\phi^\geq(v) \geq \phi^\geq(w)$ and $\phi^\not\leq(v) \geq \phi^\not\leq(w)$.
- (b) $\phi^\geq, \phi^\not\leq$, and ϕ^\leq are monotonically increasing, i.e., for $v \geq w$, $\phi^\geq(v) \geq \phi^\geq(w)$, $\phi^\not\leq(v) \geq \phi^\not\leq(w)$, and $\phi^\leq(v) \geq \phi^\leq(w)$.

- (c) $\phi^{\geq}(v+w) \geq \phi^{\geq}(v) + \phi^{\leq}(w)$.
- (d) $\phi^{\geq}(v+w) + \max\{w_i : i \in T\} \geq \phi^{\leq}(v) + \phi^{\geq}(w)$.
- (e) $\phi^{\leq}(v+w) + \max\{w_i : i \in T\} \geq \phi^{\leq}(v) + \phi^{\leq}(w)$.

With the help of Lemma 2.3 we are able to bound the lifting coefficients from below.

Theorem 2.4 For $k = 1, \dots, n - |T|$ we have

$$(5) \quad w_{\pi_k} \geq \min_{l=1, \dots, u_{\pi_k}} \frac{\phi^{\leq}(A \cdot \pi_k l - r(T))}{l} - \max\{w_i : i \in T\}.$$

Theorem 2.4 applies, in particular, if we set the coefficient of the first lifted variable w_{π_1} to the upper bound of Proposition 2.2 (a). The subsequent example shows that in this case the lower bounds given in 2.4 may be tight.

Example 2.5 Let $A = \begin{bmatrix} 1 & 1 & 6 & 6 & 8 & 6 & 1 & 3 \\ 4 & 6 & 1 & 1 & 3 & 9 & 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 14 \\ 12 \end{bmatrix}$. The set $T = \{1, 2, 3, 4\}$ is feasible for the 0/1 program $\max\{c^T x : x \in P\}$ with $P := \text{conv}\{x \in \{0, 1\}^8 : Ax \leq b\}$. For $w_i = 1, i \in T$, we obtain $\phi^{\geq} \binom{8}{3} = 3$. Moreover, $\phi^{\leq} \binom{6}{9} = 2$, because $\binom{6}{9} \geq A_i$ for $i \in T$. Accordingly we get $\phi^{\leq} \binom{1}{2} = 1 = \phi^{\leq} \binom{3}{1}$. The inequality $x_1 + x_2 + x_3 + x_4 + \phi^{\geq} \binom{8}{3} x_5 + \sum_{i=6}^8 (\phi^{\leq}(A_i - 1)x_i) \leq 4$ reads $x_1 + x_2 + x_3 + x_4 + 3x_5 + x_6 \leq 4$. It defines a facet of P .

The question remains, whether the values ϕ^{\geq}, ϕ^{\leq} and ϕ^{\leq} are easier to compute than the exact lifting coefficient. Indeed, they sometimes are. Suppose $w_i = 1$ for all $i \in T$ and consider the comparability digraph $G = (V, E)$ that is obtained by introducing a node for each column and arcs (i, j) if $A_i \geq A_j$ and $A_i \neq A_j$ or if $A_i = A_j$ and $i > j$, for $i, j \in \{1, \dots, n\}$ (where transitive arcs may be deleted). Let r denote the number of nodes with indegree zero, i. e., $\delta^-(i) = 0$. Then, ϕ^{\geq}, ϕ^{\leq} and ϕ^{\leq} can be computed in time $O(n^r + \alpha)$, where α is the time to construct the comparability digraph. For example, in case of one knapsack inequality the comparability digraph turns out to be a path, and thus ϕ^{\geq}, ϕ^{\leq} and ϕ^{\leq} can be computed in time $O(n + n \log n) = O(n \log n)$.

3 Connection to Chvátal-Gomory Cuts

So far we have been discussing feasible set inequalities for general integer programs. With the exception that with $u|_T$ also every vector $v \leq u|_T$ is valid for P we have not subsumed any assumptions on the matrix A . Thus a comparison to Chvátal-Gomory cutting planes that do not rely on any particular structure of A is natural. Recall that Chvátal-Gomory inequalities for the system $Ax \leq b, 0 \leq x \leq u, x \in \mathbf{Z}^n$ are cutting planes $d^T x \leq \delta$ such that $d_i = \lfloor \lambda^T A_i \rfloor$, $i = 1, \dots, n$, and $\delta = \lfloor \lambda^T b \rfloor$ for some $\lambda \in \mathbf{R}_+^m$.

Consider a (lifted) feasible set inequality $w^T x \leq \sum_{i \in T} w_i u_i$ associated with T , whose remaining variables $N \setminus T$ are lifted in the sequence $\pi_1, \dots, \pi_{n-|T|}$. This lifted feasible set inequality is compared to Chvátal-Gomory inequalities resulting from multipliers $\lambda \in \mathbf{R}_+^m$ that satisfy $\lfloor \lambda^T A_{\cdot i} \rfloor = w_i$ for $i \in T$.

Proposition 3.1

- (a) $\lfloor \lambda^T b \rfloor \geq \sum_{i \in T} u_i w_i$.
- (b) If $\lfloor \lambda^T b \rfloor = \sum_{i \in T} u_i w_i$, let j be the smallest index with $\lfloor \lambda^T A_{\cdot \pi_j} \rfloor \neq w_{\pi_j}$. Then, $\lfloor \lambda^T A_{\cdot \pi_j} \rfloor < w_{\pi_j}$.

As soon as the first two coefficients differ, for $k \in \{\pi_{j+1}, \dots, \pi_{n-|T|}\}$, no further statements on the relations of the coefficients are possible, in general.

Example 3.2 For $b \in \mathbf{N}$, let $P(b)$ be the convex hull of all 0/1 solutions that satisfy the knapsack inequality

$$2x_1 + 6x_2 + 8x_3 + 9x_4 + 9x_5 + 21x_6 + 4x_7 \leq b.$$

One Chvátal-Gomory cutting plane for $P(b)$ reads

$$x_1 + 3x_2 + 4x_3 + 4x_4 + 4x_5 + 10x_6 + 2x_7 \leq \lfloor b/2 \rfloor.$$

Let $b = 25$. The set $T := \{1, 2, 3, 4\}$ is a feasible set. Choosing coefficients 1, 3, 4, 4 for the items in T and lifting the items 5, 6, 7 in this order we obtain the lifted feasible set inequality that is valid for $P(25)$:

$$x_1 + 3x_2 + 4x_3 + 4x_4 + 4x_5 + 11x_6 + x_7 \leq 12.$$

The right hand side of the Chvátal-Gomory cutting plane and the lifted feasible set inequality coincide. With respect to the lifting order 5, 6, 7 the coefficient of item 6 is the first one in which the two inequalities differ. This coefficient is 11 in the feasible set inequality and 10 in the Chvátal-Gomory cutting plane. For item 7 the coefficient in the feasible set inequality is then smaller than the corresponding one in the Chvátal-Gomory cutting plane. For $b = 27$ we obtain the lifted feasible set inequality that is valid for $P(27)$:

$$x_1 + 3x_2 + 4x_3 + 4x_4 + 4x_5 + 9x_6 + x_7 \leq 12.$$

The right hand side of this inequality is by one smaller than the right hand side of the corresponding Chvátal-Gomory cutting plane. However, the coefficients of the items 6 and 7 are smaller than the corresponding coefficients of the Chvátal-Gomory cutting plane.

Under certain conditions a feasible set- and a Chvátal-Gomory cutting plane coincide.

Theorem 3.3 *Let $A \in \mathbf{N}^{m \times n}$ and T be a feasible set of the integer program $\max \{c^T x : Ax \leq b, 0 \leq x \leq \mathbf{1}, x \in \mathbf{Z}^n\}$. Let $\lambda \in \mathbf{R}_+^m$ such that $\lfloor \lambda^T b \rfloor = \sum_{i \in T} \lfloor \lambda^T A_i \rfloor$. If for all $j \in N \setminus T$, column vector A_j is a 0/1-combination of elements from $\{A_i : i \in T\}$, then*

$$\lfloor \lambda^T A_j \rfloor = \phi^{\geq}(A_j - r(T)).$$

By Proposition 2.2 the expression $\phi^{\geq}(A_j - r(T))$ is an upper bound on the exact coefficient of an item $j \in N \setminus T$ in any lifted feasible set inequality associated with the feasible set T and the weighting $\lfloor \lambda^T A_i \rfloor, i \in T$. On the other hand, the Chvátal-Gomory cutting plane $\sum_{j \in N} \lfloor \lambda^T A_j \rfloor x_j \leq \lfloor \lambda^T b \rfloor$ is valid for P . Therefore, this Chvátal-Gomory cutting plane must coincide with any lifted feasible set inequality $\sum_{i \in T} \lfloor \lambda^T A_i \rfloor x_i + \sum_{j \in N \setminus T} w_j x_j \leq \lfloor \lambda^T b \rfloor$ independent on the sequence in which the lifting coefficients w_j for the items in $N \setminus T$ are computed.

4 Consecutively Intersecting Knapsacks

So far we have been discussing a framework with which one can define and explain families of cutting planes for a general integer program. On the other hand, from a practical point of view the cutting plane phase of usual codes for integer programming relies in particular on valid inequalities for knapsack polyhedra. From both a theoretical and a practical point of view it would be desirable to understand under what conditions facets of single knapsack polyhedra define or do not define strong cutting planes of an integer program when several knapsack constraints intersect. This question is addressed in this section. In fact, here we study a special family of 0/1 programs that arises when $A \in \mathbf{N}^{m \times n}$ and $Ax \leq b$ defines a system of *consecutively intersecting knapsack constraints*. Throughout this section we assume $u = \mathbf{1}$, $A \geq 0$ and integral. For $i = 1, \dots, m$, let $N_i := \text{supp}(A_i)$ and $P_i := \text{conv} \{x \in \{0, 1\}^{N_i} : \sum_{j \in N_i} A_{ij} x_j \leq b_i\}$.

Definition 4.1 *A system of linear inequalities $Ax \leq b$ is called a system of consecutively intersecting knapsack constraints if $A \in \mathbf{N}^{m \times n}$ and $N_i \cap N_l = \emptyset$ for all $i, l \in \{1, \dots, m\}$, $|i - l| \geq 2$.*

A natural question when one starts investigating the intersection of several knapsack polyhedra is when this polyhedron inherits all the facets of the single knapsack polyhedra.

Proposition 4.2 *Let $A \in \mathbf{N}^{m \times n}$ and $Ax \leq b$ be a system of consecutively intersecting knapsack constraints. Let $i \in \{1, \dots, m\}$. If $N_i \cap N_l \cup \{k\}$ is not a cover for all $l \in \{i - 1, i + 1\}$ and $k \notin N_i$, then every facet-defining inequality of the knapsack polyhedron P_i defines a facet of P .*

The condition that, for every $k \notin N_i, l = i - 1, i + 1$, the set $(N_i \cap N_l) \cup \{k\}$ is not a cover, is essential for the correctness of the proposition as the following example shows.

Example 4.3 For $b \in \mathbf{N} \setminus \{0\}$ let A be the matrix $\begin{bmatrix} \frac{b}{3} + 1 & \frac{b}{3} + 1 & \frac{b}{3} + 1 & 0 \\ 0 & \frac{b}{3} + 1 & \frac{b}{3} + 1 & b \end{bmatrix}$ and consider $P = \text{conv}\{x \in \{0, 1\}^4 : Ax \leq b\}$. Then $N_1 \cap N_2 = \{2, 3\}$, and the set $\{2, 3, 4\}$ defines a cover.

The inequality $x_1 + x_2 + x_3 \leq 2$ defines a facet of the knapsack polyhedron

$$\text{conv}\{x \in \{0, 1\}^3 : (\frac{b}{3} + 1)x_1 + (\frac{b}{3} + 1)x_2 + (\frac{b}{3} + 1)x_3 \leq b\}.$$

On the other hand, the inequality $x_1 + x_2 + x_3 \leq 2$ is not facet-defining for P , since the face $F = \{x \in P : x_1 + x_2 + x_3 \leq 2\}$ is strictly contained in the face induced by the inequality $x_1 + x_2 + x_3 + x_4 \leq 2$.

In certain cases, a complete description of P may even be derived from the description of the single knapsack polyhedra.

Theorem 4.4 Let $m = 2$ and $A \in \mathbf{N}^{2 \times n}$. Let $Ax \leq b$ be a system of consecutively intersecting knapsack constraints such that every pair of items from $N_1 \cap N_2$ is a cover. For $i = 1, 2$ let $C^i x \leq \gamma^i$ be a system of inequalities that describes the single knapsack polyhedron P_i .

Then, P is described by the system of inequalities

$$\begin{aligned} \sum_{j \in N_1 \cap N_2} x_j &\leq 1 \\ C^i x &\leq \gamma^i \text{ for } i = 1, 2. \end{aligned}$$

The correctness of the theorem strongly relies on the fact that every pair of items from the intersection $N_1 \cap N_2$ is a cover. If this condition is not satisfied, then Example 4.3 demonstrates that the facet-defining inequalities of the two single knapsack polyhedra do not suffice in general to describe the polyhedron associated with the intersection of the two knapsack constraints. Geometrically, the fact that we intersect two knapsack constraints generates incomparabilities between the column vectors of the associated matrix. These incomparabilities give rise to cutting planes that do not define facets of one of the two single knapsack polyhedra that we intersect. In fact, incomparabilities between column vectors in a matrix make it possible to “melt” inequalities from different knapsack polyhedra. A basic situation to which the operation of melting applies is

Proposition 4.5 Let $m = 2$, $A \in \mathbf{N}^{2 \times n}$ and $Ax \leq b$ be a system of two consecutively intersecting knapsack constraints. Let $\sum_{i \in N_1 \setminus N_2} c_i x_i + \sum_{i \in N_1 \cap N_2} c_i x_i \leq \gamma$ be a valid inequality for P_1 , and let $\sum_{i \in N_2 \setminus N_1} c_i x_i + \sum_{i \in N_1 \cap N_2} c_i x_i \leq \gamma$ be a valid inequality for P_2 . Setting $\Theta := \sum_{i \in N_1 \setminus N_2} c_i$, the melted inequality

$$\sum_{i \in N_1 \setminus N_2} c_i x_i + \sum_{i \in N_2 \setminus N_1} (c_i - \Theta)^+ x_i + \sum_{i \in N_1 \cap N_2} c_i x_i \leq \gamma$$

is valid for P .

Proposition 4.5 can be extended to general upper bounds $u \in \mathbf{N}^n$. Often the inequality that results from melting valid inequalities from knapsack polyhedra as described in Proposition 4.5 does not define a facet of P . In such situations there is a good chance of strengthening the melted inequality by determining lifting coefficients for the items in $(N_1 \setminus N_2) \cup (N_2 \setminus N_1)$ with respect to a given order. However, the melting of two feasible set inequalities produces under certain conditions a facet defining inequality of P .

5 Computational Experience

In this section we report on computational experience with a separation algorithm for the feasible set inequalities. We have incorporated this algorithm in a general mixed integer programming solver, called **SIP**, and tested it on instances from the MIPLIB. Details of our separation algorithm such as how to determine a feasible set T , how to weight the variables in T , how to perform the lifting, and for which substructures of the underlying constraint matrix the separation algorithm should be called, will be given in a full version of the paper. Our test set includes all examples from the MIPLIB that contain at least one knapsack constraint, i.e., an inequality that contains at least one coefficient different from $0, \pm 1$. In addition, we restrict ourselves to inequalities whose number of non-zeros lies between 3 and 20. There are 18 instances that contain such inequalities: arki001, bell3a, bell5, fiber, flugpl, gen, gesa2, gesa3, gt2, harp2, lseu, misc03, p0033, p0201, p0548, p2756, qnet1, qnet1.o. We compared SIP with and without using feasible set inequalities. The time limit for our runs was 3600 CPU seconds.

Example	B&B Nodes	Cuts		Time		Gap %
		Others	FSs	FS	Total	
SIP without FS	948241	4435	0	0.0	16032.8	23.74
SIP with FS	817368	4363	169	2178.7	15921.6	20.45

Table 1: Comparison of SIP with and without feasible set inequalities ('FS').

Table 1 summarizes our results over all 18 problem instances. The last column gives the sum of the gaps (in percentage) between the lower and upper bounds. Table 1 shows that the time decreases slightly (5%) and the gap decreases by around 14% when adding feasible set inequalities. Based on these results we conclude that feasible set inequalities are a tool that helps solving mixed integer programs.

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