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Discrete Relaxations of Combinatorial Programs

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Abstract. This paper investigates a technique of building up *discrete relaxations* of combinatorial optimization problems. To establish such a relaxation we introduce a transformation technique —*aggregation*— that allows one to relax an integer program by means of another integer program. We show that knapsack and set packing relaxations give rise to combinatorial cutting planes in a simple and straightforward way. The constructions are algorithmic.

Keywords. Polyhedral Combinatorics, Integer Programming, Cutting Planes

Mathematics Subject Classification (MSC 1991). 90C10

1 Introduction

Integer programming is one of the most successful approaches to $\mathcal{NP}$-hard combinatorial optimization problems. Important concepts in this area are (i) *transformations* to transfer knowledge about one problem to another problem as well as (ii) *relaxations* that are algorithmically tractable. Typical relaxations of integer programs are linear or semidefinite programs. We study in this paper what we call a *discrete relaxation* of one integer program by means of another integer program.

Consider a combinatorial optimization problem in its integer programming formulation

$$\text{(IP)} \quad \max\ w^T x, \quad Ax \le b,\ x \in \mathbb{Z}^n,$$

Here, $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$, and $w \in \mathbb{Z}^n$ are an integral matrix and integer vectors, respectively. The associated linear and integer polyhedra are

$$P_{\mathrm{LP}}(A, b) := \{x \in \mathbb{R}^n \mid Ax \le b\}$$
$$P_{\mathrm{IP}}(A, b) := \operatorname{conv}\{x \in \mathbb{Z}^n \mid Ax \le b\}.$$

Where the meaning is clear, we write P_{LP} for $P_{\mathrm{LP}}(A, b)$ and P_{IP} for $P_{\mathrm{IP}}(A, b)$.

We call our method to construct discrete relaxations *aggregation*. Aggregation is a generalization of projection to arbitrary *affine functions*

$$\phi : \mathbb{R}^n \to \mathbb{R}^{n'},\ x \mapsto \Phi x - \phi^0,$$

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given by a rational matrix $\Phi \in \mathbb{Q}^{n' \times n}$ and vector $\phi^0 \in \mathbb{Q}^{n'}$; note that the image space can have a *higher* dimension than the preimage. We call such functions *aggregation schemes* or simply *schemes*. A scheme is *integer* if it maps integer points to integer points, i.e., in formulas, if $\phi(\mathbb{Z}^n) \subseteq \mathbb{Z}^{n'}$) or, equivalently, if both Φ and ϕ are integer. The image $\phi(P)$ of a polyhedron P under the scheme ϕ is called the *ϕ-aggregate* or, if there is no danger of confusion, simply the *aggregate* of P. Clearly, we are interested in suitable aggregates $\phi(P_{\text{IP}})$ of the polytope P_{IP} associated with the integer program (IP).

Our motivation for studying aggregations is that they give rise to *valid inequalities* for P_{IP}. Namely, if ${a'}^{\text{T}} x' \leq \alpha'$ is valid for the aggregate $\phi(P_{\text{IP}})$, the *expansion*

$$ {a'}^{\text{T}} \phi(x) \leq \alpha' \iff {a'}^{\text{T}} \Phi x \leq \alpha' + {a'}^{\text{T}} \phi^0 $$

of this inequality is valid for the original polyhedron P_{IP}.

The facial structure of an aggregate is, of course, in general as complicated as that of the original polyhedron. But we will see in the examples of the following sections that one can often find a *relaxation*

$$ P' \supseteq \phi(P_{\text{IP}}) $$

of the aggregate $\phi(P_{\text{IP}})$ that is of a well studied type. More precisely, we stipulate that P' is the polytope associated with some combinatorial integer program IP$'$ in the image space of the aggregation, i.e., $P' = P'_{\text{IP}}$. In this case, one can resort to known inequalities for this relaxation P' to get an approximate description of the aggregate $\phi(P)$ and, via expansion, a description of a polyhedral relaxation $\phi^{-1}(P')$ of the original polyhedron P, see Fig. 1 for an illustration. Because of this relation, we call the integer program (IP$'$) a *discrete relaxation* of the original integer program (IP).

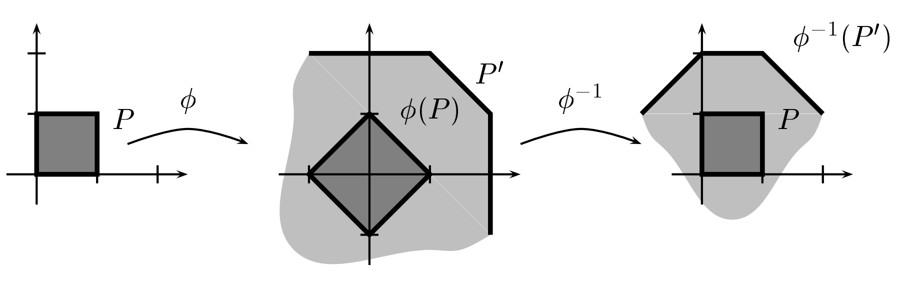

Figure 1: Constructing a Combinatorial Relaxation.

The crucial points in this procedure are the *choice* of the aggregation scheme and the *construction* of a suitable discrete relaxation. The forthcoming examples use the following simple observation. Starting with *integer* polyhedra $P = P_{\text{IP}}$ and restricting attention to likewise *integer* schemes, the resulting aggregates are integer as well (vertices map to vertices). The identification of $\phi(P_{\text{IP}})$ as a subset of some well known polyhedron P' will resort to problem specific combinatorial arguments.

Once the discrete relaxation IP$'$ is found, *separation* routines for the associated polyhedron P' carry over to the original polyhedron P_{IP} via expansion. Namely, given some point x to be tested for membership in P_{IP}, we simply (i) compute $\phi(x)$, (ii) solve the separation problem

for $\phi(x)$ and P', and, if a separating hyperplane ${a'}^{\mathrm{T}}x' \leq \alpha'$ has been found, (iii) expand it. If all of these three steps are polynomial, this yields a polynomial time separation algorithm for a class of valid inequalities for P_{IP}.
Aggregation has a good tradition in polyhedral combinatorics. *Projection techniques* have been used by Balas & Pulleyblank (1989) [2], Pulleyblank & Shepherd (1993) [14], and others to investigate matching, stable set, and other combinatorial polyhedra. Chopra & Rao (1994a;b) [5; 6] use projection to compare the strengths of directed and undirected formulations of the Steiner tree problem. Padberg & Sung (1991) [13] is the reference that is most closely related to our work: They use general aggregations to analyze the strengths of IP formulations for the travelling salesman problem.
The following sections present *applications* of discrete relaxations to a number of classical combinatorial optimization problems. We consider two types of discrete relaxations: *Set packing relaxations* and *knapsack relaxations.* These relaxations will be used to construct cutting planes. It is not our intent to give a detailed analysis of strength of these cuts here. We simply present a list of, we hope, elegant examples to advertise discrete relaxations as a novel cutting plane technique. Our constructions are simple and always algorithmic.

2 Set Packing Relaxations

This section gives examples of discrete relaxations in the form of a set packing problem. The discussion extends our earlier paper Borndörfer & Weismantel (1997) [3] that used set packing relaxations to derive polynomial time separation routines for a number of well known inequalities from the literature (and generalizations thereof), such as Möbius ladder and certain fence inequalities for the acyclic subdigraph polytope, 2-chorded cycle inequalities for the clique partitioning polytope, and several types of inequalities for the set packing polytope itself. We will use these techniques here to derive new inequalities. Our aim is to give inspiring examples how one can transfer inequalities and separation routines from the set packing to other combinatorial optimization problems.
We recall some set packing notation and results. Given a graph G with node weights $w \in \mathbb{R}_+^V$, the *set packing* or *stable set problem* (SSP) is the following integer program:

$$\text{(SSP)} \quad \max\ w^T x, \quad Ax \leq \mathbf{1},\ x \in \{0,1\}^V.$$

Here, $A = A(G) \in \{0,1\}^{E \times V}$ is the edge-node incidence matrix of G and $\mathbf{1}$ a vector of all ones of compatible dimension. We denote the associated *stable set polytope* by $P_{\mathrm{SSP}}(G)$ or P_{SSP}. For technical reasons, we will actually not work with P_{SSP} itself, but with its *anti-dominant*

$$\check{P}_{\mathrm{SSP}} := P_{\mathrm{SSP}} - \mathbb{R}_+^V = \big\{x \in \mathbb{R}^V : \exists y \in P_{\mathrm{SSP}} : x \leq y\big\}.$$

It is easy to see that the valid inequalities for $\check{P}_{\mathrm{SSP}}$ of the form $a^{\mathrm{T}}x \leq \alpha$ are exactly the valid inequalities for P_{SSP} *with non-negative coefficients.* We will make use of two well known classes of inequalities for the stable set polytope: *Clique* and *odd cycle inequalities*, see Padberg (1973) [11]. The clique inequalities are subsumed by the larger class of *orthonormal representation constraints*, see Grötschel, Lovász & Schrijver (1988) [10]. Cycle inequalities and orthonormal representation constraints can be separated in polynomial time, see again Grötschel, Lovász & Schrijver (1988) [10].

The subsequent subsections resort to the following method to construct *set packing relaxations* for a variety of combinatorial optimization problems (IP). Starting point is an integer scheme $\phi : \mathbb{R}^n \mapsto \mathbb{R}^{n'}$ that is bounded from above by one on the polyhedron P_{IP} of interest, i.e.,

$$\phi(x) \leq \mathbf{1} \qquad \forall x \in P_{\text{IP}}.$$

Such a scheme gives rise to a canonical set packing relaxation. The relaxation involves a *conflict graph* $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$. $\mathfrak{G}$ has a node for every coordinate in the scheme's image, i.e., $\mathfrak{V} = \{1, \ldots, n'\}$. We draw an edge $\mathfrak{uv}$ between two nodes if ϕ cannot attain its maximum value of one in both components simultaneously:

$$\mathfrak{uv} \in \mathfrak{E} :\Longleftrightarrow \phi_{\mathfrak{u}}(x) + \phi_{\mathfrak{v}}(x) \leq 1 \qquad \forall x \in P_{\text{IP}}.$$

In this case, we say that $\mathfrak{u}$ and $\mathfrak{v}$ are *in conflict.* By construction, we have

2.1 Lemma $\breve{P}_{\text{SSP}}(\mathfrak{G}) \supseteq \phi(P_{\text{IP}})$.

Lemma 2.1 states that the set packing problem associated with the conflict graph $\mathfrak{G}$ is a set packing relaxation of (IP). Note that it is not possible to replace $\breve{P}_{\text{SSP}}(\mathfrak{G})$ with $P_{\text{SSP}}(\mathfrak{G})$, because the scheme ϕ can attain negative values, see again Fig. 1.

Such a set packing relaxation gives rise to expanded cycle inequalities and expanded orthonormal representation constraints that can serve as cutting planes for P_{IP}. These are, however, not always automatically polynomial time separable. The set packing relaxations that come up in the applications that we have in mind often involve algorithmically intractable conflict graphs of exponential size. We cannot expect to resolve this difficulty in general. But we will see in the forthcoming examples that it is often possible to set up a significant relaxation

$$P'' \supseteq \breve{P}_{\text{SSP}}(\mathfrak{G})$$

of $\breve{P}_{\text{SSP}}(\mathfrak{G})$ that is still exponential, but has a special structure that makes it algorithmically tractable. To get a strong relaxation of set packing type, we construct this further relaxation again as a set packing relaxation associated with a large, but simply structured *subgraph* of the conflict graph $\mathfrak{G}$.

We resort to *equivalence relations* to extract an exponential conflict graph of simple structure from $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$. Namely, an equivalence relation $\sim$ on the conflict nodes $\mathfrak{V}$ gives rise to a subgraph $\widetilde{\mathfrak{G}} = (\mathfrak{V}, \widetilde{\mathfrak{E}})$ by setting

$$\mathfrak{uv} \in \widetilde{\mathfrak{E}} :\Longleftrightarrow \mathfrak{u}'\mathfrak{v}' \in \mathfrak{E} \qquad \forall \mathfrak{u}' \in [\mathfrak{u}] \neq [\mathfrak{v}] \ni \mathfrak{v}'.$$

Here, $[\mathfrak{u}]$ denotes the equivalence class that contains $\mathfrak{u}$. The maxima are taken componentwise. The equivalence relation extracts $\widetilde{\mathfrak{G}}$ from $\mathfrak{G}$ by keeping only those edges that run between two different equivalence classes that are completely in conflict. Note that, in particular, edges inside equivalence classes are deleted. Fig. 2 illustrates this construction. Our applications involve exponential conflict graphs based on cuts, cycles, and paths; typical equivalence classes are sets of such structures with properties that make them interchangeable for the purpose of constructing certain types of inequalities. To resort to $\widetilde{\mathfrak{G}}$ is to focus on class wide conflicts, ignoring possible additional conflict edges.

Our motivation for constructing a weakened set packing relaxation in this way is that this makes *separation* easier. The main property is that *most violated* cycle, clique, and orthonormal representation constraints have a very restricted support. Namely, denote for each

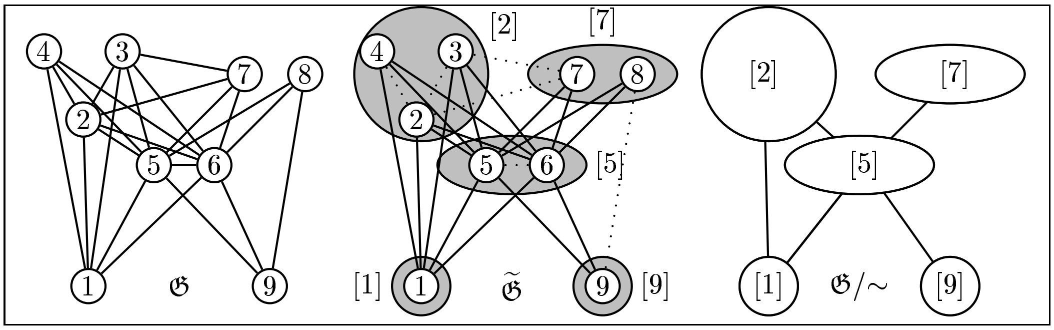

Figure 2: Constructing a Class Conflict Set Packing Relaxation.

$x' \in \mathbb{R}^{n'}$ and every equivalence class $[\mathfrak{v}]$ by

$$[\mathfrak{v}]_{x'} := \text{argmax} \;\; x'_{[\mathfrak{v}]}$$

a *representative* from $[\mathfrak{v}]$ of maximum x'-value, breaking ties, say, by index, and collect these nodes in the set

$$\mathfrak{V}_{x'} := \{[\mathfrak{v}]_{x'} : \mathfrak{v} \in \mathfrak{V}\}.$$

We claim that not every, but *a* most violated cycle, clique, or orthonormal representation constraint for $\breve{P}_{\text{SSP}}(\widetilde{\mathfrak{G}})$ must have its support contained in $\mathfrak{V}_{x'}$.

2.2 Lemma *Let $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ be a graph and $\sim$ an equivalence relation on $\mathfrak{V}$. Let $x' \in \mathbb{R}^{\mathfrak{V}}$ be a vector and $\mathfrak{V}_{x'}$ a set of equivalence class representatives of maximum x'-value. Let $\widetilde{\mathfrak{G}} = (\mathfrak{V}, \widetilde{\mathfrak{E}})$ be the subgraph of $\mathfrak{G}$ associated with $\sim$ and $\breve{P}_{\text{SSP}}(\widetilde{\mathfrak{G}})$ the corresponding set packing polyhedron.*

(i) There is a clique inequality for $\breve{P}_{\text{SSP}}(\widetilde{\mathfrak{G}})$ most violated by x' that has support only in $\mathfrak{V}_{x'}$.

(ii) There is a cycle inequality for $\breve{P}_{\text{SSP}}(\widetilde{\mathfrak{G}})$ most violated by x' that has support only in $\mathfrak{V}_{x'}$.

(iii) There is an orthonormal representation constraint for $\breve{P}_{\text{SSP}}(\widetilde{\mathfrak{G}})$ most violated by x' that has support only in $\mathfrak{V}_{x'}$.

Proof. (i) Consider a clique $\mathfrak{Q}$ in $\widetilde{\mathfrak{G}}$ such that the associated inequality $x'(\mathfrak{Q}) > 1$ has a maximum left-hand side; note that we have $>$ because the inequality was, by assumption, violated. Note also that $\mathfrak{Q}$ contains at most one node from every class $[\mathfrak{v}]$, because the classes are stable sets. Suppose the claim does not hold and there is a clique node $\mathfrak{v} \in \mathfrak{Q}$ different from $[\mathfrak{v}]_{x'}$. Exchanging $\mathfrak{v}$ and $[\mathfrak{v}]_{x'}$, however, produces a clique in $\widetilde{\mathfrak{G}}$ whose associated inequality is at least as violated as the original one, a contradiction.
(ii) The cycle inequalities are similar, but need one additional thought. Namely, it may happen that a most violated cycle inequality $x'(\mathfrak{C}) > \lfloor |\mathfrak{C}|/2 \rfloor$ contains two (or more) nodes from some class $[\mathfrak{v}]$. It is not hard to show, however, that such an inequality can be broken up into a sum of two cycle inequalities of smaller support, one even and one odd, where the odd component is at least as violated as the original inequality.

(iii) Consider an orthonormal representation $u_\mathfrak{v} \in \mathbb{R}^k$, $\mathfrak{v} \in \mathfrak{V}$, and $c \in \mathbb{R}^k$ of $\widetilde{\mathfrak{G}}$, i.e., $|u_\mathfrak{v}| = 1$, $u_\mathfrak{v}^{\mathrm{T}} u_\mathfrak{u} = 0$ for $\mathfrak{vu} \notin \widetilde{\mathfrak{E}}$, and $|c| = 1$, and an associated most violated constraint $\sum_{\mathfrak{v}\in\mathfrak{V}} (c^{\mathrm{T}} u_\mathfrak{v})^2 x'_\mathfrak{v} > 1$. Suppose there is a node $\mathfrak{v}$ in the support of this inequality different from $\mathfrak{u} := [\mathfrak{v}]_{x'}$. But then one can construct an orthonormal representation constraint with strictly smaller support that is at least as violated as the original one. The reader can verify that $u'_\mathfrak{w} := u_\mathfrak{w}$ for $\mathfrak{w} \neq \mathfrak{v}, \mathfrak{u}$, $u'_\mathfrak{v} := 0$, $u'_\mathfrak{u} := (c^{\mathrm{T}} u_\mathfrak{u} \cdot u_\mathfrak{u} + c^{\mathrm{T}} u_\mathfrak{v} \cdot u_\mathfrak{v}) / \sqrt{(c^{\mathrm{T}} u_\mathfrak{u})^2 + (c^{\mathrm{T}} u_\mathfrak{v})^2}$, $c' = c$ is the right choice. $\square$ Lemma 2.2 implies that the separation problem for cycle, clique, and orthonormal representation constraints for $\breve{P}_{\mathrm{SSP}}(\widetilde{\mathfrak{G}})$ can be solved on the graph $\widetilde{\mathfrak{G}}[\mathfrak{V}_{x'}]$ that is induced by the equivalence class representatives of maximum x'-value. The structure of this graph is invariant under x', only the x'-values of the nodes change. More precisely, $\widetilde{\mathfrak{G}}[\mathfrak{V}_{x'}]$ is isomorphic to the following *(equivalence) class conflict graph* $\mathfrak{G}/\!\sim\, = (\mathfrak{V}/\!\sim, \mathfrak{E}/\!\sim)$. $\mathfrak{G}/\!\sim$ has a node for every equivalence class, i.e., $\mathfrak{V}/\!\sim\, := \{[\mathfrak{v}] : \mathfrak{v} \in \mathfrak{V}\}$, and an edge for any two conflicting classes, i.e.,

$$[\mathfrak{u}][\mathfrak{v}] \in \mathfrak{E}/\!\sim \;:\Longleftrightarrow\; \mathfrak{u}'\mathfrak{v}' \in \widetilde{\mathfrak{E}} \qquad \forall \mathfrak{u}' \in [\mathfrak{u}] \neq [\mathfrak{v}] \ni \mathfrak{v}'.$$

We can thus resort to the class conflict graph $\mathfrak{G}/\!\sim$ to separate cycle, clique, and orthonormal representation constraints for $\breve{P}_{\mathrm{SSP}}(\widetilde{\mathfrak{G}})$. A final expansion yields what we call *inequalities from cycles, cliques, and orthonormal representations of equivalence classes* that can serve as cutting planes for P_{IP}.
We remark that Lemma 2.2 carries over to the facets of $\breve{P}_{\mathrm{SSP}}(\widetilde{\mathfrak{G}})$ in general. In fact, $\widetilde{\mathfrak{G}}$ arises from $\mathfrak{G}/\!\sim$ by a *substitution* of a stable set of size $|[\mathfrak{v}]|$ for every node $[\mathfrak{v}]$. Chvátal (1975)'s [7] polyhedral results on substitution imply that all facets of $\breve{P}_{\mathrm{SSP}}(\widetilde{\mathfrak{G}})$ can be obtained from facets of $\breve{P}_{\mathrm{SSP}}(\mathfrak{G}/\!\sim)$ by, roughly speaking, choosing some representative for any class and copying the associated coefficients, the left-hand side, and setting everything else to zero.
A convenient extension of the notion of class conflicts is to consider general sets instead of equivalence classes, i.e., to base the construction of $\widetilde{\mathfrak{G}}$ on *families* $\mathfrak{V}_i$ from a general *set system* $\mathcal{V} = \{\mathfrak{V}_1, \ldots, \mathfrak{V}_r\}$ instead of equivalence classes. It is not hard to see (though notationally awkward) that one can reduce the former case to the latter by appropriate modifications of the underlying scheme, leaving out coordinates outside $\bigcup \mathfrak{V}_i$ and duplicating others that appear in more than one family. Doing so results in a *family conflict graph* that we denote by $\mathfrak{G}/\mathcal{V}$ and its associated *inequalities from cycles, cliques, and orthonormal representations of families.* These are the inequalities that we are going to use in our examples.
We summarize the procedure to separate from P_{IP} some point x.

(i) Set up the family conflict graph $\mathfrak{G}/\mathcal{V}$.

(ii) For each family $\mathfrak{W} \in \mathcal{V}$, solve the *maximum representative problem*

$$\max\ \phi_\mathfrak{v}(x), \quad \mathfrak{v} \in \mathfrak{W}$$

to determine the set of representatives $\mathfrak{V}_{\phi(x)}$ of maximum $\phi(x)$-value.

(iii) Separate $\phi_{\mathfrak{V}_{\phi(x)}}$ from $\breve{P}_{\mathrm{SSP}}(\mathfrak{G}/\mathcal{V})$.

(iv) If a violated inequality $a''^{\mathrm{T}} x'' > \alpha''$ has been found, expand it to obtain

$$a''^{\mathrm{T}} \Phi_{\mathfrak{V}_{\phi(x)}} . x > \alpha'' + a''^{\mathrm{T}} \phi_{0\mathfrak{V}_{\phi(x)}}$$

as a cutting plane.

Whenever the family conflict graph can be constructed in polynomial time (and has, in particular, polynomial size), and the max representative problems are solvable in polynomial time, this procedure yields classes of polynomial time separable expansions of cycle, clique, and orthonormal representation constraints from set families that can serve as cutting planes for P_{IP}.
The remainder of this section discusses applications of this construction to the set covering problem, the Steiner tree problem, and the asymmetric travelling salesman problem.

2.1 The Set Covering Problem

The *set covering problem* (SCP) is the integer program

$$\text{(SCP)} \quad \min\ w^T x, \quad Ax \geq \mathbf{1},\ x \in \mathbb{Z}_+^n,$$

where $A \in \{0,1\}^{m \times n}$ and $w \in \mathbb{Z}_+^n$. The associated polyhedron is P_{SCP}.
The set packing relaxation that we suggest is based on the aggregation scheme $\phi : \mathbb{R}^n \to \mathbb{R}^{\mathfrak{V}}$ defined as

$$\phi_J(x) := 1 - \sum_{j \in J} x_j \qquad \forall J \subseteq \{1, \ldots, n\}.$$

Here, we take $\mathfrak{V} := 2^{\{1,\ldots,n\}}$ as the set of all subsets of column indices of A.
This scheme induces an exponential conflict graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ that records pairwise conflicts of column sets of the matrix A. Namely, there is an edge between two sets I and J of columns when their union covers a row of A, or, equivalently, some variable in $I \cup J$ has to be set to one:

$$IJ \in \mathfrak{E} \iff \exists A_{i\cdot} \supseteq I \cup J.$$

2.3 Lemma $\phi(P_{\mathrm{SCP}}) \subseteq \breve{P}_{\mathrm{SSP}}(\mathfrak{G})$.

This set packing relaxation has been considered by Sekiguchi (1983) [15] in a special case. He studies zero-one matrices A with the property that the column indices can be partitioned

$$\dot{\bigcup_{\mathfrak{v} \in \mathfrak{W}}} \mathfrak{v} = \{1, \ldots, n\}$$

into nonempty column sets $\mathfrak{v}$ such that (the support of) each row $A_{r\cdot}$ is the union of exactly two such column sets, i.e., $\forall A_{r\cdot} : \exists \mathfrak{u}, \mathfrak{v} \in \mathfrak{W} : \mathfrak{u} \neq \mathfrak{v} : \operatorname{supp} A_{r\cdot} = \mathfrak{u} \,\dot{\cup}\, \mathfrak{v}$. Fig. 3 shows an example of a 0/1 matrix that has such a *Sekiguchi partition* $\mathfrak{W}$.
Sekiguchi (1983) [15] shows that for a 0/1 matrix A that has a Sekiguchi partition $\mathfrak{W}$, it is not only true that $\phi(P_{\mathrm{SCP}}) = \breve{P}_{\mathrm{SSP}}(\mathfrak{G}[\mathfrak{W}])$, but, even more, that the facets of P_{SCP} are *exactly* the expansions of the facets of $\breve{P}_{\mathrm{SSP}}(\mathfrak{G}[\mathfrak{W}])$.
We mention the *odd hole inequalities* for the SCP, see, e.g., Cornuéjols & Sassano (1989) [8], as one example for a class of inequalities that can be obtained from a set packing relaxation *in the sense of Sekiguchi*. In this context of set covering, the term *odd hole* is commonly used to refer to the edge-node incidence matrix $A(2k+1, 2) = A(C(2k+1, 2)) \in \mathbb{R}^{(2k+1)\times(2k+1)}$ of the circulant graph $C(2k+1, 2)$, or, in other words, $A(2k+1, 2)_{ij} = 1$ if $j = i$ or $j = i + 1$ (mod $2k+1$) and 0 otherwise. The associated *odd hole inequality* for $P_{\mathrm{SCP}}(A(2k+1, 2))$ is

$$\sum_{i=1}^{2k+1} x_i \geq k + 1.$$

$$
A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left(\begin{array}{cc|c|cc|c|c|cc}
1 & 1 & 1 & . & . & . & . & . & . \\
. & . & 1 & 1 & 1 & . & . & . & . \\
. & . & . & 1 & 1 & 1 & . & . & . \\
1 & 1 & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & 1 & 1 & 1
\end{array}\right)
\end{array}
\qquad
\begin{array}{l}
\mathfrak{W} = \{1,2\} \cup \{3\} \cup \{4,5\} \cup \{6\} \\
\qquad\qquad \cup \{7\} \cup \{8,9\}
\end{array}
$$

Figure 3: A Sekiguchi Partitionable 0/1 Matrix.

2.4 Proposition *Every odd hole inequality for $P_{\mathrm{SCP}}(A(2k+1,2))$ is the expansion of an odd cycle inequality for $\breve{P}_{\mathrm{SSP}}(\mathfrak{G}(A(2k+1,2)))$.*

We omit the simple proof of this proposition. Turning back to the general case, we give instead an example of an expanded cycle inequality that cannot be obtained from a Sekiguchi relaxation. The matrix A on the left of Fig. 4 gives rise to a 5-cycle $\mathfrak{C}$ in $\mathfrak{G}$ formed by the

$$
A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left(\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & . & . & . & . & . \\
. & . & . & 1 & 1 & 1 & . & . & . \\
. & . & . & 1 & 1 & 1 & 1 & . & . \\
. & . & . & . & . & . & 1 & 1 & 1 \\
1 & 1 & . & . & . & . & . & 1 & 1
\end{array}\right)
\end{array}
$$

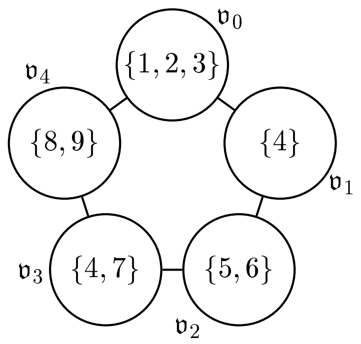

Figure 4: A Not Sekiguchi Partitionable 0/1 Matrix and a 5-Cycle of Columns.

nodes $\mathfrak{v}_0 = \{1,2,3\}$, $\mathfrak{v}_1 = \{4\}$, $\mathfrak{v}_2 = \{5,6\}$, $\mathfrak{v}_3 = \{4,7\}$, and $\mathfrak{v}_4 = \{8,9\}$. A is not Sekiguchi partitionable, because $\mathfrak{v}_3 \cup \mathfrak{v}_4 = \{4,7,8,9\} \supsetneq \{7,8,9\} = \operatorname{supp} A_4.$ and $\mathfrak{v}_4 \cup \mathfrak{v}_0 = \{1,2,3,8,9\} \supsetneq \{1,2,8,9\} = \operatorname{supp} A_1.$. An expansion of the odd cycle inequality corresponding to $\mathfrak{C}$ yields

$$
\begin{aligned}
& \sum_{i=0}^{4} \phi_{\mathfrak{v}_i}(x) \leq 2 \\
\iff \quad & (1 - x_1 - x_2 - x_3) + (1 - x_4) + (1 - x_5 - x_6) + (1 - x_4 - x_7) \\
& \qquad + (1 - x_8 - x_9) \leq 2 \\
\iff \quad & x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7 + x_8 + x_9 \geq 3.
\end{aligned}
$$

Looking at the *separation* of inequalities for P_{SCP} from the set packing relaxation $\breve{P}_{\mathrm{SSP}}(\mathfrak{G})$, we can obtain polynomially separable classes applying family conflict techniques. Namely, consider for each *ordered* 2-tupel (i,j) of different column indices i and j the family

$$
\mathfrak{F}_{ij} := \{A_{i\cdot} \setminus \{j\} : A_{i\cdot} \supseteq \{i,j\}\}.
$$

Here, we identify the rows $A_{i\cdot}$ of the matrix A with their support sets. The family $\mathfrak{F}_{ij}$ arises from collecting all rows covering columns i and j, and removing column j from these rows. We call such a family $\mathfrak{F}_{ij}$ an *(i,j)-family*. The number $n(n-1)$ of (i,j)-family is polynomial. Moreover, the maximum representation problem

$$\max\ 1 - x(F), \qquad F \in \mathfrak{F}_{ij}$$

can be solved in polynomial time simply scanning the matrix rows. This implies:

2.5 Theorem *Inequalities from cycles, cliques, and orthonormal representations of (i,j)-families for the set covering problem can be separated in polynomial time.*

These inequalities are, to the best of our knowledge, the only known *combinatorial class* of polynomial time separable inequalities for general set covering problems. Fig. 5 gives an

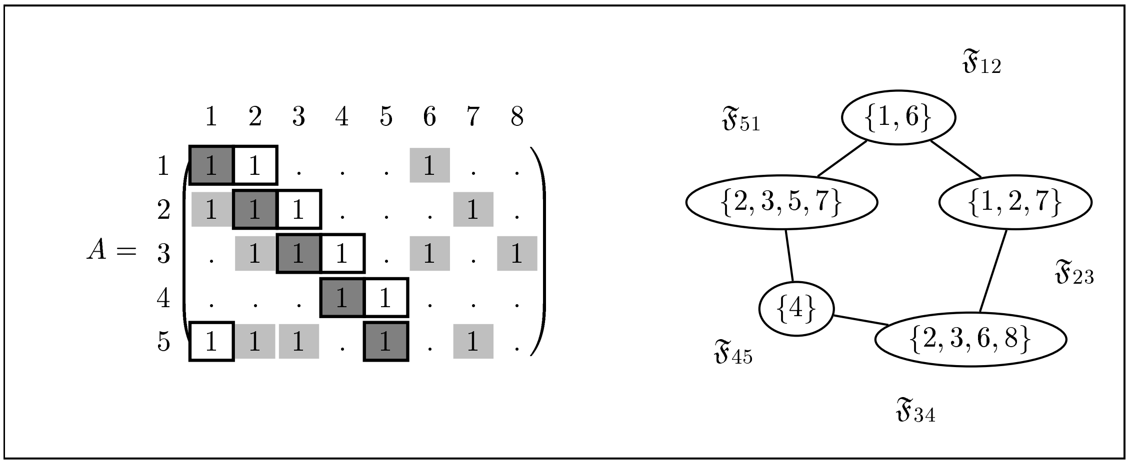

Figure 5: A 5-Cycle of (i,j)-Families.

example of this construction. The matrix A contains the circulant $A(5,2)$, which gives rise to a 5-cycle of (i,j)-families. The families are $\mathfrak{F}_{12} = \{\{1,6\},\{1,3,7\},\{1,3,5,7\}\}$, $\mathfrak{F}_{23} = \{\{1,2,7\},\{2,4,6,8\},\{1,2,5,7\}\}$, $\mathfrak{F}_{34} = \{\{2,3,6,8\}\}$, $\mathfrak{F}_{45} = \{\{4\}\}$, and $\mathfrak{F}_{51} = \{\{2,3,5,7\}\}$. The gray shaded numbers indicate one of the possible cycles of representatives that can be extracted from this meta-structure, namely, the cycle induced by the column sets $\{1,6\} \in \mathfrak{V}_{12}$, $\{1,2,7\} \in \mathfrak{V}_{23}$, $\{2,3,6,8\} \in \mathfrak{V}_{34}$, $\{4\} \in \mathfrak{V}_{45}$, and $\{2,3,5,7\} \in \mathfrak{V}_{51}$. This cycle gives rise to the inequality

$$2x_1 + 3x_2 + 2x_3 + x_4 + x_5 + 2x_6 + 2x_7 + x_8 \geq 3.$$

Depending on the value of x, choosing other representatives from the families might lead to a better inequality. Theorem 2.5 guarantees that the most violated inequality that can be constructed in this way, over all possible families, can be found in polynomial time. The reader will have noticed that this class includes, among others, a large class of lifted odd hole inequalities for the set covering problem.

The (i,j)-families do not give rise to significant clique inequalities. We remark, however, that one can construct, say, (i,j,k,l)-families involving more indices, that support clique and orthonormal representation constraints.

2.2 The Steiner Tree Problem

The Steiner tree problem in directed graphs involves a digraph $D = (V, A)$ with weights $w_a \in Q^+$ on its arcs and a node set $T \subseteq V$ of *terminals*, one of them, say $r \in T$, is the *root*. A *Steiner tree* is a set $S \subseteq A$ of arcs that contains a directed path from the root r to every terminal $t \neq r$. A *Steiner cut* is a cut of the form $\delta^+(W)$ that separates the root $r \in W$ from one or more terminals. The *Steiner tree problem* (STP) is to find a Steiner tree of minimum weight. We suggest Chopra & Rao (1994a;b) [5; 6] as references.
An IP formulation of the STP is

$$\text{(STP)} \quad \min \; w^T x, \qquad Ax \geq \mathbf{1},\; x \in \mathbb{Z}^n_+,$$

where $w \in \mathbb{Z}^n_+$ and $A \in \{0,1\}^{\mathfrak{W} \times n}$ is the cut-arc incidence matrix of all Steiner cuts. The associated polyhedron is P_{STP}.
The formulation (STP) shows that the Steiner tree problem is, in fact, a set covering problem. Its special characteristic is the exponential matrix A, which is given only implicitly. This, however, is no obstacle for the application of our results on set covering as we will see now.
The translation of the results of Subsection 2.1 into the Steiner tree context is as follows. The STP involves arc variables, giving rise to (a, b)-families

$$\mathfrak{F}_{ab} := \{\delta^+(W) \setminus \{b\} : \delta^+(W) \supseteq \{a, b\} \text{ a Steiner cut}\}.$$

One obtains such a family by collecting all Steiner cuts that contain two arcs a and b; removing from each such cut the arc b yields the family $\mathfrak{V}_{ab}$. The number of (a, b)-families is $|A|(|A|-1)$.
The maximum representation problem

$$\max \; 1 - x(F), \qquad F \in \mathfrak{F}_{ab}$$

is equivalent to a min-Steiner cut problem involving fixed arcs:

$$\min \; x(F), \qquad F \text{ is a Steiner cut containing arcs } a \text{ and } b.$$

It is not hard to see that this problem can be solved in polynomial time. Calling the members of the families $\mathfrak{F}_{ab}$ *Steiner protocuts*, we have:

2.6 Theorem *Inequalities from cycles, cliques, and orthonormal representations of Steiner protocuts for the Steiner tree problem can be separated in polynomial time.*

Fig. 6 gives a schematic drawing of an example of such a cut, as it might come up, e.g., in VLSI design. The grid sketches a planar grid digraph (each edge represents two antiparallel arcs). There are six highlighted terminal nodes, the root r and five ordinary terminals t_i. The highlighted arcs belong to five Steiner protocuts $\mathfrak{v}_1, \ldots, \mathfrak{v}_5$. Protocut $\mathfrak{v}_i$ separates the root r from terminal t_i except for a single "crossing arc" a_{i+1}, which misses in $\mathfrak{v}_i$, but is contained in the succeeding protocut $\mathfrak{v}_{i+1}$ (indices larger 5 are taken modulo 5). a_2, a_3, a_4, and a_5 are diagonal arcs, which have been added to make the drawing easier, while a_1 illustrates a "genuine grid graph crossing".
The crossing arcs $a_1, \ldots, a_5$ give rise to a 5-cycle of (a, b)-families, namely, the families $\mathfrak{F}_{a_1a_2}$, $\mathfrak{F}_{a_2a_3}$, $\mathfrak{F}_{a_3a_4}$, $\mathfrak{F}_{a_4a_5}$, and $\mathfrak{F}_{a_5a_1}$. Fig. 6 shows one of the many cycles of protocuts that are encoded in this structure. Steiner protocut $\mathfrak{v}_i$, which represents $\mathfrak{F}_{a_ia_{i+1}}$, is in conflict with

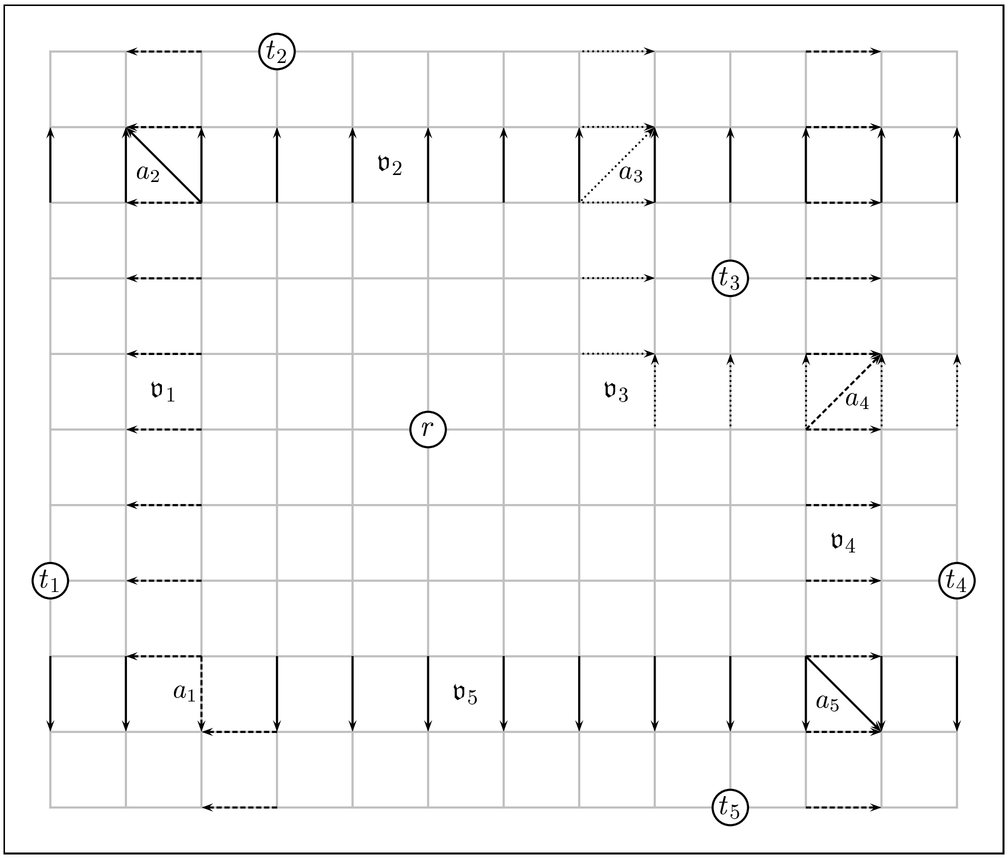

Figure 6: A 5-Cycle of Steiner Protocuts in a Planar Grid Digraph.

the succeeding protocut $\mathfrak{v}_{i+1}$, which represents $\mathfrak{F}_{a_{i+1}a_{i+2}}$. The inequality associated with this cycle of protocuts is

$$\sum_{i=1}^{5} \phi_{\mathfrak{v}_i}(x) \leq 2 \iff \sum_{i=1}^{5} \left(1 - \sum_{a \in \mathfrak{v}_i} x_a\right) \leq 2 \iff \sum_{i=1}^{5} \sum_{a \in \mathfrak{v}_i} x_a \geq 3.$$

Choosing other protocuts from the families $\mathfrak{F}_{a_i a_{i+1}}$ results in alternative cycles of protocuts and different inequalities. Theorem 2.6 states that a most violated inequality of this type, over all protocuts of all (a, b)-families, can be separated in polynomial time.

2.3 The Asymmetric Travelling Salesman Problem

Let $D = (V, A)$ be a complete graph on n nodes with weights $w_a \in \mathbb{Q}^+$ on the arcs. A *tour* is a directed hamiltonian cycle in G that visits every node exactly once. The *asymmetric travelling salesman problem* (ATSP) asks for a tour of minimum weight.
An IP formulation of the ATSP is

$$\text{(ATSP)} \;\; \min \;\; w^T x, \; Ax \geq \mathbf{1}, \; x(\delta^+(v)) = x(\delta^-(v)) = 1 \; \forall v \in V, \; x \in \mathbb{Z}^n_+,$$

where $A \in \{0, 1\}^{\mathfrak{W} \times A}$ is the cut-arc incidence matrix of all directed cuts of the form $\delta^+(W)$,

$\emptyset \neq W \subsetneq V$. The ATSP can hence be seen as a set covering problem with additional in- and out-degree constraints. As in the STP, the cut-matrix A is given implicitly.
The ATSP has several interesting set packing relaxations. A classical one, due to Balas (1989) [1], comes up directly in the space of original variables, i.e., by choosing the identity as a scheme. The conflicts of this relaxation are based on pairs of *incompatible arcs*, or, as we like to see it, on conflicts of degree constraints. The cycle inequalities of this relaxation are known as the *odd closed alternating trail inequalities.* The separation problem has been solved by Caprara & Fischetti (1996) [4].
We are now going to suggest a relaxation that is based on conflicts of *cuts.* We consider the scheme $\phi : \mathbb{R}^A \to \mathbb{R}^{\mathfrak{V}}$ defined as

$$\phi_\Delta(x) := 1 - x(\Delta) \qquad \forall \Delta \subseteq \delta^+(W), \emptyset \neq W \subsetneq V.$$

Here, we take $\mathfrak{V} := \{\Delta \subseteq \delta^+(W), \emptyset \neq W \subsetneq V\}$ as the set of all subsets of cuts $\delta^+(W)$, $\emptyset \neq W \subsetneq V$.
The scheme induces an exponential conflict graph $\mathfrak{G} = (\mathfrak{V}, \mathfrak{E})$ that is based on conflicts of pairs of arc sets whose union forms a cut, i.e.,

$$\Gamma\Delta \in \mathfrak{E} \iff \exists \emptyset \neq W \subsetneq V : \delta^+(W) \subseteq \Gamma \cup \Delta.$$

2.7 Lemma $\phi(P_{\text{ATSP}}) \subseteq \breve{P}_{\text{SSP}}(\mathfrak{G})$.

To derive polynomial time separable classes of inequalities from this relaxation, we consider for each ordered pair of arcs the (a, b)-families

$$\mathfrak{F}_{ab} := \{\delta^+(W) \setminus \{b\} : \delta^+(W) \supseteq \{a, b\}, \emptyset \neq W \subsetneq V\}.$$

These families are, in fact, exactly the (a, b)-families that one obtains from the above mentioned set covering relaxation of the ATSP, and they have the same combinatorial properties as in the STP. We call a member of such a family a *protocut.* The number of protocut families is $|A|(|A| - 1)$, and the maximum representation problem

$$\max\ 1 - x(F), \qquad F \in \mathfrak{F}_{ab}$$

is similar to the STP, in fact, a little easier, as we do not go for Steiner cuts, but for general cuts.

2.8 Theorem *Inequalities from cycles, cliques, and orthonormal representations of protocuts for the asymmetric travelling salesman problem can be separated in polynomial time.*

As far as we know, these classes and separation algorithms are new.

3 Knapsack Relaxations

For various combinatorial optimization problems one can construct natural *knapsack relaxations* in complete analogy to the set packing case. Our aim in this section is to give examples of how one can use such a relaxation to produce cutting planes as expansions of inequalities from the knapsack polytope.
The (0/1) *single knapsack problem* can be stated as

$$\text{(SKP)} \quad \max\ w^{\mathrm{T}}x, \quad a^{\mathrm{T}}x \leq \alpha,\ x \in \{0,1\}^I.$$

Here, $a, w \in \mathbb{Z}_+^I$ are vectors of non-negative integer weights and values, respectively, of the set of items I, and $\alpha \in \mathbb{Z}_+$ is the knapsack capacity. We denote the associated knapsack polytope by $P_{\text{SKP}} = P_{\text{SKP}}(a, \alpha)$.
Basic inequalities for P_{SKP} are *cover inequalities*, see, e.g., Wolsey (1975) [18], $(1,k)$-*configuration inequalities*, see Padberg (1980) [12], and *weight* and *extended weight inequalities*, see Weismantel (1997) [16].
The remainder of this section gives two examples of knapsack relaxations for the multiple knapsack and the node capacitated graph partitioning problem.

3.1 The Multiple Knapsack Problem

The *multiple knapsack problem* is the integer program

$$
\begin{array}{llrcl}
\text{(MKP)} & \max & \sum_{i \in I} \sum_{k \in K} w_i x_{ik} & & \\
& \text{(i)} & \sum_{i \in I} a_i x_{ik} & \leq & \alpha \quad \forall k \in K \\
& \text{(ii)} & \sum_{k \in K} x_{ik} & = & 1 \quad \forall i \in I \\
& \text{(ii)} & x & \in & \{0,1\}^{I \times K}.
\end{array}
$$

Here, $I = \{1, \ldots, n\}$ is a set of items of nonnegative integer weights and profits $a, w \in \mathbb{Z}_+^I$, that can be stored in a *set* K of knapsacks of capacity α each. Associated with the MKP is the polytope P_{MKP}.
One class of valid inequalities for the multiple knapsack polytope are the *multiple cover inequalities* of Wolsey (1990) [17]: Given a subset $K' \subseteq K$ of knapsacks and a subset $I' \subseteq I$ of items such that $\sum_{i \in I'} a_i > |K'|\alpha$, the multiple cover inequality reads

$$\sum_{i \in I'} \sum_{k \in K'} x_{ik} \leq |I'| - 1.$$

It can be derived from a single knapsack relaxation of the MKP.
The appropriate aggregation scheme $\phi : \mathbb{R}^{I \times K} \to \mathbb{R}^I$ is defined as

$$\phi_i(x) = \sum_{k \in K'} x_{ik} \qquad \forall i \in I.$$

3.1 Lemma $\phi(P_{\text{MKP}}) \subseteq P_{\text{SKP}}(a, |K'|\alpha)$.

3.2 Theorem *Every multiple cover inequality for P_{MKP} is an expansion of a cover inequality for $P_{\text{SKP}}(a, |K'|\alpha)$.*

Lemma 3.1 suggests to apply not only covers, but also $(1,k)$-configurations, weight and extended weight inequalities with their separation routines to the MKP. These classes are new.

3.2 The Node Capacitated Graph Partitioning Problem

We study in this subsection the *node capacitated graph partitioning problem* (cap-MCP). Given a graph $G = (V, E)$ with node and edge weights $f \in \mathbb{Z}_+^V$ and $w \in \mathbb{Z}^E$, respectively, a fixed number k, and a clique capacity $F \in \mathbb{Z}_+$, the feasible solutions of the node capacitated graph partitioning problem are the k-multicuts of the complete graph K_n such that the constraints

$$\sum_{ij \in T} x_e \geq 1 \qquad \text{for all trees } T \subseteq E \text{ such that } \sum_{v \in V(T)} f_v > F$$

hold. An IP formulation reads

$$\begin{array}{llll}
\max & \sum_{ij \in E} w_{ij} x_{ij} & & \\
\text{(i)} & \sum_{ij \in E(W)} x_{ij} \leq |E(W)| - 1 & & \forall\, W \subseteq V : |W| = k+1 \\
\text{(ii)} & x_{ij} - x_{jk} - x_{ik} \leq 0 & & \forall\, \{i,j,k\} \subseteq V \\
\text{(iii)} & \sum_{ij \in T} x_e \geq 1 & & \forall \text{ trees } T \subseteq E : f(V(T)) > F \\
\text{(iv)} & -x_{ij} \leq 0 & & \forall\, ij \in E \\
\text{(v)} & x_{ij} \leq 1 & & \forall\, ij \in E \\
\text{(vi)} & x_{ij} \in \mathbb{Z} & & \forall\, ij \in E.
\end{array}$$

$$(\text{cap} - \text{MCP})$$

A class of inequalities for the $P_{\text{cap}-\text{MCP}}$ that takes node weights into account are the *knapsack tree inequalities* of Ferreira et. al. (1996) [9]. These constraints are based on a tree T_r rooted at some node r. If we denote the (unique) path from r to every node $v \in T_r$ by P_v, a knapsack tree inequality reads

$$\sum_{v \in V(T_r)} a_v \Big(1 - \sum_{ij \in P_v} x_{ij}\Big) \leq \alpha,$$

where $a^{\mathrm{T}} y \leq \alpha$ is valid for the single knapsack polytope $P_{\text{SKP}}(f, F)$. This inequality can be derived with the scheme $\phi^{T_r} : \mathbb{R}^E \to \mathbb{R}^V$ defined as

$$\phi_v^{T_r}(x) := \begin{cases} 1 - \sum_{ij \in P_v} x_{ij} & \forall v \in V(T_r) \\ 0 & \text{otherwise.} \end{cases}$$

3.3 Lemma $\phi^{T_r}(P_{\text{cap}-\text{MCP}}) \subseteq P_{\text{SKP}}(f, F)$.

3.4 Theorem *Every knapsack tree inequality for $P_{\text{cap}-\text{MCP}}$ is an expansion of a valid inequality for $P_{\text{SKP}}(f, F)$.*

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