

Killing Pairs in Flat Space

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Abstract

In this paper the integrability conditions for Killing pairs in flat spaces are investigated and it is shown that only trivial Killing pairs exist.

KEY WORDS: Conservation laws, symmetries, equation of motion

1 Introduction

The concept of Killing pairs (KPs) considers two symmetric tensors $A_{a_1 a_2 \dots a_p} = A_{(a_1 a_2 \dots a_p)}$, $B_{b_1 b_2 \dots b_q} = B_{(b_1 b_2 \dots b_q)}$ which define a constant of geodesic motion with velocity u_m as

$$\left(\frac{A_{a_1 a_2 \dots a_p} u^{a_1} u^{a_2} \dots u^{a_p}}{B_{b_1 b_2 \dots b_q} u^{b_1} u^{b_2} \dots u^{b_q}} \right)_{;m} u^m = 0.$$

With geodesic motion being described through $0 = u^r_{;s} u^s$, the equivalent condition for \mathbf{A}, \mathbf{B} is

$$A_{(a_1 a_2 \dots a_p; m} B_{b_1 b_2 \dots b_q)} = A_{(a_1 a_2 \dots a_p} B_{b_1 b_2 \dots b_q; m)}.$$

A KP is called trivial either if \mathbf{A}, \mathbf{B} are both Killing tensors referred to below as trivial case TC1, or if \mathbf{A}, \mathbf{B} are of same rank and are proportional $\mathbf{A} = \text{const} \cdot \mathbf{B}$, referred to below as trivial case TC2. If (\mathbf{A}, \mathbf{B}) is a KP then as well is $(\mathbf{A} \cap \mathbf{C}, \mathbf{B} \cap \mathbf{C})$ where \mathbf{C} is any symmetric tensor and \cap stands for the completely symmetric tensor product.

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Integrability conditions for \mathbf{A}, \mathbf{B} both of rank one have been formulated by Collinson [1]. As shown there it follows from

$$A_{i,m}u^i u^m B_j u^j = A_i u^i B_{j,m} u^j u^m \quad (1)$$

and $A_i u^i$ and $B_j u^j$ being coprime that there is a vector P_m such that $A_{(i;k)} = A_{(i} P_{k)}$ and $B_{(i;k)} = B_{(i} P_{k)}$. Analogously there follows for higher rank tensors the existence of a vector \mathbf{P} such that $A_{(a_1 a_2 \dots a_p; m)} = A_{(a_1 a_2 \dots a_p} P_{m)}$ and $B_{(b_1 b_2 \dots b_q; m)} = B_{(b_1 b_2 \dots b_q} P_{m)}$.

The gauge freedom of multiplying \mathbf{A} and \mathbf{B} with a scalar function Ψ shifts \mathbf{P} by a gradient:

$$A'_{a_1 \dots a_p} = \Psi A_{a_1 \dots a_p}, \quad B'_{b_1 \dots b_q} = \Psi B_{b_1 \dots b_q}, \quad P'_m = P_m + \frac{\Psi_{,m}}{\Psi}. \quad (2)$$

The trivial case that \mathbf{A}, \mathbf{B} are Killing tensors (KTs) is equivalent to $P_m = 0$ and they are multiples of KT's iff $P_{[l;m]} = 0$. Both cases are referred to as TC1.

Collinson, Vaz and O'Donnell have given metrics with nontrivial KPs with \mathbf{A}, \mathbf{B} of rank 1 but unfortunately none of these describes vacuum or is known to have an energy momentum tensor of physical interest ([2] - [5]). So far an exhaustive search for KPs has not been carried out for any metric, even for flat space.

2 Flat space

The algorithm to formulate integrability conditions as given in [1] is complex and provides nonlinear conditions according to the nonlinear nature of the problem. The first step towards a better understanding of their structure is to solve them for flat space.

As the geodesic motion in flat space is known, all conservation laws have to depend functionally on the conservation laws linear in the momentum based on Killing vectors (KVs) of flat space. The situation is somehow comparable with geodesic motion in a 3-dimensional spherically symmetric $1/r$ -potential where the geodesic motion is explicitly known. Nevertheless, from this alone it is not obvious that there are first integrals *quadratic* in the momentum, (the Runge-Lenz vector,

the vector pointing from the center of mass to the perihelion), which are not just linear combinations of products of first integrals linear in the momentum.

In the following we want to show that Killing pairs in flat space have to be trivial, i.e. they have to belong to cases TC1 and TC2.

For the beginning we restrict ourselves to rank 1 tensors A_i, B_j . In the next three subsections we will treat the cases of 2, 3 and n dimensions.

2.1 Two dimensions

The system of equations to solve is

$$A_{i,j} + A_{j,i} = P_i A_j + P_j A_i \quad (3)$$

$$B_{i,j} + B_{j,i} = P_i B_j + P_j B_i, \quad (4)$$

with $i, j = 1, 2$ where we exclude the case TC2, i.e. $B_i = \varphi A_i$ for some constant φ . None of the components A_i, B_i can be identically zero. If, for example, $A_1 = 0$ then $A_2 \neq 0$ and due to (3), $P_i = (\log A_2)_{,i}$ would be a gradient \Rightarrow TC1. For the rest of the paper we will not apply the summation convention. With $A_i, B_i \neq 0$, (3),(4) give

$$P_i = A_{i,i}/A_i = B_{i,i}/B_i \quad (5)$$

and substituted back into (3),(4) the equivalent conditions for functions F, H defined as

$$F = A_1/A_2, \quad H = B_1/B_2, \quad (6)$$

are

$$F_{,1} = F F_{,2}, \quad (7)$$

$$H_{,1} = H H_{,2}. \quad (8)$$

From (5) follows $A_i = c_i B_i$ for some functions c_i with $c_{i,i} = 0$ and consequently

$$\left[\log \left(\frac{c_1}{c_2} \right) \right]_{,12} = \left[\log \left(\frac{F}{H} \right) \right]_{,12} = 0. \quad (9)$$

The overdetermined system (7) - (9) for the functions F, H was first solved with the computer algebra program CRACK [6]. An alternative way (giving the same

solution) is as follows. Substitute $F_{,2} = F_{,1}/F = (\log F)_{,1}$ from (7) and similarly $H_{,2}$ from (8) into (9) to obtain

$$F_{,22} - H_{,22} = 0. \quad (10)$$

$\partial_1(10)$ simplified with (8),(10) gives

$$0 = 3F_{,22} (F_{,2} - H_{,2}) + F_{,222} (F - H). \quad (11)$$

$\partial_2(11)$ simplified with (10) gives

$$0 = 4F_{,222} (F_{,2} - H_{,2}) + F_{,2222} (F - H). \quad (12)$$

As $F - H \neq 0$ (otherwise TC2) the determinant of the coefficient matrix of $(F_{,2} - H_{,2}, F - H)$ in (11),(12) must vanish, which gives

$$0 = 3F_{,22} F_{,2222} - 4(F_{,222})^2. \quad (13)$$

$\partial_1(13)$ simplified with (7), $\partial_2(13)$ and (13) gives

$$0 = F_{,22} F_{,222}. \quad (14)$$

In the case $F_{,22} \neq 0$ it follows that $F_{,222} = 0 = F_{,2} - H_{,2}$ which integrated and substituted into (7),(8) gives the contradiction $F_{,22} = 0$.

The remaining case $0 = F_{,22}$ integrated and substituted into (7) provides finally

$$F = \frac{\varepsilon x^2 + \alpha}{-\varepsilon x^1 + \beta}, \quad \varepsilon = 0, 1, \quad \alpha, \beta = \text{const} \quad (15)$$

and a similar expression for H .

For $A_i \neq 0 \neq A_j$, (5) gives the general formula

$$P_{[i,j]} = [\log(A_i/A_j)]_{,ij} \quad (16)$$

and with $F = A_1/A_2$ and (15) further $P_{[1,2]} = 0, \Rightarrow$ TC1. Thus we have proved the following lemma.

Lemma 1. *In two dimensions two vectors \mathbf{A}, \mathbf{B} satisfying eq.s (3),(4) satisfy one of the following two properties.*

1. There is a function Ψ such that $\mathbf{A}/\Psi, \mathbf{B}/\Psi$ are KVs with $P_i = \Psi_{,i}/\Psi$ and A_1/A_2 and B_1/B_2 having the shape (15) (resp. $A_2/A_1, B_2/B_1$ if $A_2 = 0$ or $B_2 = 0$). (TC1)
2. $\mathbf{A} = \varphi\mathbf{B}$ with $\varphi = \text{const.}$ (TC2)

The two-dimensional case has been solved independently by Laura Johnson (University of Hull) [7].

2.2 Three dimensions

Lemma 2. In three dimensions two vectors \mathbf{A}, \mathbf{B} satisfying eq.s (3), (4) belong to the trivial cases TC1 or TC2.

Proof.

In the following we have the notation convention that lower case greek letters with an index i denote functions of x^i and lower case greek letters without index denote constants.

Applying Lemma 1 to the index pairs (1,2), (2,3) and (3,1) we only have to deal with the case that for only two indices there is proportionality $B_i = \varphi A_i$ and for only two index pairs there is $P_{[i,j]} = 0$. We therefore assume

$$B_1 = \varphi_3 A_1, \quad B_2 = \varphi_3 A_2, \quad B_3 \neq \varphi_3 A_3, \quad (17)$$

$$P_{[1,2]} \neq 0, \quad P_{[2,3]} = 0, \quad P_{[3,1]} = 0, \quad (18)$$

for some function $\varphi_3 = \varphi_3(x^3)$. We distinguish the following cases.

Case 1. $A_1 = 0$.

Then (17) gives $B_1 = 0 \Rightarrow$ two-dimensional case.

Case 2. $A_1 \neq 0 \neq A_2$.

Case 2.1. $B_1 = 0$.

$\Rightarrow \varphi_3 = 0 \Rightarrow B_2 = 0$. Multiplication of \mathbf{A}, \mathbf{B} with $1/B_3$ gives $\mathbf{B} = (0, 0, 1)$ and $P_i = 0 \Rightarrow$ TC1.

Case 2.2. $B_1 \neq 0 \neq B_2$.

It follows that

$$\varphi_{3,3} \neq 0$$

because if $\varphi_3 = \varphi = \text{const}$, then the Killing pair (\mathbf{A}, \mathbf{B}) would be equivalent to the pair $(\mathbf{A}, \mathbf{B} - \varphi\mathbf{A})$ with the new $B_1, B_2 = 0 \Rightarrow$ case 2.1. In the following we will assume $B_3 \neq 0$ w.l.o.g. as otherwise $A_3 = B_3 = 0$ would be the two-dimensional case.

After multiplication of \mathbf{A}, \mathbf{B} with $1/B_3$ we get $B_3 = 1 \Rightarrow P_3 = 0$. With $B_3 = 1$ and Lemma 1 applied to the index pairs (1,3) and (2,3) we get

$$B_1 = \frac{\varepsilon x^3 + \alpha_2}{-\varepsilon x^1 + \beta_2}, \quad (19)$$

$$B_2 = \frac{\rho x^3 + \gamma_1}{-\rho x^2 + \delta_1}, \quad (20)$$

with $\varepsilon, \rho = 0, 1$ and appropriate functions $\alpha_2, \beta_2, \gamma_1, \delta_1$.

Case 2.2.1. $\varepsilon = \rho = 0$.

Due to $B_{1,1} = P_1 = 0$, $B_{2,2} = P_2 = 0$ and $P_3 = 0$ we have case TC1.

Case 2.2.2. $\varepsilon = 1, \rho = 0$.

In this case we have $B_{2,3} = 0, B_{2,2} = 0 \Rightarrow P_2 = 0$, (19) $\Rightarrow P_1 = 1/(-x^1 + \beta_2)$ and with (4) for $i = 1, j = 2$ further $B_{1,23} = 0$. Together with (19) we get $\beta_{2,2} = 0 \Rightarrow P_{1,2} = 0 \Rightarrow P_{[1,2]} = 0 \Rightarrow$ TC1.

Case 2.2.3. $\varepsilon = \rho = 1$

Case 2.2.3.1. $A_3 = 0$.

With (3) and $j = 3$ follows further $A_{i,3} = 0$. Equation (17) gives $[\log(\varphi_3 A_1)]_{,2,3} = [\log B_1]_{,2,3} = 0 \Rightarrow \alpha_{2,2} = 0$ and similarly $\gamma_{1,1} = 0$. From this follows

$$A_1 = \frac{1}{\beta_2 - x^1}, \quad A_2 = \frac{1}{\delta_1 - x^2}.$$

Equation (4) for $i, j = 1, 2$ reads

$$\beta_{2,2}(\delta_1 - x^2)^2 + \delta_{1,1}(\beta_2 - x^1)^2 + 2(\delta_1 - x^2)(\beta_2 - x^1) = 0 \quad (21)$$

which is solved in the appendix. The only solution $\beta_2 = \kappa x^2 - \sigma, \delta_1 = (x^1 - \sigma)/\kappa$ gives $P_{[1,2]} = 0 \Rightarrow$ TC1.

Case 2.2.3.2. $A_i \neq 0 \neq B_i, i = 1, 2, 3$.

With $B_3 = 1$ and (4) for $i, j = 3$ follows $P_3 = 0$ and with (3) further $A_{3,3} = 0$.

Applying Lemma 1 to the index pair (1,3) we get

$$A_1 = A_3 \frac{\nu x^3 + \zeta_2}{-\nu x^1 + \xi_2} \quad (22)$$

$$A_2 = A_3 \frac{\mu x^3 + \chi_1}{-\mu x^2 + \eta_1}. \quad (23)$$

As B_1, B_2 are linear in x^3 and $B_1 = \varphi_3(x^3)A_1, B_2 = \varphi_3(x^3)A_2$ we can only have either $\nu = \mu = 0$ or $\nu = \mu = 1$.

Case 2.2.3.2.1. $\nu = \mu = 0$.

For $\nu = \mu = 0$ we have $[\log A_1]_{,1} = [\log A_3]_{,1}, [\log A_2]_{,2} = [\log A_3]_{,2}$ and therefore $P_{[1,2]} = (\log A_1 - \log A_2)_{,12} = 0$ i.e. TC1.

Case 2.2.3.2.2. $\nu \neq 0 \neq \mu$.

In this case $B_1 = \varphi_3(x^3)A_1, B_2 = \varphi_3(x^3)A_2$ demand $\alpha_2 = \gamma_1 = \text{const}$ and $\zeta_2 = \chi_1 = \text{const}$. Condition (6) for the index pair (1,2) becomes eq. (21) solved in the appendix. The only solution $\beta_2 = \kappa x^2 + \sigma, \delta_1 = (x^1 - \sigma)/\kappa$ gives $P_{[1,2]} = 0 \Rightarrow$ TC1.

This completes the proof of Lemma 2.

2.3 N dimensions

To extend lemmas 1,2 to n dimensions we assume that there is no constant φ such that $\mathbf{B} = \varphi \mathbf{A}$ in order to avoid TC2. We will show that for any pair of indices p, q we have $P_{[p,q]} = 0$ and therefore TC1.

Case 1. $A_p B_q \neq A_q B_p$

Due to lemma 1 we have $P_{[p,q]} = 0$.

Case 2. $A_p B_q = A_q B_p$

Case 2.1. $A_p B_q = A_q B_p = 0 \Rightarrow$ w.l.o.g. $A_p = 0$

If $B_q = 0$ then $n - 1$ - dimensional problem else $A_q = 0 \Rightarrow P_{[p,q]} = 0$.

Case 2.2. $A_p B_q = A_q B_p \neq 0$

In that case there must exist a third index r such that either $B_r = 0$ or not all three ratios $A_p/B_p, A_q/B_q, A_r/B_r$ are equal and constant. In both cases lemma 2 gives $P_{[p,q]} = 0$.

3 Appendix

The condition (21) which is to be solved for the two functions $\delta_1 = \delta_1(x^1), \beta_2 = \beta_2(x^2)$ in this appendix is rewritten as

$$0 = [\beta_2'(x^2)^2 - 2x^2\beta_2] + [\delta_1'(x^1)^2 - 2x\delta_1] + \beta_2(\delta_1 - 2x^1\delta_1') + \delta_1(\beta_2 - 2x^2\beta_2') + \beta_2'\delta_1^2 + \delta_1'\beta_2^2 + 2x^1x^2 \quad (24)$$

where ' denotes differentiation. As the case $\beta_2' = 0$ leads to a contradiction after substitution in (24), we can assume w.l.o.g. $\beta_2' \neq 0 \neq \delta_1'$. From $\partial_1\partial_2 1/(2\beta_2'\delta_1')$ $\partial_1\partial_2$ (24) we get the condition

$$0 = (\log \beta_2')'\delta_1' + (\log \delta_1')'\beta_2' + \left(\frac{1}{\beta_2'}\right)' \left(\frac{1}{\delta_1'}\right)' \quad (25)$$

A further division and differentiation provides

$$0 = \left[\frac{1}{\beta_2'} \left(\frac{1}{\beta_2'}\right)'\right]' \left[\frac{1}{\delta_1'} \left(\frac{1}{\delta_1'}\right)'\right]'$$

which w.l.o.g. gives

$$\theta(\beta_2)^2 + \vartheta\beta_2 + y + \omega = 0.$$

After substitution in (25) we get $\theta = 0$ and from (24) finally

$$\beta_2 = \kappa x^2 + \sigma, \quad \delta_1 = (x^1 - \sigma)/\kappa. \quad (26)$$

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