



MARTIN HENK AND ROBERT WEISMANTEL

**Hilbert bases of cones related to
simultaneous Diophantine
approximations and linear Diophantine
equations**

Hilbert bases of cones related to simultaneous Diophantine approximations and linear Diophantine equations

Martin Henk* Robert Weismantel†

Abstract

This paper investigates properties of the minimal integral solutions of a linear diophantine equation. We present best possible inequalities that must be satisfied by these elements which improves on former results. We also show that the elements of the minimal Hilbert basis of the dual cone of all minimal integral solutions of a linear diophantine equation yield best approximations of a rational vector “from above”. Relations between these cones are applied to the knapsack problem.

Keywords: Diophantine equations, simultaneous Diophantine approximation, knapsack problem, Hilbert basis, test sets.

1 Introduction

Throughout this paper we resort to the following notation. For integral points $z^1, \dots, z^m \in \mathbb{Z}^n$, the set

$$C := \text{pos} \{z^1, \dots, z^m\} = \left\{ \sum_{i=1}^m \lambda_i z^i : \lambda \in \mathbb{R}_{\geq 0}^m \right\}$$

is called a *rational polyhedral cone*. It is called *pointed* if there exists a hyperplane $\{x \in \mathbb{R}^n : a^T x = 0\}$ such that $\{0\} = \{x \in C : a^T x \leq 0\}$. Here we are interested in generating systems of the integral points contained in such a cone.

Definition 1.1. *Let $C \subseteq \mathbb{R}^n$ be a rational polyhedral cone. A finite subset $H = \{h^1, \dots, h^t\} \subseteq C \cap \mathbb{Z}^n$ is called a Hilbert basis of C if every $z \in C \cap \mathbb{Z}^n$ has a representation of the form*

$$z = \sum_{i=1}^t \lambda_i h^i,$$

with non-negative integral multipliers $\lambda_1, \dots, \lambda_t$. A minimal Hilbert basis w.r.t. inclusion is also called an integral basis of the cone C and it is denoted by $\mathcal{H}(C)$.

*Supported by a “Leibniz Preis” of the German Science Foundation (DFG) awarded to M. Grötschel.

†Supported by a “Gerhard-Hess-Forschungsförderpreis” of the German Science Foundation (DFG).

The name Hilbert basis was introduced by Giles and Pulleyblank [GP79] in the context of totally dual integral systems. It was shown by Gordan [G1873] that every rational polyhedral cone has an integral basis and for pointed cones we have the following result due to van der Corput [Cor31]: *The integral basis $\mathcal{H}(C)$ of a rational, pointed cone $C \subseteq \mathbb{R}^n$ is uniquely determined by*

$$\mathcal{H}(C) = \left\{ z \in C \cap \mathbb{Z}^n \setminus \{0\} : z \text{ can not be written as the sum} \right. \quad (1.1)$$

$$\left. \text{of two other elements of } C \cap \mathbb{Z}^n \setminus \{0\} \right\}.$$

Although Hilbert bases play a role in various fields of mathematics, like combinatorial convexity, geometry of numbers, special desingularizations of toric varieties, or in integer programming, their geometrical structure is not very well understood yet.

This paper studies Hilbert bases (integral bases) of cones associated with the problem of simultaneous Diophantine approximation of rationals and the non-negative integer solutions of a linear Diophantine equation.

The problem of *simultaneous Diophantine approximation* and a constrained version of it that we investigate here reads as follows. Let $A \in \mathbb{Z}^{m \times n}$ be an integral matrix of rank n and let $f_A : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ be the norm on \mathbb{R}^n defined by

$$f_A(x) = |Ax|_1 = \sum_{j=1}^m |a^j x|,$$

where $|\cdot|_1$ denotes the l_1 -norm and a^j denotes the j -th row of A . With respect to that norm we study the

Constrained Simultaneous Dioph. Approximation Problem (CSDAP)

Let $p_1, \dots, p_n, p_{n+1} \in \mathbb{Z}$, $p_{n+1} > 0$, $N \in \mathbb{N} > 0$. Find integers $q_1, \dots, q_n \in \mathbb{Z}$ and an integer $q_{n+1} \in \mathbb{N} > 0$ such that $q_{n+1} \leq N$ and

1. $a^j (q_1/q_{n+1}, \dots, q_n/q_{n+1})^T \geq a^j (p_1/p_{n+1}, \dots, p_n/p_{n+1})^T$, $j = 1, \dots, m$,
2. $f_A(q_1/q_{n+1} - p_1/p_{n+1}, \dots, q_n/q_{n+1} - p_n/p_{n+1})$ is minimal.

Observe, that by neglecting the restrictions of the form 1., the problem reduces to the “standard” simultaneous Diophantine approximation problem of n rationals w.r.t. the norm f_A .

It is known for a long time that the two-dimensional simultaneous Diophantine approximation problem ($n=1$) can be solved in polynomial time by the method of *continued fractions* as described in Khintchine [Khi56], Perron [Per13] and Grötschel, Lovász and Schrijver [GLS88]. Moreover, in the two-dimensional case best approximations have a nice geometric structure. More precisely, for a given $p = (p_1, p_2)^T \in \mathbb{Z}^2$ and $N \in \mathbb{N}$ let $C(I_1, p) = \text{pos} \{(1, 0)^T, p\}$ and $C(-I_1, p) = \text{pos} \{(-1, 0)^T, p\}$. Then it was shown by Klein [K1895] that a point $(q_1, q_2)^T$ on the lower convex hull of one of the two Klein polyhedra $K^+ = \text{conv} \{C(I_1, p) \cap \mathbb{Z}^2 \setminus \{0\}\}$, $K^- = \text{conv} \{C(-I_1, p) \cap \mathbb{Z}^2 \setminus \{0\}\}$ yields a best approximation of p_1/p_2 , i.e., $|q_1/q_2 - p_1/p_2|$ is minimal among all rationals whose denominator is bounded by N . In particular, an appropriate point $(q_1^+, q_2^+)^T$ ($(q_1^-, q_2^-)^T$) lying on the lower convex hull of K^+ (K^-) gives a best approximation of the constrained problem: $\min |q_1/q_2 - p_1/p_2|$, $q_2 \leq N$ and $q_1/q_2 \geq p_1/p_2$ ($q_1/q_2 \leq p_1/p_2$) (see also [BP94], [Fin93], [DS82]).

Whereas the two dimensional case of the problem has been very well understood, this much insight could not be gained for higher dimensions. We show in this paper that for every dimension and every $N \in \mathbb{N}$, a solution to the constrained simultaneous approximation problem with smallest denominator belongs to the minimal Hilbert basis of the cone

$$C(A, p) = \left\{ x \in \mathbb{R}^{n+1} : p_{n+1} a^j(x_1, \dots, x_n)^T - x_{n+1} a^j(p_1, \dots, p_n)^T \geq 0, \right. \\ \left. j = 1, \dots, m, \quad x_{n+1} \geq 0 \right\}. \quad (1.2)$$

Since the integral basis of any 2-dimensional rational pointed C cone consists of the integral points lying on the lower convex hull of $\text{conv}\{C \cap \mathbb{Z}^2 \setminus \{0\}\}$ (see e.g. [Oda88] and the references within), Klein's result follows indeed.

In the particular case when A coincides with the $(n \times n)$ identity matrix I_n , we have $C(I_n, p) = \text{pos}\{e^1, \dots, e^n, p\}$, where $e^i \in \mathbb{R}^{n+1}$ denotes the i -th unit vector and $p = (p_1, \dots, p_{n+1})^T$. The dual cone $C^*(I_n, p)$ of $C(I_n, p)$ is given by

$$C^*(I_n, p) = \{y \in \mathbb{R}^{n+1} : p^T y \geq 0, y_i \geq 0, 1 \leq i \leq n\}. \quad (1.3)$$

The non-negative integer points on the facet $\{y \in C^*(I_n, p) : p^T y = 0\}$ may be interpreted as the set of all non-negative integer solutions of a linear Diophantine equation. Such an equation reads

$$K_{n_1, n_2} = \left\{ (x, y) \in \mathbb{R}_{\geq 0}^{n_1} \times \mathbb{R}_{\geq 0}^{n_2} : a^T x = b^T y \right\}, \quad (1.4)$$

for suitable numbers n_1, n_2 and positive integer vectors $a \in \mathbb{N}^{n_1}$ and $b \in \mathbb{N}^{n_2}$. The minimal solutions of such an equation constitute the integral basis of the *knapsack cone* K_{n_1, n_2} and have been studied for a long time in various different contexts, see e.g. [Ehr79], [FT95], [Gre88] and the references within.

We generalize a theorem of Lambert [Lam87] and Diaconis, Graham and Sturmfels [DGS94] by proving that the minimal integer solutions of such a linear Diophantine equation satisfy a system of inequalities that is stated in Section 3 and that allow to bound the norm of such elements..

In Section 4 we deal with the integer version of Caratheodory's theorem. We show that it applies to the families of cones of Sections 2 and 3 when all the numbers are divisible. Section 5 discusses connections between cones associated with approximations of a rational vector and knapsack polyhedra.

2 Simultaneous Diophantine Approximation

This section deals with the constrained simultaneous Diophantine approximation problem CSDAP that we introduced in the previous section. We remark that if $N > p_{n+1}$ then $q := (p_1, \dots, p_{n+1})$ is always an optimal solution of this problem. It is, however, not clear how one can characterize a solution if $N < p_{n+1}$.

Theorem 2.1. *Among all solutions of CSDAP let q_1, \dots, q_{n+1} be one with denominator q_{n+1} as small as possible. Then the vector $q = (q_1, \dots, q_{n+1})^T$ is an element of the integral basis of $C(A, p)$, i.e., $q \in \mathcal{H}(C(A, p))$.*

Proof. On account of the restrictions 1., the vector q is an element of the cone $C(A, p)$ and conversely, for each vector $x \in C(A, p)$ with $x_{n+1} > 0$ the associated rational vector $(x_1/x_{n+1}, \dots, x_n/x_{n+1})^T$ satisfies the restrictions 1. Furthermore, since $\text{rank}(A) = n$, the cone $C(A, p)$ is pointed.

Suppose that q is not an element of the integral basis of $C(A, p)$. Then we can find two nontrivial vectors $v, w \in C(A, p) \setminus \{0\}$ such that $q = v + w$. For abbreviation we write \bar{x} for the first n components of a vector $x \in \mathbb{R}^{n+1}$ and \tilde{x} for the ‘‘rational’’ vector $(x_1/x_{n+1}, \dots, x_n/x_{n+1})$ if $x_{n+1} \neq 0$. Let $v_{n+1} \geq w_{n+1} \geq 0$ and let $j \in \{1, \dots, m\}$. We have

$$a^j (\tilde{q} - \tilde{p}) = a^j \left(\frac{\bar{v} + \bar{w}}{v_{n+1} + w_{n+1}} - \tilde{p} \right). \quad (2.1)$$

We have to distinguish two cases.

I. Let $w_{n+1} = 0$. Then we may write (cf. (2.1) and (1.2))

$$a^j (\tilde{q} - \tilde{p}) = a^j \tilde{v} + \frac{1}{v_{n+1}} a^j \bar{w} - a^j \tilde{p} \geq a^j (\tilde{v} - \tilde{p}).$$

Since $v, w \in C(A, p) \setminus \{0\}$ and $\text{rang}(A) = n$ summation over all indices $j = 1, \dots, m$ yields $f_A(\tilde{q} - \tilde{p}) > f_A(\tilde{v} - \tilde{p})$. This is a contradiction that (q_1, \dots, q_{n+1}) is a solution of CSDAP.

II. Let $v_{n+1} \geq w_{n+1} > 0$. Then we may write (cf. (2.1))

$$a^j (\tilde{q} - \tilde{p}) = \frac{v_{n+1}}{v_{n+1} + w_{n+1}} a^j (\tilde{v} - \tilde{p}) + \frac{w_{n+1}}{v_{n+1} + w_{n+1}} a^j (\tilde{w} - \tilde{p}).$$

Again summation over all inequalities yields

$$f_A(\tilde{q} - \tilde{p}) = \frac{v_{n+1}}{v_{n+1} + w_{n+1}} f_A(\tilde{v} - \tilde{p}) + \frac{w_{n+1}}{v_{n+1} + w_{n+1}} f_A(\tilde{w} - \tilde{p}).$$

Thus, by the minimality of q we conclude that $f_A(\tilde{v} - \tilde{p}) = f_A(\tilde{w} - \tilde{p}) = f_A(\tilde{q} - \tilde{p})$. This contradicts the minimality of the denominator q_{n+1} and completes the proof. \square

We remark that for $A = I_n$, $f_A(\cdot)$ is the l_1 -norm. Then CSDAP is the problem to find a best approximation ‘‘from above’’ of the given rationals by other rationals whose common denominator is bounded. As pointed out in the introduction, the solutions of this problem for $n = 1$ can be interpreted as the lattice points lying on the lower convex hull of the conv $\{C(I_1, p) \cap \mathbb{Z}^2 \setminus \{0\}\}$. However, in general it is not sufficient to consider only the lattice points on the lower convex hull of $C(I_1, p)$. To see this, let $p = (1, \dots, 1, r)^T \in \mathbb{Z}^{n+1}$, $n \geq 2$, $r > 1$ and let $N = r - 1$. Then $C(I_n, p) = \text{pos}\{e^1, \dots, e^n, p\}$ and the lower convex hull of conv $\{C(I_n, p) \cap \mathbb{Z}^{n+1} \setminus \{0\}\}$ is given by conv $\{e^1, \dots, e^n, p\}$. Obviously, $1/(r-1), \dots, 1/(r-1)$ is a solution of CSDAP, but the vector $(1, \dots, 1, r-1)^T$ is not contained in conv $\{e^1, \dots, e^n, p\}$.

3 The knapsack cone

As mentioned in the introduction the dual of the cone $C(I_n, p)$ is closely related to the so called knapsack cone $K_{n,m}$, i.e., the set of all non-negative solutions

of a linear diophantine equation of the form

$$K_{n,m} = \left\{ (x, y)^T \in \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{\geq 0}^m : \sum_{i=1}^n a_i x_i = \sum_{j=1}^m b_j y_j \right\},$$

where we always assume that $a = (a_1, \dots, a_n)^T \in \mathbb{N}^n$, $b = (b_1, \dots, b_m)^T \in \mathbb{N}^m$, $n \geq m \geq 1$, and $a_1 \leq a_2 \leq \dots \leq a_n$, $b_1 \leq b_2 \leq \dots \leq b_m$. It is easy to see that

$$K_{n,m} = \text{pos} \{ b_j e^i + a_i e^{n+j} : 1 \leq i \leq n, 1 \leq j \leq m \}, \quad (3.1)$$

where $e^i \in \mathbb{R}^{n+m}$ denotes the i -th unit vector. The integral basis of $K_{n,m}$ is denoted by $\mathcal{H}_{n,m}$, for abbreviation.

One of the major results of this paper is to show that every element in $\mathcal{H}_{n,m}$ satisfies $n + m$ special inequalities that generalize the two inequalities

$$\sum_{i=1}^n x_i \leq b_m, \text{ and } \sum_{j=1}^m y_j \leq a_n \quad (3.2)$$

proved by Lambert ([Lam87]) and independently by Diaconis, Graham & Sturmfels [DGS94].

Theorem 3.1. *Every $(x, y)^T \in \mathcal{H}_{n,m}$ satisfies the inequalities*

$$\begin{aligned} [J_l]: \quad & \sum_{i=1}^n x_i + \sum_{j=1}^{l-1} \left\lfloor \frac{b_l - b_j}{a_n} \right\rfloor y_j \leq b_l + \sum_{j=l+1}^m \left\lceil \frac{b_j - b_l}{a_1} \right\rceil y_j, \quad l = 1, \dots, m, \\ [I_k]: \quad & \sum_{j=1}^m y_j + \sum_{i=1}^{k-1} \left\lfloor \frac{a_k - a_i}{b_m} \right\rfloor x_i \leq a_k + \sum_{i=k+1}^n \left\lceil \frac{a_i - a_k}{b_1} \right\rceil x_i, \quad k = 1, \dots, n. \end{aligned}$$

Observe, that $[J_m]$ and $[I_n]$ are generalizations of the inequalities stated in (3.2). From an algorithmic point of view Theorem 3.1 allows to assert that an integral point in $K_{n,m}$ does not belong to a minimal Hilbert basis of this cone. This problem is in general \mathcal{NP} -complete, see Sebö [Seb90].

Let us point out that, although Theorem 3.1 gives the best inequalities known so far to assert that an integral point in $K_{n,m}$ does not belong to the minimal Hilbert basis, we believe that a much stronger and more general statement is true: every element in the integral basis of $K_{n,m}$ is a convex combination of 0 and the generators $b_j e^i + a_i e^{n+j}$ of $K_{n,m}$. More formally, let

$$P_{n,m} = \text{conv} \{ 0, b_j e^i + a_i e^{n+j} : 1 \leq i \leq n, 1 \leq j \leq m \}.$$

One might conjecture that

Conjecture 3.1. $\mathcal{H}_{n,m} \subset P_{n,m}$.¹

For $m = 1$ Theorem 3.1 implies the inclusion $\mathcal{H}_{n,1} \subset P_{n,1}$. This can easily be read off from the representation

$$P_{n,1} = \left\{ (x, y)^T \in \mathbb{R}^n \times \mathbb{R} : a^T x = b_1 y, y \geq 0, \sum_{i=1}^n x_i \leq b_1 \right\}.$$

¹This conjecture was independently made by Hosten and Sturmfels, private communication

We remark that it was shown by Hosten and Sturmfels that the “more natural” conjecture $\mathcal{H}_{n,m} \subset \text{conv} \{0, (b_j e^i + a_i e^{n+j}) / \gcd(b_j, a_i) : 1 \leq i \leq n, 1 \leq j \leq m\}$ is false.

One approach to prove Conjecture 3.1 would be to find all the facet-defining inequalities of $P_{n,m}$ and check that these inequalities are satisfied by the elements of $\mathcal{H}_{n,m}$. A subset of all the facet defining inequalities of $P_{n,m}$ is given by

Proposition 3.1. *For $l = 1, \dots, m$ let*

$$J_l = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \sum_{i=1}^n x_i + \sum_{j=1}^{l-1} \frac{b_l - b_j}{a_n} y_j \leq b_l + \sum_{j=l+1}^m \frac{b_j - b_l}{a_1} y_j \right\}$$

and for $k = 1, \dots, n$ let

$$I_k = \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^m : \sum_{j=1}^m y_j + \sum_{i=1}^{k-1} \frac{a_k - a_i}{b_m} x_i \leq a_k + \sum_{i=k+1}^n \frac{a_i - a_k}{b_1} x_i \right\}.$$

Then we have $P_{n,m} \subset J_l$, $P_{n,m} \subset I_k$. Moreover, $P_{n,m} \cap J_l$ and $P_{n,m} \cap I_k$ are facets of $P_{n,m}$, $1 \leq l \leq m$, $1 \leq k \leq n$.

Proof. It is quite easy to check that all vectors $b_j e^i + a_i e^{n+j}$, $1 \leq i \leq n$, $1 \leq j \leq m$, are contained in J_l , $l = 1, \dots, m$. Moreover, the inequality corresponding to J_l is satisfied with equality by the $n + m - 1$ linearly independent points $b^l e^i + a_i e^{n+l}$, $1 \leq i \leq n$, $b_j e^n + a_n e^{n+j}$, $1 \leq j \leq l-1$, $b_j e^1 + a_1 e^{n+j}$, $l+1 \leq j \leq m$. The halfspaces I_k can be treated in the same way. \square

Remark 3.1. *Since $P_{n,2} = \{(x, y)^T \in \mathbb{R}^n \times \mathbb{R}^2 : a^T x = b^T y; x, y \geq 0, (x, y)^T \in I_k, 1 \leq k \leq n\}$, Theorem 3.1 shows that the conjecture is “almost true” when $m = 2$.*

The proof of Theorem 3.1 relies on the following observation, which was also the key of Lambert’s proof of (3.2).

Lemma 3.1. *Let $(\hat{x}, \hat{y})^T \in K_{n,m}$ and let $(x^1, y^1)^T, (x^2, y^2)^T \in \mathbb{N}^{n+m}$ such that $0 <_{\text{lex}} (x^2 - x^1, y^2 - y^1)^T <_{\text{lex}} (\hat{x}, \hat{y})^T$ and $a^T x^1 - b^T y^1 = a^T x^2 - b^T y^2$. Then $(\hat{x}, \hat{y})^T$ is not an element of $\mathcal{H}_{n,m}$.*

Proof. Let $(z_x, z_y) = (x^2 - x^1, y^2 - y^1)$. By assumption we have $(z_x, z_y)^T, (\hat{x} - z_x, \hat{y} - z_y)^T \in K_{n,m} \setminus \{0\}$. Thus $(\hat{x}, \hat{y}) = (\hat{x} - z_x, \hat{y} - z_y) + (z_x, z_y)$ can be written as a non trivial combination of two elements of $K_{n,m}$. \square

Proof of Theorem 3.1. Let $(\tilde{x}, \tilde{y})^T \in \mathcal{H}_{n,m}$. By symmetry it suffices to consider only the inequalities $[J_l]$, $l = 1, \dots, m$. Let us fix an index $l \in \{1, \dots, m\}$ and let $\xi = \sum_{i=1}^n \tilde{x}_i$, $v = \sum_{j=1}^m \tilde{y}_j$. We choose a sequence of points $x^i \in \mathbb{N}^n$, $0 \leq i \leq \xi$, such that

$$0 = x^0 <_{\text{lex}} x^1 <_{\text{lex}} x^2 <_{\text{lex}} \dots <_{\text{lex}} x^\xi = \tilde{x}. \quad (3.3)$$

Next we define recursively a sequence of points $y^j \in \mathbb{N}^m$, $0 \leq j \leq v$, by $y^0 = 0$ and $y^j = y^{j-1} + e^{d(j)}$, $j \geq 1$, where the index $d(j)$ is given by $d(j) = \min\{1 \leq$

$d \leq m : y_d^{j-1} + e^d \leq \tilde{y}_d$. Observe that here e^d denotes the d -th unit vector in \mathbb{R}^m . Obviously, we have

$$0 = y^0 <_{\text{lex}} y^1 <_{\text{lex}} y^2 <_{\text{lex}} \cdots <_{\text{lex}} y^v = \tilde{y}. \quad (3.4)$$

For two points $x \in \mathbb{N}^n$, $y \in \mathbb{N}^m$ let $r(x, y) = a^T x - b^T y$ and for a given point x^i let $y^{\mu(i)}$ be the unique point such that

$$r(x^i, y^{\mu(i)}) = \min \{ r(x^i, y^j) : r(x^i, y^j) \geq 0, 0 \leq j \leq v \}.$$

For abbreviation we set $r(i) = r(x^i, y^{\mu(i)})$. It is easy to see that $r(i) \in \{0, b_m - 1\}$ and

$$0 = y^{\mu(0)} <_{\text{lex}} y^{\mu(1)} <_{\text{lex}} \cdots <_{\text{lex}} y^{\mu(\xi)} = \tilde{y}. \quad (3.5)$$

Moreover, by definition of y^j we have the relation

$$r(i) \geq b_t \implies y_j^{\mu(i)} = \tilde{y}_j, 1 \leq j \leq t. \quad (3.6)$$

So we have assigned each $i \in \{0, \dots, \xi - 1\}$ its residue $r(i)$ and now we count the number of different residues which may occur. To this end let

$$R_l = \{i \in \{0, \dots, \xi - 1\} : r(i) < b_l\},$$

and for $l + 1 \leq j \leq m$ let

$$R_j = \{i \in \{0, \dots, \xi - 1\} : b_l \leq r(i) < b_j, y_{j-1}^{\mu(i)} = \tilde{y}_{j-1}, y_j^{\mu(i)} < \tilde{y}_j\}.$$

Since $\{0, \dots, \xi - 1\} = \bigcup_{j=l}^m R_j$ we have

$$\sum_{i=1}^n x_i \leq \#R_l + \sum_{j=l+1}^m \#R_j. \quad (3.7)$$

By Lemma 3.1, (3.3), (3.4) we may assume

$$\#R_l = \#\{r(i) : i \in R_l\} \leq b_l. \quad (3.8)$$

We claim that for $j = l + 1, \dots, m$

$$\#R_j \leq \left\lceil \frac{b_j - b_l}{a_1} \right\rceil \tilde{y}_j. \quad (3.9)$$

To show this let $\zeta \in \{0, \dots, \tilde{y}_j - 1\}$ and let $x^{i_1} <_{\text{lex}} \cdots <_{\text{lex}} x^{i_\tau}$ be all vectors of the x -sequence (cf. (3.3)) satisfying $y_j^{\mu(i)} = \zeta$ and $i \in R_j$. By construction we have $y^{\mu(i_1)} = y^{\mu(i_2)} = \cdots = y^{\mu(i_\tau)}$ and so

$$(\tau - 1)a_1 \leq a^T x^{i_\tau} - a^T x^{i_1} = r(i_\tau) - r(i_1) \leq (b_j - 1) - b_l.$$

Hence $\tau \leq \lceil (b_j - b_l)/a_1 \rceil$ and we get (3.9).

So far we have proved (cf. (3.7), (3.9))

$$\sum_{i=1}^n x_i \leq \#R_l + \sum_{j=l+1}^m \left\lceil \frac{b_j - b_l}{a_1} \right\rceil \tilde{y}_j. \quad (3.10)$$

In the following we estimate the number of residues $\in \{0, \dots, b_l - 1\}$ which are not contained in $\{r(i) : i \in R_l\}$.

To do this we have to extend our x -sequence. For $v \in \mathbb{N}$ let $r_v, q_v \in \mathbb{N}$ be the uniquely determined numbers with $v = r_v \xi + q_v$, $0 \leq q_v < \xi$, and let

$$\bar{x}^v = r_v x^\xi + x^{q_v}.$$

Observe that $r(\bar{x}^v, y) = r_v b^T \tilde{y} - b^T y + a^T x^{q_v}$. For $s \in \{1, \dots, l-1\}$ and $t \in \{0, \dots, \tilde{y}_s - 1\}$ let $y^{s,t}$ be the point of the y -sequence (cf. (3.4)) with coordinates

$$y_s^{s,t} = t, \quad y_j^{s,t} = \tilde{y}_j, \quad 1 \leq j \leq s-1, \quad \text{and} \quad y_j^{s,t} = 0, \quad s+1 \leq j \leq m.$$

For such a vector $y^{s,t}$ let $\bar{x}^{\delta(s,t)}$ be the point of the \bar{x} -sequence such that

$$r(\bar{x}^{\delta(s,t)}, y^{s,t}) = \min \{r(\bar{x}^i, y^{s,t}) : r(\bar{x}^i, y^{s,t}) \geq b_s, i \in \{0, \dots, \xi\}\}.$$

Observe that such a point $\bar{x}^{\delta(s,t)}$ exists, because $t \in \{0, \dots, \tilde{y}_s - 1\}$. Moreover, $\bar{x}^{\delta(s,t)}$ belongs to the ‘‘original’’ x -sequence. In particular, we have

$$b_s \leq r(x^{\delta(s,t)}, y^{s,t}) < b_s + a_n. \quad (3.11)$$

Let $r_{s,t} = \{\bar{x}^i : b_s \leq r(\bar{x}^i, y^{s,t}) < b_l\}$. Obviously, by (3.11) we have

$$\#r_{s,t} \geq \lfloor (b_l - b_s)/a_n \rfloor. \quad (3.12)$$

Now we study the cardinality of

$$\bar{R} = \bigcup_{s=1}^{l-1} \left\{ \bigcup_{t=0}^{\tilde{y}_s-1} \{r(\bar{x}^i, y^{s,t}) : b_s \leq r(\bar{x}^i, y^{s,t}) < b_l\} \right\}$$

and we show

$$\#\bar{R} \geq \sum_{s=1}^{l-1} \left\lfloor \frac{b_l - b_s}{a_n} \right\rfloor \tilde{y}_s. \quad (3.13)$$

Suppose the contrary. Then, by (3.12), we can find $s, s' \in \{1, \dots, l-1\}$, $t \in \{0, \dots, \tilde{y}_s - 1\}$, $t' \in \{0, \dots, \tilde{y}_{s'} - 1\}$ and vectors \bar{x}^v, \bar{x}^w of the \bar{x} -sequence such that $r(\bar{x}^v, y^{s,t}) = r(\bar{x}^w, y^{s',t'})$. We may assume $y^{s,t} <_{\text{lex}} y^{s',t'}$ and therefore $\bar{x}^v <_{\text{lex}} \bar{x}^w$, i.e., $v \leq w$. Since

$$r(\bar{x}^v, y^{s,t}) = r_v b^T \tilde{y} - b^T y^{s,t} + a^T x^{q_v} = r_w b^T \tilde{y} - b^T y^{s',t'} + a^T x^{q_w} = r(\bar{x}^w, y^{s',t'})$$

we get $r_w \in \{r_v, r_v + 1\}$.

a) If $r_w = r_v$ then $0 <_{\text{lex}} \bar{x}^w - \bar{x}^v = x^{q_w} - x^{q_v} <_{\text{lex}} x^\xi$ and we can apply Lemma 3.1 to $(\bar{x}^w, y^{s',t'})^T, (\bar{x}^v, y^{s,t})^T$ which yields the contradiction $(\tilde{x}, \tilde{y}) \notin \mathcal{H}_{n,m}$.

b) If $r_w = r_v + 1$ then $0 <_{\text{lex}} \bar{x}^w - \bar{x}^v = x^\xi + x^{q_w} - x^{q_v}$. Since

$$a^T (x^{q_v} - x^{q_w}) = b^T \tilde{y} + b^T y^{s,t} - b^T y^{s',t'} > 0$$

we have $x^{q_w} <_{\text{lex}} x^{q_v}$ and thus $0 <_{\text{lex}} \bar{x}^w - \bar{x}^v <_{\text{lex}} x^\xi$. Hence, also in this case we can apply Lemma 3.1 and obtain a contradiction. Next we claim that

$$\bar{R} \cap \{r(i) : i \in R_l\} = \emptyset. \quad (3.14)$$

Otherwise there exist $\bar{x}^v, y^{s,t}$ with $b_s \leq r(\bar{x}^v, y^{s,t}) < b_l$ and $\bar{x}^i, y^{\mu(i)}, 0 \leq i \leq \xi - 1$, such that $r(\bar{x}^v, y^{s,t}) = r(\bar{x}^i, y^{\mu(i)})$. Since $r(\bar{x}^v, y^{s,t}) \geq b_s$ but $y^{s,t} < \tilde{y}_s$ we have $y^{s,t} \neq y^{\mu(i)}$ (cf. (3.6)). Hence, we may assume $y^{s,t} <_{\text{lex}} y^{\mu(i)}$ or $y^{\mu(i)} <_{\text{lex}} y^{s,t}$.

a) If $y^{s,t} <_{\text{lex}} y^{\mu(i)}$ then $\bar{x}^v <_{\text{lex}} \bar{x}^i$ and thus $v < i < \xi$. Again, by Lemma 3.1 we find $(\tilde{x}, \tilde{y}) \notin \mathcal{H}_{n,m}$.

b) If $y^{\mu(i)} <_{\text{lex}} y^{s,t}$ then $\bar{x}^i <_{\text{lex}} \bar{x}^v$. As above, it is easy to see that $r_v \in \{0, 1\}$ and that in both cases Lemma 3.1 can be applied in order to get a contradiction.

Finally, we note that (3.8), (3.14) and (3.13) imply

$$\#R_l \leq b_l - \sum_{s=1}^{l-1} \left\lfloor \frac{b_l - b_s}{a_n} \right\rfloor \tilde{y}_s,$$

which proves inequality $[J_l]$ (cf. (3.10)). \square

4 Carathéodory Property

Every integral vector in a polyhedral cone can be decomposed by vectors of a Hilbert basis. In fact one can write every integral vector in any pointed cone of dimension n as the non-negative integer combination of at most $2n - 2$ vectors from the integral basis. This was shown by Sebö [Seb90] and gives currently the best bound in general; it improves the bound given by Cook, Fonlupt & Schrijver [CFS86] by 1, yet is still quite far from what many researchers conjecture to be true, namely: every integral vector in a pointed cone is the non-negative integer combination of at most n vectors of the integral basis. So far it has been verified only for dimensions less or equal than three, see [Seb90].

We show below that this *integer Version of Caratheodory's Theorem* holds for the knapsack cone of Section 2 and the cone of best approximations studied in Section 3 when the numbers are divisible. To this end we use the notation $a|b$ if a is a divisor of b .

Theorem 4.1. *Let positive integers a_1, \dots, a_n and b_1, \dots, b_m be given such that $a_{i-1}|a_i, i = 2, \dots, n, a_n|b_1$ and $b_{i-1}|b_i, i = 2, \dots, m$. Every integral point in*

$$K_{n,m} = \left\{ (x, y)^T \in \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{\geq 0}^m : \sum_{i=1}^n a_i x_i = \sum_{j=1}^m b_j y_j \right\}$$

can be written as the non-negative integer combination of at most $n + m - 1 = \dim(K_{n,m})$ elements of $\mathcal{H}_{n,m} = \mathcal{H}(K_{n,m})$.

Proof. Let $(\tilde{x}, \tilde{y})^T \in K_{n,m}$. We have to show that there exist $(x^i, y^i)^T \in \mathcal{H}_{n,m}, n_i \in \mathbb{N}, 1 \leq i \leq n + m - 1$ such that $(\tilde{x}, \tilde{y})^T = \sum_{i=1}^{n+m-1} n_i (x^i, y^i)^T$.

In order to show this statement we use induction w.r.t. n . W.l.o.g. we may assume that $a_1 = 1$. If $n = 1$ then we see by (3.2) that $b_i e^1 + e^{n+i}, i = 1, \dots, m$, is the integral basis of $K_{n,m}$ and we are done. So let $n \geq 2, a_i^* = a_i/a_2, 2 \leq i \leq n, b_j^* = b_j/a_2, 1 \leq j \leq m$, and let

$$K_{n,m}^* = \left\{ (x, y)^T \in \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{\geq 0}^m : x_1 + x_2 + \sum_{i=3}^n a_i^* x_i = \sum_{j=1}^m b_j^* y_j \right\}.$$

Let $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}$ be the map given by $f((x_1, \dots, x_n, y_1, \dots, y_m)^T) = (x_1/a_2, x_2, \dots, x_n, y_1, \dots, y_m)^T$. It is easy to see that the linear map f induces a bijection between $K_{n,m} \cap \mathbb{Z}^{n+m}$ and $K_{n,m}^* \cap \mathbb{Z}^{n+m}$. Therefore it suffices to prove that $f((\tilde{x}, \tilde{y})^T)$ can be written as an integral combination of at most $n+m-1$ elements of the integral basis of $K_{n,m}^*$, which is denoted by $\mathcal{H}_{n,m}^*$.

For abbreviation we set $(x^*, y^*) = f((\tilde{x}, \tilde{y})^T)$. Now let

$$K_{n-1,m} = \left\{ (x, y)^T \in \mathbb{R}_{\geq 0}^{n-1} \times \mathbb{R}_{\geq 0}^m : x_1 + \sum_{i=2}^{n-1} a_{i+1}^* x_i = \sum_{j=1}^m b_j^* y_j \right\}$$

and let $g : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n-1+m}$ defined by $g((x_1, \dots, x_n, y_1, \dots, y_m)^T) = (x_1 + x_2, x_3, \dots, x_n, y_1, \dots, y_m)^T$. Next we make use of our induction hypothesis w.r.t. to the point $g((x^*, y^*)^T) \in K_{n-1,m}$, i.e., there exist $n_i \in \mathbb{N}$, $(\bar{x}^i, \bar{y}^i)^T \in \mathcal{H}_{n-1,m}$, $1 \leq i \leq n+m-2$, such that

$$g((x^*, y^*)^T) = \sum_{i=1}^{n+m-2} n_i (\bar{x}^i, \bar{y}^i)^T,$$

where $\mathcal{H}_{n-1,m}$ denotes the integral basis of $K_{n-1,m}$ w.r.t. \mathbb{Z}^{n-1+m} . In particular, we have

$$x_1^* + x_2^* = \sum_{i=1}^{n+m-2} n_i \bar{x}_1^i. \quad (4.1)$$

W.l.o.g. let $n_1 \bar{x}_1^1 \geq n_2 \bar{x}_1^2 \geq \dots \geq n_{n+m-2} \bar{x}_1^{n+m-2}$. From the identity (4.1) we see that we can find $\epsilon_i \in \{0, 1\}$, $2 \leq i \leq n+m-2$ such that

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} - \sum_{i=2}^{n+m-2} n_i \left(\epsilon_i \begin{pmatrix} \bar{x}_1^i \\ 0 \end{pmatrix} + (1 - \epsilon_i) \begin{pmatrix} 0 \\ \bar{x}_1^i \end{pmatrix} \right) \geq 0. \quad (4.2)$$

Obviously, we have $v_1 + v_2 = n_1 \bar{x}_1^1$. Now if we consider the 2-dimensional knapsack cone $C = \{(x, y)^T \in \mathbb{R}_{\geq 0}^2 \times \mathbb{R}_{\geq 0} : x_1 + x_2 = \bar{x}_1^1 y_1\}$ then all elements of the integral basis have the form $(\alpha_1, \alpha_2, 1)^T$ (cf. (3.2)). Further, for a 2-dimensional cone it is well known that each element can be written as an integral combination of at most two elements of the minimal Hilbert basis. Therefore, since $(v_1, v_2)^T \in C$, there exist $n_0^*, n_1^* \in \mathbb{N}$, $w_0, w_1 \in \mathbb{N}^2$, such that

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = n_0^* w_0 + n_1^* w_1, \quad n_0^* + n_1^* = n_1, \quad w_1^0 + w_2^0 = w_1^1 + w_2^1 = \bar{x}_1^1. \quad (4.3)$$

Finally, for $i = 2, \dots, n+m-2$ let $(x^i, y^i)^T \in \mathbb{R}^n \times \mathbb{R}^m$ be defined by

$$(x^i, y^i) = (\epsilon_i \bar{x}_1^i, (1 - \epsilon_i) \bar{x}_1^i, \bar{x}_2^i, \dots, \bar{x}_{n-1}^i, \bar{y}_1^i, \dots, \bar{y}_m^i).$$

and for $i = 0, 1$ let

$$(x^i, y^i) = (w_1^i, w_2^i, \bar{x}_2^1, \dots, \bar{x}_{n-1}^1, \bar{y}_1^1, \dots, \bar{y}_m^1).$$

By definition and (4.3) we have $(x^i, y^i)^T \in K_{n,m}$ and indeed it is easy to check that $(x^i, y^i)^T \in \mathcal{H}_{n,m}$, $0 \leq i \leq n+m-2$. By (4.2) and (4.3) we get

$$(x^*, y^*)^T = n_0^* (x^0, y^0)^T + n_1^* (x^1, y^1)^T + \sum_{i=2}^{n+m-2} n_i (x^i, y^i)^T.$$

□

The Carathéodry property also holds for a special family of cones that we investigated in Section 2.

Theorem 4.2. *Let $C = \text{pos}\{e^1, \dots, e^{n-1}, p\}$ with $p = (p_1, \dots, p_n)^T \in \mathbb{N}^n$ satisfying $p_1 = 1$ and $p_i | p_{i+1}$, $1 \leq i \leq n-1$, i.e., the numbers p_i are successively divisible. Let $\mathcal{H}(C)$ be the integral basis of C . Then for each $z \in C \cap \mathbb{Z}^n$ there exist at most n elements b^1, \dots, b^n of $\mathcal{H}(C)$ such that $z = \sum_{i=1}^n v_i b^i$ with $v_i \in \mathbb{N}$.*

Proof. We use double induction w.r.t. the dimension n and the last coordinate p_n of the vector p . For $n = 2$ the theorem follows from the more general result of Sebö [Seb90].

Let $n \geq 3$. If $p_n = 1$ then the generators of C constitute a basis of \mathbb{Z}^n and the result follows. Let $p_n \geq 2$ and let

$$\begin{aligned} P(C) &= \left\{ z \in \mathbb{Z}^n : z = \sum_{i=1}^{n-1} \lambda_i e^i + \lambda_n p, \quad 0 \leq \lambda_i < 1 \right\} \\ &= \left\{ \left(1, \left\lfloor \frac{jp_2}{p_n} \right\rfloor, \dots, \left\lfloor \frac{jp_{n-1}}{p_n} \right\rfloor, j \right)^T : 1 \leq j \leq p_n - 1 \right\}. \end{aligned}$$

It is quite easy to check that

$$\mathcal{H}(C) = \{e^1, \dots, e^{n-1}, p\} \cup P(C).$$

Next we have to distinguish two cases.

I. $p_2 = 1$.

Let $z \in C \cap \mathbb{Z}^n$. We first analyze the case when $z_1 \geq z_2$. For a vector $x \in \mathbb{R}^n$ let $\tilde{x} = (x_2, \dots, x_n)^T$ be its orthogonal projection onto the plane $\{x \in \mathbb{R}^n : x_1 = 0\}$ (identified with \mathbb{R}^{n-1}). Then \tilde{z} is an integral vector of the $(n-1)$ -dimensional cone $\tilde{C} = \text{pos}\{\tilde{e}^2, \dots, \tilde{e}^{n-1}, \tilde{p}\}$ which is of the same type as the cone C . Hence, by induction hypothesis w.r.t. the dimension we can find $\tilde{b}^1, \dots, \tilde{b}^{n-1} \in \mathcal{H}(\tilde{C})$ such that $\tilde{z} = \sum_{i=1}^{n-1} v_i \tilde{b}^i$, $v_i \in \mathbb{N}$. Now, it is easy to check that $b^i = (1, \tilde{b}^i)^T \in \mathcal{H}(C)$ and since $\tilde{b}_2^i = 1$ we get

$$z = \sum_{i=1}^{n-1} v_i b^i + (z_1 - z_2) e^1.$$

The case $z_2 \leq z_1$ can be treated in the same way w.r.t. the orthogonal projection onto the plane $\{x \in \mathbb{R}^n : x_2 = 0\}$.

II. $p_2 > 1$.

Let $v = (1, 1, p_3/p_2, \dots, p_n/p_2)^T \in P(C)$. Then we may write $v = (1/p_2)p + ((p_2 - 1)/p_2)e^1$ and thus v is contained in the relative interior of a 2-face of the cone C . Hence, $C = C_1 \cup C_2$ with

$$C_1 = \text{pos}\{e^1, \dots, e^{n-1}, v\} \quad \text{and} \quad C_2 = \text{pos}\{e^2, \dots, e^{n-1}, v, p\}.$$

Obviously, C_1 is of the same type as C but with $v_n < p_n$ and therefore, we may assume that the Carathéodry property holds for this cone.

Since v, e^2, \dots, e^n constitute a basis of the lattice \mathbb{Z}^n there exists a unimodular matrix U such that $Uv = e^1$, $Ue^i = e^i$, $i = 2, \dots, n$. Let

$$\bar{p} = Up = \left(1, p_2 - 1, \frac{p_3}{p_2}(p_2 - 1), \dots, \frac{p_n}{p_2}(p_2 - 1) \right)^T.$$

Then $UC_2 = \text{pos}\{e^1, \dots, e^{n-1}, \bar{p}\}$. This cone is of the same type as C but with $\bar{p}_n < p_n$. Therefore, we can assume that the Carathéodory property holds for the cone UC_2 and hence, also for C_2 .

Next we study the minimal Hilbert bases of the cones C_1 and C_2 . Obviously,

$$\begin{aligned} \mathcal{H}(C_1) &= \{e^1, \dots, e^{n-1}, v\} \cup \left\{ \left(1, \left\lceil j \frac{v_2}{v_n} \right\rceil, \dots, \left\lceil j \frac{v_{n-1}}{v_n} \right\rceil, j \right) : 1 \leq j \leq v_n \right\} \\ &= \{e^1, \dots, e^{n-1}, v\} \cup \left\{ \left(1, \left\lceil j \frac{p_2}{p_n} \right\rceil, \dots, \left\lceil j \frac{p_{n-1}}{p_n} \right\rceil, j \right)^T : 1 \leq j \leq \frac{p_n}{p_2} \right\}. \end{aligned}$$

Thus $\mathcal{H}(C_1) \subset \mathcal{H}(C)$.

For the cone UC_2 we get

$$\mathcal{H}(UC_2) = \{e^1, \dots, e^{n-1}, \bar{p}\} \cup \left\{ w^j : j = 1, \dots, \frac{p_n}{p_2}(p_2 - 1) \right\},$$

where $w^j = (1, \lceil jp_2/p_n \rceil, \dots, \lceil jp_{n-1}/p_n \rceil, j)^T$, $1 \leq j \leq p_n/p_2(p_2 - 1)$. Now,

$$U^{-1}w^j = \left(1, \left\lceil \left(j + \frac{p_n}{p_2} \right) \frac{p_2}{p_n} \right\rceil, \dots, \left\lceil \left(j + \frac{p_n}{p_2} \right) \frac{p_{n-1}}{p_n} \right\rceil, j + \frac{p_n}{p_2} \right)^T$$

and this shows $\mathcal{H}(C_2) \subset \mathcal{H}(C)$.

Since $C = C_1 \cup C_2$ and each subcone satisfies the Carathéodory property we get the desired result. \square

Remark 4.1. *With small modifications of the above proof one can show that a cone C as in Theorem 4 admits a unimodular partition, i.e., one can find subcones C_i generated by the elements of $\mathcal{H}(C)$, such that i) the generators of C_i form a basis of \mathbb{Z}^n , ii) the union of the subcones C_i covers C and iii) the intersection of two distinct subcones is a face of both. Of course, this property implies the Carathéodory property.*

5 A relation to polyhedral combinatorics

In this section we discuss a relation of the cone investigated in Section 2 to families of knapsack polyhedra. For $k, n, m, b \in \mathbb{N}$, $k \leq n$ and given vectors $p \in \mathbb{N}^{n+m}$ and $u \in \mathbb{N}^n$, let

$$P_I = \text{conv} \{x \in \mathbb{Z}^{n+m} : p^T x \leq b, 0 \leq x_i \leq u_i \text{ for all } 1 \leq i \leq k, x_i \leq u_i \text{ for all } k+1 \leq i \leq n\}.$$

In the sequel we investigate the inequalities that are needed to describe P_I . To this end we resort to the Chvátal-Gomory rounding procedure. The following notation is needed. Let

$$P^0 = \{x \in \mathbb{R}^{n+m} : p^T x \leq b, 0 \leq x_i \leq u_i \text{ for all } 1 \leq i \leq k, x_i \leq u_i \text{ for all } k+1 \leq i \leq n\},$$

and define for $t \in \mathbb{N}$,

$$P^{t+1} := \{x \in \mathbb{R}^{n+m} : c^T x \leq \lfloor \delta \rfloor \text{ for all supporting hyperplanes } c^T x \leq \delta \text{ of } P^t \text{ with } c \text{ integral.}\}$$

It is well known that in the case when $n = 0$ and $\gcd(p_{n+1}, \dots, p_{n+m}) = 1$, $P_I = P^0$ holds, see Proposition 1.2 on page 211 in [NW88].

In fact the following generalization holds.

Proposition 5.1. *If $\gcd(p_{n+1}, \dots, p_{n+m}) = 1$, then $P_I = P^0$.*

Proof. We use the Chvátal-Gomory rounding procedure to prove the statement. In fact, [Gom60], and later [Chv73] and [Sch80] have shown that there exists a finite number t_0 such that $P_I = P^{t_0}$. Let $c^T x \leq \delta$ be a supporting hyperplane of P^0 with c integral. Then, c is an integer element in the cone generated by the row vectors describing P^0 , i.e.,

$$c \in \text{pos}(-e^1, \dots, -e^k, e^1, \dots, e^n, p) \cap \mathbb{Z}^{n+m}.$$

Hence, there exist $\mu_1, \dots, \mu_k \geq 0$, $\lambda_1, \dots, \lambda_n \geq 0$ and $\tau \geq 0$ such that

$$\begin{aligned} c_i &= -\mu_i + \lambda_i + \tau p_i, \text{ for all } 1 \leq i \leq k, \\ c_i &= \lambda_i + \tau p_i, \text{ for all } k+1 \leq i \leq n, \\ c_i &= \tau p_i, \text{ for all } n+1 \leq i \leq n+m. \end{aligned}$$

On account of the condition that $\gcd(p_{n+1}, \dots, p_{n+m}) = 1$ we conclude that $\tau \in \mathbb{N}$. This further implies that $\lambda_i \in \mathbb{N}$ for all $k+1 \leq i \leq n$ and that $-\mu_i + \lambda_i \in \mathbb{N}$ for all $1 \leq i \leq k$. Moreover,

$$\begin{aligned} \lfloor \delta \rfloor &= \lfloor \sum_{i=1}^n \lambda_i u_i + \tau b \rfloor \\ &\geq \lfloor \sum_{i=1}^k (\lambda_i - \mu_i)^+ u_i + \sum_{i=k+1}^n \lambda_i u_i + \tau b \rfloor \\ &= \sum_{i=1}^k (\lambda_i - \mu_i)^+ u_i + \sum_{i=k+1}^n \lambda_i u_i + \tau b. \end{aligned}$$

Because $c = \sum_{i=1}^k (\lambda_i - \mu_i)^+ e^i + \sum_{i=k+1}^n \lambda_i e^i + \tau p$, $c^T x \leq \lfloor \delta \rfloor$ is a non-negative combination of inequalities from P^0 . Therefore, $P^1 = P^0$, implying that $P^{t+1} = P^t$ for all $t \in \mathbb{N}$ and finally, $P_I = P^0$. \square

On account of Proposition 5.1, the only case in which inequalities might have to be added to the system describing P^0 in order to describe P_I is the case when $\gcd(p_{n+1}, \dots, p_{n+m}) > 0$. When $m = 1$ and $p_{n+1} > 1$, then the arguments used in the proof Proposition 5.1 show that the supporting hyperplanes $\{x \in \mathbb{R}^n : c^T x = \delta\}$ of P^0 with c integral can be reconstructed from a Hilbert basis of $\text{pos}(-e^1, \dots, -e^k, e^1, \dots, e^n, p)$. For $k = 0$, we have to analyze the minimal Hilbert basis of the cone of all best approximations, $\text{pos}(e^1, \dots, e^n, p)$. By similar arguments one may verify that for $k > 0$, we need to inspect the Hilbert bases of the family of cones

$$\text{pos}(\sigma_1 e^1, \dots, \sigma_k e^k, e^{k+1}, \dots, e^n, p) : \sigma \in \{-1, 0, +1\}^k.$$

Denoting by $b(h) = \sum_{i=1}^n \lambda_i u_i + \lambda_{n+1} b$ for $h = \sum_{i=1}^n \lambda_i e^i + \lambda_{n+1} p$, we obtain

Proposition 5.2. *Let $k = 0$ and $m = 1$. Then the system of inequalities*

$$h^t x \leq \lfloor b(h) \rfloor \text{ for all } h \text{ in the integral basis of } \text{pos}(e^1, \dots, e^n, p) \quad (5.1)$$

describes P^1 .

In the special case when $u_i = 1$ for all $i = 1, \dots, n$ and $b = 1$, $p_1 \leq \dots \leq p_{n+1}$ and p_i divides p_{i+1} for all $i = 1, \dots, n$, it may be checked that the system (5.1) is totally dual integral.

References

- [BP94] A.D. Bryuno and V.I. Parusnikov, *Klein polyhedrals for two cubic Davenport forms*, Mathematical Notes, vol. 46, no. 3-4, 994 – 1007 (1994).
- [Gom60] R. Gomory, *Solving linear programming problems in integers*, in R. Bellman and M. Hall (eds), Combinatorial analysis, Proc. of Symposia in Applied Mathematics, vol 10, Providence RI (1960).
- [CT91] P. Conti, C. Traverso, *Buchberger algorithm and integer programming*, Proceedings AAECC-9 (New Orleans), Springer LNCS 539, 130 - 139 (1991).
- [Chv73] V. Chvátal, *Edmond's polytopes and a hierarchy of combinatorial problems*, Discrete Mathematics 4, 305–337 (1973).
- [CFS86] W. Cook, J. Fonlupt, and A. Schrijver, *An integer analogue of Caratheodory's theorem*, J. Comb. Theory (B) **40**, 1986, 63–70.
- [Cor31] J.G. van der Corput, *Über Systeme von linear-homogenen Gleichungen und Ungleichungen*, Proceedings Koninklijke Akademie van Wetenschappen te Amsterdam 34, 368 - 371 (1931).
- [DS82] A. Dress and R. Scharlau, *Indecomposable totally positive numbers in real quadratic orders*, Journal of Number Theory, vol. 14 no. 3, 292 – 306 (1982).
- [DGS94] P. Diaconis, R. Graham, and B. Sturmfels, *Primitive partition identities*, Paul Erdős is 80, Vol. II, Janos Bolyai Society, Budapest, 1-20 (1995).
- [Ehr79] E. Ehrhart, *Sur les équations diophantiennes linéaires*, C. R. Acad. Sc. Paris, t. 288, Série A, 785 – 787 (1979).
- [FT95] M. Filgueiras and A.P. Tomás, *A fast method for finding the basis of non-negative solutions to a linear Diophantine equation*, J. Symbolic Computation **19**, 507 – 526 (1995).
- [Fin93] Yu.Yu. Finkel'shtein, *Klein polygons and reduced regular continued fractions*, Russ. Math. Surveys **48** (3), 198 – 200 (1993).
- [GP79] F.R. Giles and W.R. Pulleyblank, *Total dual integrality and integer polyhedra*, Lineare Algebra Appl. 25, 191 - 196 (1979).
- [G1873] P. Gordan, *Über die Auflösung linearer Gleichungen mit reellen Coefficienten*, Math. Ann. 6, 23 - 28 (1873).
- [Gra75] J. E. Graver, *On the foundations of linear and integer programming I*, Mathematical Programming 8, 207 - 226 (1975).
- [Gre88] H. Greenberg, *Solution to a linear Diophantine equation for nonnegative integers*, J. of Algorithms **9**, 343 – 353 (1988).
- [GLS88] M. Grötschel, L. Lovász and A. Schrijver, *Geometric algorithms and combinatorial optimization*, Springer Verlag Berlin (1988).

- [Khi56] A. Khintchine, *Kettenbrüche*, Teubner Verlag, Leipzig (1956).
- [K1895] F. Klein, *Über eine geometrische Auffassung der gewöhnlichen Kettenbruchentwicklung*, Nachr. Ges. Wiss. Göttingen, Math.-Phys. 3, 357 - 359 (1895).
- [Lam87] J.L. Lambert, *Une borne pour les générateurs des solutions entières positives d'une équation diophnatiennne linéaire*, C.R. Acad. Sci. Paris **305**, Série I, 1987, 39–40.
- [NW88] G. L. Nemhauser, L. Wolsey, *Integer and Combinatorial Optimization*, Wiley, Chichester (1988).
- [Oda88] T. Oda, *Convex bodies and algebraic geometry*, Springer-Verlag, (1988).
- [Per13] O. Perron, *Die Lehre von den Kettenbrüchen*, Teubner Verlag, Leipzig (1913).
- [Sca86] H. E. Scarf, *Neighborhood systems for production sets with indivisibilities*, *Econometrica* 54, 507 - 532 (1986).
- [Seb90] A. Sebö, *Hilbert bases, Caratheodory's Theorem and combinatorial optimization*, in Proc. of the IPCO conference, Waterloo, Canada, 431–455 (1990).
- [Sch80] A. Schrijver, *On cutting planes*, *Annals of Discrete Mathematics* 9, 291–296 (1980).