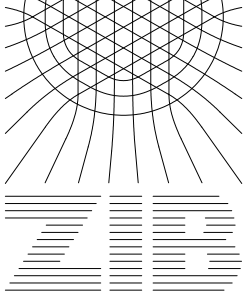
Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany



ZSOLT TUZA

ANNEGRET WAGLER

Minimally non-preperfect graphs of small maximum degree

Preprint SC 97-28 (June 1997)

Minimally non-preperfect graphs of small maximum degree

Zsolt Tuza *

Computer and Automation Institute
Hungarian Academy of Sciences
H–1111 Budapest, Kende u. 13–17.
Hungary

e-mail: tuza@lutra.sztaki.hu

Annegret Wagler

Konrad–Zuse–Zentrum für
Informationstechnik Berlin
Takustr. 7, D–14195 Berlin
Germany

e-mail: wagler@zib.de

July 16, 1997

Abstract

A graph G is called preperfect if each induced subgraph $G' \subseteq G$ of order at least 2 has two vertices x, y such that either all maximum cliques of G' containing x contain y, or all maximum independent sets of G' containing y contain x, too. Giving a partial answer to a problem of Hammer and Maffray [Combinatorica 13 (1993), 199–208], we describe new classes of minimally non-preperfect graphs, and prove the following characterizations:

(i) A graph of maximum degree 4 is minimally non-preperfect if and only if it is an odd cycle of length at least 5, or the complement of a cycle of length 7, or the line graph of a 3-regular 3-connected bipartite graph.

(ii) If a graph G is not an odd cycle and has no isolated vertices, then its line graph is minimally non-preperfect if and only if G is bipartite, 3-edge-connected, regular of degree d for some $d \geq 3$, and contains no 3-edge-connected d'-regular subgraph for any $3 \leq d' < d$.

*guest at ZIB 1997

1 Introduction

In his classic paper [1], Berge proposed to call a graph G *perfect* if, for each of its induced subgraphs G', the chromatic number $\chi(G')$ equals the clique number $\omega(G')$; and call G *imperfect* otherwise. Chordless cycles of length at least four have been termed *holes* and their complements *antiholes*. Obviously, any graph that contains an odd hole or an odd antihole is imperfect. The famous Strong Perfect Graph Conjecture of Berge states that a graph is perfect if and only if it contains no odd hole and no odd antihole as an induced subgraph.

Investigating subclasses of perfect graphs, Hammer and Maffray [2] have introduced the notions of predomination and preperfectness. Following their terminology, we say that a vertex x of a graph $G = (V, E)$ *predominates* a vertex y if one of the following three situations occurs:

(o) $V = \{x\} = \{y\}$,

(i) $x \neq y$, $xy \in E$, and every maximum clique containing y contains x,

(ii) $x \neq y$, $xy \notin E$, and every maximum independent set containing x contains y.

A graph is called *preperfect* if each of its subgraphs has a predominant vertex (i.e., an $x \in V$ that predominates some $y \in V$). It follows by definition that preperfectness is closed under induced subgraphs and complementation. Hammer and Maffray have shown that every preperfect graph is perfect, and that the class of preperfect graphs contains all bipartite, i-triangulated, and parity graphs (see [2] for the definitions of those graph classes).

In order to understand a graph property $\mathcal{P}$ better, it is often useful to investigate extremal cases with respect to $\mathcal{P}$, i.e., graphs having $\mathcal{P}$ but losing it by a small modification. Two interesting types of such graphs are the *minimal* graphs (they possess $\mathcal{P}$ but lose it by the deletion of any *vertex*) and the *critical* graphs (they have $\mathcal{P}$ but lose it by the deletion of any *edge*). In the context of preperfect graphs, the former approach seems to be more fruitful so far.

Hammer and Maffray observed that every odd hole and odd antihole is minimally non-preperfect, moreover they presented an infinite sequence of perfect but minimally non-preperfect graphs (one graph for every even order $n \equiv 2 \pmod 4$, cf. Section 1.2). Later a different sequence of minimally non-preperfect graphs, including a graph for every even n, was found (but not published) by Hougardy, Maffray and Sebő [3].

Our goal here is to exhibit a much larger family of such graphs, hence answering in a way an open question of [2, p. 205]. Our two main results are the *complete characterization of minimally non-preperfect graphs of maximum degree 4* (Theorem 1), and a *strong property of non-preperfect line graphs* (Theorem 2) that can be turned to a necessary and sufficient condition to describe all the minimal ones. These results are formulated in Section 2 and proved in the later sections.

One of the key concepts in our investigations is the Matching Independence Property (MIP, for short), see the definition below. We prove in Section 3 that for regular bipartite graphs the MIP is equivalent to 3-edge-connectivity (Lemma 4). We also present a characterization of perfect line graphs which is equivalent to — but in some sense stronger than — a corresponding result of Maffray [4].

Our intuition says that the Matching Independence Property would deserve a closer look on its own as well, but this will not be done in the present paper. On the other hand, we mention that the MIP has some relationships with the *critically perfect graphs*, a concept introduced and studied by the second author in [6]. This connection will be described in a greater detail in the forthcoming manuscript [5]. We should note at this point that all perfect, minimally non-preperfect graphs found so far are *critically perfect line graphs or their complements*.

1.1 Definitions and notation

As usual, the *minimum degree* of graph G will be denoted by $\delta(G)$, the *degree* of vertex x by $d(x)$ or $d_G(x)$ if the graph in question has to be emphasized. The *open neighborhood* of x (i.e., the set of vertices adjacent to x) is denoted by $N(x)$, and $N(x) \cup \{x\}$ is called the *closed neighborhood* of x; $\langle N(x) \rangle$ stands for the subgraph induced by $N(x)$. Moreover, ω denotes the clique number of G.

Standard notation is applied for the particular types of graphs, too, such as K_n (complete graph), P_n (path), C_n (cycle), S_n (star), each on n vertices. Finally, the *line graph* of G will be denoted by $L(G)$. (The vertices of $L(G)$ are the edges of G, and two vertices are adjacent in $L(G)$ if and only if the corresponding two edges share a vertex in G.)

Operations on graphs. In order to simplify the description of small graphs, we write $G' + G''$ for the (vertex-disjoint) *union* of G' and G'', and mG' for the union of m graphs each of which is isomorphic to G' ($m \geq 2$). The difference $G - G'$ denotes a graph obtained from G by removing the edges of a subgraph isomorphic to G'. (In our cases, $G - G'$ will be unique up to isomorphism.) For a *vertex* x, $G - x$ means the subgraph obtained by removing x and all edges incident to it; and for an *edge* of $G = (V, E)$, $G - e$ is the graph $(V, E \setminus \{e\})$. The complement $K_{|V|} - G$ of G is denoted by $\overline{G}$.

Minimally non-preperfect graphs. For $d \in Z^+$, we denote by NPR_d the class of those graphs of *maximum degree* d which are not preperfect but all of their proper induced subgraphs are. Set $\mathsf{NPR} := \bigcup_{i \geq 0} \mathsf{NPR}_i$ for the class of *minimally non-preperfect* graphs.

Matching Independence Property (MIP). In a graph $G = (V, E)$, an ordered pair (e, e') of edges will be called *matching independent* if there exists a maximum matching M in G such that $e \in M$ and $e' \notin M$. The graph G is said to have the *Matching Independence Property* if each ordered pair of its edges is matching independent.

1.2 An example

It is immediately seen that the odd holes and odd antiholes are minimally non-preperfect; but these graphs are minimally non-perfect as well (perhaps the unique ones, by the Strong Perfect Graph Conjecture [1]). Hammer and Maffray [2] observed that, beside these, there exist also some perfect graphs which are minimally non-preperfect. Next, we recall this construction.

Example 1. For $k \geq 1$, let C^*_{4k+2} denote the graph obtained from the cycle $C_{4k+2} = v_1v_2 \cdots v_{4k+2}$ by adding the chords v_iv_{i+2k+1} (the diagonals) for each $i = 1, \ldots, 2k+1$; and let $H_k := L(C^*_{4k+2})$ be the line graph of C^*_{4k+2}. Then H_k satisfies the following three properties:

(i)' Each vertex of H_k belongs to precisely two maximum cliques (triangles), and these two cliques have no further vertex in common.

(ii)' Each pair (e, e') of edges in C^*_{4k+2} is matching independent, i.e., some matching of $2k+1$ edges contains e but not e'.

(iii)' When some vertices of H_k are removed, the remaining subgraph is either bipartite or contains a vertex belonging to only one maximum clique.

It follows that these graphs H_k ($k \geq 1$), as well as their complements $\overline{H}_k$, are minimally non-preperfect. Indeed, they are non-preperfect by (i)' and (ii)', and the minimality follows by (iii)' (on applying the result that every bipartite graph is preperfect, see [2, Theorem 2.1]).

This example already exhibits some important features, namely that if a line graph $L(G)$ is minimally non-preperfect, then G has to be regular (property (i)'), and if in addition G is bipartite, then it has to satisfy the Matching Independence Property (property (ii)'), since it always contains a perfect matching by Hall's Marriage Theorem.

2 Characterizations

Here we briefly summarize the theorems that will be proved in the sequel. Our results on minimal non-preperfect graphs of *given minimum or maximum degree* are collected in the following assertions.

Claim 0. There is no $H \in \mathsf{NPR}$ with $\delta(H) = 0$.

Claim 1. There is no $H \in \mathsf{NPR}$ with $\delta(H) = 1$.

Claim 2. The graphs $H \in \mathsf{NPR}$ with $\delta(H) = 2$ are precisely the odd holes.

Claim 3. There is no $H \in \mathsf{NPR}$ with $\delta(H) = 3$.

Claim 4. If H is 4-regular, then $H \in \mathsf{NPR}$ if and only if $H = \overline{C}_7$ or H is the line graph of a 3-connected 3-regular bipartite graph.

Turning from H to its complement, analogous results can be obtained for minimally non-preperfect graphs with vertices of degree at least $|V(H)| - 5$, too.

The combination of the above claims yields the complete description of NPR_d for $d \leq 4$, as follows.

Theorem 1 A graph H of maximum degree 4 is minimally non-preperfect if and only if either $H = C_{2t+1}$ $(t \geq 2)$, or $H = \overline{C}_7$, or H is the line graph of a 3-connected 3-regular bipartite graph.

We also have a necessary condition for the minimal non-preperfectness of *line graphs*, but it is not sufficient in general:

Theorem 2 Let G be a graph without isolated vertices. If its line graph $L(G)$ is minimally non-preperfect, then either $G = C_{2t+1}$ $(t \geq 2)$, or G is a 3-edge-connected, regular bipartite graph.

All graphs listed in the conclusion of this theorem are non-preperfect, and therefore the following set of conditions can be seen to be *necessary and sufficient* for all line graphs $L(G) \neq C_{2t+1}$ to be minimally non-preperfect:

- G is bipartite,
- G is 3-edge-connected,
- G is d-regular for some $d \geq 3$,
- G contains no 3-edge-connected d'-regular subgraph for any d', $3 \leq d' < d$.

We do not know at present, however, how to test the last condition, neither have we constructions for minimally non-preperfect line graphs with larger vertex degrees.

A consequence of the above two theorems extends Example 1 as follows.

Corollary 1 The line graph of a 3-regular graph G is minimally non-preperfect if and only if G is bipartite and 3-connected.

3 Preliminaries

In this section we prove some facts that we shall need in the proofs of the main results. The section is concluded with the proof of Theorem 2.

The first simple observation, to be applied several times later, follows directly by definition: If $H \in \mathsf{NPR}$, then

($*$) H contains no predominated vertex, and therefore

- each vertex is contained in some cliques of size $\omega(H)$, and
- the intersection of the ω-cliques containing any one $x \in V(H)$ consists of the single vertex x itself.

Another fact that we shall need is the following characterization of 2-connected graphs without (not necessarily induced) odd cycles of length at least 5. It may be viewed as an equivalent formulation of a result due to Maffray [4, Theorem 2] characterizing perfect line graphs; nevertheless, we shall need the assertion in the stronger form given below.

Lemma 1 If a 2-connected graph G contains no C_{2t+1} ($t \geq 2$) as a subgraph, then one of the following cases occurs:

(i) G is bipartite, or

(ii) $G = K_n - K_{n-2}$, $n \geq 4$, or

(iii) $G = K_n$, $n \leq 4$.

Proof. If G is not bipartite, then it contains some odd cycle C. This C must be a triangle, since the longer odd cycles are excluded. If $G \neq C$, choose any vertex y outside C. The 2-connectedness of G implies that there exist two internally disjoint paths P, P' starting at y and ending in *distinct* vertices of C. Since the endpoints of P and P' on C are adjacent and also are connected by a path of length 2, the exclusion of longer odd cycles implies that both P and P' consist of a single edge. Thus, the subgraph G' induced by $C \cup \{y\}$ is either K_4 or $K_4 - e$.

If $G' = K_4$, then each pair of its vertices is joined by a path of length 3, and also by a path of length 2. In this way, if G contains at least one further vertex, then the argument above leads to the contradiction that G contains an odd cycle of length greater than 3.

Finally, if $G' = K_4 - e$, then any two vertices of G' are joined by a path of length 2, and the degree-3 vertices of G' form the unique pair which is not connected by a path of length 3. Thus, by the condition on odd cycles, each vertex outside G' must have degree 2, and has to be adjacent to the two degree-3 vertices of G'. Consequently, $G = K_n - K_{n-2}$, as claimed. □

From this result we deduce

Lemma 2 Let $d \geq 1$, G a connected d-regular graph, and suppose that its line graph $L(G)$ contains no odd holes.

(i) If $d \geq 4$, then G is bipartite.

(ii) If $d = 3$, then either $G = K_4$, or G is bipartite.

(iii) If $d = 2$, then either $G = K_3$, or G is an even cycle.

Proof. By Lemma 1 — where $K_n - K_{n-2}$ is the unique non-regular graph — it suffices to show that G is 2-connected. Suppose on the contrary that G contains some cut-vertices. Consider and endblock B of G incident to a cut-vertex x. Since G is supposed to be regular, and the degree of x is greater in G than in B, the block B itself cannot be regular. Consequently, $B = K_n - K_{n-2}$ should hold. But B is an endblock, and therefore x is the unique cut-vertex of G in B. Thus, $B - x$ contains at least two vertices of unequal degree in G, a contradiction to regularity. □

3.1 Connectivity and matching independence

In the context of our paper, the Matching Independence Property is needed to ensure that the line graph of a *regular, bipartite* graph is not preperfect. Below, the MIP is analyzed in regular bipartite graphs.

The first lemma describes how the *dependent* pairs of edges are structurally related.

Lemma 3 Let G be a regular bipartite graph, and suppose that $e, e' \in E(G)$ are two edges such that every perfect matching of G containing e also contains e'. Then $\{e, e'\}$ is an edge cut in G.

Proof. Suppose G has vertex classes X and Y, and let $e = xy$, $e' = x'y'$ with $x, x' \in X$, $y, y' \in Y$. Since G is regular, it has a 1-factorization, so that e is contained in some perfect matching M. By assumption, $e' \in M$ also holds. What is more, the graph $G' := G - x - y - e'$ has no perfect matching, therefore it contains a set $X' \cup Y'$ ($X' \subset X$, $Y' \subset Y$), $|X' \cup Y'| = |X| - 2$, that meets all edges of G'. In particular, all neighbors of x' and y' belong to Y' and X', respectively, but y' and x' do not.

We denote $X'' := X \setminus (X' \cup \{x, x'\})$ and $Y'' := Y \setminus (Y' \cup \{y, y'\})$. (One may note that X' and Y'', as well as X'' and Y', are matched along $M \setminus \{e, e'\}$.) Then, setting

$$T' := X' \cup Y' \cup \{x, y\}\,, \qquad T'' := X'' \cup Y'' \cup \{x', y'\}\,,$$

each edge of the graph $G'' := G - e - e'$ has at least one vertex in T' and at most one vertex in T''. On the other hand, the degree sums in both T' and T'' are equal to

$d|X|-2$, where d is the degree of regularity in G, as we have removed just the unique edges e from T' and e' from T''. Consequently, both T' and T'' are independent. Recalling that X and Y are independent as well, it follows that G'' contains no edge between $X' \cup Y'' \cup \{x, y'\}$ and $X'' \cup Y' \cup \{x', y\}$; i.e., $\{e, e'\}$ indeed is an edge cut. □

With the help of this lemma, we can transform the MIP to the more transparent and better understood property of edge connectivity.

Lemma 4 Let G be a connected, bipartite, regular graph of degree $d \geq 3$. Then, G satisfies the Matching Independence Property if and only if G is 3-edge-connected.

Proof. The "if" part follows from Lemma 3. The other direction will be proved by contradiction.

Suppose that G has a cut of two edges e, e'. Let V', V'' be the vertex sets of the two components in $G - e - e'$. Since G is d-regular and bipartite, the assumption $d \geq 3$ implies that the endpoints of e and e' belong to distinct vertex classes of V'. Indeed, the number of edges induced by V' equals the degree sum in either vertex class of V', therefore those two degree sums, say s_1 and s_2, must be equal. On the other hand, precisely two vertices of V' (the endpoints of e and e') have degree $d-1$ in the induced subgraph, and all the other vertices have degree d. Thus, if the two endpoints were in the same vertex class, then we would obtain $|s_1 - s_2| \equiv 2 \pmod{d}$, i.e., $s_1 \neq s_2$ for $d \geq 3$, a contradiction.

It follows further, again by comparing s_1 and s_2, that V' contains the same number of vertices in both of its classes; i.e., $|V'|$ is even. Consequently, every perfect matching of G that contains e, has to contain a further edge joining V' and V''. Since e' is the unique such edge, we obtain the contradiction to the MIP that there is no perfect matching in G containing e but not e'. □

It is worth noting that in 3-regular graphs the concepts of 3-edge-connectivity and 3-vertex-connectivity mean the same thing. Moreover, the standard argument on vertex degrees yields that if a regular bipartite graph is connected, then it is 2-vertex-connected as well.

One may note further that the 2-regular bipartite graphs (i.e., the vertex-disjoint unions of even cycles) do not satisfy the Matching Independence Property, because each of their components has only two perfect matchings. Therefore, the condition $d \geq 3$ might be omitted from Lemma 4.

At the end of this section we prove the characterization theorem of minimally non-preperfect line graphs.

Proof of Theorem 2. Suppose that $L(G)$ is minimal non-preperfect, $G \neq C_{2t+1}$. The minimality with respect to non-preperfectness implies that G is connected, and regularity follows by $(*)$, because the maximum clique size in $L(G)$ equals the

maximum degree in G (for $G \neq K_3$). Thus, G has to be one of the graphs listed in Lemma 2. Now, K_3 and K_4 are excluded because they are preperfect. Consequently, G is bipartite, and hence also 3-edge-connected, by Lemma 4. □

4 The easy cases

In this short section we prove Claim i for $i = 0, 1, 2, 3$. We shall assume throughout that $H \in \mathsf{NPR}$.

Proof of Claim 0. By $(*)$, H should have no edges, but then it would be preperfect. □

Proof of Claim 1. No vertex of degree 1 can satisfy $(*)$. □

Proof of Claim 2. Suppose $d(x) = 2$. The two neighbors of x are nonadjacent, otherwise we get a contradiction to $(*)$, independently of whether $\omega(H) = 3$ or $\omega(H) \geq 4$. Thus, $\omega(H) = 2$, again by $(*)$; and, by a theorem of Hammer and Maffray [2], H is nonbipartite as it is not preperfect. Consequently, H contains an odd hole which must then be identical to H, by minimality. □

Proof of Claim 3. Suppose $d(x) = 3$. We shall deduce the contradiction that x is predominated. Note that $\omega(H) \geq 3$, for otherwise either H contains (but is not identical to) an odd hole, or H is bipartite, both contradicting the assumption $H \in \mathsf{NPR}$.

Consider now the subgraph $\langle N(x) \rangle$ induced by the three neighbors of x. It should be one of the following four graphs:

1. $3K_1$
2. $K_1 + K_2$
3. P_3
4. K_3

For $\langle N(x) \rangle \simeq 3K_1$, x cannot be contained in any maximum clique. In the other three situations, x is adjacent to a vertex contained in all the largest cliques sitting on x. Thus, in either case, we obtain a contradiction to $(*)$. □

5 Regular graphs of degree 4

Here we make the main preparations for the proof of Claim 4. The final argument will then be presented in the next section.

Assume that $H \in \mathsf{NPR}$ is 4-regular, and, similarly to the proof of Claim 3, consider the subgraphs $\langle N(x) \rangle$ induced by the vertex neighborhoods. Now, those subgraphs on four vertices can be of the following 11 types:

1. $4K_1$
2. $2K_1 + K_2$
3. $2K_2$
4. $K_1 + P_3$
5. $K_1 + K_3$
6. P_4
7. S_4
8. $K_4 - P_3$
9. C_4
10. $K_4 - e$
11. K_4

Eight of these eleven graphs are easily ruled out along the lines of the proof of Claim 3, on applying $(*)$. (In the present situation, $4K_1$ does not admit any maximum clique on x, and in seven further cases there exists at least one vertex in the neighborhood of x which belongs to all largest cliques containing x.)

The hard cases are the 3rd, 6th, and 9th ones. We can classify the vertices according to them, as follows. A vertex x of H is said to be a

M-vertex if its neighborhood induces a matching, i.e., $\langle N(x) \rangle \simeq 2K_2$,

P-vertex if its neighborhood induces a path, i.e., $\langle N(x) \rangle \simeq P_4$,

C-vertex if its neighborhood induces a cycle, i.e., $\langle N(x) \rangle \simeq C_4$.

We shall see in the next subsections that each type behaves differently. The M-vertices will provide us with the line graphs, the P-vertices create the "isolated" example of $\overline{C}_7$, and it will turn out that the C-vertices cannot occur in any $H \in \mathsf{NPR}_4$. Actually, we do not know whether or not there exists any minimal non-preperfect graph containing at least one C-vertex.

5.1 C-vertices

In this subsection we prove the following

Lemma 5 If x is a C-vertex in a 4-regular graph $H \in \mathsf{NPR}$, then the block containing x is not an endblock of H.

Proof. Suppose that x is a C-vertex and that its neighbors v_1, v_2, v_3, v_4 form a C_4 in this order. If x is not predominated, then the clique number of H equals 3, and if x does not predominate any of its neighbors, then each v_i $(1 \leq i \leq 4)$ is incident to at least one further triangle. Since each v_i is adjacent to precisely one vertex outside the closed neighborhood of x, there are two possible situations. Either there is a vertex y adjacent to all the four v_i, or there are two distinct vertices z, u such that both $\{v_1, v_2, z\}$ and $\{v_3, v_4, u\}$ induce K_3 in H. (It may be necessary to renumber the v_i to ensure that $\{v_1, v_2, z\}$ is indeed a triangle, but this can always be done since $N(x)$ induces a cycle.)

In the first case, involving the single vertex y, the vertices v_1, v_2, v_3, v_4, x, y already induce a 4-regular subgraph in H, and since H is connected, this graph must be the entire H. Consequently, $H \simeq K_6 - 3K_2$, and therefore H is preperfect, contrary to our assumptions.

In the second case, we claim that both z and u are cut-vertices of H. This will immediately imply that x does not belong to an endblock.

Observe that all the four neighbors of x and of the v_i are inside the set $\{v_1, v_2, v_3, v_4, x, z, u\}$, therefore the other two neighbors of z and u not yet detected cannot belong to the closed neighborhood of x. Moreover, z and u are nonadjacent, for otherwise zv_1xv_3u is an induced C_5 in H.

Choose a neighbor z' of z and u' of u, both outside of $N(x)$, and consider the graph $H' = H - z - u$. Clearly, $\{x, v_1, v_2, v_3, v_4\}$ induces a connected component in H', not containing any of z' and u'. We claim that z' and u' belong to distinct components of H'. Indeed, otherwise we can choose a chordless path $P = y_0y_1 \cdots y_t$ joining $z = y_0$ and $u = y_t$ in $H' \cup \{z, u\}$. If P has odd length, then $y_0y_1 \cdots y_tv_3xv_1$ is an induced odd cycle, and if P has even length, then $y_0y_1 \cdots y_tv_3v_2$ is an induced odd cycle. In either case, we have arrived at a contradiction. □

5.2 P-vertices

The goal of this subsection is to analyze the possible presence of P-vertices in 4-regular, minimally non-preperfect graphs. One important case will result in the "isolated" example $\overline{C}_7$.

Lemma 6 If an endblock of a 4-regular graph $H \in \mathsf{NPR}$ contains a P-vertex, then $H = \overline{C}_7$.

Proof. Let x be a P-vertex, and suppose that its neighbors v_1, v_2, v_3, v_4 induce the path P_4 in this order. The general viewpoint of the analysis below will be to interpret the subgraph under consideration as the union of two vertex-disjoint paths

$$P' \;=\; x_0x_1x_2\cdots x_t$$

and

$$P'' \;=\; y_1y_2\cdots y_t$$

joined by the edges

$$y_ix_{i-1}\;,\quad y_ix_i \qquad\qquad \forall\;\; 1\le i\le t\,.$$

Already the starting configuration can be expressed in this way as $P' = v_1xv_4$ and $P'' = v_2v_3$ ($t = 2$), but the small values up to $t = 3$ deserve separate attention, therefore we treat them first.

To fix terminology, in a (not necessarily induced) subgraph H' of H, the *missing degree* of a vertex x is defined as $4 - d_{H'}(x)$, i.e., it is meant to express how many further edges of H are incident to x. Those edges of $H - H'$ will be referred to as *missing edges*. For instance, in the starting configuration of $N(x)\cup\{x\}$, the missing degree is 0 for x, 1 for v_2 and v_3, and 2 for v_1 and v_4. Generally, in $P'\cup P''$, it is 2 for x_0 and x_t, 1 for y_1 and y_t, and zero for all the other vertices.

The method of the proof below is to investigate an increasing part of the block containing x, in order to show that either the block itself is isomorphic to $\overline{C}_7$ (and hence so is H, too), or it contains more than one cut-vertex in H (i.e., is not an endblock), or contains an odd hole or a predominated vertex (these cases are excluded by the minimality of non-preperfectness).

Case 1. Every triangle containing v_2 (or v_3) also contains x.

In this case, x predominates v_2 (or v_3). $\diamond$

Therefore, we may assume in the sequel that the missing edge e_i incident to v_i ($i = 2, 3$) is contained in a triangle. We distinguish between two situations, depending on whether or not e_2 and e_3 share a vertex.

Case 2. v_2 and v_3 have a common neighbor $y \neq x$.

In this situation, the role of v_1, v_4, and y is symmetric. Observe first that if those three vertices are not mutually nonadjacent, then v_2 or v_3 is a C-vertex, and therefore the 2-connected component of x is not an endblock, by Lemma 5. Hence, assume that $yv_1, yv_4 \notin E(H)$. Now, any two of v_1, v_4, and y are joined by induced paths of both lengths 2 and 3, therefore we can apply an argument similar to the one in the last paragraph of the proof of Lemma 5. Namely, if v_1, v_4, y are not all cut-vertices, then any shortest path in $H - x - v_2 - v_3$ that joins two of them, and does not pass through the third one, can be completed to an odd hole in H. $\diamond$

We may now suppose $e_2 \cap e_3 = \emptyset$. Assuming that Case 1 does not apply, each of e_1 and e_2 creates a triangle in H ; i.e., there exist (distinct) vertices z_2, z_3 such that both $\{v_1, v_2, z_2\}$ and $\{v_4, v_3, z_3\}$ induce K_3 in H.

Case 3. $z_2z_3 \in E(H)$.

Observe that in this case both $xv_2z_2z_3v_4$ and $xv_3z_3z_2v_1$ are cycles of length 5. The only possibility to make them non-induced is to draw the edges v_1z_3 and v_4z_2, because the missing degree of x, v_2, and v_3 is zero. Hence, we have obtained $H = \overline{C}_7$ (in the order $xz_2v_3v_1v_4v_2z_3$ along the complementary cycle $\overline{H}$). $\diamond$

Hence, we may assume $z_2z_3 \notin E(H)$. We can argue, again as in the first case, that the (currently unique) missing edge at v_1 has to be contained in a further triangle $v_1z_1z_2$, or else v_2 predominates v_1. Similarly, a triangle $z_3z_4v_4$ is needed to avoid the predominatedness of v_4. Observe further that $z_1 \neq z_4$, for otherwise $z_1v_1v_2v_3v_4$ induces a C_5 in H.

At this point we arrived at the general case, mentioned at the very beginning of the proof, viewing $z_1v_1xv_4z_4$ as P' and $z_2v_2v_3z_3$ as P''. In order to derive a final contradiction, let us choose now the pair (P', P'') in the block containing x in such a way that the value of t is as large as possible. By what has been said, we may assume $t \geq 4$.

To simplify notation, we denote $x = x_0$, $x' = x_t$, $y = y_1$, $y' = y_t$. The missing degree is 2 for x and x', 1 for y and y', and zero for all the other vertices of $P' \cup P''$. Let yz and $y'z'$ be the missing edges at y and y', respectively (where these two edges may or may not coincide).

We may assume that $zx, z'x' \in E(H)$, otherwise y or y' would be predominated by x_1 or x_{t-1}, respectively. We now distinguish between four situations, according to the positions of z and z'.

Case 4.1. $y_1y_t \in E(H)$.

This case is equivalent to assuming $z = y'$, and also $z' = y$. Then

$$y_1y_2 \cdots y_t \ , \qquad y_1x_1x_2 \cdots x_{t-1}y_t$$

are induced cycles of H, of lengths t and $t+1$. Since $t \geq 4$, one of these cycles is an odd hole in H.

Case 4.2. $x_0x_t \in E(H)$ and $y_1y_t \notin E(H)$.

This assumption includes, as particular case, the situations where $z = x'$ and/or $z' = x$. Now,

$$x_0x_1 \cdots x_t \ , \qquad x_0x_1y_2 \cdots y_{t-1}x_{t-1}x_t$$

are induced cycles of lengths $t+1$ and $t+2$, one of them inducing an odd hole in H.

Case 4.3. $z = z' \notin \{x, x', y, y'\}$, and $x_0x_t \notin E(H)$.

Now the subgraphs induced by $P' \cup \{z\}$ and $P'' \cup \{z\}$ are chordless cycles of lengths $t+2$ and $t+1$, respectively, one of them being an odd hole.

Case 4.4. $z \neq z'$, $z, z' \notin \{x, x', y, y'\}$, and $x_0 x_t \notin E(H)$.

Extending P'' with the two new vertices z and z', we obtain an induced path P''' of length $t+1$. Rename the original P' as P'', and the new P''' as P'. In this way, the corresponding value of t gets increased, contradicting the choice of the original pair (P', P''). □

6 Proof of the characterization theorem

Here we combine the results of the previous sections, to complete the proof of Claim 4, and so of Theorem 1 as well. We shall need one further observation, summarized in the next assertion, that establishes the link between line graphs and those 4-regular graphs in which every vertex is an M-vertex.

Lemma 7 Suppose that H' is a connected graph such that each of its vertices has degree 2 or 4, and $\langle N(x) \rangle$ is a matching of $d(x)/2$ edges, for each $x \in V(H')$. Then H' is the line graph of a graph of maximum degree 3.

Proof. If $H' = K_3$, then $H' = L(S_4)$, and the assertion is obvious. For larger H', denote by $\mathcal{T}$ the set of triangles (subgraphs isomorphic to K_3) in H'. The conditions above imply that the members of $\mathcal{T}$ are mutually edge-disjoint, and so $E(H')$ is partitioned into the triangles of H'.

Let $G_{\mathcal{T}}$ be the graph whose vertices are the triangles of H', two of its vertices being adjacent if and only if the corresponding two triangles share a vertex in H'. (In other words, $G_{\mathcal{T}}$ is the line graph of the 3-uniform hypergraph $\mathcal{T}$.) Moreover, for each degree-2 vertex x of H', we take a distinct new vertex and join it in $G_{\mathcal{T}}$ to the vertex that represents the unique triangle of H' containing x. In particular, if two vertices of degree 2 in H' are adjacent, then the new pendant vertices corresponding to them will have the same neighbor in $G_{\mathcal{T}}$. We denote by $G'_{\mathcal{T}}$ the graph obtained.

One can easily see that $G'_{\mathcal{T}}$ is a graph with all vertex degrees equal to 1 or 3, and that $H' = L(G'_{\mathcal{T}})$. □

Returning to the proof of Claim 4, let $H \in \mathsf{NPR}$ be a 4-regular graph. We know that H is connected. By the results of Section 5, the vertices of H are classified as M-vertices, P-vertices, and C-vertices.

If H is 2-connected, then, by Lemmas 5 and 6, either $H = \overline{C}_7$ or all vertices of H are M-vertices. In the latter case, the conditions of Lemma 7 are fulfilled for $H' = H$, so that $H = L(G)$ is a line graph of some graph G of maximum degree 3. Applying Theorem 2, we conclude that G is bipartite, 3-regular, and 3-edge-connected. This completes the proof for 2-connected H.

Suppose that H has some cut-vertices. Consider an endblock B of H, incident to a cut-vertex y. Since the neighborhood of a C-vertex and also of a P-vertex is connected, y has to be an M-vertex, having two of its (adjacent) neighbors inside

B and the other two of them outside B. Moreover, Lemmas 5 and 6 imply that each vertex $z \neq y$ of B is an M-vertex, with all its neighbors inside B. Thus, it follows by Lemma 7 that $B = L(G_B)$ for some graph G_B. From the proof of the lemma we also see that G_B has precisely one vertex z of degree 1, adjacent to the vertex u representing the triangle incident to y, and all the other vertices in G_B have degree 3.

Consider the graph $G_B - z$. It has just one vertex of degree 2 (namely, u), and all its other vertices are of degree 3. Consequently, for each of its vertex partitions into two classes, the partial degree sums are unequal. Thus, G_B cannot be bipartite. But, on the other hand, G_B can contain no triangles (otherwise the endblock B would contain a P-vertex or a C-vertex), and no odd cycles either (as an odd cycle on at least 5 vertices in G_B yields a cycle of the same length in B, and so is excluded because H is supposed to be minimal with respect to non-preperfectness). This final contradiction completes the proof.

7 Concluding remarks and open problems

In connection with the results presented above, some problems arise in a natural way. First, in order to complete the characterization of minimally non-preperfect line graphs, an answer to the following question would be needed in combination with Theorem 2.

Problem 1 Describe the structure of those regular, 3-edge-connected bipartite graphs which do not contain any regular, 3-edge-connected proper subgraph. In particular, do there exist such graphs of arbitrarily large vertex degree?

Another problem of central importance in the present context is the following one, a negative answer to which has been conjectured by Hougardy, Maffray and Sebő [3].

Problem 2 Does there exist any minimally non-preperfect graphs such that neither the graph nor its complement is a line graph?

By Theorem 2, a negative answer would have several consequences; but we are not convinced that no such graph exists. On the other hand, it may be the case that at least under some further assumptions, the line graphs are the really relevant structures, e.g. if the clique number is bounded above by a fairly small integer. Here we formulate the following subproblem.

Problem 3 Characterize those graphs $H \in \mathsf{NPR}$ which have $\omega(H) = 3$.

If all these graphs turn out to be line graphs, then they get characterized by Theorem 1, since H has to be 4-regular in that case. At this point it may be worth

noting that in the class of the line graphs of connected 3-regular bipartite graphs, non-preperfectness and minimal non-preperfectness are equivalent properties.

A more general question is

Problem 4 Do there exist non-regular graphs in NPR ?

Relating NPR to the class of critically perfect graphs [6], we ask

Problem 5 Does there exist a perfect, minimally non-preperfect graph such that neither the graph nor its complement is critically perfect ?

Problems 2 and 5 are related in the following way. It is shown in [5] that the line graph of every 2-connected bipartite graph satisfying the Matching Independence Property is critically perfect. On the other hand, for every minimally non-preperfect line graph $L(G)$, the graph G is connected and regular (if we exclude isolated vertices), therefore G is 2-connected as well. This argument shows that a negative answer for Problem 2 would imply that for Problem 5, too. It may be the case, however, that $\mathsf{NPR} \setminus \{C_{2t+1}, \overline{C}_{2t+1} \mid t \geq 2\}$ is a subclass of the critically perfect graphs even if not all $H \in \mathsf{NPR}$ are line graphs or complements of line graphs.

Problem 6 Prove that every minimally imperfect graph is minimally non-preperfect.

Certainly, this would follow immediately from the validity of the Strong Perfect Graph Conjecture, but we expect that there is an independent proof to it.

Finally, it may be noted that Claim 3 remains valid in a stronger form, namely its proof exhibits *clique*-predominated vertices, without any need for maximum independent sets. In this way one can show that if a vertex of degree 3 is not predominated by any of its neighbors, then the graph either is bipartite or contains an odd hole. This is not the case anymore in 4-regular graphs, however, because e.g. in the graph $K_6 - 3K_2$ only the three non-edges establish predomination.

Acknowledgement. The first author thankfully acknowledges support from the Konrad-Zuse-Zentrum für Informationstechnik Berlin, where this research was carried out.

References

[1] C. Berge, *Färbungen von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind.* Wiss. Zeitschrift der Martin-Luther-Universität Halle-Wittenberg (1961), 114–115.

[2] P. L. Hammer and F. Maffray, *Preperfect graphs.* Combinatorica 13 (2) (1993), 199–208.

[3] S. Hougardy, F. Maffray, and A. Sebő, unpublished, 1993. (Private communication with S. Hougardy, June 1997.)

[4] F. Maffray, *Kernels in perfect line-graphs.* J. Combinatorial Theory, Ser. B, 55 (1) (1992), 1–8.

[5] Zs. Tuza and A. Wagler, *Some properties of critically perfect line graphs.* In preparation.

[6] A. Wagler, *On critically perfect graphs.* Preprint SC-96-50, Konrad-Zuse-Zentrum für Informationstechnik Berlin, December 1996.