

Uncolorable mixed hypergraphs

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Abstract. A mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ consists of the vertex set X and two families of subsets: the family \mathcal{E} of edges and the family \mathcal{A} of co-edges. In a coloring every edge $E \in \mathcal{E}$ has at least two vertices of different colors, while every co-edge $A \in \mathcal{A}$ has at least two vertices of the same color. The largest (smallest) number of colors for which there exists a coloring of a mixed hypergraph \mathcal{H} using all the colors is called the upper (lower) chromatic number and is denoted $\bar{\chi}(\mathcal{H})$ ($\chi(\mathcal{H})$). A mixed hypergraph is called uncolorable if it admits no coloring.

We show that there exist uncolorable mixed hypergraphs $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ with arbitrary difference between the upper chromatic number $\bar{\chi}(\mathcal{H}_{\mathcal{A}})$ of $\mathcal{H}_{\mathcal{A}} = (X, \mathcal{A})$ and the lower chromatic number $\chi(\mathcal{H}_{\mathcal{E}})$ of $\mathcal{H}_{\mathcal{E}} = (X, \mathcal{E})$. Moreover, for any $k = \bar{\chi}(\mathcal{H}_{\mathcal{A}}) - \chi(\mathcal{H}_{\mathcal{E}})$, the minimum number $v(k)$ of vertices of an inclusionwise minimal uncolorable mixed hypergraph is exactly $k + 4$.

We introduce a measure of uncolorability which is called the vertex uncolorability number. We propose an algorithm that finds an estimate on it and is a greedy mixed hypergraph coloring algorithm at the same time. We also show that the colorability problem can be expressed in terms of integer linear programming.

Concerning particular cases, we describe those complete (l, m) -uniform mixed hypergraphs which are uncolorable, and observe that for given (l, m) almost all complete (l, m) -uniform mixed hypergraphs are uncolorable, whereas generally almost all complete mixed hypergraphs are colorable.

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1 Introduction

We use the terminology of [19, 20]. A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, where $X = \{x_1, x_2, \dots, x_n\}$ ($n \geq 1$) is the *vertex set*, $\mathcal{A} = \{A_1, A_2, \dots, A_l\}$ ($A_i \subseteq X$, $i = 1, \dots, l$, $l \geq 1$) is the family of *co-edges*, and $\mathcal{E} = \{E_1, E_2, \dots, E_m\}$ ($E_j \subseteq X$, $j = 1, \dots, m$, $m \geq 1$) is the family of *edges*. In the context of the present paper, we restrict our attention to mixed hypergraphs satisfying $|A_i| \geq 2$ and $|E_j| \geq 2$ for all $1 \leq i \leq l$ and $1 \leq j \leq m$, and assume that no edge (co-edge) is contained in any other edge (co-edge). (In hypergraph terminology, these conditions mean that the *lower rank* is at least 2, and both \mathcal{A} and \mathcal{E} are supposed to be *Sperner systems*. For standard notions concerning hypergraphs, we refer to [1].)

We use the following definition [20] of colorings of a mixed hypergraph with $\lambda \geq 1$ colors.

Definition 1 A coloring of a mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ with λ colors is a mapping $c : X \rightarrow \{1, 2, \dots, \lambda\}$ such that the following two conditions hold:

- (1) each co-edge $A \in \mathcal{A}$ has at least two vertices of the same color;
- (2) each edge $E \in \mathcal{E}$ has at least two vertices colored differently.

We shall also need to generalize this notion for the more general situation where some of the vertices may not get colored:

Definition 2 A partial coloring of a mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ with λ colors is a mapping $c : Y \rightarrow \{1, 2, \dots, \lambda\}$, $Y \subseteq X$, $Y \neq \emptyset$, such that the following two conditions hold:

- (1) each co-edge $A \in \mathcal{A}$ with all the vertices colored, has at least two vertices of the same color;
- (2) each edge $E \in \mathcal{E}$ with all the vertices colored, has at least two vertices colored differently.

The hypergraphs $\mathcal{H}_{\mathcal{E}} = (X, \mathcal{E})$ and $\mathcal{H}_{\mathcal{A}} = (X, \mathcal{A})$ are called the *partial hypergraph* and the *partial co-hypergraph* of the initial mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, respectively. We can view the partial hypergraph and the partial co-hypergraph of a mixed hypergraph as the partial cases of mixed hypergraphs (when $\mathcal{A} = \emptyset$ and $\mathcal{E} = \emptyset$).

For any subset $Y \subseteq X$, we call the mixed hypergraph $\mathcal{H}/Y = (Y, \mathcal{A}', \mathcal{E}')$ the *induced subhypergraph* of \mathcal{H} if \mathcal{A}' and \mathcal{E}' consist of all those members of \mathcal{A} and of \mathcal{E} , respectively, which are entirely contained in Y .

Definition 3 The largest (smallest) number of colors for which there exists a coloring of \mathcal{H} when all the colors are used, is called the upper (lower) chromatic number and is denoted by $\bar{\chi}(\mathcal{H})$ ($\chi(\mathcal{H})$).

Definition 4 *A mixed hypergraph is called uncolorable if it admits no coloring. Otherwise it is called colorable. The colorability problem takes a mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ as input, and asks whether \mathcal{H} admits at least one coloring.*

The colorability problem represents a new type of problems in coloring theory. It contains the problem to find the coloring of a graph using a fixed number of colors as a particular case. It is closely related, but not limited, to the problem to characterize all uncolorable mixed hypergraphs. The latter was first formulated in [20]. Particular cases of the colorability problem appeared in [2, 11].

The aim of this paper is to begin a systematic study of the colorability problem in mixed hypergraphs. We show that, together with a general approach, quite different methods are required to determine the conditions for colorability in different classes of mixed hypergraphs. Nevertheless, one of the basic goals is to find the list of all minimal uncolorable mixed hypergraphs from some given class, in order to describe the colorable structures in terms of forbidden subhypergraphs with respect to the class in question.

The paper is organized as follows. In the next section we show that there exist uncolorable mixed hypergraphs $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ with arbitrarily large difference between the upper chromatic number $\bar{\chi}(\mathcal{H}_{\mathcal{A}})$ of the partial co-hypergraph $\mathcal{H}_{\mathcal{A}} = (X, \mathcal{A})$ and the lower chromatic number $\chi(\mathcal{H}_{\mathcal{E}})$ of the partial hypergraph $\mathcal{H}_{\mathcal{E}} = (X, \mathcal{E})$. We also describe uncolorable mixed hypergraphs of smallest order in the following sense: for any $k = \bar{\chi}(\mathcal{H}_{\mathcal{A}}) - \chi(\mathcal{H}_{\mathcal{E}})$, the minimum number $v(k)$ of vertices of an inclusionwise minimal uncolorable mixed hypergraph without isolated vertices is exactly $k + 4$. (The isolated vertices have to be excluded here, otherwise the problem becomes trivial by taking the mixed uncolorable hypergraph with just one edge and one co-edge, $A_1 = E_1 = \{x_1, x_2\}$, on $n = k + 3$ vertices; then the lower chromatic number is 2 and the upper chromatic number is $n - 1$.)

In Section 3 we introduce a measure of uncolorability (so called vertex uncolorability number) that is the minimum number of vertices to be deleted in such a way that the mixed hypergraph obtained is colorable. A greedy algorithm to find an estimate on the vertex uncolorability number is developed. It is related to such known parameters as the coloring number of graphs introduced by Erdős and Hajnal [5], the Szekeres–Wilf number [14] (see also [8]), and the resistance (originality) of a co-hypergraph introduced in [20]. It is the first greedy mixed hypergraph coloring algorithm at the same time.

In Section 4 we show how the colorability problem can be formulated as an integer programming Problem. The main point here is that the number of constraints need not grow much faster than that of the independent sets in $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{E}}$.

In the last section we consider some particular cases of uncolorable mixed hypergraphs, and investigate the asymptotic behavior of uncolorability in one special case. Namely, we describe those complete (l, m) -uniform mixed hypergraphs (where every

l vertices form a co-edge and every m vertices form an edge) which are uncolorable, and show that for given (l, m) almost all complete (l, m) -uniform mixed hypergraphs are uncolorable. In contrast, we prove that generally almost all complete mixed hypergraphs are colorable. Some results on uncolorability are derived for constructions using graphs, too. At last, the necessary and sufficient conditions for the uncolorability of mixed hypertrees is obtained.

2 Minimal uncolorable mixed hypergraphs

The following problem was formulated in [20]:

Let $v(k)$, $k \geq 0$, be the smallest natural number n such that there exists an inclusionwise minimal uncolorable mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, $|X| = n$, for which

$$\bar{\chi}(\mathcal{H}_{\mathcal{A}}) - \chi(\mathcal{H}_{\mathcal{E}}) = k.$$

Determine $v(k)$ for $k = 0, 1, 2, \dots$

The theorem below gives the characterization of these numbers.

Theorem 1 *For every $k \geq 0$, $v(k) = k + 4$.*

Proof. Let $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ be an uncolorable mixed hypergraph such that $|X| = n = v(k)$, and $\bar{\chi}(\mathcal{H}_{\mathcal{A}}) - \chi(\mathcal{H}_{\mathcal{E}}) = k$. We have to prove that $n = v(k) = k + 4$.

We show first that $n \geq k + 4$. Assume on the contrary that $n < k + 4$. Since \mathcal{H} is uncolorable, $\chi(\mathcal{H}_{\mathcal{E}}) \geq 2$. If $\chi(\mathcal{H}_{\mathcal{E}}) \geq 3$, then $\bar{\chi}(\mathcal{H}_{\mathcal{A}}) \geq k + 3$, that implies $n = k + 3$, $\chi(\mathcal{H}_{\mathcal{A}}) = n$, therefore \mathcal{H} contains no co-edges and thus it is colorable, a contradiction.

Hence, $\chi(\mathcal{H}_{\mathcal{E}}) = 2$. Then we have only two possibilities for the number of vertices: $n = k + 2$ or $n = k + 3$.

Similarly to the previous case, for $n = k + 2$ and $\bar{\chi}(\mathcal{H}_{\mathcal{A}}) = k + 2$ it follows that the mixed hypergraph \mathcal{H} contains no co-edges, and therefore is not uncolorable. Hence, consider the last case $n = k + 3$. Since $\bar{\chi}(\mathcal{H}_{\mathcal{A}}) = k + 2 = n - 1$, the partial co-hypergraph $\mathcal{H}_{\mathcal{A}} = (X, \mathcal{A})$ is a co-bistar [20], i.e., a co-hypergraph having two vertices, say x_1 and x_2 , that belong to all co-edges. If this pair (x_1, x_2) were not an edge in $\mathcal{H}_{\mathcal{E}}$, then we could color x_1, x_2 with the first color and the remaining vertices all differently, that contradicts again to the uncolorability of \mathcal{H} . Consequently, the pair (x_1, x_2) is an edge in \mathcal{H} . Since \mathcal{H} is an uncolorable hypergraph minimal under inclusion, no co-edge may coincide with (x_1, x_2) , and therefore the cardinality of each co-edge is at least 3.

Consider an arbitrary 2-coloring of $\mathcal{H}_{\mathcal{E}} = (X, \mathcal{E})$. It is at the same time a coloring of the initial mixed hypergraph \mathcal{H} , because each co-edge contains at least three vertices, and hence \mathcal{H} again is colorable. This contradiction shows that $v(k) = n \geq k + 4$.

Now, in order to prove the converse inequality $v(k) \leq k + 4$, we construct a series of examples of minimal uncolorable mixed hypergraphs with $\bar{\chi}(\mathcal{H}_{\mathcal{A}}) - \chi(\mathcal{H}_{\mathcal{E}}) = k$ and $n = k + 4$, $k = 0, 1, 2, \dots$. The construction will depend on the parity of k ; we first describe the particular cases $k = 0, 1$ that can be verified directly.

$k = 0$.

Let $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, where $X = \{1, 2, 3, 4\}$, $\mathcal{A} = \{(1, 2, 3), (1, 2, 4)\}$, $\mathcal{E} = \{(1, 2), (2, 3), (2, 4), (3, 4)\}$.

$k = 1$.

Consider $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, where $X = \{1, 2, 3, 4, 5\}$, $\mathcal{A} = \{(1, 2, 3), (1, 2, 4), (1, 2, 5)\}$, $\mathcal{E} = \{(1, 2), (3, 4), (4, 5), (3, 5)\}$.

$k = 2l$, $l \geq 1$.

Construct the mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, where $X = \{1, 2, 3, \dots, k + 4\}$, $\mathcal{A} = \{(1, 2, i) \mid 3 \leq i \leq k + 4\}$, and $\mathcal{E} = \{(i, i + 1) \mid 1 \leq i \leq k + 3\} \cup \{(k + 4, 2)\}$.

In other words $\mathcal{H}_{\mathcal{A}} = (X, \mathcal{A})$ represents a 3-uniform co-bistar in which the vertices 1, 2 belong to all co-edges, and therefore $\bar{\chi}(\mathcal{H}_{\mathcal{A}}) = n - 1 = k + 3$.

Moreover, $\mathcal{H}_{\mathcal{E}} = (X, \mathcal{E})$ is the odd cycle $(2, 3, 4, \dots, k + 4, 2)$ with the pendant edge $(1, 2)$, so that $\chi(\mathcal{H}_{\mathcal{E}}) = 3$.

Let $c(i)$ denote the color of the vertex i , $i = 1, 2, \dots, n$. In any possible coloring of \mathcal{H} , the vertices 1, 2 are colored differently, say $c(1) = 1$, $c(2) = 2$.

Because of the edge $(2, 3)$ we have $c(3) \neq c(2)$ and, because of the co-edge $(1, 2, 3)$, the unique possibility for vertex 3 to be colored is $c(3) = c(1) = 1$. In the same way, $c(4) \neq c(3)$ and, because of the co-edge $(1, 2, 4)$, the unique possibility for vertex 4 to be colored is $c(4) = c(2) = 2$.

It is clear now that the colors have to alternate on the cycle $(2, 3, 4, \dots, k + 4)$. Since $c(k + 3) = 1$ and $c(2) = 2$, we can color the vertex $k + 4$ neither with color 1, nor with color 2. However, any other color $c(k + 4)$ is infeasible on the co-edge $(1, 2, k + 4)$. Consequently, \mathcal{H} is uncolorable. One can easily check that it is minimal under inclusion.

$k = 2l + 1$, $l \geq 1$.

Construct the mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, where $X = \{1, 2, 3, \dots, k + 4\}$, $\mathcal{A} = \{(1, 2, i) \mid 3 \leq i \leq k + 4\}$, and $\mathcal{E} = \{(1, 2)\} \cup \{(i, i + 1) \mid 3 \leq i \leq k + 3\} \cup \{(k + 4, 3)\}$.

Again, $\mathcal{H}_{\mathcal{A}} = (X, \mathcal{A})$ represents a 3-uniform co-bistar with the vertices 1, 2 shared by all the co-edges, so that $\bar{\chi}(\mathcal{H}_{\mathcal{A}}) = n - 1 = k + 3$. In the present case $\mathcal{H}_{\mathcal{E}} = (X, \mathcal{E})$ is a disconnected graph having the edge $(1, 2)$ as the first component and the odd cycle $(3, 4, \dots, k + 4, 3)$ as the second component, yielding again $\chi(\mathcal{H}_{\mathcal{E}}) = 3$.

Let $c(1) = 1$, $c(2) = 2$. For $c(3)$ there are only two possibilities: $c(3) = 1$, or $c(3) = 2$. By symmetry reasons, we may assume $c(3) = 1$. Then, similarly to the argument above, we obtain $c(4) = 2$, $c(5) = 1$, $c(6) = 2$, and so on, i.e., the colors have to alternate along the odd cycle. Since the vertex $k + 4$ cannot be colored with any color (because of the co-edge $(1, 2, k + 4)$ and the edges $(k + 3, k + 4)$ and $(k + 4, 3)$), we conclude that \mathcal{H} is uncolorable. Minimality is also easily seen. Hence, the theorem follows. \square

3 Uncolorability measure and greedy algorithm

In this section we introduce several concepts and parameters related to (un)colorability, and apply them to develop an algorithm that colors a mixed hypergraph or finds a fairly large colorable part of it.

Definition 5 For a mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ the vertex uncolorability number $\Omega(\mathcal{H})$ is the minimum number of vertices to be deleted in such a way that the hypergraph obtained is colorable.

Clearly, $0 \leq \Omega(\mathcal{H}) \leq n - 1$ holds for any mixed hypergraph, and $\Omega(\mathcal{H}) = 0$ holds if and only if \mathcal{H} is colorable.

Denote by $\tau(\mathcal{H}_{\mathcal{E}})$ the *transversal number* of a hypergraph $\mathcal{H} = (X, \mathcal{E})$ [1, p. 53], i.e., the minimum cardinality of a subset of vertices that contains at least one vertex from every edge. The next upper bound is immediate.

Proposition 1 For every mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ the following inequality holds:

$$\Omega(\mathcal{H}) \leq \min\{\tau(\mathcal{H}_{\mathcal{A}}), \tau(\mathcal{H}_{\mathcal{E}})\}. \quad \square$$

In the algorithmic sense, the vertex uncolorability number is a hard-to-determine parameter, already for the smallest particular case:

Theorem 2 The problem to decide for an arbitrary mixed hypergraph \mathcal{H} whether $\Omega(\mathcal{H}) = 0$ is NP-complete.

Proof. We will prove that the recognition problem of colorable mixed hypergraphs is at least as hard as the problem of hypergraph 2-colorability. Since the latter is NP-complete [10], the same will follow for the former, too.

For an arbitrary hypergraph $\mathcal{H} = (X, \mathcal{E})$ with vertex set X , construct the mixed hypergraph

$$\mathcal{H}^3 = (X, \binom{X}{3}, \mathcal{E}),$$

where $\binom{X}{3}$ denotes the collection of all 3-element subsets of X , i.e., each triple of vertices forms a co-edge in \mathcal{H}^3 . Every assignment of the vertices to at most two colors is a feasible coloring of $\mathcal{H}_{\mathcal{A}}^3$, while more than two colors would yield an unfeasible co-edge. This fact implies that \mathcal{H}^3 is colorable if and only if \mathcal{H} is 2-colorable. \square

Let $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ be a mixed hypergraph, and let $\mathcal{A}(x)$ ($\mathcal{E}(x)$) denote the set of co-edges (edges) containing the vertex $x \in X$.

Definition 6 *The mono-degree $m(x, \mathcal{H})$ of a vertex $x \in X$ in a mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ is the maximum cardinality of a subfamily $\mathcal{E}_1(x) \subseteq \mathcal{E}(x)$ such that*

$$E_i \cap E_j = \{x\} \quad \forall E_i, E_j \in \mathcal{E}_1(x), E_i \neq E_j.$$

We denote

$$M(\mathcal{H}_{\mathcal{E}}) = \max_{Y \subseteq X} \min_{x \in Y} m(x, \mathcal{H}/Y).$$

If applied to graphs, $M(\mathcal{H}_{\mathcal{E}})+1$ is equal to the so called *coloring number*, introduced and studied by Erdős and Hajnal [5], and also to the Szekeres–Wilf number [14], see [8, p. 8]. The latter was used implicitly by Vizing in [17]. If $\omega(G)$ means the maximum cardinality of a clique of the graph G , then the Szekeres–Wilf number is at least $\omega(G) - 1$. It was shown in [18] (see also [21, p. 268]), that it attains this smallest possible value for a graph and for all its induced subgraphs if and only if the graph G is chordal (i.e., if every cycle of length at least four has two non-consecutive adjacent vertices; also called triangulated or rigid circuit graphs, introduced by Hajnal and Surányi [7] and characterized by Dirac [4]).

In [1, p. 116], $m(x, \mathcal{H})$ was termed β -degree, and the following theorem was proved:

Theorem 3 *For any hypergraph $\mathcal{H} = (X, \mathcal{E})$,*

$$\chi(\mathcal{H}) \leq 1 + M(\mathcal{H}_{\mathcal{E}}).$$

In hypergraph coloring theory this value plays an important role. It shows how effective the consecutive greedy hypergraph coloring algorithm described in [1] can be.

Call the value

$$b(x, \mathcal{H}) = \max \{ |\mathcal{A}(x) \cap \mathcal{A}(y)| : y \in X, y \neq x \}$$

the *bi-degree* of a vertex x .

The value

$$o(x, \mathcal{H}) = |\mathcal{A}(x)| - b(x, \mathcal{H})$$

was introduced in [20] and was called the *originality* of vertex x in the co-hypergraph $\mathcal{H}_{\mathcal{A}}$. It expresses the similarity of a vertex to its neighbors in $\mathcal{H}_{\mathcal{A}}$.

Clearly, $o(x, \mathcal{H}) = o(x, \mathcal{H}_{\mathcal{A}})$, $o(x, \mathcal{H}) \geq 0$, and $o(\mathcal{H}, x) = 0$ means that there exists some another vertex $y \in X$ belonging to all those co-edges which contain x .

As it was shown in [20], $o(x, \mathcal{H})$ plays an important role in the greedy consecutive co-hypergraph coloring algorithm. Namely,

Theorem 4 *The number of colors that may be lost at any step of the consecutive greedy co-hypergraph coloring algorithm does not exceed the value*

$$O(\mathcal{H}_{\mathcal{A}}) + 1 = \max_{Y \subseteq X} \min_{x \in Y} o(\mathcal{H}/Y, x) + 1.$$

Call the value $O(\mathcal{H}_{\mathcal{A}})$ the *resistance* of the co-hypergraph $\mathcal{H}_{\mathcal{A}}$. (In [20] it was called the originality of the co-hypergraph, but we shall see later that ‘resistance’ is a better word for it.)

It follows from the above observations that the resistance of a co-hypergraph is dual in a combinatorial sense to the value of $M(\mathcal{H}_{\mathcal{E}})$.

If $M(\mathcal{H}_{\mathcal{E}})$ is small, then the lower chromatic number of the hypergraph $\mathcal{H}_{\mathcal{E}}$ is small, too. It means that at each coloring step of the greedy algorithm there exist several possibilities to color the next vertex. And the less $M(\mathcal{H}_{\mathcal{E}})$ is, the more such possibilities we have.

If $O(\mathcal{H}_{\mathcal{A}})$ is large, then $\mathcal{H}_{\mathcal{A}}$ necessarily contains a subhypergraph which causes the large lose of colors as a result of re-colorings of monochromatic components [20]. Then the upper chromatic number of the co-hypergraph $\mathcal{H}_{\mathcal{A}}$ is small on the average. On the other hand, if $O(\mathcal{H}_{\mathcal{A}})$ is small, then the greedy algorithm for the upper chromatic number does not lose many colors. Again, at each coloring step there exist several possibilities to color the next vertex. The less $O(\mathcal{H}_{\mathcal{A}})$ is, the more such possibilities we have.

As a result, for $O(\mathcal{H}_{\mathcal{A}})$ small, the upper chromatic number of the co-hypergraph $\mathcal{H}_{\mathcal{A}}$ is large. This explains the term ‘resistance’ for $O(\mathcal{H}_{\mathcal{A}})$. It shows how the structure of a co-hypergraph ‘resists’ to the greedy algorithm.

However, these values have very different nature. $M(\mathcal{H}_{\mathcal{E}})$ shows directly an estimate on the lower chromatic number of $\mathcal{H}_{\mathcal{E}}$. In contrast, $O(\mathcal{H}_{\mathcal{A}})$ does not give an estimate on the upper chromatic number of $\mathcal{H}_{\mathcal{A}}$. It is necessary to implement the greedy algorithm in order to obtain such an estimate.

Now we will combine these two values in order to form a parameter that expresses the possibility to color (with some approximation) the mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, or, at least, to color as many vertices as possible using local information (based on vertex degrees). In this way we obtain some estimate on the vertex uncolorability number.

We need the following notions introduced in [16].

Let $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$. Assume that c is a coloring of the mixed hypergraph \mathcal{H} . Now consider a mixed hypergraph \mathcal{H}' constructed by adding a vertex y to the vertex set X , and adding a family \mathcal{A}_y of co-edges to \mathcal{A} , where each co-edge $A \in \mathcal{A}_y$ contains y , and a family \mathcal{E}_y of edges to \mathcal{E} , where each edge $E \in \mathcal{E}_y$ contains y .

The co-edge $A \in \mathcal{A}_y$ is called *influencing* with respect to the coloring c of \mathcal{H} if all its vertices except y are colored with *mutually different* colors in the coloring c of \mathcal{H} . Analogously, the edge $E \in \mathcal{E}_y$ is called *influencing* with respect to the coloring c of \mathcal{H} if all its vertices except y are colored with the *same* color in the coloring c . Note that all vertices (but y itself) of an influencing edge or co-edge are supposed to have been colored. Influencing co-edges and edges define all constraints concerning the possibilities for extending the coloring c of \mathcal{H} to the vertex y .

Let $c(A)$ ($c(E)$) be the set of colors used by the vertices in the co-edge A (edge E) in the coloring c of \mathcal{H} . Let

$$FS(y) = \bigcap \{c(A) : A \in \mathcal{A}_y, A \text{ is an influencing co-edge}\}.$$

It means that $FS(y)$ is the set of colors one of which must be used on y in extending c to y . We call $FS(y)$ the *Forcing Set* of y .

Let

$$VS(y) = \bigcup \{c(E) : E \in \mathcal{E}_y, E \text{ is an influencing edge}\}.$$

It means that $VS(y)$ is the set of colors which must *not* be used on y in extending the coloring c to y . We call $VS(y)$ the *Veto Set* of y .

Definition 7 *The vertex y is called uncolorable in extending the coloring c , if*

1. *there exists at least one influencing co-edge, and*
2. $FS(y) \setminus VS(y) = \emptyset$.

Definition 8 *The first free color is:*

1. *the smallest number in the list $FS(y) \setminus VS(y)$ when there exists at least one influencing co-edge and $FS(y) \setminus VS(y) \neq \emptyset$;*
2. *the first natural number missing from $VS(y)$ otherwise.*

Definition 9 *For a vertex $x \in X$ of a mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, the value*

$$\phi(x, \mathcal{H}) = |\mathcal{A}(x)| - b(x, \mathcal{H}_{\mathcal{A}}) + m(x, \mathcal{H}_{\mathcal{E}})$$

is called the risk of x .

Clearly, $\phi(x, \mathcal{H}) \geq 0$, and $\phi(x, \mathcal{H}) = 0$ implies that $\mathcal{E}(x) = \emptyset$ (since we consider mixed hypergraphs without loops and co-loops).

Next we introduce the value for which we preserve the term ‘resistance.’

Definition 10 For a mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, the value

$$\Phi(\mathcal{H}) = \max_{Y \subseteq X} \min_{x \in Y} \phi(x, \mathcal{H}/Y)$$

is called the resistance of \mathcal{H} .

Hence, $\Phi(\mathcal{H}) \geq 0$, and $\Phi(\mathcal{H}) = 0$ means that $\mathcal{E} = \emptyset$ holds and, thus, \mathcal{H} is colorable. If \mathcal{H} is a simple graph, then $\Phi(\mathcal{H})$ equals the coloring number, if \mathcal{H} is a co-hypergraph, then $\Phi(\mathcal{H})$ equals the resistance (introduced as ‘originality’ in [20]).

Now, in order to find an estimate on the vertex uncolorability number $\Omega(\mathcal{H})$, we propose a greedy coloring algorithm for an arbitrary mixed hypergraph \mathcal{H} . The idea is to find a fairly good ordering of the vertices and then greedily color \mathcal{H} successively, using the local information as much as possible. First we decompose the hypergraph by consecutively eliminating the vertices with minimal risk. In this way we obtain some ordering of the vertex set. Then we start coloring, sequentially reconstructing the initial mixed hypergraph by adding vertices in the reverse order. At each coloring step we use the Veto Set and the Forcing Set in order to choose the most appropriate color for the next vertex.

ALGORITHM (greedy mixed hypergraph coloring)

INPUT: An arbitrary mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, $X = \{1, 2, \dots, n\}$.

OUTPUT: A partial coloring $C = (c(1), c(2), \dots, c(n))$ of \mathcal{H} , and the list U of uncolored vertices.

STEP 1. Set $i := n$; declare $\mathcal{H}_n := \mathcal{H}$; find a vertex of minimum risk and label it x_n .

STEP 2. Set $i := i - 1$; if $i = 0$, then go to STEP 5.

STEP 3. Form a subhypergraph $\mathcal{H}_i := \mathcal{H}/\{X - \{x_n, \dots, x_{i+1}\}\} = \mathcal{H}_{i+1} - x_{i+1}$.

STEP 4. Find a vertex of minimum risk in \mathcal{H}_i and label it x_i ; go to STEP 2.

STEP 5. Set the list of uncolored vertices $U := \{1, 2, \dots, n\}$, set the color vector $C := (0, 0, \dots, 0)$.

STEP 6. Color the vertex x_1 with the first color: $c(x_1) := 1$; set $U := U \setminus \{x_1\}$; set $i := 1$.

STEP 7. Set $i := i + 1$; if $i = n + 1$, then go to STEP 12.

STEP 8. If there are no influencing co-edges, then let $c(x_i) := \{\text{the first free color}\}$, $U := U \setminus \{x_i\}$ and go to STEP 7.

STEP 9. Construct $FS(x_i)$ and $VS(x_i)$.

STEP 10. If $FS(x_i) \setminus VS(x_i) = \emptyset$, then go to STEP 7.

STEP 11. $c(x_i) = \{\text{the first free color}\}$, $U := U \setminus \{x_i\}$; go to STEP 7.

STEP 12. Output: C, U . End.

Remark 1. In graph theory the bi-chromatic chain recoloring method by Kempe [9] is well known. It can be generalized to hypergraphs. It is not possible to use it for any recoloring in the *mixed hypergraphs*, however, because some co-edges may get colored unfeasibly.

In [20] it was developed the so called monochromatic component re-coloring method. It is dual in combinatorial sense to the method by Kempe. However, again, it is not possible to use it for any monochromatic component re-coloring in the mixed hypergraphs, because the wrongly colored edges may appear as a result.

Therefore, when encountering an uncolorable vertex in a mixed hypergraph, we are able to use neither the recoloring method by Kempe [9], nor its opposite developed in [20]. Re-colorings in mixed hypergraphs deserve a separate study.

Remark 2. At every coloring step the algorithm chooses the vertex of minimum risk that greedily decreases the sum of the cardinalities of the Veto Set and Forcing Set. It minimizes the possible conflict of constraints. (This explains the term ‘risk’ for $\phi(x, \mathcal{H})$.) There may be other ways to minimize the possibility of such conflicts.

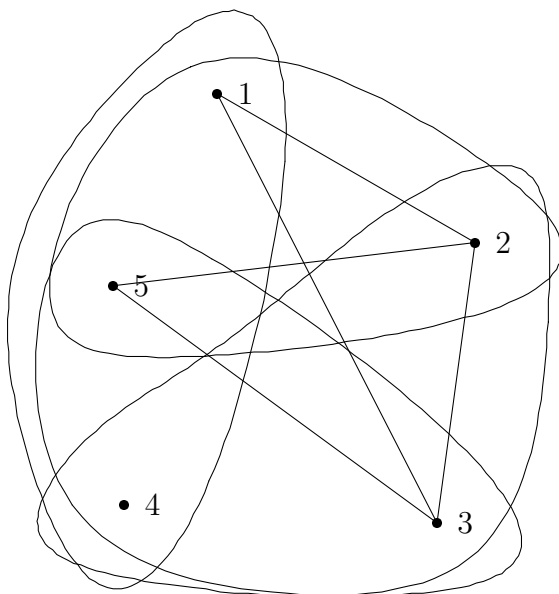
Remark 3. If $\Phi(\mathcal{H})$ is big, then \mathcal{H} necessarily contains a subhypergraph which is ‘very hard to color’ by the algorithm. So, one can think that with high probability this subhypergraph remains uncolored. (In fact, it is so when the respective mono-degrees and bi-degrees have the same cardinality; the difference between them shows the ‘roughness’ of the greedy algorithm). This justifies the term ‘resistance’ for $\Phi(\mathcal{H})$. In other words, $\Phi(\mathcal{H})$ shows how the structure of \mathcal{H} ‘may resist to the greedy coloring algorithm.’

Remark 4. In the greedy coloring algorithm for the upper chromatic number the recoloring of colored vertices is unavoidable [20]. In contrast, in the greedy coloring algorithm for the lower chromatic number no recoloring of colored vertices is required [1]. From this viewpoint, recalling Remark 1, the co-edges in a mixed hypergraph are less favorable with respect to colorability. At STEP 8 of the algorithm one could use any color, including a new one. However, we use the ‘first free color,’ with the aim to use as few colors as possible. Using fewer colors on the average leads to a larger

number of vertices colored with the same color. Hence, more co-edges and, thus, more vertices have a chance to get colored.

EXAMPLE.

Consider the mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$, $|X| = 5$, such that (see the picture below): $X = (1, 2, 3, 4, 5)$, $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\}$, $A_1 = (1, 3)$, $A_2 = (2, 3, 4)$, $A_3 = (3, 4, 5)$, $A_4 = (4, 5, 1)$, $A_5 = (5, 1, 2)$, $\mathcal{E} = \{E_1, E_2, E_3, E_4\}$, $E_1 = (1, 2)$, $E_2 = (2, 5)$, $E_3 = (2, 3)$, $E_4 = (3, 5)$.



Declare $\mathcal{H}_5 = \mathcal{H}$. Since the vertex 4 has minimal risk, start with the fourth vertex: $x_5 = 4$.

Form the subhypergraph $\mathcal{H}_4 = (X_4, \mathcal{A}_4, \mathcal{E}_4)$ with $X_4 = (1, 2, 3, 5)$, $\mathcal{A}_4 = \{A_1, A_5\}$, $\mathcal{E}_4 = \{E_1, E_2, E_3, E_4\}$, $E_1 = (1, 2)$, $E_2 = (2, 5)$, $E_3 = (2, 3)$, $E_4 = (3, 5)$.

In \mathcal{H}_4 , the first vertex with minimal risk is $x_4 = 1$. Form the subhypergraph $\mathcal{H}_3 = (X_3, \mathcal{A}_3, \mathcal{E}_3)$ with $X_3 = (2, 3, 5)$, $\mathcal{A}_3 = \emptyset$, $\mathcal{E}_3 = \{E_2, E_3, E_4\}$, $E_2 = (2, 5)$, $E_3 = (2, 3)$, $E_4 = (3, 5)$.

In \mathcal{H}_3 the first vertex with minimal risk is $x_3 = 2$. Form the subhypergraph $\mathcal{H}_2 = (X_2, \mathcal{A}_2, \mathcal{E}_2)$ with $X_2 = (3, 5)$, $\mathcal{A}_2 = \emptyset$, $\mathcal{E}_2 = \{E_4\}$, $E_4 = (3, 5)$.

In \mathcal{H}_2 the first vertex with minimal risk is $x_2 = 3$. Form the subhypergraph $\mathcal{H}_1 = (X_1, \mathcal{A}_1, \mathcal{E}_1)$ with $X_1 = (5)$, $\mathcal{A}_3 = \emptyset$, $\mathcal{E}_3 = \emptyset$.

These were the results of steps 1–4. Now start coloring.

STEP 5. Set the list $U := \{1, 2, 3, 4, 5\}$ of uncolored vertices, set the color vector $C = (0, 0, 0, 0, 0)$.

STEP 6. $C = (0, 0, 0, 0, 1)$, $U := \{1, 2, 3, 4\}$.

STEP 7. $i = 2$.

STEP 8. $C = (0, 0, 2, 0, 1)$, $U = \{1, 2, 4\}$.

STEP 7. $i = 3$.

STEP 8. $C = (0, 3, 2, 0, 1)$, $U = \{1, 4\}$.

STEP 7. $i = 4$. STEP 8.

STEP 9. $FS(1) = \{1, 3\} \cap \{2\} = \emptyset$, $VS(1) = \{3\}$.

STEP 10. STEP 7. $i = 5$. STEP 8.

STEP 9. $FS(4) = \{1, 2\} \cap \{2, 3\} = \{2\}$, $VS(4) = \emptyset$.

STEP 10. STEP 11. $C = (0, 3, 2, 2, 1)$, $U = \{1\}$.

STEP 7. $i = 6$. STEP 12. Output $C = (0, 3, 2, 2, 1)$, $U = \{1\}$. End.

Hence we can conclude $\Omega(\mathcal{H}) \leq 1$. One can check, actually, that indeed $\Omega(\mathcal{H}) = 1$.

Theorem 5 *The greedy mixed hypergraph coloring algorithm finds the resistance $\Phi(\mathcal{H})$ for any mixed hypergraph \mathcal{H} .*

Proof. Let t be the largest value of the minimum risk over all the vertices in the order generated by STEPS 1–5. It is clear that $t \leq \Phi(\mathcal{H})$.

Suppose that $\Phi(\mathcal{H}) \geq t + 1$, i.e., in some subhypergraph \mathcal{H}^* of \mathcal{H} , there exists a vertex y such that

$$\phi(y, \mathcal{H}^*) = \min_z o(z, \mathcal{H}^*) = \Phi(\mathcal{H}) \geq t + 1.$$

The resistance of any vertex is a monotone function with respect to subhypergraph inclusion. This implies that the first vertex of \mathcal{H}^* that was deleted in STEP 3 by the algorithm had resistance $\geq t + 1$, and this contradicts the definition of t . Consequently, $t = \Phi(\mathcal{H})$. \square

The following two assertions are obvious.

Theorem 6 *If U is the list of vertices uncolored by the algorithm, then*

$$\Omega(\mathcal{H}) \leq |U|. \quad \square$$

Theorem 7 *The total sum of Forcing Set and Veto Set cardinalities at every coloring step of the algorithm does not exceed the value $\Phi(\mathcal{H})$.* \square

4 Colorability as an Integer Programming Problem

There are several ways to formulate the colorability problem for mixed hypergraphs as an integer programming problem. In this section we describe one possible approach that seems to us the most promising one for future applications. We will show that not only the colorability but also the upper and lower chromatic numbers of a mixed hypergraph can be determined by the solutions of an integer programming problem.

Let $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ be a mixed hypergraph, where $X = \{x_1, x_2, \dots, x_n\}$, $n \geq 1$, $\mathcal{A} = \{A_1, A_2, \dots, A_l\}$, $l \geq 1$, and $\mathcal{E} = \{E_1, E_2, \dots, E_m\}$, $m \geq 1$.

Definition 11 *A set $S \subset X$ is stable if it contains no edge $E \in \mathcal{E}$; and S is called co-stable if it contains no $A \in \mathcal{A}$ as a subset. We denote by $\mathcal{S}_{\mathcal{A}}$ and $\mathcal{S}_{\mathcal{E}}$ the collection of all co-stable sets and all stable sets of \mathcal{H} , respectively.*

By definition, a mapping $c : X \rightarrow \{1, 2, \dots, \lambda\}$ is a coloring of \mathcal{H} if and only if every $S \subset X$ satisfies the following two requirements:

- (1) if S is monochromatic, then $S \in \mathcal{S}_{\mathcal{E}}$, and
- (2) if S is totally multicolored, then $S \in \mathcal{S}_{\mathcal{A}}$.

For our purpose, it will be convenient to view colorings from another side, namely as vertex partitions into stable sets satisfying condition (2). Based on this idea, we now introduce a more general coloring/covering concept, assigning stable sets to *real weights* in the half-open interval $(0, 1]$ as follows.

Definition 12 *A fractional coloring of \mathcal{H} with t colors is a collection $\mathcal{S} = \{S_1, \dots, S_t\} \subseteq \mathcal{S}_{\mathcal{E}}$ of t distinct stable sets together with a weight function*

$$w : \mathcal{S} \rightarrow (0, 1]$$

satisfying the following properties:

- (i) For each vertex $x \in X$,

$$\sum_{\substack{S_i \in \mathcal{S} \\ x \in S_i}} w(S_i) = 1,$$

- (ii) For each co-edge $A \in \mathcal{A}$,

$$\sum_{\substack{S_i \in \mathcal{S} \\ A \cap S_i \neq \emptyset}} w(S_i) \leq |A| - 1.$$

It is convenient to extend the domain of w to the entire $\mathcal{S}_\mathcal{E}$, by defining

$$w(S) = 0 \quad \forall S \in \mathcal{S}_\mathcal{E} \setminus \mathcal{S}.$$

Then the extended w on $\mathcal{S}_\mathcal{E}$ and its restriction to \mathcal{S} can be considered equivalent, without ambiguity. Actually, the latter becomes important only in contexts where the number of colors assigned to fractional weights is relevant.

The value of a fractional coloring (\mathcal{S}, w) is defined as

$$w(\mathcal{S}) = \sum_{i=1}^t w(S_i).$$

The quantities

$$\chi^*(\mathcal{H}) = \min_{(\mathcal{S}, w)} w(\mathcal{S})$$

and

$$\bar{\chi}^*(\mathcal{H}) = \max_{(\mathcal{S}, w)} w(\mathcal{S})$$

are termed the fractional lower chromatic number and the fractional upper chromatic number of \mathcal{H} , respectively, where the corresponding minimum or maximum is taken over all t and all feasible fractional t -colorings (\mathcal{S}, w) .

It is readily seen that the following sequence of inequalities is valid for every colorable mixed hypergraph \mathcal{H} :

$$\chi^*(\mathcal{H}) \leq \chi(\mathcal{H}) \leq \bar{\chi}(\mathcal{H}) \leq \bar{\chi}^*(\mathcal{H}).$$

Moreover, by what has been said, the problem of determining χ^* and $\bar{\chi}^*$ can be solved by linear programming on an $|\mathcal{S}_\mathcal{E}|$ -dimensional polyhedron defined by $|X| + |\mathcal{H}_\mathcal{A}|$ constraints. As a consequence, we obtain

Theorem 8 *The fractional upper and lower chromatic numbers of a mixed hypergraph \mathcal{H} can be determined by an algorithm whose running time is a polynomial of the numbers of vertices, co-edges, and stable sets.*

Unfortunately, $\mathcal{S}_\mathcal{E}$ can be exponentially large with respect to X , partly because we have to consider *all* stable sets, not only the maximal ones. On the other hand, it is worth noting that *fractional colorings may be* — though are not always — *feasible for uncolorable mixed hypergraphs* as well.

EXAMPLE.

Consider the mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ with $X = \{1, 2, 3, 4, 5\}$ where the co-edges form the co-cycloid

$$\mathcal{A} = \{(1, 2, 3), (2, 3, 4), (3, 4, 5), (4, 5, 1), (5, 1, 2)\}$$

and the edges induce the 5-cycle in the ‘diagonal’ form,

$$\mathcal{E} = \{(1, 3), (2, 4), (3, 5), (4, 1), (5, 2)\}.$$

One can check that the collection

$$\mathcal{S}_{\mathcal{E}} = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\} \cup \{(1), (2), (3), (4), (5)\}$$

of stable sets admits a unique fractional coloring that assigns $1/2$ to each stable pair. On the other hand, \mathcal{H} is uncolorable because $\bar{\chi}(\mathcal{H}_{\mathcal{A}}) = 2 < 3 = \chi(\mathcal{H}_{\mathcal{E}})$.

Observe further that restricting the range of the weight function w to the integers $0, 1$ (with the same convention $w(S) = 0$ for all $S \notin \mathcal{S}$ as above), the minimum and maximum values of the objective function coincide with $\chi(\mathcal{H})$ and $\bar{\chi}(\mathcal{H})$, respectively. Indeed, choosing the sets S_i from $\mathcal{S}_{\mathcal{E}}$ means that no edge becomes monochromatic, the condition (i) ensures that every vertex is assigned to precisely one color, and (ii) implies the presence of at least one monochromatic pair of vertices inside each co-edge $A \in \mathcal{A}$. Moreover, $\sum_{S \in \mathcal{S}_{\mathcal{E}}} w(S)$ equals the number of colors used in the coloring. In this way we obtain the following result.

Theorem 9 *The upper and lower chromatic number of a mixed hypergraph can be determined by the solution of an integer programming problem. Moreover, \mathcal{H} is colorable if and only if the integer programming problem associated to it admits at least one feasible $(0, 1)$ -solution.*

5 Partial cases of uncolorable mixed hypergraphs

In this section we investigate the conditions of uncolorability in various types of well-structured mixed hypergraphs.

5.1 Complete (l, m) -uniform uncolorable mixed hypergraphs

Let $\mathcal{K}(l, m, n) = (X, \mathcal{A}, \mathcal{E})$, where $|X| = n$, $\mathcal{A} = \binom{X}{l}$ is the family of all l -element subsets of X , and $\mathcal{E} = \binom{X}{m}$ is the collection of all m -element subsets of X . Hence, $|\mathcal{A}| = \binom{n}{l}$ and $|\mathcal{E}| = \binom{n}{m}$. Call $\mathcal{K}(l, m, n)$ the *complete (l, m) -uniform mixed hypergraph* of order n .

Theorem 10 *$\mathcal{K}(l, m, n)$ is uncolorable if and only if $n \geq (l - 1)(m - 1) + 1$.*

Proof. \Rightarrow We prove the contrapositive. Suppose that $n \leq (l - 1)(m - 1)$. Color $m - 1$ vertices with the first color, the next $m - 1$ vertices with the second color, etc.

Since $n \leq (l-1)(m-1)$, this procedure requires at most $l-1$ colors, and a strict coloring of $\mathcal{K}(l, m, n)$ is obtained, i.e., the hypergraph is colorable.

\Leftarrow Let $n \geq (l-1)(m-1) + 1$. Since each m -tuple is an edge and each l -tuple is a co-edge, the number of vertices in any one color does not exceed $m-1$ and the number of colors does not exceed $l-1$. It follows that in any partial coloring at least one vertex remains uncolored. \square

Hence, for any fixed l and m , the total number of all colorable $\mathcal{K}(l, m, n)$ equals $(l-1)(m-1) - \min(l, m)$, i.e., it is finite.

Corollary 1 *For fixed (l, m) almost all $\mathcal{K}(l, m, n)$ are uncolorable.* \square

A completely different conclusion is obtained, however, if we do not fix the values l and m . In the analysis below it will turn out that the proportion of uncolorable complete mixed hypergraph of order n tends to zero as n gets large. Let us recall that the definition of coloring excludes singletons as edges and co-edges.

Theorem 11 *Almost all $\mathcal{K}(l, m, n)$ are colorable.*

Proof. In order to simplify the formulas, let us make the calculation for mixed hypergraphs of order $n+1$ instead of n . Since $l=1$ and $m=1$ are excluded by definition, we have n^2 possibilities to choose the pair (l, m) in the range $2 \leq l \leq n+1$, $2 \leq m \leq n+1$. Applying Theorem 10, we obtain that $\mathcal{K}(l, m, n+1)$ is uncolorable if and only if

$$(l-1)(m-1) \leq n.$$

Here the smallest possible value of $m-1$ is 1. Thus, for each $l \geq 2$, there are precisely $\lfloor \frac{n}{l-1} \rfloor$ uncolorable complete mixed hypergraphs of order $n+1$. Consequently, the total number N_n of complete uncolorable mixed hypergraphs on $n+1$ vertices equals

$$N_n = \sum_{k=1}^n \lfloor \frac{n}{k} \rfloor \simeq n \log n$$

where the asymptotic equation is meant as n tends to infinity. Thus, the proportion of uncolorable complete mixed hypergraphs equals

$$\lim_{n \rightarrow \infty} \frac{N_n}{n^2} = \lim_{n \rightarrow \infty} \frac{\log n}{n} = 0,$$

implying that almost all large complete mixed hypergraphs are colorable. \square

5.2 Constructions from graphs

Let $G = (X, \mathcal{E})$ be a graph with $\chi(G) = k$. Construct a mixed hypergraph $\mathcal{H}_G = (X, \mathcal{A}, \mathcal{E})$, where $\mathcal{A} = \{A \subset V : |A| = k, A = V(P) \text{ for some path } P \subset G\}$. (Here $V(P)$ denotes the vertex set of P .) We call \mathcal{H}_G the *k-path mixed hypergraph* on G .

Theorem 12 *The k-path mixed hypergraph $\mathcal{H}_G = (X, \mathcal{A}, \mathcal{E})$ is uncolorable for every graph G .*

Proof. Suppose on the contrary that \mathcal{H}_G is colorable. Clearly, $\chi(\mathcal{H}_G) \geq \chi(G) = k$; let $c : X \rightarrow \{1, 2, \dots, k'\}$ be a feasible coloring for some $k' \geq k$. For each $uv \in \mathcal{E}$, orient uv from the vertex of smaller color to the larger one. Since $\chi(G) = k$, the Gallai–Roy theorem ([6, 13], see also [15] for a short proof and generalizations) implies that G contains a directed path P on k vertices. By definition, $V(P) \in \mathcal{A}$, and since the colors are increasing along P , the vertices of P have mutually distinct colors. Thus, \mathcal{A} contains a totally multicolored co-edge. This contradiction proves that \mathcal{H} is uncolorable. \square

Remarks. If $\chi(G) < k$, then the k -path mixed hypergraph is colorable because in any coloring of \mathcal{H} with fewer than k colors every $A \in \mathcal{A}$ contains a monochromatic pair of vertices. On the other hand, for every $k < \chi(G)$, the mixed hypergraph is uncolorable, as the above proof works also for these cases.

A subclass of uncolorable mixed hypergraphs of this type is constructed by taking $G = C_{2t+1}$; then we obtain the odd cycle (graph) with the 3-uniform co-cycloid [20].

5.3 Uncolorable mixed hypertrees

Throughout this subsection we assume that the graphs considered are connected.

Definition 13 *A mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ is called a mixed hypertree if there exists a tree $T = (X, F)$ such that every $A \in \mathcal{A}$ and every $E \in \mathcal{E}$ induces a subtree in T .*

For $\mathcal{A} = \emptyset$, we obtain the classic concept of hypertrees, the structural properties of which are well investigated, see for example [1] (‘arboreal hypergraphs’). Some chromatic properties of co-hypertrees (mixed hypertrees with $\mathcal{E} = \emptyset$) have been investigated in [20]. Here we find the value of resistance and give the criteria of colorability for mixed hypertrees.

Theorem 13 *If $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ is a mixed hypertree, then*

$$\Phi(\mathcal{H}) \leq 1.$$

Proof. Recall that we consider mixed hypergraphs without loops and co-loops, and also assume that no (co-)edge contains any other (co-)edge. We shall apply induction on $|X| = n$. For $n = 1, 2, 3$ the assertion is obvious. Assume it holds true for any mixed hypertree with fewer than n vertices. Consider a vertex x that is a leaf in the corresponding tree T . Since every edge and every co-edge of \mathcal{H} has cardinality at least 2, $o(x, \mathcal{H}) = 0$ and $m(x, \mathcal{H}) \leq 1$. Therefore $\phi(x, \mathcal{H}) \leq 1$. Further, it is clear that for every $Y \subset X$ the induced subhypergraph \mathcal{H}/Y represents a mixed hypertree. By the induction hypothesis, we have then that $\Phi(\mathcal{H}/Y) \leq 1$. Consequently, $\Phi(\mathcal{H}) \leq 1$ holds, too. \square

We recall the following notion defined in [2]:

Definition 14 *In a mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ an edge $E_j \in \mathcal{E}$, $|E_j| \geq 2$, is called evidently uncolorable if for each pair of vertices $x, y \in E_j$ there exists a sequence $(xA_1z_1A_2z_2 \dots A_{l-1}z_{l-1}A_ly)$ such that*

- 1) $z_1, z_2, \dots, z_{l-1} \in E_j$,
- 2) $A_i \in \mathcal{A}$ for every $i = 1, \dots, l$,
- 3) $A_1 = \{x, z_1\}$, $A_2 = \{z_1, z_2\}, \dots, A_l = \{z_{l-1}, y\}$.

Theorem 14 *A mixed hypertree $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ is colorable if and only if it does not contain any evidently uncolorable edge.*

Proof. \Rightarrow Obvious.

\Leftarrow Let $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ be a mixed hypertree without evidently uncolorable edges. Observe that if it contains no co-edges of size 2, then it is colorable. Indeed, consider the corresponding tree T and color it as usually, alternating with colors 1 and 2, starting at any vertex. The coloring obtained is at the same time a coloring of \mathcal{H} . If $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ contains co-edges of size 2, then each of them coincides with some edge of T . Now we repeat the previous procedure with the following exception: if we encounter a co-edge of size 2, then we do not change color along this edge of T . (I.e., an edge of T is properly colored if and only if it is not a co-edge in \mathcal{H} .) Since there are no evidently uncolorable edges in \mathcal{H} , we again obtain the coloring of \mathcal{H} . \square

5.4 Uncolorable block designs

Finally, we mention a different type of constructions studied in [11, 12]. Among other results, it is proven there that for any Steiner Triple System $\mathcal{S} = STS(n)$ on a point set X of cardinality $n \leq 2^k - 1$, the co-hypergraph $\mathcal{H} = (X, \mathcal{A})$ with $\mathcal{A} = \mathcal{S}$ (i.e., viewing each block as a co-edge) has upper chromatic number at most k , i.e.,

$$\bar{\chi}(\mathcal{H}) \leq \lceil \log_2(n + 1) \rceil.$$

As a consequence, if n tends to infinity and the independence number of \mathcal{S} becomes smaller than $\frac{n}{\log_2(n/2)}$, we obtain that there exists an infinite family of uncolorable Steiner Triple Systems viewed as the mixed hypergraphs $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ with $\mathcal{A} = \mathcal{E} = \mathcal{S}$. Similar ideas work for Steiner systems $S(t, t+1, n)$ with larger block size, $t+1 > 3$, as well.

Moreover, it is found that some Steiner Triple Systems (where each block is considered again as an edge and co-edge at the same time) are uncolorable already for $n=15$ (B. Ganter, private communication, 1997).

6 Open problems

We conclude this paper with some problems that remain open for future research.

Problem 1 The following notion was introduced in [20]. For a mixed hypergraph $\mathcal{H} = (X, \mathcal{A}, \mathcal{E})$ call a hypergraph $\mathcal{H}_1 = (X, \mathcal{A}_1, \mathcal{E}_1)$ the *chromatic inversion* of \mathcal{H} if $\mathcal{A}_1 = \mathcal{E}$ and $\mathcal{E}_1 = \mathcal{A}$.

When are both \mathcal{H} and $\bar{\mathcal{H}}$ colorable (uncolorable)?

Problem 2 Let us call \mathcal{H} a *bi-hypergraph* if $\mathcal{A} = \mathcal{E}$. We consider r -uniform bi-hypergraphs, i.e. when all the (co)-edges are of a size r , $r = 2, 3, 4, \dots$

What is the minimum $n = n(r)$ ($m = m(r)$) for which there exist inclusionwise minimal uncolorable r -uniform mixed bi-hypergraphs on n vertices (with m (co)-edges, respectively)?

Evidently, $n(2) = 2, m(2) = 1, n(3) = 5, m(3) \leq 10$. Theorem 10 implies that $n(r) \leq (r-1)^2 + 1$ and $m(r) \leq \binom{(r-1)^2+1}{r}$.

Problem 3 Characterize the critical uncolorable mixed hypergraphs, i.e. those becoming colorable if we delete any vertex (or any edge, or any co-edge).

Problem 4 Let \mathcal{H} be a minimal uncolorable mixed hypergraph. Characterize the maximal colorable subhypergraphs of \mathcal{H} . Is every colorable mixed hypergraph a maximal colorable subhypergraph of some minimal uncolorable mixed hypergraph?

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