

On-line rankings of graphs

Dedicated to professor Horst Sachs
on the occasion of his 70th birthday

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Abstract

A (vertex) k -ranking of a graph $G = (V, E)$ is a proper vertex coloring $\varphi : V \rightarrow \{1, \dots, k\}$ such that each path with endvertices of the same color i contains an internal vertex of color $\geq i+1$. In the on-line coloring algorithms, the vertices v_1, \dots, v_n arrive one by one in an unrestricted order, and only the edges inside the set $\{v_1, \dots, v_i\}$ are known when the color of v_i has to be chosen. We characterize those graphs for which a 3-ranking can be found on-line. We also prove that the greedy (First-Fit) on-line algorithm, assigning the smallest feasible color to the next vertex at each step, generates a $(3 \log n)$ -ranking for the path with $n \geq 2$ vertices, independently of the order in which the vertices are received.

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E . A *vertex k -ranking* of G is a proper vertex coloring $\varphi : V \rightarrow \{1, \dots, k\}$ such that every path in G with endvertices x and y of the same color $\varphi(x) = \varphi(y)$ contains a vertex z with higher color $\varphi(z) > \varphi(x)$. The *ranking number* $\chi_r(G)$ is the smallest integer k for which there exists a vertex k -ranking of G .

The investigation of ranking problems has importance in the parallel Cholesky factorization of matrices [2, 6] and also in the field of VLSI-design [5, 7]. In the latest years, some investigations concerning the computational complexity of ranking problems were also done; see [4] for references.

In this paper we consider the problem of finding rankings of graphs *on-line*. This problem, among other questions, has been raised in [8]. Here we investigate two versions of on-line ranking.

In the first version, the graph G to be colored is specified before the on-line procedure. The vertices of G arrive one by one in an unrestricted order. The *input sequence* will be denoted by v_1, v_2, v_3, \dots , indicating the actual order by increasing subscripts. In the i -th step, the induced subgraph $G[v_1, \dots, v_i]$ is known (without any information on the position of this subgraph in G), and a color (rank) has to be assigned to v_i . The assigned colors must not be changed later.

The graph G is called *on-line k -rankable* if there is an algorithm that generates a vertex k -ranking of G for each possible input sequence of the vertices of G . The *on-line ranking number* $\chi_r^*(G)$ is the smallest integer k such that G is on-line k -rankable. The class of all on-line k -rankable graphs is denoted by \mathcal{R}_k^* .

In the second version, a class of graphs is given from which the graph G is chosen; but at the beginning of the on-line process the graph G itself is not specified. A class \mathcal{G} of graphs is called *on-line k -rankable* if there is an algorithm which gives a vertex k -ranking for each graph G which belongs to \mathcal{G} and for each possible input sequence of the vertices of G , without any preliminary information on the choice of $G \in \mathcal{G}$.

The paper consists of two main parts. In Section 2 we characterize the on-line 3-rankable graphs, and in Section 3 we prove an upper bound on the on-line ranking number of paths.

For a more precise formulation, we shall denote by S_n the star with center of degree $n - 1$, by P_n the path of length (number of edges) $n - 1$, and by C_n the cycle of length n . In general, the number of vertices of the graph under consideration is denoted by n throughout the paper.

Obviously, a graph G is on-line 1-rankable if and only if G is edgeless, and G is on-line 2-rankable if and only if the edges of G are pairwise non-adjacent. Hence, for $k \leq 2$ the entire class \mathcal{R}_k^* is on-line k -rankable. As shown by the results below, this property does not remain valid for $k = 3$.

Theorem 1 A connected graph G of order $n \geq 4$ has on-line ranking number 3 if and only if $G = S_n$, n arbitrary, or $n = 4$ and $G = P_4$ or $G = C_4$. Moreover, a graph of order at most 3 has on-line ranking number 3 if and only if it is connected and has precisely three vertices.

Without assuming connectivity, the characterization becomes slightly more complicated.

Theorem 2 A graph G has on-line ranking number 3 if and only if

- (i) each connected component of G is on-line 3-rankable,
- (ii) at least one connected component of G has more than two vertices, and
- (iii) if G contains a star S_n ($n \geq 4$) component, then it contains no C_4 and no P_4 .

We shall prove the ‘if’ part (sufficiency) in a stronger form, for *graph classes* instead of single graphs. The notation $F \not\subseteq G$ means that G contains no subgraph isomorphic to F .

Theorem 3 The graph classes

$$\{G \in \mathcal{R}_3^* \mid P_4 \not\subseteq G \wedge C_4 \not\subseteq G\}$$

and

$$\{G \in \mathcal{R}_3^* \mid S_n \not\subseteq G \forall n \geq 4\}$$

are on-line 3-rankable.

Our last result concerns the on-line ranking number of paths. For the (off-line) ranking number $\chi_r(P_n)$ it is well known that $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$. The situation for on-line rankings is different, however, as it is easily seen (at least for infinitely many n) that some input sequences require more than $\chi_r(P_n)$ colors.

Let us consider a special algorithm assigning the colors to the vertices. The *greedy algorithm*, also called *First-Fit*, chooses in each step the smallest feasible color to the arriving vertex v_i such that the coloring of the known induced subgraph $G_i = G[v_1, \dots, v_i]$ is a vertex ranking of G_i .

Theorem 4 The First-Fit ranking algorithm uses fewer than $3 \log_2 n$ colors on P_n ($n \geq 2$), independently of the order the vertices arrive in.

The exact determination of $\chi_r^*(P_n)$ remains an open problem. Also, it seems to us that the characterization of on-line k -rankable graphs for $k \geq 4$ becomes increasingly complicated.

2 On-line 3-ranking

In this section we prove the results concerning graphs with on-line ranking number 3. The proof of Theorem 1 and the necessity for Theorem 2 is shown next, while the sufficiency is deduced from the more general Theorem 3, discussed in the second subsection.

Before giving those details, we note that the graphs P_4 , C_4 , and S_n belong to \mathcal{R}_3^* . Indeed, one can easily check that the First-Fit algorithm generates a 3-ranking for both P_4 and C_4 , for an arbitrary input sequence of the vertices. This is not valid for stars, but in S_n one can assign color 2 to all v_1, \dots, v_i as long as these vertices form an independent set. If v_{i+1} is adjacent to the previous vertex ($i = 1$) or vertices ($i \geq 2$), let it have rank 3. Obviously, all the remaining vertices (if any) may get color 1, no matter whether v_{i+1} is the center or a leaf of S_n .

2.1 On-line 3-rankable graphs

In order to prove Theorem 1, we are going to verify some simple structural properties of on-line 3-rankable graphs.

Lemma 5 If a connected graph $G \in \mathcal{R}_3^*$ has maximum degree at least three, then G is a star.

Proof. Assume, as a bad case, that the first two vertices v_1, v_2 in the input sequence are adjacent. If one of them, say v_1 , is assigned to color 1, it may turn out in the next steps that v_1 has degree ≥ 3 in G , and then all vertices adjacent to v_1 must get distinct colors. In this way the procedure would require more than three colors, therefore any 3-ranking on-line algorithm must assign $\varphi(v_1) = 2$ and $\varphi(v_2) = 3$ (or vice versa).

Suppose that G is not a star; i.e., the vertex v of degree ≥ 3 does not cover all edges of G . Let now e be an edge of G incident to a neighbor of v but not incident to v itself. Define $v_1v_2 := e$ and assume that v_1 is colored by 2 and v_2 is colored by 3. Let $v_3 := v$ be a neighbor of v_1 . In this situation, v_3 cannot get color 2 or 3, therefore $\varphi(v_3) = 1$ should hold. But then, again, the (at least three) neighbors of v_3 must get mutually distinct ranks, implying the contradiction that G requires more than three colors on-line. \square

Lemma 6 $P_5, C_5 \notin \mathcal{R}_3^*$.

Proof. The assertion is clearly valid for the 5-cycle, because even the (off-line) ranking number of C_5 is 4. As regards P_5 , observe that its two endvertices cannot get color 3, and the middle vertex cannot get color 2, in any ranking with three colors. On the other hand, assuming that v_1v_2 is an edge, color 2 or 3 must occur, and then the corresponding vertex can happen to be the middle or first/last vertex of P_5 . \square

Proof of Theorem 1. Suppose that $G \in \mathcal{R}_3^*$ is a connected graph. By Lemma 5, if G is not a star, then it is a path or a cycle. But P_n and C_n are excluded for $n > 5$, by Lemma 6, because already P_5 itself excludes all longer paths and cycles. This proves the assertion for $n \geq 4$, while the smaller cases are obvious. \square

Turning now to disconnected graphs, it is clear that the (on-line as well as off-line) ranking number of any one connected component is a lower bound for the entire graph. Therefore, the first condition in Theorem 2 is obviously necessary. Also, matchings are on-line 2-rankable, implying the necessity of the second condition.

Proof of the necessity of (iii) in Theorem 2. Suppose that S_n ($n \geq 4$) is a connected component of a graph $G \in \mathcal{R}_3^*$. We have to prove that neither P_4 nor C_4 is a component of G .

Let v_1v_2 be an edge. This edge may belong to a star component, therefore — as we have seen in the proof of Lemma 5 — the only chance to color it with an extendible partial 3-ranking is that the on-line algorithm assigns $\varphi(v_1) = 2$ and $\varphi(v_2) = 3$. Such an edge cannot occur in a C_4 , however, because C_4 has a unique 3-ranking, and colors 2 and 3 must be antipodal in it. Also, in any 3-ranking of P_4 , color 3 has to appear on one of the two internal vertices. But if P_4 is a component of G , then v_2 may be an endvertex of this P_4 , contradicting 3-rankability. \square

That the two conditions together are also sufficient will be proved in the next subsection.

2.2 On-line 3-rankable classes

Here we prove that the two subclasses of graphs $G \in \mathcal{R}_3^*$ containing either no P_4 and C_4 or no S_n ($n \geq 4$) are on-line 3-rankable, even if the exact structure of the input graph is not known. We handle the two cases separately, in the two lemmas below.

Lemma 7 If $G \in \mathcal{R}_3^*$ contains no S_n , $n \geq 4$, then the First-Fit algorithm finds a 3-ranking on G .

Proof. As the First-Fit algorithm proceeds componentwise independently, it suffices to verify the validity of the assertion for P_4 and C_4 . In either case, color 4 is not needed unless the first three vertices get colors 1, 2, 3. Moreover, one can check that the color set $\{1, 2, 3\}$ occurs only if the vertices of colors 2, 1, 3 form an induced path in this order. But then the last vertex can get color 1. \square

Lemma 8 The class $\{G \in \mathcal{R}_3^* \mid P_4 \not\subseteq G \wedge C_4 \not\subseteq G\}$ is on-line 3-rankable.

Proof. The *on-line degree* of the vertex v_i , denoted $d^-(v_i)$, is the number of vertices adjacent to v_i in the set $\{v_j \mid 1 \leq j < i\}$. Consider the following algorithm.

1. Every vertex of on-line degree 0 gets color 2.
2. If $d^-(v_i) > 0$, then $\varphi(v_i) = 3$ if and only if all of its neighbors v_j with $j < i$ have on-line degree 0.
3. All the other vertices get color 1.

We have seen already before Subsection 2.1 that the above procedure finds a 3-ranking on each star component, and it obviously ranks properly the components of orders at most 3 as well. \square

Lemmas 7 and 8 together imply the validity of Theorems 2 and 3.

3 Greedy on-line rankings of paths

Here we prove an upper bound on the on-line ranking number of paths. Instead of the input sequence, it will be more convenient to consider the vertices in their order along the path. For this purpose, we shall denote them by u_1, u_2, \dots, u_n . The color assigned to u_i by the First-Fit algorithm will be denoted by $c(u_i)$. (This color depends on the actual input sequence, but we do not indicate the latter in the notation.)

Proof of Theorem 4. Let $f(k)$ be the smallest integer such that the First-Fit algorithm uses k colors on $P_{f(k)}$ for some input sequence. It is immediately seen that $f(1) = 1, f(2) = 2, f(3) = 3$, and $f(k+1) > f(k)$ for every $k \geq 4$; in particular, $f(k) \geq k$ for every k . For convenience, we set $f(0) = 0$. Our goal is to prove that $f(k)$ grows with an exponential function of k .

Consider $P_n = u_1 u_2 \cdots u_n$ for $n = f(k)$. We assume that the vertices arrive in a ‘worst’ input sequence, i.e., each of the colors $1, 2, \dots, k$ gets assigned to some vertex of P_n by First-Fit. Let p be the subscript with $c(u_p) = k$. Then color k occurs exactly once, i.e., $c(u_i) \leq k - 1$ for all $1 \leq i \leq n, i \neq p$. Denoting by $\tilde{P}_1 := u_1 \cdots u_{p-1}$ and $\tilde{P}_2 := u_{p+1} \cdots u_n$ the two subpaths of $P_n - u_p$, let u_q (u_r) be the vertex of \tilde{P}_1 (of \tilde{P}_2) colored with the largest number. We may assume, without loss of generality, that $c(u_q) = k - 1$. In this case, \tilde{P}_2 may or may not be empty.

The following lemma will be of major importance. Its proof follows from the strategy of the First-Fit ranking algorithm.

Cascade Lemma. Suppose $c(u_r) = k - d$ for some d with $3 \leq d \leq k$. Then there are vertices $u_{i_2}, u_{i_3}, \dots, u_{i_{d-1}}$ such that $q < i_2 < i_3 < \dots < i_{d-1} < p$ and for $2 \leq j \leq d - 1$ we have $c(u_{i_j}) = k - j$, and $c(u_\ell) \leq k - j - 1$ for all $i_j < \ell < p$. \square

Moreover, we derive the following estimations from the Cascade Lemma.

Lemma 9 Set $d := k - c(u_r)$ or $d := 0$ if \tilde{P}_2 is empty. It follows that

- (i) $p - 1 \geq f(k - 1)$.
- (ii) If $d \neq 0$, then $n - p \geq f(k - d)$.
- (iii) If $d \notin \{1, 2, 3\}$, then $p \geq 2f(k - 3) + 3$.

Proof.

- (i) Notice that the colors of the vertices u_p, \dots, u_n have no influence on the coloring of $\tilde{P}_1 = u_1, \dots, u_{p-1}$ by the First-Fit algorithm. Thus $|\tilde{P}_1| = p - 1 \geq f(k - 1)$ because \tilde{P}_1 is colored with $k - 1$ colors by the algorithm.
- (ii) The colors of u_1, \dots, u_p have no influence on the coloring of $\tilde{P}_2 = u_{p+1}, \dots, u_n$. Thus $|\tilde{P}_2| = n - p \geq f(k - d)$ because \tilde{P}_2 is colored with $k - d$ colors by the algorithm.
- (iii) There are integers s, t with $q < s < t < p$ such that $c(u_s) = k - 2$ and $c(u_t) = k - 3$, otherwise u_p gets color $k - 2$ or color $k - 3$ by the algorithm. Again the coloring of u_{q+1}, \dots, u_{p-1} does not depend on the coloring of the remaining vertices. It follows that $p - 1 - q \geq f(k - 2)$. By a similar argument, for the coloring of u_{s+1}, \dots, u_{p-1} we obtain $p - 1 - s \geq f(k - 3)$.

Let u_x be the vertex of $u_1 \cdots u_{q-1}$ which is colored with the largest color in this subpath.

If $c(u_x) = k - j$ for some $j \in \{2, 3\}$, then $q - 1 \geq f(k - 3)$ and together with $p - 1 - q \geq f(k - 2)$ we have $p \geq f(k - 2) + f(k - 3) + 2 \geq 2f(k - 3) + 3$.

If $c(u_x) \leq k - 4$, then there is a vertex u_y between u_q and u_s colored with $k - 3$ because otherwise u_q gets color $k - 3$ by the algorithm. It follows that $s - 1 - q \geq f(k - 3)$, and together with $p - 1 - s \geq f(k - 3)$ we have $p - q - 2 \geq 2f(k - 3)$ and $p \geq 2f(k - 3) + 3$. \square

If $d \in \{1, 2, 3\}$, it follows from Lemma 9 (i)–(ii) that

$$f(k) = n \geq f(k - 1) + f(k - 3) + 1 \geq 2f(k - 3) + 3.$$

The same inequality follows from (iii) for the case $d \notin \{1, 2, 3\}$. We conclude

$$f(k) + 3 \geq 2 \cdot (f(k - 3) + 3),$$

and therefore

$$f(k) \geq 3 \cdot (2^{k/3} - 1) > 2^{k/3}$$

for $k \geq 2$, by induction on k . Hence, if P_n ($n \geq 4$) is colored with k colors by the First-Fit algorithm, then $k < 3 \log_2 n$. This completes the proof of Theorem 4. \square

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