Konrad-Zuse-Zentrum für Informationstechnik Berlin

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Symbolic solution of polynomial equation systems with symmetry

Preprint SC 90–3 (March 1990)

Veröffentlichungen des Konrad-Zuse-Zentrum für Informationstechnik Berlin
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Herausgegeben vom
Konrad-Zuse-Zentrum für Informationstechnik Berlin
Heilbronner Strasse 10
1000 Berlin 31
Verantwortlich: Dr. Klaus André
Umschlagsatz und Druck: Rabe KG Buch- und Offsetdruck Berlin

ISSN 0933-7911

Symbolic solution of polynomial equation systems with symmetry

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February 28, 1990

Abstract

Systems of polynomial equations often have symmetry. The Buchberger algorithm which may be used for the solution ignores this symmetry. It is restricted to moderate problems unless factorizing polynomials are found leading to several smaller systems. Therefore two methods are presented which use the symmetry to find factorizing polynomials, decompose the ideal and thus decrease the complexity of the system a lot.

In a first approach projections determine factorizing polynomials as input for the solution process, if the group contains reflections with respect to a hyperplane. Two different ways are described for the symmetric group S_m and the dihedral group D_m. While for S_m subsystems are ignored if they have the same zeros modulo G as another subsystem, for the dihedral group D_m polynomials with more than two factors are generated with the help of the theory of linear representations and restrictions are used as well. These decomposition algorithms are independent of the finally used solution technique. We used the REDUCE package Groebner to solve examples from Caprasse, Demaret and Noonburg which illustrate the efficiency of our REDUCE program. A short introduction to the theory of linear representations is given.

In a second approach problems of another class are transformed such that more factors are found during the computation; these transformations are based on the theory of linear representations.

Examples illustrate these approaches. The range of solvable problems is enlarged significantly.

Contents

1 Introduction

We are concerned with systems of polynomial equations with symmetry. Such problems often arise in physics, chemistry and mathematical models (see [2], [7], [8], [10]). Because one is interested in exact solutions the program Groebner is used which is available in Computer Algebra Systems like REDUCE (see [5]) and which computes Groebner bases with the Buchberger algorithm (for a brief introduction see [1]). This may need a lot of cpu time and storage. But if factorizations of polynomials are used the computation needs less time and storage. That's why even larger systems become solvable. Such a version of Groebner using factorizing polynomials was developed at the Konrad-Zuse-Zentrum (see [6],[7]).

Unfortunately, the classical Buchberger algorithm cannot use the special structure of a system of polynomial equations with symmetry. But the symmetry can be used to find factorizing polynomials. For groups with reflections with respect to a hyperplane factorizing polynomials are generated with projections. Thus only subsystems need to be solved. This reduces the computing time enormously. Two different ways are described for the symmetric group S_m and the dihedral group D_m. Groups without reflections have to be treated in a different way. The coordinates are transformed based on the theory of linear representations. During the computation of the Groebner basis the transformed problem leads to much more factorizing polynomials than the original problem and thus saves a lot of time.

Some group theoretical facts are given, but more theoretical background may be found in SERRE [11] or STIEFEL and FÄSSLER [13].

The ideas are coded in a REDUCE program which was used for the calculation of the examples given here.

2 Definitions concerning symmetry

In this section we define systems with symmetry. A typical example is

$$\begin{aligned} 1 - cx_1 - x_1x_2^2 - x_1x_3^2 &= 0 \\ 1 - cx_2 - x_2x_1^2 - x_2x_3^2 &= 0 \\ 1 - cx_3 - x_3x_1^2 - x_3x_2^2 &= 0 \end{aligned} \tag{1}$$

which is given in NOONBURG [8]. c is an independent parameter and x_1, x_2, x_3 are the variables. The variables x_1, x_2, x_3 may be cyclic rotated or two of them may be interchanged without changing the system itself. The solution is given in section 4.2.2.

Whenever we are concerned with problems with symmetry there is a *group* G. In the following G is the dihedral group

$$D_m = \{id, r, r^2, \dots, r^{m-1}, s, s \cdot r, \dots, s \cdot r^{m-1}\}$$

or S_m, the group of permutations of m elements. We are interested in realizations of these groups. However that means an example of a linear representation

$$\begin{aligned} \rho : G \to GL(\mathbb{C}^n) \quad \text{with} \quad & \rho(t_1)\rho(t_2) = \rho(t_1 t_2) \\ \text{and} \quad & t_1 \neq t_2 \Longrightarrow \rho(t_1) \neq \rho(t_2), \end{aligned} \tag{2}$$

which is treated in section 4.2.1. $GL(\mathbb{C}^n)$ is the group of isomorphisms. Because of condition (2), the group $\{\rho(t) : t \in G\}$ may be identified with G. The linear mappings $\rho(t)$ may be represented (concerning the canonical basis) by matrices T. This gives a group of matrices T which we identify with G itself. In the system of equations (1) the group of interest is

$$D_3 = \{I, R, R^2, S, S \cdot R, S \cdot R^2\}$$

with

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A tuple $v \in \mathbb{C}^n$ is called *G–invariant* if $Tv = v \, \forall T \in G$. A vector space $V \subset \mathbb{C}^n$ is called *G–invariant* if

$$Tv \in V \quad \forall T \in G \; \forall v \in V.$$

For example the tuple $v = (1,1,1)^T$ is D_3–invariant and the vector spaces span(v) and $\mathbb{C}^3$ itself are D_3–invariant.

The group G operates also on the ring of polynomials $\mathbb{C}[x_1, \dots, x_n]$. For $p(\underline{x}) \in \mathbb{C}[\underline{x}]$ a polynomial $p(T\underline{x})$ is given for every $T \in G$. This is another example of a linear representation which will be used in sections 4.2.2 and 5.

Definition 1 *:*
A vector space $V = span(p_1, \dots, p_d) \subset \mathbb{C}[\underline{x}]$ and the corresponding ideal $A = (p_1, \dots, p_d) \subset \mathbb{C}[\underline{x}]$ are called G–invariant if

$$\begin{aligned} p(T\underline{x}) \in V \quad & \forall T \in G \, \forall p \in V, \\ p(T\underline{x}) \in A \quad & \forall T \in G \, \forall p \in A, \end{aligned}$$

respectively.

In general it is hard to verify that an ideal is G–invariant. But obviously if $V = < p_1, \ldots, p_d > \subset \mathbb{C}[\underline{x}]$ is a G–invariant vector space then the generated corresponding ideal $A = (p_1, .., p_d)$ is G–invariant.

The important fact is that a G–invariant ideal $A = (p_1, \ldots, p_d)$ has a G–invariant set of zeros.

Now we are able to define a problem with symmetry. A system of polynomial equations

$$\begin{array}{rcl} p_1(x_1, \ldots, x_n) & = & 0 \\ p_2(x_1, \ldots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ p_d(x_1, \ldots, x_n) & = & 0 \end{array} \tag{3}$$

$p_i \in \mathbb{C}[\underline{x}], i = 1, \ldots, d$ is called *G–invariant* if $p_i, i = 1, \ldots, d$ span a G–invariant vector space. The solutions of (3) are the zeros of the G–invariant ideal $A = (p_1, \ldots, p_d)$ and thus build a G–invariant set. The system (1) for example is D_3–invariant.

A special case of G–invariant systems are the G–equivariant systems. Let G be the group of matrices $T \in \mathbb{C}^{n,n}$. Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ with $f_i \in \mathbb{C}[\underline{x}], i = 1, \ldots, n$ and

$$f(T\underline{x}) = Tf(\underline{x}).$$

Then the system $f(\underline{x}) = 0$ is called *G–equivariant.*

3 Efficient check for symmetry

First we have to make sure that the system of polynomial equations is G–invariant. But the check of a vector space V to be symmetric may be time consuming. Thus the following well–known proposition is useful.

Proposition 2 *: Let G be a group which is generated by $T_1, \ldots, T_\nu$. Let $V \subset \mathbb{C}[\underline{x}]$ be a vector space of polynomials and $p_i, i = 1, \ldots, d$ a basis of V. If*

$$p_i(T_j\underline{x}) \in V \quad \forall i = 1, \ldots, d \quad \forall j = 1, \ldots, \nu$$

holds, then V is G–invariant.

In table 1 the generators are given for some groups.

G		Generators
S_m		the $m-1$ neighboured transpositions $(1,2),(2,3),\ldots,(m-1,m)$
	or	all $m \cdot (m-1)/2$ transpositions
	or	the transposition (1,2) and the cyclic permutation (1,2,3... m)
D_m		the rotation r and the reflection s

Table 1: Generators of some groups G

4 Splitting the system

The main idea is the following. Because factorizing polynomials in the system of polynomial equations (3) lead to a decomposition of the system into several smaller systems factorizing polynomials in the corresponding ideal are searched systematically. This is done with the help of symmetry and the following well-known lemma.

Lemma 3 : *Let $T \in \mathbb{C}^{n,n}$ be a reflection with respect to a hyperplane. Let $p \in \mathbb{C}[\underline{x}]$ be a polynomial with*

$$p(T\underline{x}) = -p(\underline{x}).$$

then exists $q_1 \in \mathbb{C}[\underline{x}]$ of degree one and $q_2 \in \mathbb{C}[\underline{x}]$ with $p = q_1 \cdot q_2$.

Proof : As T is a reflection there exist linear independent vectors $v_1, \ldots, v_n \in \mathbb{C}^n$ with $Tv_i = v_i, i = 1, \ldots, n-1$ and $Tv_n = -v_n$.

Thus for $v \in < v_1, \ldots, v_{n-1} >$ we have

$$p(v) = p(Tv) = -p(v) \quad \Longrightarrow \quad p(v) = 0.$$

Thus either $p \equiv 0$ or p is linear or has a linear factor. ∎

The projection $\Pi_T : \mathbb{C}[\underline{x}] \longrightarrow \mathbb{C}[\underline{x}]$

$$\Pi_T\big(p(\underline{x})\big) := \tfrac{1}{2}\big(p(\underline{x}) - p(T\underline{x})\big) \tag{4}$$

computes polynomials $P := \Pi_T(p)$ with $P(T\underline{x}) = -P(\underline{x})$. From Lemma 3 follows that $P \equiv 0$ or P is linear or has a linear factor.

Although Lemma 3 predicts a linear factorization in the domain $\mathbb{C}$ the examples in practice have factorizations in $\mathbb{Z}$.

4.1 Splitting in case of the symmetric group

Assume a system (3) which is S_m-invariant with transpositions corresponding to reflections. A factorizing polynomial $P_1 = q_1^1 q_2^1 \in A = (p_1, \ldots, p_d)$ is computed with the help of the projection corresponding to the transposition $(1,2) \in S_m$. Other factorizing polynomials are computed by symmetry:

$$P_i := t_i(P_{i-1}) = q_1^i q_2^i, \quad i = 2, \ldots, m-1$$

where $t_i \in S_m$ are the $m-2$ remaining neighbouring transpositions. Then

$$A = \bigcap_{\underline{i}} A^{\underline{i}},$$

where $A^{\underline{i}}, \underline{i} = (i_1, \ldots, i_{m-1})$ are 2^{m-1} ideals with the bases

$$p_1, \ldots, p_d, q_{i_1}^1, q_{i_2}^2, \ldots, q_{i_{m-1}}^{m-1} \quad, i_j \in \{1,2\}$$

Thus the system of polynomial equations is split into 2^{m-1} smaller systems.

If the zeros of $A^{\underline{i}}$ are known, the zeros of

$$t(A^{\underline{i}}) = \big(p_1(\underline{x}), \ldots, p_d(\underline{x}),\ q_{i_1}^1(T\underline{x}), \ldots, q_{i_{m-1}}^{m-1}(T\underline{x})\big)$$

are known by symmetry as well. This reduces the number of ideals $A^{\underline{i}}$ which have to be investigated. Such a symmetry check can't be done without a Groebner basis in general and might be time consuming. We implemented the following check: Let

$$V^{\underline{i}} := < q_{i_1}^1, \ldots, q_{i_{m-1}}^{m-1} >, \quad i_j \in \{1,2\}.$$

If we find a transposition $t \in S_m$ and a space $V^{\underline{k}}$ with

$$q_{k_j}^j(T\underline{x}) \in V^{\underline{i}} \quad j = 1, \ldots, m-1$$

then $t(A^{\underline{k}}) \subseteq A^{\underline{i}}$ and $A^{\underline{i}}$ is ignored. For the remaining ideals a Groebner basis is computed with factorizing Buchberger algorithm.

Example : CAPRASSE, DEMARET [2] gave the following S_5-invariant system:

$$\begin{aligned}
p_1(\underline{x}) &:= 3x_1^3 - x_1(3x_1a_1 + 3a_1 + 3a_2 - 6) + a_2 - a_3 = 0, \\
p_2(\underline{x}) &:= 3x_2^3 - x_2(3x_2a_1 + 3a_1 + 3a_2 - 6) + a_2 - a_3 = 0, \\
p_3(\underline{x}) &:= 3x_3^3 - x_3(3x_3a_1 + 3a_1 + 3a_2 - 6) + a_2 - a_3 = 0, \\
p_4(\underline{x}) &:= 3x_4^3 - x_4(3x_4a_1 + 3a_1 + 3a_2 - 6) + a_2 - a_3 = 0, \\
p_5(\underline{x}) &:= 3x_5^3 - x_5(3x_5a_1 + 3a_1 + 3a_2 - 6) + a_2 - a_3 = 0,
\end{aligned}$$

$$a_i := x_1^i + x_2^i + x_3^i + x_4^i + x_5^i \ , \quad i = 1, 2, 3 \ .$$

There are 6 subsystems generated which split further during the Groebner computation. There are still Groebner bases left which are equal modulo G (see figure 1). The Groebner bases were computed for generalizations of this problem with 6,7,8 and 9 variables, which were unsolvable before.

4.2 Splitting in case of the dihedral group

For the dihedral group D_m polynomials with more than two factors may be generated with linear representations. First we give a short survey on the theory of linear representations of a group G. The details can be found in [11] and [13].

4.2.1 Linear representations

Definition 4 *: Let G be a finite group. A linear representation ϑ is a mapping*

$$\vartheta : G \to GL(V) \ , \ \ \vartheta(t_1)\vartheta(t_2) = \vartheta(t_1 t_2) \ ,$$

where V is a complex vector space and $GL(V)$ is the group of isomorphisms. The dimension of V is the dimension of ϑ.

An example was already mentioned in section 2:

$$\vartheta : G \to GL\big(\mathbb{C}[\underline{x}]\big) \ , \quad \vartheta(t)\big(p(\underline{x})\big) = p(T\underline{x}) \ , \tag{5}$$

where $T \in \mathbb{C}^{n,n}$ are the matrices corresponding to a linear representation $\rho : G \to GL(\mathbb{C}^n)$ with $t_1 \neq t_2 \Longrightarrow \rho(t_1) \neq \rho(t_2)$.

$\{x_1 - x_5,$
$932x_2 - 923x_5^5 + 11835x_5^4 - 32526x_5^3 + 10724x_5^2$
$+2474x_5 - 76, 932x_3 - 923x_5^5 + 11835x_5^4 - 32526x_5^3 + 10724x_5^2 + 2474x_5 - 76,$
$x_4 - x_5,$
$13x_5^6 - 172x_5^5 + 528x_5^4 - 356x_5^3 + 36x_5^2 + 8x_5 + 4\}$
$\{2x_1 + 5x_5^2 - 6, x_2 + x_5, x_3 - x_5, x_4 + x_5,$
$5x_5^4 - 10x_5^2 + 8\}$
$\{x_1 - 1, x_2, x_3, x_4, x_5\}$
$\{2x_1 + 5x_5^2 - 6,$
$x_2 - x_5, x_3 + x_5, x_4 + x_5, 5x_5^4 - 10x_5^2 + 8\}$
$\{2x_1 + 5x_5^2 - 6,$
$x_2 + x_5, x_3 + x_5, x_4 - x_5, 5x_5^4 - 10x_5^2 + 8\}$
$\{6x_1 + 3x_5^2 - 32x_5 + 14, x_2 - x_5, x_3 - x_5, x_4 - x_5,$
$9x_5^4 - 120x_5^3 + 358x_5^2 - 224x_5 + 40\}$
$\{x_1, x_2 - 1, x_3, x_4, x_5\}$
$\{x_1 + x_5, x_2 + x_5, x_3 - x_5,$
$2x_4 + 5x_5^2 - 6, 5x_5^4 - 10x_5^2 + 8\}$
$\{x_1 + x_4, x_2 + x_4, x_3 - x_4,$
$5x_4^2 + 2x_5 - 6, x_5^2 - x_5 + 4\}$
$\{x_1 + x_5, x_2 + x_5, 2x_3 + 5x_5^2 - 6, x_4 - x_5,$
$5x_5^4 - 10x_5^2 + 8\},$
$\{x_1, x_2, x_3 - 1, x_4, x_5\}$
$\{932x_1 - 923x_5^5 + 11835x_5^4 - 32526x_5^3 + 10724x_5^2$
$+2474x_5 - 76,$
$932x_2 - 923x_5^5 + 11835x_5^4 - 32526x_5^3 + 10724x_5^2 + 2474x_5 - 76,$
$x_3 - x_5, x_4 - x_5,$
$13x_5^6 - 172x_5^5 + 528x_5^4 - 356x_5^3 + 36x_5^2 + 8x_5 + 4\}$
$\{x_1 - x_5, x_2 - x_5, x_3 - x_5, x_4 - x_5, x_5^2 - 10x_5 + 3\}$
$\{x_1, x_2, x_3, x_4, x_5\}$
$\{1281x_1 - 1560x_5^5 + 20067x_5^4 - 55916x_5^3 + 12773x_5^2 + 10350x_5 + 2346,$
$1281x_2 - 1560x_5^5 + 20067x_5^4 - 55916x_5^3 + 12773x_5^2 + 10350x_5 + 2346,$
$1281x_3 - 1560x_5^5 + 20067x_5^4 - 55916x_5^3 + 12773x_5^2 + 10350x_5 + 2346,$
$x_4 - x_5,$
$13x_5^6 - 176x_5^5 + 580x_5^4 - 432x_5^3 - 12x_5^2 + 36x_5 + 18\}$
$\{x_1, x_2, x_3, x_4, x_5 - 1\}$
$\{x_1, x_2, x_3, x_4 - 1, x_5\}$
$\{x_1 - x_5, x_2 - x_5, x_3 - x_5, 6x_4 + 3x_5^2 - 32x_5 + 14,$
$9x_5^4 - 120x_5^3 + 358x_5^2 - 224x_5 + 40\}.$

Figure 1: Solution of the system from Caprasse, Demaret

The system (1) in section 2 is D_3-invariant. For $G = D_3$ there are some special polynomials:

$$p_1(\underline{x}) := x_1 x_2 x_3 \,,$$
$$p_2(\underline{x}) := (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \,,$$

which behave like

$$\vartheta(t)(p_1) = p_1 \,, \quad \forall t \in D_3 \text{ but}$$
$$\vartheta(sr^i)(p_2) = -p_2 \,, \quad i = 1, 2, 3$$

for the reflections sr^i. These are examples for the following definitions.

Let ϑ be a linear representation of a finite group G on a d–dimensional vector space V. A subspace $W_1 \subseteq V$ is called *G–invariant,* if $\vartheta(t)(w) \in W_1$, $\forall w \in W_1, t \in G$.

There exists always a second G–invariant space W_2 with $V = W_1 + W_2$.

$$\vartheta^{W_1} : G \to GL(W_1) \,, \quad \vartheta^{W_1}(t) = \vartheta(t)_{|W_1}$$

is called a *subrepresentation.* Of course, two representations may be isomorphic. If the representation ϑ has only the two trivial subrepresentations $\vartheta^{\{0\}}, \vartheta^V = \vartheta$ then ϑ and V are called *irreducible.*

Theorem 5 *(see [11],[13]) : For every finite group G, there is a finite number h of nonisomorphic irreducible representations ϑ^j. For every linear representation $\vartheta : G \to GL(V)$ of dimension d there is the so–called* **canonical decomposition**

$$V = \sum_{j=1}^{h} V^j \,, \tag{6}$$

such that the subrepresentations ϑ^{V^j}, $V^j \neq \{0\}$ have only irreducible subrepresentations isomorphic to ϑ^j. The canonical decomposition is unique but the further decomposition into irreducible spaces is not unique in general.

We are interested in the isotypical components V^j of the canonical decomposition. They may be computed by projections

$$\begin{aligned} &\Pi^j : V \to V^j, \quad j = 1, \ldots, h, \\ &\Pi^j(v) = \tfrac{n_j}{|G|} \sum_{t \in G} \vartheta(t)(v) \psi^j(t^{-1}) \,. \end{aligned} \tag{7}$$

n_j is the dimension of the corresponding irreducible representation ϑ^j and $\psi^j : G \to \mathbb{C}$ are *characters.* For the usual groups the characters ψ^j corresponding to the irreducible representations are known (see for instance [11], [13]).

4.2.2 Subsystems in case of the dihedral group

Now we are able to explain how a system with D_m–symmetry may be split into smaller ones. Recall the following linear representation:

$$\rho : D_m \to GL(\mathbb{C}^n) \quad \rho(t_1) \neq \rho(t_1)$$

with corresponding matrices R, S, where SR^i, $i = 1, \ldots, m$ are reflections with respect to a hyperplane. Then

$$\vartheta : D_m \to GL\big(\mathbb{C}[\underline{x}]\big),\ \vartheta(t)\big(p(\underline{x})\big) = p\big(\rho(t)(\underline{x})\big)$$

describes the corresponding group operations on the polynomials.

Let $p_1, \ldots, p_d$ of the system (3) span a D_m–invariant vector space V. Our aim is to find factorizing polynomials in the isotypical components of V using the theory of linear representations. According to Lemma 3 a polynomial p has a linear factor if there is a reflection SR^i with $p(SR^i\underline{x}) = -p(\underline{x})$. The most important irreducible representations for D_m are the trivial representation

$$\vartheta^1 : D_m \to GL\big(\langle p_1\rangle\big), \quad \vartheta^1(t)(p_1) = p_1 \quad \forall t \in D_m$$

and

$$\begin{aligned}\vartheta^2 : D_m \to GL\big(\langle p_2\rangle\big), \quad &\vartheta^2(r^i)(p_2) = p_2\,,\\ &\vartheta^2(sr^i)(p_2) = -p_2\,, i = 1, \ldots, m\,.\end{aligned}$$

Thus by Lemma 3 p_2 either is zero or has m or more factors.

V has the canonical decomposition

$$V = \sum_{j=1}^{h} V^j$$

and thus a decomposition

$$V = V^1 \oplus V^2 \oplus V_r\,, \quad V_r = \sum_{j=3}^{h} V^j\,. \tag{8}$$

As we have seen polynomials $p_2 \in V_2$ factorizes. There exist factorizing polynomials $p_r \in V_r \setminus \{0\}$, because any $p \in V_r$ with the help of one of the projections Π_T in (4) gives a factorizing polynomial. Otherwise with Proposition 2 the D_m–invariance of $p \in V_r$ is proven and with (8) $p = 0$ follows.

The computation of V^1, V^2, V_r in (8) may be done by the projections (7). But we have chosen a better way with

$$\begin{aligned}
p_c &:= \Pi^c(p) := \tfrac{1}{m}\sum_{i=1}^{m} p(R^i \underline{x}) \,, \\
\Pi^1\big(p(\underline{x})\big) &:= \tfrac{1}{2}\big(p_c(\underline{x}) + p_c(S\underline{x})\big) \,, \\
\Pi^2\big(p(\underline{x})\big) &:= \tfrac{1}{2}\big(p_c(\underline{x}) - p_c(S\underline{x})\big) \,, \\
\Pi^r(p) &:= p - \Pi^1(p) - \Pi^2(p) \,.
\end{aligned}$$

Proposition 6 *: Π^1, Π^2, Π^r are projections onto V^1, V^2, V_r respectively.*

When a basis of V_r is computed by Π_T factorizing polynomials are found. The Groebner bases of those ideals $A^{\underline{i}}$ are computed with the program Groebner in REDUCE. The used version of Groebner uses factorization and has a third parameter, a list for restrictions.

The mathematical idea is the following. Let

$$\begin{array}{ll}
A = A(p_1, \ldots, p_d), & P = q_1 \cdot q_2 \in A \,, \\
A = A_1 \cap A_2, & A_1 = (p_1, \ldots, p_d, q_1), \\
 & A_2 = (p_1, \ldots, p_d, q_2).
\end{array}$$

If $A_1 = A_2$, it is nonsense to compute the Groebner basis twice. But $A_1 = A_2$ normally is not obvious. If q_1 is detected in the computation of the Groebner basis of A_2 then $A_1 \subseteq A_2$ and the zeros of A_2 are zeros of A_1. So the computation for A_2 is cancelled. Therefore a new parameter was defined for the operator Groebner in REDUCE.

Example : In [8] there is given a class of systems of polynomial equations with different symmetries. We mentioned already (1). The solution is

$$\begin{aligned}
&\{\{C \cdot x_1 + 2 \cdot C \cdot x_3 + 2 \cdot x_3^3 - 1, x_2 - x_3, \\
&C^2 + 3 \cdot C \cdot x_3^2 + 2 \cdot x_3^4 - x_3\}, \\
&\{x_1 + x_2 + x_3, \\
&C + x_2^2 + x_2 \cdot x_3 + x_3^2, C \cdot x_3 + x_3^3 + 1\}, \\
&\{x_1 - x_3, \\
&C \cdot x_2 + 2 \cdot C \cdot x_3 + 2 \cdot x_3^3 - 1, C^2 + 3 \cdot C \cdot x_3^2 + 2 \cdot x_3^4 - x_3\}, \\
&\{x_1 - x_3, x_2 - x_3, C \cdot x_3 + 2 \cdot x_3^3 - 1\}, \\
&\{-C^3 \cdot x_3 - 2 \cdot C^2 \cdot x_3^3 - C^2 + 4 \cdot C \cdot x_3^2 + 2 \cdot x_1 - 2 \cdot x_3, \\
&-C^3 \cdot x_3 - 2 \cdot C^2 \cdot x_3^3 - C^2 + 4 \cdot C \cdot x_3^2 + 2 \cdot x_2 - 2 \cdot x_3, \\
&C^2 \cdot x_3^2 + 2 \cdot C \cdot x_3^4 + 2 \cdot C \cdot x_3 - 2 \cdot x_3^3 + 1\}\},
\end{aligned}$$

which is computed in 64 sec on a SUN4.

5 Transformation of the problem

In [9] a model of a chemical reaction in a brusselator with several cells is introduced which is often treated numerically in bifurcation theory in several versions (see for example [3],[4]). For a brusselator with two cells the polynomials are

$$\begin{aligned} p_1(\underline{x},\tau) &:= 2 - 7x_1 + x_1^2 x_2 + \tau(x_3 - x_1), \\ p_2(\underline{x},\tau) &:= 6x_1 - x_1^2 x_2 + 10\tau(x_4 - x_2), \\ p_3(\underline{x},\tau) &:= 2 - 7x_3 + x_3^2 x_4 + \tau(x_1 - x_3), \\ p_4(\underline{x},\tau) &:= 6x_3 - x_3^2 x_4 + 10\tau(x_2 - x_4). \end{aligned} \tag{9}$$

The system (9) is S_2-equivariant, that means you may interchange x_1 with x_3 and x_2 with x_4. But the corresponding matrix $T \in \mathbb{R}^{4,4}$ is not a reflection with respect to a hyperplane. Thus the methods of section 4 don't work.

Our suggestion is a linear transformation of the coordinates and the input polynomials for an arbitrary G–invariant system like (3). Remember the linear representations

$$\rho : G \longrightarrow GL(\mathbb{C}^n)\,, \quad \rho(t)(v) = Tv$$

with representation matrices $T \in \mathbb{C}^{n,n}$ and

$$\vartheta : G \longrightarrow GL\big(\mathbb{C}[\underline{x}]\big)\,, \quad \vartheta(t)\big(p(\underline{x})\big) = p(T\underline{x}).$$

The variables $x_1, \ldots, x_n$ correspond to the canonical basis of $\mathbb{C}^n$. But $\mathbb{C}^n$ has a basis which fits better the symmetry. Compute the bases of the isotypical components W^j of the canonical decomposition

$$\mathbb{C}^n = \sum W^j.$$

This may be done with the projections (7). Put the bases into the columns of $M \in \mathbb{C}^{n,n}$. Thus the matrices $M^{-1}TM$ are blockdiagonal (see [11],[13]). They are the representation matrices of ρ with respect to the basis of $\mathbb{C}^n$ consisting of the bases of the isotypical components. If G has only onedimensional irreducible representations the matrices $M^{-1}TM$ are diagonal.

Substitute $\underline{x} = M\underline{y}$ into $p_1, \ldots, p_d$ of the system (3). The group G operates also on the polynomials $P_i(\underline{y}) := p_i(M\underline{y}) \in \mathbb{C}[\underline{y}]$ in a way that the linear representation ϑ correspond to a new linear representation $\hat{\vartheta} : G \to GL\big(\mathbb{C}[\underline{y}]\big)$:

$$\begin{aligned} \hat{\vartheta}(t)\big(P(\underline{y})\big) &= \hat{\vartheta}(t)\big(p(M\underline{y})\big) &&= \vartheta(t)\big(p(\underline{x})\big) \\ &= p(T\underline{x}) &&= p\big(MM^{-1}TM\underline{y}\big) \;=\; P\big(M^{-1}TM\underline{y}\big). \end{aligned}$$

The vector space $V =< p_1, \ldots, p_d >\subseteq \mathbb{C}[\underline{x}]$, G–invariant with respect to ϑ, correspond to the space $\hat{V} =< P_1, \ldots, P_d >\subseteq \mathbb{C}[\underline{y}]$ which is G–invariant with respect to $\hat{\vartheta}$ and which has a canonical decomposition as well. Compute the bases of the isotypical components with the projections (7).

This is a transformation of the system (3) $p_i(\underline{x}) = 0, i = 1, \ldots, d$ to a system $Q_i(\underline{y}) = 0, i = 1, \ldots, d$ which have by $\underline{x} = M\underline{y}$ the same solutions.

Theorem 7 *: Let G be a finite group with onedimensional irreducible representations only. Let $Q_i(\underline{y}), i = 1, \ldots, d$ be a system which is invariant with respect to*

$$\hat{\vartheta} : G \to GL\big(\mathbb{C}[\underline{y}]\big), \quad \hat{\vartheta}(t)\big(P(\underline{y})\big) = P\big(T\underline{y}\big).$$

Assume that $T \in \mathbb{C}^{n,n}$ are the representation matrices of $\rho : G \to GL(\mathbb{C}^n)$ with respect to a basis which is the union of the bases of the isotypical components $W^j \subseteq \mathbb{C}^n$. Let the polynomials Q_i be elements of the isotypical components of $V^j \subseteq \mathbb{C}[\underline{y}]$.

If the system $Q_i = 0, i = 1, \ldots, d$ is treated with the Buchberger algorithm, all intermediate polynomials are elements of the isotypical components V^j.

Proof: The assumption that all irreducible representations of G are of dimension one gives that ρ operates on a vector $w^j \in W^j$ of an isotypical component W^j like $\rho(t)(w^j) = \psi^j(t)w^j$ with numbers $\psi^j(t) \in \mathbb{C}\ \forall t \in G$. Because the matrices T are the representation matrices of ρ with respect to the special basis they are diagonal and thus each monomial is an element of one of the isotypical components $V^i$of $\mathbb{C}[\underline{y}]$.

If $p \in V^i$ and $m \in V^j$ then the product $p \cdot m$ is an element of another component V^k. The argumentation is done with

$$\hat{\vartheta}(t)\big(p \cdot m\big) = \hat{\vartheta}(t)\big(p\big) \cdot \hat{\vartheta}(t)\big(m\big) = \psi^i(t) \cdot p \cdot \psi^j(t) \cdot m = \big(\psi^j(t) \cdot \psi^j(t)\big) \cdot p \cdot m\,.$$

Thus $< p \cdot m >$ ist G–invariant and by Theorem 5 the subrepresentation $\hat{\vartheta}^{<pm>}$ is isomorphic to one of the irreducible representations ϑ^k. This meens $p \cdot m \in V^k$. If $p_1 \in V^i, p_2 \in V^j$ and m_1, m_2 are monomials such that $p_1 m_1$ and $p_2 m_2$ have the same leading monomial then $m_1 p_1$ and $m_2 p_2$ are elements of the same component V^k and thus also $m_1 p_1 - m_2 p_2 \in V^k$. As the input polynomials Q_i of the Buchberger algorithm are elements of the isotypical components and the algorithm consists of computations like $p_1 m_1 - p_2 m_2$ with cancelling leading term all intermediate polynomials are elements of the isotypical components. ∎

A consequence of Theorem 7 is that for the transformed problem all intermediate polynomials are sparse. An intermediate polynomial is a linear combination of monomials in one of the components V^i. Monomials of components different from V^i do not appear.

Because the polynomials are homogenous in the isotypical components W^j except W^1 which contains the G–invariant polynomials, monomial factors are found during the Groebner computation.

Example : For the S_2-invariant system (9) we choose

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

The polynomials are either even or odd in y_3, y_4. The Groebner basis is received in 2 sec on a SUN4:

$$\begin{aligned}
&\{\{y_4, y_3, y_2 - 3, y_1 - 2, \tau\},\\
&\{y_4, y_3, y_2 - 3, y_1 - 2\},\\
&\{-100 \cdot y_2^2 \cdot y_3 \cdot \tau + 50 \cdot y_2 \cdot y_3 \cdot \tau^2\\
&+215 \cdot y_2 \cdot y_3 \cdot \tau + 20 \cdot y_2 \cdot y_3 - 20 \cdot y_3 \cdot \tau^2 - 54 \cdot y_3 \cdot \tau - 22 \cdot y_3 + 20 \cdot y_4,\\
&-400 \cdot y_2^2 \cdot \tau + 5 \cdot y_2 \cdot y_3^2 + 200 \cdot y_2 \cdot \tau^2\\
&+860 \cdot y_2 \cdot \tau + 20 \cdot y_2 - 80 \cdot \tau^2 - 176 \cdot \tau - 8,\\
&-80 \cdot y_2 \cdot \tau + 2 \cdot y_3^2 \cdot \tau + y_3^2 + 40 \cdot \tau^2 + 148 \cdot \tau + 4,\\
&100 \cdot y_2^2 \cdot \tau^2 - 50 \cdot y_2 \cdot \tau^3 - 215 \cdot y_2 \cdot \tau^2 - 20 \cdot y_2 \cdot \tau + 20 \cdot \tau^3 + 54 \cdot \tau^2 + 24 \cdot \tau + 1,\\
&y_1 - 2\}\}
\end{aligned}$$

The advantage of the transformation is that the system split into subsystems during the computation which saves time. We computed the basis of the brusselator with 3 cells in a linear array as well. This is even for numerical algorithm not an easy problem.

Example : Speer [12] gave the following system of polynomial equations:

$$\begin{aligned}
p_1(\underline{x}) &:= 4 \cdot \beta \cdot (n + 2 \cdot a_1 - 8 \cdot x_1) \cdot (a_2 - a_3) - x_2 \cdot x_3 \cdot x_4 + x_2 + x_4 = 0\\
p_2(\underline{x}) &:= 4 \cdot \beta \cdot (n + 2 \cdot a_1 - 8 \cdot x_2) \cdot (a_2 - a_3) - x_1 \cdot x_3 \cdot x_4 + x_1 + x_3 = 0\\
p_3(\underline{x}) &:= 4 \cdot \beta \cdot (n + 2 \cdot a_1 - 8 \cdot x_3) \cdot (a_2 - a_3) - x_2 \cdot x_1 \cdot x_4 + x_2 + x_4 = 0\\
p_4(\underline{x}) &:= 4 \cdot \beta \cdot (n + 2 \cdot a_1 - 8 \cdot x_4) \cdot (a_2 - a_3) - x_1 \cdot x_3 \cdot x_2 + x_1 + x_3 = 0\\
a_1 &:= x_1 + x_2 + x_3 + x_4\\
a_2 &:= x_1 \cdot x_2 \cdot x_3 \cdot x_4\\
a_3 &:= x_1 \cdot x_2 + x_2 \cdot x_3 + x_3 \cdot x_4 + x_4 \cdot x_1
\end{aligned}$$

It is invariant with respect to the Kleinian group consisting of

$$(x_1, x_2, x_3, x_4) \longrightarrow (x_3, x_4, x_1, x_2) \quad (x_1, x_2, x_3, x_4) \longrightarrow (x_4, x_3, x_2, x_1)$$
$$(x_1, x_2, x_3, x_4) \longrightarrow (x_2, x_1, x_4, x_3)$$

and the identity. For the transformation we choose

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}.$$

During the computation of the Groebner bases with respect to the ordering *revgradlex* factors are detected. Then each Groebner basis is the input of a computation with the ordering *lex* (for explaination of the orderings see [6]).

The final solution is given in figure 2.

Acknowledgement : I like to thank Prof. H. M. Möller who draw my attention to this subject and H. Melenk, W. Neun for the discussions about this subject and Prof. P. Deuflhard for encouraging for this research.

$$
\begin{array}{l}
\{\{y_4, y_2 + y_3, y_1\}, \\
\{-1400 \cdot N^2 \cdot y_1 \cdot \beta - 768 \cdot N^2 \cdot y_4 \cdot \beta + 16512 \cdot N \cdot y_1^2 \cdot \beta - 12288 \cdot N \cdot \beta \\
+79 \cdot N - 32256 \cdot y_1^3 \cdot \beta + 3584 \cdot y_1 \cdot \beta - 952 \cdot y_1 + 75264 \cdot y_4 \cdot \beta - 784 \cdot y_4, \\
y_2 + y_3, \\
616 \cdot N^3 \cdot y_1 \cdot \beta^2 - 9600 \cdot N^2 \cdot y_1^2 \cdot \beta^2 - 6144 \cdot N^2 \cdot y_2^2 \cdot \beta^2 + 6144 \cdot N^2 \cdot \beta^2 \\
-29 \cdot N^2 \cdot \beta + 23040 \cdot N \cdot y_1^3 \cdot \beta^2 + 35072 \cdot N \cdot y_1 \cdot \beta^2 + 288 \cdot N \cdot y_1 \cdot \beta \\
-215040 \cdot y_1^2 \cdot \beta^2 + 2240 \cdot y_1^2 \cdot \beta + 602112 \cdot y_2^2 \cdot \beta^2 - 6272 \cdot y_2^2 \cdot \beta + 258048 \cdot \beta^2 - 5376 \cdot \beta + 28, \\
24 \cdot N^3 \cdot y_1 \cdot \beta^2 - 1216 \cdot N^2 \cdot y_1^2 \cdot \beta^2 - 3 \cdot N^2 \cdot \beta + 16896 \cdot N \cdot y_1^3 \cdot \beta^2 \\
-24576 \cdot N \cdot y_1 \cdot \beta^2 + 144 \cdot N \cdot y_1 \cdot \beta - 36864 \cdot y_1^4 \cdot \beta^2 + 65536 \cdot y_1^2 \cdot \beta^2 \\
-1728 \cdot y_1^2 \cdot \beta - 73728 \cdot \beta^2 + 1536 \cdot \beta - 8\}, \\
\{y_4, y_3, y_2, y_1\}, \\
\{y_4, y_3, y_2, -4 \cdot N \cdot y_1^3 \cdot \beta + 16 \cdot N \cdot y_1 \cdot \beta + y_1^2 - 2\}, \\
\{-N^4 \cdot \beta + 128 \cdot N^2 \cdot y_4^2 \cdot \beta + 256 \cdot N^2 \cdot \beta + 2 \cdot N^2 - 4096 \cdot y_4^4 \cdot \beta - 16384 \cdot y_4^2 \cdot \beta \\
-128 \cdot y_4^2 - 256, \\
y_3, y_2, \\
-N + 8 \cdot y_1\}, \\
\{y_4, -y_2 + y_3, y_1\}, \\
\{1400 \cdot N^2 \cdot y_1 \cdot \beta - 768 \cdot N^2 \cdot y_4 \cdot \beta - 16512 \cdot N \cdot y_1^2 \cdot \beta + 12288 \cdot N \cdot \beta \\
-79 \cdot N + 32256 \cdot y_1^3 \cdot \beta - 3584 \cdot y_1 \cdot \beta + 952 \cdot y_1 + 75264 \cdot y_4 \cdot \beta - 784 \cdot y_4, \\
-y_2 + y_3, \\
-616 \cdot N^3 \cdot y_1 \cdot \beta^2 + 9600 \cdot N^2 \cdot y_1^2 \cdot \beta^2 + 6144 \cdot N^2 \cdot y_2^2 \cdot \beta^2 - 6144 \cdot N^2 \cdot \beta^2 \\
+29 \cdot N^2 \cdot \beta - 23040 \cdot N \cdot y_1^3 \cdot \beta^2 - 35072 \cdot N \cdot y_1 \cdot \beta^2 - 288 \cdot N \cdot y_1 \cdot \beta \\
+215040 \cdot y_1^2 \cdot \beta^2 - 2240 \cdot y_1^2 \cdot \beta - 602112 \cdot y_2^2 \cdot \beta^2 + 6272 \cdot y_2^2 \cdot \beta - 258048 \cdot \beta^2 + 5376 \cdot \beta - 28, \\
24 \cdot N^3 \cdot y_1 \cdot \beta^2 - 1216 \cdot N^2 \cdot y_1^2 \cdot \beta^2 - 3 \cdot N^2 \cdot \beta + 16896 \cdot N \cdot y_1^3 \cdot \beta^2 \\
-24576 \cdot N \cdot y_1 \cdot \beta^2 + 144 \cdot N \cdot y_1 \cdot \beta - 36864 \cdot y_1^4 \cdot \beta^2 + 65536 \cdot y_1^2 \cdot \beta^2 \\
-1728 \cdot y_1^2 \cdot \beta - 73728 \cdot \beta^2 + 1536 \cdot \beta - 8\}, \\
\{y_4, y_3, y_2, y_1\}, \\
\{y_4, y_3, y_2, -4 \cdot N \cdot y_1^3 \cdot \beta + 16 \cdot N \cdot y_1 \cdot \beta + y_1^2 - 2\}, \\
\{y_4, y_3, y_2, y_1\}, \\
\{y_4, 8 \cdot N \cdot y_1 \cdot \beta - 32 \cdot y_1^2 \cdot \beta - 32 \cdot y_3^2 \cdot \beta - 1, \\
y_2, \\
8 \cdot N^2 \cdot y_1 \cdot \beta - 128 \cdot N \cdot y_1^2 \cdot \beta - N + 512 \cdot y_1^3 \cdot \beta - 512 \cdot y_1 \cdot \beta + 8 \cdot y_1\}, \\
\{y_4, y_3, y_2, y_1\}, \\
\{y_4, y_3, \\
8 \cdot N \cdot y_1 \cdot \beta - 32 \cdot y_1^2 \cdot \beta - 32 \cdot y_2^2 \cdot \beta - 1, \\
8 \cdot N^2 \cdot y_1 \cdot \beta - 128 \cdot N \cdot y_1^2 \cdot \beta - N + 512 \cdot y_1^3 \cdot \beta - 512 \cdot y_1 \cdot \beta + 8 \cdot y_1\}, \\
\{-N \cdot y_4 + 24 \cdot y_2 \cdot y_3, \\
-4 \cdot N^2 \cdot \beta + 1152 \cdot y_2^2 \cdot \beta + 1152 \cdot y_3^2 \cdot \beta - 864 \cdot \beta + 9, \\
2 \cdot N^4 \cdot \beta - 2304 \cdot N^2 \cdot y_2^2 \cdot \beta - 288 \cdot N^2 \cdot \beta - 9 \cdot N^2 + 663552 \cdot y_2^4 \cdot \beta \\
-497664 \cdot y_2^2 \cdot \beta + 5184 \cdot y_2^2, \\
-N + 24 \cdot y_1\}\}
\end{array}
$$

Figure 2: Solution of the system from Speer

References

[1] B. Buchberger: *Groebner-basis: an Algorithmic Method in Polynomial Ideal Theory.* In: Multidimensional Systems Theory (ed.: N. K. Bose), D. Reidel Publishing Company, Dordrecht, pp. 184–232 (1985).

[2] H. Caprasse, J. Demaret : private communication.

[3] M. Dellnitz, B. Werner: *Computational Methods for Bifurcation Problems with Symmetries - with special Attention to Steady State and Hopf Bifurcation Points.* Hamburger Beiträge zur Angewandten Mathematik, Reihe A, Preprint 15 (1988).

[4] P. Deuflhard, B. Fiedler, P. Kunkel: *Efficient Numerical Pathfollowing beyond Critical Points.* SIAM J. Numer. Anal. 24, pp. 912–927 (1987).

[5] A. C. Hearn: *REDUCE User's Manual, Version 3.3.* The Rand Corporation, Santa Monica (1987).

[6] H. Melenk, H. M. Möller, W. Neun: *GROEBNER A Package for Calculating Groebner Bases.* (1990).

[7] H. Melenk, H. M. Moeller, W. Neun: *Symbolic Solution of Large Stationary Chemical Kinetics Problems.* Impact of Computing in Science and Engineering 1, pp. 138–167 (1989).

[8] V. W. Noonburg: *A Neural Network Modeled by an Adaptive Lotka-Volterra System.* accepted for publication in SIAM J. Appl. Math. (1988).

[9] I. Prigogine, R. Lefever: *Symmetry breaking Instabilities in Dissipative Systems II.* J. Chem. Phys. 48, pp. 1695–1701 (1968).

[10] K. Rimey: *Problem Section (Rotating Fluids): A System of Polynomial Equations and a Solution by an Unusual Method.* ACM SIGSAM Bulletin 18(1), pp. 30–32 (1984).

[11] J.-P. Serre: *Linear Representations of Finite Groups.* Springer–Verlag, New York (1977).

[12] E. R. Speer: private communication.

[13] E. Stiefel, A. Fässler: *Gruppentheoretische Methoden und ihre Anwendung.* B. G. Teubner, Stuttgart (1979).

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