Konrad-Zuse-Zentrum für Informationstechnik Berlin

Harry Yserentant *

Two Preconditioners Based on the Multi-Level Splitting of Finite Element Spaces

* Institut für Mathematik
Universität Dortmund
Postfach 500 500
D-4600 Dortmund 50
Federal Republic of Germany

Fellow of the Konrad-Zuse-Zentrum

Preprint SC 89-9 (January 1990)

Herausgegeben vom
Konrad-Zuse-Zentrum für Informationstechnik Berlin
Heilbronner Strasse 10
1000 Berlin 31
Verantwortlich: Dr. Klaus André
Umschlagsatz und Druck: Rabe KG Buch- und Offsetdruck Berlin

ISSN 0933-7911

Contents

Abstract

The hierarchical basis preconditioner and the recent preconditioner of BRAMBLE, PASCIAK and XU are derived and analyzed within a joint framework. This discussion elucidates the close relationship between both methods. Special care is devoted to highly nonuniform meshes; our theory is based exclusively on local properties like the shape regularity of the finite elements.

Subject Classifications: 65F10, 65F35, 65N20, 65N30

The author would like to thank Mrs. E.C. Körnig for her quick and careful writing of the manuscript.

1 Introduction

An ideal preconditioner B for a discretized second-order elliptic boundary value problem

$$Au = b \tag{1.1}$$

(A and B symmetric positive definite) should have the following properties:

- The spectral condition number of the operator $B^{-1/2}AB^{-1/2}$ should remain bounded independently of the dimension of the problem (i.e. the gridsize) or should grow only logarithmically.
- The cost for computing $B^{-1}r$ should be proportional to the dimension of the problem.
- The algorithm should be easily and efficiently realizable on scalar as well as on parallel machines.
- These properties should not depend (severely) on the shape of the domain under consideration, on jumps in the coefficient functions or even on the quasiuniformity of the grid.

Last but not least,

- the algorithm should be simple.

There are two preconditioners that meet these requirements especially well. The first is the hierarchical basis preconditioner [11] together with its variant [2], the hierarchical basis multigrid method. The second has been developed recently by BRAMBLE, PASCIAK & XU [4] and XU [10].

Both preconditioners utilize a multi-level structure. Assume that $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \ldots \subseteq \mathcal{S}_j$ is a usual family of nested finite element spaces corresponding to finer and finer subdivisions and let the discrete problem be the finite element discretization with respect to $\mathcal{S} = \mathcal{S}_j$. Let $I_k u \in \mathcal{S}_k$ be the function interpolating $u \in \mathcal{S}$ at the nodes defining $\mathcal{S}_k$. Then the hierarchical basis preconditioner is based on the splitting

$$u = I_0 u + \sum_{k=1}^{j} (I_k u - I_{k-1} u) \tag{1.2}$$

of the functions $u \in \mathcal{S}$. The Bramble–Pasciak–Xu preconditioner relies on a related splitting

$$u = Q_0 u + \sum_{k=1}^{j} (Q_k u - Q_{k-1} u) \tag{1.3}$$

of $\mathcal{S}$ where the Q_k are L_2-like orthogonal projections onto $\mathcal{S}_k$. In their final form both preconditioners have a very similar structure. Contrary to the hierarchical basis preconditioner, which deteriorates in the three-dimensional case, the Bramble–Pasciak–Xu preconditioner works equally well for two- as for three-dimensional problems.

One aim of this paper is to develop and to analyze the hierarchical basis and the Bramble–Pasciak–Xu preconditioner in parallel and within a joint framework. We hope that this discussion will improve the understanding of both preconditioners, of their relationship and of their common roots.

Secondly, special care is devoted to nonuniformly refined grids. We attempt and prove estimates which rely only on local properties like the shape regularity of the finite elements but do not depend on the global quasiuniformity of the initial or any following mesh. This requires a careful treatment of the initial level and the corresponding subdivision of the domain under consideration. Compared to the original papers [10], [4], one has to modify the L_2–like inner product defining the orthogonal projectors Q_k. In our version this inner product depends also on the sizes of the finite elements of the initial subdivision.

The remainder of this paper is organized as follows:

- In Section 2 we discuss a special finite element discretization and give a formal definition of the interpolation operators I_k and the L_2–like projections Q_k. We introduce discrete norms corresponding to the splittings (1.2) and (1.3).

- In Sections 3 and 4 it is shown that these discrete norms are nearly equivalent to the energy norm induced by the boundary value problem. These results form the mathematical background of the hierarchical basis preconditioner and of the Bramble–Pasciak–Xu preconditioner, respectively. In Section 4 we utilize a simple, but apparently new technique for deriving error estimates and H^1–norms for L_2–like projections onto finite element spaces.

- In Section 5 we derive the preconditioners and discuss some algorithmic aspects.

2 A Finite Element Discretization

In this paper a two-dimensional model problem is studied in detail. We remark that the theory developed here can be extended in a straightforward fashion to other types of refinement procedures. Many arguments are dimension independent and can be transferred to the three-dimensional case. If a result cannot be generalized to three dimensions, this will be pointed out explicitly.

Let $\overline{\Omega} \subseteq \mathbb{R}^2$ be a bounded polygonal domain. As a model problem, we consider the differential equation

$$-\sum_{i,j=1}^{2} D_j(a_{ij} D_i u) = f \tag{2.1}$$

on Ω with homogeneous Dirichlet boundary conditions on the boundary piece Γ and homogeneous natural boundary conditions on the remaining part $\partial\Omega \setminus \Gamma$ of the boundary of Ω. We assume that Γ is composed of straight lines. The solution space of this boundary value problem is

$$\mathcal{H} = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma\} \tag{2.2}$$

where the zero boundary conditions have to be understood in the sense of the trace operator. The weak formulation is to find a function $u \in \mathcal{H}$ satisfying

$$a(u, v) = \int_{\Omega} f v \, dx \ , \quad v \in \mathcal{H} \ , \tag{2.3}$$

where the bilinear form a is defined by

$$a(u, v) = \int_{\Omega} \sum_{i,j=1}^{2} a_{ij} D_i u D_j v \, dx \ . \tag{2.4}$$

We assume that the a_{ij} are measurable and bounded functions and that

$$a_{ij} = a_{ji} \ . \tag{2.5}$$

By a triangulation $\mathcal{T}$ of the polygonal domain $\overline{\Omega}$, we mean a set of triangles such that the union of these triangles is $\overline{\Omega}$ and such that the intersection of two such triangles either consists of a common side or a common vertex of both triangles or is empty. Here we start with an intentionally coarse triangulation $\mathcal{T}_0$ of $\overline{\Omega}$ with the property that the boundary piece Γ is composed of edges of triangles $T \in \mathcal{T}_0$. We assume that there are positive constants $0 < \delta \leq 1 \leq M$ and $\omega(T) > 0$ with

$$\delta\omega(T)\sum_{i=1}^{2}\eta_i^2 \leq \sum_{i,j=1}^{2} a_{ij}(x)\eta_i\eta_j \leq M\omega(T)\sum_{i=1}^{2}\eta_i^2 \tag{2.6}$$

for all $T \in \mathcal{T}_0$, almost all $x \in T$ and all $\eta \in \mathbb{R}^2$. Clearly, the constants M and δ will enter into our estimates, but we try to keep the estimates as independent of the $\omega(T)$ as possible.

By (2.5) and (2.6) a is a symmetric, bounded and coercive bilinear form on $\mathcal{H}$.

$$\|u\|^2 = a(u,u) \tag{2.7}$$

defines a norm on $\mathcal{H}$, the energy norm induced by the given boundary value problem. This norm is equivalent to the norm

$$|u|^2_{1,2;\Omega} = \sum_{i=1}^{2} \int_{\Omega} |(D_i u)(x)|^2 dx \tag{2.8}$$

on $\mathcal{H}$. Since $\mathcal{H}$ is a Hilbert space under the norm (2.8), the Riesz representation theorem guarantees that the boundary value problem (2.3) has a unique solution.

In addition to the (semi–)norms (2.7) and (2.8), we use the weighted H^1–seminorm

$$|u|^2_{1;G} = \sum_{i=1}^{2} \sum_{T \in \mathcal{T}_0} \omega(T) \int_{G \cap T} |(D_i u)(x)|^2 dx \tag{2.9}$$

(G a subset of $\overline{\Omega}$) and the weighted L_2–norm

$$\|u\|^2_{0;G} = (u,u)_G \tag{2.10}$$

which is induced by the inner product

$$(u,v)_G = \sum_{T \in \mathcal{T}_0} \omega(T) h(T)^{-2} \int_{G \cap T} u(x)v(x)dx \ . \tag{2.11}$$

$h(T)$ denotes the diameter of the triangle T. Note that the (semi–)norm (2.9) and the norm (2.10) depend on the initial triangulation $\mathcal{T}_0$ and on the coefficients of the boundary value problem. They have to be distinguished from the seminorm (2.8) and the usual L_2–norm

$$\|u\|^2_{0,2;G} = \int_G |u(x)|^2 dx \ , \tag{2.12}$$

respectively. For $G = \Omega$ or $G = \overline{\Omega}$, we omit the subscript G and write (u,v) instead of $(u,v)_G$, for example.

The triangulation $\mathcal{T}_0$ is refined several times, giving a family of nested triangulations $\mathcal{T}_0$, $\mathcal{T}_1$, $\mathcal{T}_2$, ... A triangle of $\mathcal{T}_{k+1}$ is either a triangle of $\mathcal{T}_k$ or is generated by subdividing a triangle of $\mathcal{T}_k$ into four congruent triangles or into two triangles by connecting one of its vertices with the midpoint of the opposite side. The first case is called a regular refinement and the resulting triangles as well as the triangles of the initial triangulation are regular triangles. The second case is an irregular refinement and results in two irregular triangles.

The irregular refinement is potentially dangerous because interior angles are reduced. *Therefore, we add the rule that irregular triangles may not be further refined.* This rule insures that every triangle of any triangulation $\mathcal{T}_k$ is geometrically similar to a triangle of the initial triangulation $\mathcal{T}_0$ or to an irregular refinement of a triangle in $\mathcal{T}_0$.

The triangles in $\mathcal{T}_0$ are level 0 elements, and the regular and irregular triangles created by the refinement of level $k-1$ elements are level k elements. It is important to recognize that not all elements in $\mathcal{T}_{k-1}$ need to be refined in creating $\mathcal{T}_k$. The mesh $\mathcal{T}_k$ may contain unrefined elements from all lower levels, and thus it may be a highly nonuniform mesh. *We require that only level $k-1$ elements are refined in the construction of $\mathcal{T}_k$.*

The described triangulations are meanwhile standard; we refer to [1] and to [3] . We remark that our levels usually do not reflect the dynamic refinement process in an adaptive algorithm, although the final triangulations can be decomposed a-posteriori as described above; see [6] for a detailed discussion. Due to the last rule, this decomposition is unique.

Corresponding to the triangulations $\mathcal{T}_k$, we have finite element spaces $\mathcal{S}_k$. $\mathcal{S}_k$ consists of all functions which are continuous on $\overline{\Omega}$ and linear on the triangles $T \in \mathcal{T}_k$ and which vanish on the boundary piece Γ. Clearly, $\mathcal{S}_k$ is a subspace of $\mathcal{S}_l$ for $l \geq k$.

Let $\mathcal{N}_k = \{x_i, \ldots, x_{n_k}\}$ be the set of vertices of the triangles in $\mathcal{T}_k$ not lying on the boundary piece Γ. Then $\mathcal{S}_k$ is spanned by the nodal basis functions $\psi_i^{(k)}$, $i = 1, \ldots, n_k$, which are defined by

$$\psi_i^{(k)}(x_l) = \delta_{il} \ , \ x_l \in \mathcal{N}_k \ . \tag{2.13}$$

The hierarchical basis functions are

$$\widehat{\psi}_i = \psi_i^{(0)} \ , \ x_i \in \mathcal{N}_0 \ , \tag{2.14}$$

and

$$\widehat{\psi}_i = \psi_i^{(k)} \ , \ x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1} \ . \tag{2.15}$$

$\widehat{\psi}_i$, $i = 1, \ldots, n_k$, is the hierarchical basis of $\mathcal{S}_k$.

We fix a final level j and set $\mathcal{S} = \mathcal{S}_j$. The interpolation operators $I_k : \mathcal{S} \to \mathcal{S}_k$ are defined by

$$(I_k u)(x_i) = u(x_i) \ , \ x_i \in \mathcal{N}_k \ . \tag{2.16}$$

Because of $u = I_j u$, one has the splitting (1.2)

$$u = I_0 u + \sum_{k=1}^{j} (I_k u - I_{k-1} u) \tag{2.17}$$

of the functions $u \in \mathcal{S}$. The L_2-like projections $Q_k : \mathcal{S} \to \mathcal{S}_k$ are given by

$$(Q_k u, v) = (u, v) \ , \ v \in \mathcal{S}_k \ . \tag{2.18}$$

The corresponding splitting (1.3) of $\mathcal{S}$ is

$$u = Q_0 u + \sum_{k=1}^{j} (Q_k u - Q_{k-1} u) \ . \tag{2.19}$$

With the splittings (2.17) and (2.19) we associate the discrete norms

$$|||u|||_{II}^2 = \|I_0 u\|^2 + \sum_{k=1}^{j} 4^k \|I_k u - I_{k-1} u\|_0^2 \tag{2.20}$$

and

$$|||u|||_X^2 = \|Q_0 u\|^2 + \sum_{k=1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2 \tag{2.21}$$

on $\mathcal{S}$. In the next two sections we show that these discrete norms essentially behave like the energy norm (2.7) or the norm (2.9).

The generic factors 4^k result from the fact that a triangle of level k has half the size of his father of level $k-1$. They depend on the refinement strategy and have to be replaced, for example, by the factors 9^k, if the diameters of the triangles are reduced by the factor 3 from one level to the next. These factors are not dimension dependent and replace the spectral radii λ_k in [4], [10].

3 Estimates for the Interpolation Operators

In this section it is shown that for two-dimensional problems the discrete norm (2.20) essentially behaves like the energy norm (2.7).

The basic tools are the following norm estimates for the interpolation operators I_k.

Theorem 3.1 *There are constants C_0 and C_1 with*

$$\|I_k u\|_{0;T}^2 \leq C_0(j-k+1)\{4^{-k}|u|_{1;T}^2 + \|u\|_{0;T}^2\} \tag{3.1}$$

and

$$|I_k u|_{1;T}^2 \leq C_1(j-k+1)|u|_{1;T}^2 \tag{3.2}$$

for all functions $u \in \mathcal{S}$, for $k = 0, \ldots, j$, and for all triangles $T \in \mathcal{T}_k$. These constants depend only on the shape regularity of the triangles T, that means a lower bound for their interior angles, but not on j.

Proof. By [11], Lemma 2.3, for $u \in \mathcal{S}$

$$\|I_k u\|_{0,2;T}^2 \leq \overline{C}_0 \left(\log \frac{h(T)}{h_*} + \frac{1}{4}\right) \left\{h(T)^2 |u|_{1,2;T}^2 + \|u\|_{0,2;T}^2\right\} \tag{3.3}$$

holds, and by [11], Lemma 2.2, one has

$$|I_k u|_{1,2;T}^2 \leq \overline{C}_1 \left(\log \frac{h(T)}{h_*} + \frac{1}{4}\right) |u|_{1,2;T}^2 \tag{3.4}$$

where

$$h_* = \min\{h(T') | T' \in \mathcal{T}_j\ ,\ T' \subseteq T\}\ .$$

Because of

$$\left(\tfrac{1}{2}\right)^{j-k} h(T) \leq h(T')$$

for all $T' \in \mathcal{T}_j$, $T' \subseteq T$, the logarithmic term can be estimated by

$$\log \frac{h(T)}{h_*} + \frac{1}{4} \leq j - k + 1\ . \tag{3.5}$$

With (3.4) and this estimate, by construction of the weighted (semi-)norms (2.9) and (2.10), the proposition (3.2) is already proved.

If $T \in \mathcal{T}_k$ is a triangle of level k and if $T \subseteq T''$, $T'' \in \mathcal{T}_0$,

$$h(T) \leq \left(\tfrac{1}{2}\right)^{k-1} h(T'')\ . \tag{3.6}$$

By (3.3), (3.5) and (3.6) the proposition (3.1) follows.

If $T \in \mathcal{T}_k$ is a triangle of a level less than k, by the rules given in Section 2 it will not be refined any more in the transition to $\mathcal{T}_j$. Therefore, $I_k u|T = u|T$ for all $u \in \mathcal{S}$ and (3.1) becomes trivial. ∎

An easy consequence of (3.1) and the Poincaré-inequality is the following error estimate.

Theorem 3.2 *There is a constant C_2 with*

$$\|u - I_k u\|_{0;T}^2 \leq C_2(j - k + 1)4^{-k}|u|_{1;T}^2 \tag{3.7}$$

for all functions $u \in \mathcal{S}$, for $k = 0, \ldots, j$, and for all triangles $T \in \mathcal{T}_k$. C_2 does not depend on j, but only on the shape regularity of the triangles T.

Proof. As in the proof of (3.1), it is sufficient to consider a triangle $T \in \mathcal{T}_k$ of level k. The proof relies on the fact that for all constants α

$$u - I_k u = (u + \alpha) - I_k(u + \alpha)\ .$$

Because of

$$\inf_\alpha \|u + \alpha\|_{0,2;T} \leq c\, h(T)|u|_{1,2;T}$$

and since T is a triangle of level k, by (3.6) one obtains

$$\inf_\alpha \|u + \alpha\|_{0;T} \leq c\left(\tfrac{1}{2}\right)^{k-1}|u|_{1;T}\ .$$

Utilizing (3.1) the proposition follows. ■

Finally we need the following inverse estimate:

Lemma 3.3 *For all functions $v \in \mathcal{S}_k$ and all triangles $T \in \mathcal{T}_k$*

$$|v|_{1;T}^2 \leq K_0 4^k \|v\|_{0;T}^2 \tag{3.8}$$

with a constant $K_0 \geq 1$. This constant depends only on the shape regularity of the finite elements T.

Proof. (3.8) is an immediate consequence of the usual inverse estimate

$$|v|_{1,2;T}^2 \leq c\, h(T)^{-2}\|v\|_{0,2;T}^2$$

and of the fact that for $T \subseteq T'$, $T \in \mathcal{T}_k$, $T' \in \mathcal{T}_0$, one has

$$\left(\tfrac{1}{2}\right)^k h(T') \leq h(T)\ .$$

■

It should be noted that (3.1), (3.2), (3.7) and (3.8) are local estimates which refer to a single triangle. This is the reason why the constants in the corresponding global estimates

$$\|I_k u\|_0^2 \leq C_0(j - k + 1)\{4^{-k}|u|_1^2 + \|u\|_0^2\}\ , \tag{3.9}$$

$$|I_k u|_1^2 \leq C_1(j - k + 1)|u|_1^2 \tag{3.10}$$

and

$$\|u - I_k u\|_0^2 \leq C_2(j - k + 1)4^{-k}|u|_1^2 \tag{3.11}$$

for the interpolation operators I_k and the functions $u \in \mathcal{S}$ and in the inverse estimate

$$|v|_1^2 \leq K_0 4^k \|v\|_0^2 \tag{3.12}$$

for the functions $v \in \mathcal{S}_k$ do not depend on the constants $\omega(T)$ in (2.6) and are absolutely insensitive to jumps of the coefficient functions across the boundaries of the initial triangles.

Now we are able to prove the theorem which forms the mathematical background of the hierarchical basis preconditioner.

Theorem 3.4 *There are positive constants K_1 and K_2 with*

$$\frac{\delta}{M}\frac{K_1}{(j+1)^2}|||u|||_H^2 \leq \|u\|^2 \leq \frac{M}{\delta}K_2|||u|||_H^2 \tag{3.13}$$

for all $u \in \mathcal{S}_j$. These constants depend only on the shape regularity of the triangles of the initial triangulation and are independent of the number j of refinement levels.

Proof. Utilizing (3.2) and (3.7) for $u \in \mathcal{S}$ and the triangles $T \in \mathcal{T}_k$ one obtains

$$|I_0 u|_{1;T}^2 + \sum_{k=1}^{j} 4^k \|I_k u - I_{k-1}u\|_{0;T}^2 \leq \frac{(j+1)^2}{K_1}|u|_{1;T}^2 \, .$$

By (2.6) the first estimate

$$|||u|||_H^2 \leq \frac{M}{\delta}\frac{(j+1)^2}{K_1}\|u\|^2 \tag{3.14}$$

follows.

The splitting (2.17) and the Cauchy-Schwarz inequality lead to

$$\|u\|^2 \leq (j+1)\left\{\|I_0 u\|^2 + \sum_{k=1}^{j}\|I_k u - I_{k-1}u\|^2\right\} \, .$$

With (2.6) and the inverse estimate (3.12) and utilizing $M \geq 1$ and $K_0 \geq 1$ one obtains

$$\|u\|^2 \leq M K_0 (j+1)|||u|||_H^2 \, . \tag{3.15}$$

With respect to j, (3.15) is a slightly weaker estimate than the right-hand side

$$\|u\|^2 \leq \frac{M}{\delta}K_2|||u|||_H^2 \tag{3.16}$$

of (3.13). (3.16) itself is proved using certain orthogonality properties of the spaces $V_k =$ range $(I_k - I_{k-1})$. We refer to [11], [12]. ∎

The estimates (3.1) and (3.2) of Theorem 3.1 and (3.7) of Theorem 3.2 are dimension dependent. In the three-dimensional case the factor $j - k + 1$ has to be replaced by an exponentially growing factor. Therefore, the estimate (3.14) is restricted to two-dimensional problems and therefore the hierarchical basis preconditioner deteriorates in the three-dimensional case. (3.15) and (3.16) can be generalized to the three-dimensional case. A detailed discussion of the 3D-hierarchical basis preconditioner can be found in [8] and a discussion from a more general point of view in [9].

4 Estimates for the L_2–like Projections

The proof that the discrete norm (2.21) essentially behaves like the energy norm (2.7) corresponds completely to the proof of Theorem 3.4: It is based on a norm estimate

$$|Q_k u|_1^2 \le C_1 |u|_1^2 \tag{4.1}$$

for the projections (2.18), on an error estimate

$$\|u - Q_k u\|_0^2 \le C_2 4^{-k} |u|_1^2 \tag{4.2}$$

and on the inverse estimate (3.12).

Proofs of estimates like (4.2) and indirectly also of estimates like (4.1) usually are based on the Aubin–Nitsche lemma. As we have in mind a theory which applies not only for regular problems, such proofs have to be ruled out here. A careful discussion of the general case can be found in Xu's thesis [10]. For stability results like (4.1), we refer also to [5].

Here, we utilize a simple technique which is based on the linear operators $M_k : L_2(\Omega) \to \mathcal{S}_k$ given by

$$M_k u = \sum_{i=1}^{n_k} \frac{(u, \psi_i^{(k)})}{(1, \psi_i^{(k)})} \psi_i^{(k)} \tag{4.3}$$

where the nodal basis functions $\psi_i^{(k)}$ are defined by (2.13) and the inner product is given by (2.11) with $G = \Omega$. The main property of the M_k is that they locally reproduce locally constant functions. For every triangle $T \in \mathcal{T}_k$ let

$$U(T,k) = \bigcup \{T' \in \mathcal{T}_k | T \cap T' \neq \emptyset\} \tag{4.4}$$

be the union of the triangles $T' \in \mathcal{T}_k$ intersecting T. Then for $u|U(T,k) = \alpha$, α constant, and for $T \cap \Gamma = \emptyset$ one has $M_k u | T = \alpha$.

To get estimates for the M_k the local quasiuniformity of the triangulations $\mathcal{T}_k$ will be utilized. We assume that for all levels k and all triangles $T, T' \in \mathcal{T}_k$ with $T \cap T' \neq \emptyset$

$$\frac{h(T')}{h(T)} \le K \tag{4.5}$$

holds. Remembering (2.6), for $T \in \mathcal{T}_k$ we define

$$\sigma(T) = \frac{\max\{\omega(T') | T' \in \mathcal{T}_0\ ,\ \ T' \cap T \neq \emptyset\}}{\min\{\omega(T') | T' \in \mathcal{T}_0\ ,\ \ T' \cap T \neq \emptyset\}} \tag{4.6}$$

and set

$$\overline{\sigma} = \max_{T \in \mathcal{T}_0} \sigma(T)\,. \tag{4.7}$$

Note that

$$\max_{T \in \mathcal{T}_k} \sigma(T) \le \overline{\sigma}\,. \tag{4.8}$$

Lemma 4.1 *For all triangles $T \in \mathcal{T}_k$ of level k and all functions $u \in \mathcal{S}$*

$$\|u - M_k u\|_{0;T}^2 \le c\sigma(T) 4^{-k} |u|_{1;U(T,k)}^2\,. \tag{4.9}$$

Proof. Because of (4.12), it is sufficient to prove (4.13) for the triangles $T \in \mathcal{T}_k$ of level k; for the other triangles T of $\mathcal{T}_k$, one has $\overline{M}_k u|T = u|T$. For this purpose, we first prove an estimate like (4.10) for the operators $\overline{M}_k$

Let x_i be a common vertex of a triangle $T \in \mathcal{T}_k$ of level k with triangle $T' \in \mathcal{T}_k$ of a level less than k, that means a node with $(\overline{M}_k u)(x_i) = u(x_i)$. Assume that x_i is also a vertex of $T'' \in \mathcal{T}_j$, $T'' \subseteq T$. As $u \in \mathcal{S}$ is linear on T'',

$$h(T'')|u(x_i)| \leq \hat{c}\|u\|_{0,2;T''} \leq \hat{c}\|u\|_{0,2;T} .$$

Because of $T' \in \mathcal{T}_j$ (see above) and $x_i \in T' \cap T''$ by (4.5) we have

$$h(T') \leq K\, h(T'')$$

and because of $x_i \in T \cap T'$

$$h(T) \leq K\, h(T') .$$

Therefore

$$h(T)|u(x_i)| \leq \tilde{c}\|u\|_{0,2;T} .$$

It follows that

$$|(\overline{M}_k u)(x_i)|\, \|\psi_i^{(k)}\|_{0;T} \leq \bar{c}\|u\|_{0;T} .$$

For the nodes $x_i \in T \cap \mathcal{N}_k \setminus \overline{\Omega}_k$, one has

$$(\overline{M}_k u)(x_i) = (u, \psi_i^{(k)})/(1, \psi_i^{(k)})$$

and therefore likewise

$$|(\overline{M}_k u)(x_i)|\, \|\psi_i^{(k)}\|_{0;T} \leq \|u\|_{0;U(T,k)} .$$

Together

$$\|\overline{M}_k u\|_{0;T} \leq c\|u\|_{0;U(T,k)} . \tag{4.14}$$

As $\overline{M}$ reproduces locally constant functions in the same sense as M_k, one can proceed as in the proof of Lemma 4.1 and gets (4.13). ∎

Theorem 4.3 *For all functions $u \in \mathcal{S}$ the error estimate*

$$\|u - Q_k u\|_0^2 \leq C_2^* \overline{\sigma}\, 4^{-k} |u|_1^2 \tag{4.15}$$

holds. The constant C_2^* depends only on the local geometry of the initial triangulation.

Proof. By definition of an orthogonal projection and by Lemma (4.2)

$$\|u - Q_k u\|_0^2 \leq \|u - \overline{M}_k u\|_0^2 \leq c\overline{\sigma} 4^{-k} \sum_{T \in \mathcal{T}_k} |u|_{1;U(T,k)}^2 .$$

∎

To discuss the stability (4.1) of the projections Q_k, we utilize an additional result on the operators $\overline{M}_k$.

Lemma 4.4 *For all functions $u \in \mathcal{S}$ and all triangles $T \in \mathcal{T}_k$*

$$|\overline{M}_k u|_{1;T}^2 \leq c\sigma(T)|u|_{1;U(T,k)}^2 . \tag{4.16}$$

Proof. By (4.12) it is sufficient to prove (4.16) for the triangles $T \in \mathcal{T}_k$ of level k. For all such triangles by Lemma 3.3 one gets

$$|\overline{M}_k u|_{1;T}^2 \le K_0 4^k \|\overline{M}_k u\|_{0;T}^2 .$$

By (4.14)

$$\|\overline{M}_k u\|_{0;T}^2 \le \bar{c}\|u\|_{0;U(T,k)}^2 ,$$

yielding

$$|\overline{M}_k u|_{1;T}^2 \le c4^k \|u\|_{0;U(T,k)}^2 .$$

For those triangles T with $T \cap \Gamma = \emptyset$ one can use

$$|\overline{M}_k u|_{1;T} = |\overline{M}_k(u+\alpha)|_{1;T}$$

for constant α and obtains

$$|\overline{M}_k u|_{1;T}^2 \le c4^k \inf_\alpha \|u+\alpha\|_{0;U(T,k)}^2 .$$

Then one proceeds as in the proof of Lemma 4.1. ∎

Theorem 4.5 *There exist a constant C_1^* with*

$$|Q_k u|_1^2 \le C_1^* \overline{\sigma} |u|_1^2 \tag{4.17}$$

for all functions $u \in \mathcal{S}$. This constant depends only on local geometric properties of the initial triangulation.

Proof. The proof relies on an old trick from approximation theory. For all functions $u \in \mathcal{S}$, by the inverse estimate (3.10) and the properties of an orthogonal projection, we have

$$\begin{aligned} |Q_k u|_1 &\le |Q_k u - \overline{M}_k u|_1 + |\overline{M}_k u|_1 \\ &\le K_0^{1/2} 2^k \|Q_k u - \overline{M}_k u\|_0 + |\overline{M}_k u|_1 \\ &\le K_0^{1/2} 2^k \|u - \overline{M}_k u\|_0 + |\overline{M}_k u|_1 . \end{aligned}$$

Applying Lemmas 4.2 and 4.4, the proposition follows. ∎

Now we are in position to prove the second main theorem of this paper which corresponds to Theorem 3.4 and which forms the basis of the Bramble–Pasciak–Xu preconditioner.

Theorem 4.6 *There are positive constants K_1^* and K_2^* with*

$$\frac{\delta}{M\overline{\sigma}} \frac{K_1^*}{j+1} |||u|||_X^2 \le \|u\|^2 \le M K_2^* (j+1) |||u|||_X^2 \tag{4.18}$$

for all functions $u \in \mathcal{S}$. These constants depend only on the local geometry of the initial triangulation and are independent of the number j of refinement levels.

Proof. By Theorem 4.3 and Theorem 4.5 one gets

$$|Q_0 u|_1^2 + \sum_{k+1}^{j} 4^k \|Q_k u - Q_{k-1} u\|_0^2 \leq \overline{\sigma} \frac{j+1}{K_1^*} |u|_1^2 .$$

(2.6) yields

$$|\!|\!| u |\!|\!|_X^2 \leq \frac{M\overline{\sigma}}{\delta} \frac{j+1}{K_1^*} \|u\|^2 . \tag{4.19}$$

Corresponding to the proof of Theorem 3.4, the other estimate

$$\|u\|^2 \leq M K_2^* (j+1) |\!|\!| u |\!|\!|_X^2 \tag{4.20}$$

follows from the splitting (2.19) and the inverse inequality (3.12). The constant $K_2^* = K_0$ is the same as in (3.15). ■

Compared to the Theorem 3.4, the left–hand side inequality (4.19) behaves better in terms of j and the right–hand side estimate (4.20) worse. The asymptotic growth of the quotient of the constants on the right– and on the left–hand side for $j \to \infty$ remains the same and is $\mathcal{O}(j^2)$.

Under the same strong regularity assumptions which are utilized in the theory of ordinary multigrid algorithms [7], in the extreme case one can improve (4.19) to

$$|\!|\!| u |\!|\!|_X^2 \leq \widehat{K}_1^{-1} \|u\|^2 \tag{4.21}$$

reducing the growth of the quotient of the optimal constants to $\mathcal{O}(j)$. We do not discuss this topic here and refer to the original literature [4], [10] or to the appendix of this report.

In contrast to Theorem 3.4, where the lower estimate in (3.13) is restricted to two–dimensional applications, Theorem 4.6 and the other results of this section can be generalized to three–dimensional problems or even to higher dimensional cases. Therefore, for such applications, the Bramble–Pasciak–Xu preconditioner is superior to the hierarchical basis method.

Contrary to the estimate (3.13) in Theorem 3.4, which is totally independent of jumps in the coefficient functions across the boundaries of the initial triangles, the constant (4.7) enters into (4.18), (4.19) via the estimates (4.15) and (4.17). This can be a serious drawback.

At the price of a slightly worse behavior in j, one can avoid this dependence at least for two–dimensional problems. Using the interpolation operators I_k of Section 3 instead of the operators $\overline{M}_k$, one finds

$$\|u - Q_k u\|_0^2 \leq C_2 (j-k+1) 4^{-k} |u|_1^2 \tag{4.22}$$

and

$$|Q_k u|_1^2 \leq C_1' (j-k+1) |u|_1^2 \tag{4.23}$$

for $u \in \mathcal{S}$ so that (4.19) can be replaced by

$$|\!|\!| u |\!|\!|_X^2 \leq \frac{M}{\delta} \frac{(j+1)^2}{K_1'} \|u\|^2 . \tag{4.24}$$

5 The Preconditioners

The discrete boundary value problem to be solved is to find a function $u \in \mathcal{S}$ satisfying

$$a(u,v) = f^*(v)\ ,\quad v \in \mathcal{S}\ . \tag{5.1}$$

f^* is a linear functional representing the right–hand side of the differential equation. Introducing the selfadjoint and positive definite operators $A : \mathcal{S} \to \mathcal{S}$ by

$$(Au,v) = a(u,v)\ ,\quad v \in \mathcal{S}\ . \tag{5.2}$$

and a vector $b \in \mathcal{S}$ by

$$(b,v) = f^*(v)\ ,\quad v \in \mathcal{S}\ , \tag{5.3}$$

(5.1) can be reformulated as

$$Au = b\ . \tag{5.4}$$

For the solution of (5.4), we consider iterations

$$u \leftarrow u + \omega C(b - Au) \tag{5.5}$$

with selfadjoint and positive definite operators $C : \mathcal{S} \to \mathcal{S}$ and with properly chosen constants $\omega > 0$, and conjugate gradient type accelerations of such iterations, respectively. Then the speed of convergence is governed by the spectral condition number $\kappa(C^{1/2}AC^{1/2})$, which is the quotient of the maximum and the minimum eigenvalue of the operator $C^{1/2}AC^{1/2}$. (5.5) can be rewritten as

$$r \leftarrow b - Au\ ,\quad u \leftarrow u + \omega Cr\ . \tag{5.6}$$

To realize (5.6) efficiently, the right representation of the vectors u, $r \in \mathcal{S}$ is essential. We store u by the values

$$u(x_i)\ ,\quad i = 1,\ldots,n\ , \tag{5.7}$$

whereas r is represented by

$$(r,\psi_i)\ ,\quad i = 1,\ldots,n\ , \tag{5.8}$$

where for simplicity $n = n_j$ and $\psi_i = \psi_i^{(j)}$. We get

$$(r,\psi_i) = f^*(\psi_i) - \sum_{l=1}^{n} a(\psi_i,\psi_l)u(x_l)\ . \tag{5.9}$$

Therefore only the usual residual has to be computed; neither an explicit representation of the operator A nor of the right–hand side b is needed. In addition, we restrict our attention to such operators C for which $(Cr)(x_i)$, $i = 1,\ldots,n$, can be computed easily from the values (5.8). As all other reasonable methods the hierarchical basis- and the Bramble–Pasciak–Xu preconditioner are of this type.

For $u \in \mathcal{S}$ we have

$$\|u\|^2 = (u, Au)\ . \tag{5.10}$$

Correspondingly, there are selfadjoint and positive definite operators B_H, B_X : $\mathcal{S} \to \mathcal{S}$ with

$$|||u|||_H^2 = (u,\, B_H u)\ , \tag{5.11}$$

$$|||u|||_X^2 = (u,\, B_X u) \tag{5.12}$$

for $u \in \mathcal{S}$.

By Theorem 3.4 for all $u \in \mathcal{S}$

$$\frac{\delta}{M}\,\frac{K_1}{(j+1)^2}(u, B_H u) \le (u, Au) \le \frac{M}{\delta}K_2(u, B_H u) \tag{5.13}$$

and by Theorem 4.6

$$\frac{\delta}{M\overline{\sigma}}\,\frac{K_1^*}{(j+1)}(u, B_X u) \le (u, Au) \le MK_2^*(j+1)(u, B_X u)\ . \tag{5.14}$$

(5.13) and (5.14) imply the condition number estimates

$$\kappa(B_H^{-1/2} A B_H^{-1/2}) \le \left(\frac{M}{\delta}\right)^2 \frac{K_2}{K_1}(j+1)^2 \tag{5.15}$$

and

$$\kappa(B_X^{-1/2} A B_X^{-1/2}) \le \frac{M^2\overline{\sigma}}{\delta}\,\frac{K_2^*}{K_1^*}(j+1)^2\ . \tag{5.16}$$

This is a very moderate growth in the number j of refinement levels. For a uniformly refined grid with gridsize $h \sim h_0/2^j$, the right-hand side of (5.15), (5.16) behaves like

$$(j+1)^2 \sim |\log h|^2\ .$$

The estimates (4.21) and (4.24), respectively, lead to corresponding modifications of (5.14) and the condition number estimate (5.16).

Unfortunately, $Cr = B_H^{-1}r$ and $Cr = B_X^{-1}r, r \in \mathcal{S}$, cannot be computed with a tolerable amount of work, so that B_H^{-1} and B_X^{-1} must be replaced by simpler operators $C_H = H^{-1}$ and C_X.

Assume that a selfadjoint positive definite operator $H : \mathcal{S} \to \mathcal{S}$ satisfies the estimate

$$\mu_1(u, Hu) \le (u, B_H u) \le \mu_2(u, Hu) \tag{5.17}$$

for all $u \in \mathcal{S}$ where μ_1 and μ_2 are positive constants which depend only on the shape regularity of the triangles. Then

$$\kappa(H^{-1/2} A H^{-1/2}) \le \left(\frac{M}{\delta}\right)^2 \frac{K_2}{K_1}\,\frac{\mu_2}{\mu_1}(j+1)^2 \tag{5.18}$$

so that H is a good preconditioner for A provided that it can be handled easily.

The construction of H is based on the following lemma.

Lemma 5.1 *There are positive constants μ_1 and μ_2 which depend only on a lower bound for the interior angles of the triangles $T \in \mathcal{T}_k$, with*

$$\frac{1}{\mu_2}\|v\|_0^2 \le \sum_{i=1}^{n_k}(1, \psi_i^{(k)})|v(x_i)|^2 \le \frac{1}{\mu_1}\|v\|_0^2 \tag{5.19}$$

for all functions $v \in \mathcal{S}_k$.

Proof. By the usual arguments, one shows that for all triangles $T \in \mathcal{T}_k$ and all linear functions v

$$\frac{1}{\mu_2}\|v\|_{0;T}^2 \leq \sum_{x_i \in \mathcal{N}_k \cap T} (1, \psi_i^{(k)})_T |v(x_i)|^2 \leq \frac{1}{\mu_1}\|v\|_{0;T}^2$$

The summation over all triangles $T \in \mathcal{T}_k$ gives (5.19) ∎

It follows that with

$$d_i = 4^k(1, \psi_i^{(k)}) , \quad x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1} , \tag{5.20}$$

the discrete norm on $\mathcal{S} = \mathcal{S}_j$ given by

$$|\!|\!| u |\!|\!|^2 = \|I_0 u\|^2 + \sum_{k=1}^{j} \sum_{x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} d_i |(I_k u - I_{k-1}u)(x_i)|^2 \tag{5.21}$$

is equivalent to the norm (2.20)

$$|\!|\!| u |\!|\!|_H^2 = \|I_0 u\|^2 + \sum_{k=1}^{j} 4^k \|I_k u - I_{k-1} u\|_0^2 .$$

Of course, it is possible to replace the weights (5.20) by other weights which can be estimated from above and below by the weights (5.20), for example by

$$d_i = 4^k(\psi_i^{(k)}, \psi_i^{(k)}) \tag{5.22}$$

or by

$$d_i = a(\psi_i^{(k)}, \psi_i^{(k)}) . \tag{5.23}$$

In [11] a discrete norm like (5.21) has been treated directly.

H is defined by

$$|\!|\!| u |\!|\!|^2 = (u, Hu) . \tag{5.24}$$

The matrix $\left((\widehat{\psi}_i, H\widehat{\psi}_l)\right)$ representing H with respect to the hierarchical basis (2.14), (2.15) of $\mathcal{S}$ is diagonal up to a small block of the dimension n_0 of $\mathcal{S}_0$. Utilizing this fact the hierarchical basis preconditioner can be realized in less than $7n_j$ floating point operations up to the solution of a linear system with the level 0 discretization matrix. For the algorithmic details we refer to [11].

XU [10] gives an explicit representation of the hierarchical basis preconditioner in terms of the operator

$$C_H = H^{-1}. \tag{5.25}$$

This representation fits into the framework above and is essential for its comparison with the Bramble–Pasciak–Xu preconditioner. Generalizing (5.2) we introduce selfadjoint, positive definite operators $A_k : \mathcal{S}_k \to \mathcal{S}_k$ by

$$(A_k u, v) = a(u, v) , \quad v \in \mathcal{S}_k \tag{5.26}$$

Lemma 5.2 *For all $r \in \mathcal{S}$*

$$C_H r = A_0^{-1} Q_0 r + \sum_{k=1}^{j} \sum_{x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} d_i^{-1} (r, \widehat{\psi}_i) \widehat{\psi}_i . \tag{5.27}$$

Proof. Defining C_H by the right-hand side of (5.27), for all $v \in \mathcal{S}_0$

$$\begin{aligned}(HC_Hr, v) &= a(I_0C_Hr, v) &&= a(A_0^{-1}Q_0r, v)\\ &= (A_0^{-1}Q_0r, A_0v) &&= (r, v)\end{aligned}$$

and for all $v \in$ range $(I_k - I_{k-1})$

$$\begin{aligned}(HC_Hr, v) &= \sum_{x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} d_i(I_kC_Hr - I_{k-1}C_Hr)(x_i)v(x_i)\\ &= \sum_{x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} d_i d_i^{-1}(r, \widehat{\psi}_i)\, v(x_i) = (r, v)\,.\end{aligned}$$

Thus $HC_Hr = r$ for all $r \in \mathcal{S}$, or $H^{-1} = C_H$. ■

Note that the values $(r, \widehat{\psi}_i)$ can be computed recursively beginning with the values $(r, \psi_i) = (r, \psi_i^{(j)})$ and that the summation of the single terms in (5.27) can be formulated as a recursive process, too.

We remark that for a given $r \in \mathcal{S}$ the function

$$u_0 = A_0^{-1}Q_0r \in \mathcal{S}_0 \tag{5.28}$$

satisfies

$$a(u_0, v) = (r, v)\,, \quad v \in \mathcal{S}_0\,. \tag{5.29}$$

To compute u_0 therefore one needs only $(r, \psi_i^{(0)})$, $i = 1, \ldots, n_0$, but not Q_0r itself, and one has to solve a linear system with the level 0 discretization matrix. Note that for $\mathcal{N}_0 \subseteq \partial\Omega$ and $\partial\Omega = \Gamma$ one has $\mathcal{S}_0 = \{0\}$. For this case, the first term on the right-hand side of (5.27) vanishes.

The Bramble–Pasciak–Xu preconditioner is not based immediately on a norm equivalence and is less suggestive, because no simple associated basis exists. Its inventors replace B_X^{-1} directly by a selfadjoint positive definite operator $C_X : \mathcal{S} \to \mathcal{S}$ with

$$\frac{1}{\mu_2^*}(r, C_Xr) \le (r, B_X^{-1}r) \le \frac{1}{\mu_1^*}(r, C_Xr) \tag{5.30}$$

for all $r \in \mathcal{S}$. Contrary to $C_H^{-1} = H$ the operator C_X^{-1} is only implicitly known. (5.30) is equivalent to

$$\mu_1^*(u, C_X^{-1}u) \le (u, B_Xu) \le \mu_2^*(u, C_X^{-1}u) \tag{5.31}$$

for all $u \in \mathcal{S}$. Therefore, by (5.14),

$$\kappa(C_X^{1/2}AC_X^{1/2}) \le \frac{M^2\overline{\sigma}}{\delta}\,\frac{K_2^*}{K_1^*}\,\frac{\mu_2^*}{\mu_1^*}(j+1)^2\,. \tag{5.32}$$

Due to the construction of the weighted (semi–)norms (2.9) and (2.10), in our version the positive constants μ_1^* and μ_2^* depend only on the shape regularity of the triangles, but not on the quasiuniformity of the initial triangulation.

A first step towards the construction of C_X is an explicit representation of B_X^{-1} as given in [10]:

Lemma 5.3

$$B_X = A_0 Q_0 + \sum_{k=1}^{j} 4^k (Q_k - Q_{k-1}) \tag{5.33}$$

$$B_X^{-1} = A_0^{-1} Q_0 + \sum_{k=1}^{j} 4^{-k} (Q_k - Q_{k-1}) \,. \tag{5.34}$$

Proof. (5.33) is obvious. The proof of (5.34) is an easy consequence of the fact that the Q_k are orthogonal projectors and that $\mathcal{S}_{k-1}$ is a subspace of $\mathcal{S}_k$. ∎

Conceptionally, the next step represents the main difference to the derivation of the hierarchical basis method. Because of the monotonely decreasing, even exponentially decaying forefactors, according to BRAMBLE–PASCIAK & XU [4] and XU [10] one is able to replace B_X^{-1} by

$$\widehat{C}_X = A_0^{-1} Q_0 + \sum_{k=1}^{j} 4^{-k} Q_k \,. \tag{5.35}$$

Lemma 5.4 *For all* $r \in \mathcal{S}$

$$(r, B_X^{-1} r) \le (r, \widehat{C}_X r) \le (1 + \tfrac{1}{3} M K_0)(r, B_X^{-1} r) \tag{5.36}$$

where M *is defined in* (2.6) *and where* K_0 *is the constant from the inverse inequality* (3.8), (3.12).

Proof. The left–hand side is trivial. Utilizing

$$\sum_{k=1}^{j} 4^{-k} Q_k = \tfrac{4}{3} \sum_{k=1}^{j} 4^{-k} (Q_k - Q_{k-1}) + \tfrac{1}{3} Q_0 - \tfrac{1}{3} 4^{-j} Q_j \,,$$

one obtains

$$(r, \widehat{C}_X r) \le \|A_0^{-1/2} Q_0 r\|_0^2 + \tfrac{4}{3} \sum_{k=1}^{j} 4^{-k} \|Q_k r - Q_{k-1} r\|_0^2 + \tfrac{1}{3} \|Q_0 r\|_0^2 \,.$$

By (2.6) and the inverse inequality (3.12)

$$\begin{aligned} \|Q_0 r\|_0^2 &= \|A_0^{-1/2} Q_0 r\|^2 \le M |A_0^{-1/2} Q_0 r|_1^2 \\ &\le M K_0 \|A_0^{-1/2} Q_0 r\|_0^2 \,, \end{aligned} \tag{5.37}$$

yielding the right–hand side of (5.36) if we use again $M K_0 \ge 1$. ∎

(5.35) is a representation of $\widehat{C}_X$ as a sum of selfadjoint positive semidefinite operators. Therefore BRAMBLE, PASCIAK & XU can replace each of the Q_k separately by a spectrally equivalent selfadjoint positive semidefinite operator $R_k : \mathcal{S} \to \mathcal{S}$, leading to the final preconditioners

$$C_X = A_0^{-1} Q_0 + \sum_{k=1}^{j} 4^{-k} R_k \,. \tag{5.38}$$

Following the ideas in [4] and [10] here we discuss a special choice of the R_k. We begin with an observation concerning the operators M_k introduced in Section 4, which is an algebraic reformulation of Lemma 5.1.

Lemma 5.5 *For all $v \in \mathcal{S}_k$*

$$\frac{1}{\mu_2}(M_k v, v) \le \|v\|_0^2 \le \frac{1}{\mu_1}(M_k v, v) \,. \tag{5.39}$$

Proof. If we define the symmetric positive definite matrix G by

$$G|_{il} = (\psi_i^{(k)}, \psi_l^{(k)})$$

and the diagonal matrix D by

$$D|_{ii} = (1, \psi_i^{(k)}) \,,$$

by Lemma 5.1 for all coefficient vectors x we have

$$\frac{1}{\mu_2} x^T G x \le x^T D x \le \frac{1}{\mu_1} x^T G x \,.$$

Equivalently, for all y

$$\frac{1}{\mu_2} y^T D^{-1} y \le y^T G^{-1} y \le \frac{1}{\mu_1} y^T D^{-1} y$$

or, with $y = Gz$, for all z

$$\frac{1}{\mu_2} z^T G D^{-1} G z \le z^T G z \le \frac{1}{\mu_1} z^T G D^{-1} G z$$

which is another formulation of (5.39). ∎

Because of $M_k Q_k = M_k$, Lemma 5.5, applied to $v = Q_k r$, yields

$$\frac{1}{\mu_2}(M_k r, r) \le (Q_k r, r) \le \frac{1}{\mu_1}(M_k r, r) \tag{5.40}$$

for all $r \in \mathcal{S}$. Therefore, the operators B_X^{-1}, $\widehat{C}_X$ and

$$C_X = A_0^{-1} Q_0 + \sum_{k=1}^{j} 4^{-k} M_k \tag{5.41}$$

are spectrally equivalent. C_X^{-1} is the wanted preconditioner for A.

An explicit representation of the operator C_X is

$$C_X r = A_0^{-1} Q_0 r + \sum_{k=1}^{j} 4^{-k} \sum_{i=1}^{n_k} \frac{(r, \psi_i^{(k)})}{(1, \psi_i^{(k)})} \psi_i^{(k)} \,. \tag{5.42}$$

Compared to the representation (5.27)

$$C_H r = A_0^{-1} Q_0 r + \sum_{k=1}^{j} 4^{-k} \sum_{i=n_{k-1}+1}^{n_k} \frac{(r, \psi_i^{(k)})}{(1, \psi_i^{(k)})} \psi_i^{(k)} \tag{5.43}$$

of the hierarchical basis preconditioner only additional terms have been added in (5.42). As the single terms

$$r \longrightarrow \frac{(r, \psi_i^{(k)})}{(1, \psi_i^{(k)})} \psi_i^{(k)} \tag{5.44}$$

represent selfadjoint positive semidefinite operators, one has $(r, C_H r) \leq (r, C_X r)$ for all $r \in \mathcal{S}$ or equivalently

$$(u, C_X^{-1} u) \leq (u, C_H^{-1} u) \tag{5.45}$$

for all $u \in \mathcal{S}$. By the min–max characterization of the eigenvalues of a selfadjoint operator it follows that all eigenvalues of $C_X^{1/2} A C_X^{1/2}$ are greater than or equal to the corresponding eigenvalues of $C_H^{1/2} A C_H^{1/2}$.

To evaluate $C_X r$, first the inner products $(r, \psi_i^{(k)})$, $i = 1, \ldots, n_k$, $k = 0, 1 \ldots, j$, have to be computed. This can be done recursively, beginning with the final level j. According to (5.29), the computation of $A_0^{-1} Q_0 r$ additionally requires the solution of a linear system of dimension n_0 with the level 0 discretization matrix. Finally, all terms must be summed up.

The number of terms (5.44) in (5.42), which are different from each other, is bounded by the dimension n_j of $\mathcal{S}$ independently of the dimensions of the spaces $\mathcal{S}_k$. Therefore, with a proper rearrangement, the expression (5.42) can be evaluated in $\mathcal{O}(n_j)$ operations regardless of the dimensions of the spaces $\mathcal{S}_k$.

Another, probably simpler possibility is to replace the operator (5.42) by

$$C r = A_0^{-1} Q_0 r + \sum_{k=1}^{j} 4^{-k} \sum_{\psi_i^{(k)} \neq \psi_i^{(k-1)}} \frac{(r, \psi_i^{(k)})}{(1, \psi_i^{(k)})} \psi_i^{(k)} \tag{5.46}$$

where

$$\sum_{\psi_i^{(k)} \neq \psi_i^{(k-1)}} = \sum_{\substack{i=1 \\ \psi_i^{(k)} \neq \psi_i^{(k-1)}}}^{n_{k-1}} + \sum_{i=n_{k-1}+1}^{n_k} .$$

This is possible because the single terms (5.44) represent selfadjoint positive semidefinite operators and because the forefactors 4^{-k} decay exponentially. Hence, it is sufficient that every such term occurs only once, with the largest forefactor 4^{-k}. The eventually remaining terms with basis functions $\psi_i^{(1)} = \psi_i^{(0)}$ can be treated using (5.40) and (5.37), that means

$$(M_0 r, r) \leq \mu_2 \|Q_0 r\|_0^2 \leq \mu_2 M K_0 (A_0^{-1} Q_0 r, r) .$$

By the same reason it is possible to modify the scaling factors

$$d_i^{(k)} = 4^k (1, \psi_i^{(k)}) \tag{5.47}$$

appropriately. As the remaining basis function $\psi_i^{(k)}$ in (5.46) are associated with vertices of level k triangles and because of (4.5), one can replace the scaling factors (5.47) by

$$d_i^{(k)} = a(\psi_i^{(k)}, \psi_i^{(k)}) \tag{5.48}$$

for example, similar as in the hierarchical basis case. The quotient of the constants (5.47) and (5.48) remains bounded from above and below. The double sum in (5.46) consists of $\mathcal{O}(n_j)$-terms regardless the dimensions of the spaces $\mathcal{S}_k$.

Note that for the application of C_X and C_H, respectively, in the iteration (5.5) or a preconditioned conjugate gradient type method, the values $(r, \psi_i^{(j)}) = (r, \psi_i), i = 1, \ldots, n_j$, are already known and do not need to be computed. Therefore the inner product (,) enters into the final algorithms only indirectly via the scaling factor d_i and $d_i^{(k)}$, respectively. The correct choice of these scaling factors is essential for the performance of both methods.

Acknowledgement. The author was supported by a Konrad Zuse Fellowship.

References

[1] Bank, R. E.: *PLTMG User's Guide.* Edition 5.0, Tech. Rep., Department of Mathematics, University of California at San Diego (1988).

[2] Bank, R. E., Dupont, T., Yserentant, H.: *The Hierarchical Basis Multigrid Method.* Numer. Math. **52**, pp. 427–458 (1988).

[3] Bank, R. E., Sherman, A. H., Weiser, A.: *Refinement Algorithms and Data Structures for Regular Local Mesh Refinement.* In: Scientific Computing (eds.: R. Stepleman et al.), Amsterdam: IMACS/North Holland, pp. 3–17 (1983).

[4] Bramble, J. H, Pasciak, J. E., Xu, J.: *Parallel Multilevel Preconditioners.* To appear in Math. Comp.

[5] Crouzeix, M., Thomée, V.: *The Stability in L_p and W_p^1 of the L_2-Projection onto Finite-Element Function Spaces.* Math Comp. **48**, pp. 521–532 (1987).

[6] Deuflhard, P., Leinen, P., Yserentant, H.: *Concepts of an Adaptive Hierarchical Finite Element Code.* IMPACT of Computing in Science and Engineering **1** , pp. 3–35 (1989).

[7] Hackbusch, W.: *Multigrid Methods and Applications.* Berlin, Heidelberg, New York: Springer (1985).

[8] Ong, M. E. G.: *Hierarchical Basis Preconditioners for Second Order Elliptic Problems in Three Dimensions.* Technical Report No. 89–3, Department of Applied Mathematics, University of Washington, Seattle (1989).

[9] Oswald, P. : *On Estimates for Hierarchic Basis Representations of Finite Element Functions.* Technical Report N/89/16, Sektion Mathematik, Friedrich-Schiller Universität Jena (1989).

[10] Xu, J.: *Theory of Multilevel Methods.* Report No. AM48, Department of Mathematics, Pennsylvania State University (1989).

[11] Yserentant, H.: *On the Multi-Level Splitting of Finite Element Spaces.* Numer. Math **49**, pp. 379–412 (1986).

[12] Yserentant, H.: *Hierarchical Bases Give Conjugate Gradient Type Methods a Multigrid Speed of Convergence.* Applied Mathematics and Computation **19**, pp. 347–358 (1986).

Appendix: The Regular Case

In this appendix we show that under the same regularity assumptions, which are used in the theory of ordinary multigrid methods, the estimate (4.19) or equivalently

$$(u, \widehat{C}_X^{-1} u) \le \frac{j+1}{K^*}(u, Au) \tag{1}$$

can be improved to

$$(u, \widehat{C}_X^{-1} u) \le \widehat{K}_1^{-1}(u, Au) \tag{2}$$

which means

$$\kappa(C_X^{1/2} A C_X^{1/2}) = \mathcal{O}(j) \ . \tag{3}$$

For this purpose we introduce the finite element projections $P_k : \mathcal{H} \longrightarrow \mathcal{S}_k$ by

$$a(P_k u, v) = a(u, v) \ , \quad v \in \mathcal{S}_k \ . \tag{4}$$

Note that P_k projects the solution of the boundary value problem (2.3) onto the finite element solution with respect to the space $\mathcal{S}_k$.

Our main assumption is that for all $u \in \mathcal{H}$

$$\|u - P_k u\|_0^2 \le \widehat{K} 4^{-k} \|u\|^2 \tag{5}$$

or equivalently

$$\|u - P_k u\|_0^2 \le \widehat{K} 4^{-k} \|u - P_k u\|^2 \ . \tag{6}$$

This assumption is fulfilled for convex domains Ω, smooth coefficient functions a_{ij} and quasiuniform triangulations. For this case, the boundary value problem is H^2-regular, and (6) is the Aubin–Nitsche Lemma. We remark that (4.2) and, using the trick in the proof of Theorem 4.5, (4.1) are immediate consequences of (5).

With this strong assumption, one obtains the following theorem which is a special case of Theorem 2 in [4] where $H^{1+\alpha}$–regularity is covered.

Theorem: *Provided that assumption* (5) *holds, for all* $u \in \mathcal{S}$

$$(u, \widehat{C}_X^{-1} u) \le \max\{4\widehat{K}, 1\}(u, Au) \ . \tag{7}$$

Proof. We follow the proof of Theorem 2 in [4]. First we state that (7) follows from

$$\|u\|^2 \le \max\{4\widehat{K}, 1\} a(\widehat{C}_X A u, u) \ . \tag{8}$$

By the orthogonality of the P_k and because of $\mathcal{S}_{k-1} \subseteq \mathcal{S}_k$, we have

$$\|u\|^2 = \|P_0 u\|^2 + \sum_{k=1}^{j} \|P_k u - P_{k-1} u\|^2 \ . \tag{9}$$

For all $v \in \mathcal{S}_k$ by (5) one gets

$$\begin{aligned}\|v - P_{k-1}v\|^2 &= ((I - P_{k-1})(v - P_{k-1}v), A_k v) \\ &\le \|(I - P_{k-1})(v - P_{k-1}v)\|_0 \|A_k v\|_0 \\ &\le \widehat{K}^{1/2} 2^{-(k-1)} \|v - P_{k-1}v\| \|A_k v\|_0\end{aligned}$$

or

$$\|v - P_{k-1}v\|^2 \le 4\widehat{K} 4^{-k} \|A_k v\|_0^2 .$$

Therefore, for $v = P_k u$

$$\|P_k u - P_{k-1}u\|^2 \le 4\widehat{K} 4^{-k} \|A_k P_k u\|_0^2 .$$

With (9)

$$\|u\|^2 \le \|P_0 u\|^2 + 4\widehat{K} \sum_{k=1}^{j} 4^{-k} \|A_k P_k u\|_0^2 \tag{10}$$

follows. Since for all $u \in \mathcal{S}$ and all $v \in \mathcal{S}_k$

$$\begin{aligned}(A_k P_k u, v) &= a(P_k u, v) = a(u, v) \\ &= (Au, v) = (Q_k A u, v)\end{aligned}$$

we have $A_k P_k = Q_k A$. Thus

$$\begin{aligned}\|P_0 u\|^2 &= a(P_0 u, u) = a(A_0^{-1} A_0 P_0 u, u) \\ &= a(A_0^{-1} Q_0 A u, u)\end{aligned}$$

and

$$\begin{aligned}\|A_k P_k u\|_0^2 &= \|Q_k A u\|_0^2 = (Q_k A u, A u) \\ &= a(Q_k A u, u)\end{aligned}$$

so that (10) implies (8). ∎

Veröffentlichungen des Konrad-Zuse-Zentrum für Informationstechnik Berlin
Preprints Januar 1990

SC 86-1. P. Deuflhard; U. Nowak. *Efficient Numerical Simulation and Identification of Large Chemical Reaction Systems.* (vergriffen) In: Ber. Bunsenges. Phys. Chem., vol. 90, 1986, 940-946
SC 86-2. H. Melenk; W. Neun. *Portable Standard LISP for CRAY X-MP Computers.*

SC 87-1. J. Anderson; W. Galway; R. Kessler; H. Melenk; W. Neun. *The Implementation and Optimization of Portable Standard LISP for the CRAY.*
SC 87-2. Randolph E. Bank; Todd F. Dupont; Harry Yserentant. *The Hierarchical Basis Multigrid Method.* (vergriffen) In: Numerische Mathematik, 52, 1988, 427-458.
SC 87-3. Peter Deuflhard. *Uniqueness Theorems for Stiff ODE Initial Value Problems.*
SC 87-4. Rainer Buhtz. *CGM-Concepts and their Realizations.*
SC 87-5. P. Deuflhard. *A Note on Extrapolation Methods for Second Order ODE Systems.*
SC 87-6. Harry Yserentant. *Preconditioning Indefinite Discretization Matrices.*

SC 88-1. Winfried Neun; Herbert Melenk. *Implementation of the LISP-Arbitrary Precision Arithmetic for a Vector Processor.*
SC 88-2. H. Melenk; H. M. Möller; W. Neun. *On Gröbner Bases Computation on a Supercomputer Using REDUCE.* (vergriffen)
SC 88-3. J. C. Alexander; B. Fiedler. *Global Decoupling of Coupled Symmetric Oscillators.*
SC 88-4. Herbert Melenk; Winfried Neun. *Parallel Polynomial Operations in the Buchberger Algorithm.*
SC 88-5. P. Deuflhard; P. Leinen; H. Yserentant. *Concepts of an Adaptive Hierarchical Finite Element Code.*
SC 88-6. P. Deuflhard; M. Wulkow. *Computational Treatment of Polyreaction Kinetics by Orthogonal Polynomials of a Discrete Variable.* (vergriffen)
SC 88-7. H. Melenk; H. M. Möller; W. Neun. *Symbolic Solution of Large Stationary Chemical Kinetics Problems.* I
SC 88-8. Ronald H. W. Hoppe; Ralf Kornhuber. *Multi-Grid Solution of Two Coupled Stefan Equations Arising in Induction Heating of Large Steel Slabs.*
SC 88-9. Ralf Kornhuber; Rainer Roitzsch. *Adaptive Finite-Element-Methoden für konvektions-dominierte Randwertprobleme bei partiellen Differentialgleichungen.*
SC 88-10. S -N. Chow; B. Deng; B. Fiedler. *Homoclinic Bifurcation at Resonant Eigenvalues.*

SC 89-1. Hongyuan Zha. *A Numerical Algorithm for Computing the Restricted Singular Value Decomposition of Matrix Triplets.*
SC 89-2. Hongyuan Zha. *Restricted Singular Value Decomposition of Matrix Triplets.*
SC 89-3. Wu Huamo. *On the Possible Accuracy of TVD Schemes.*
SC 89-4. H. Michael Möller. *Multivariate Rational Interpolation: Reconstruction of Rational Functions.*
SC 89-5. Ralf Kornhuber; Rainer Roitzsch. *On Adaptive Grid Refinement in the Presence of Internal or Boundary Layers.*
SC 89-6. Wu Huamo; Yang Shuli. *MmB-A New Class of Accurate High Resolution Schemes for Conservation Laws in Two Dimensions.*
SC 89-7. U. Budde; M. Wulkow. *Computation of Molecular Weight Distributions for Free Radical Polymerization Systems.*
SC 89-8. Gerhard Maierhöfer. *Ein paralleler adaptiver Algorithmus für die numerische Integration.*
SC 89-9. Harry Yserentant. *Two Preconditioners Based on the Multi-Level Splitting of Finite Element Spaces.*
SC 89-10. Ronald H. W. Hoppe. *Numerical Solution of Multicomponent Alloy Solidification by Multi-Grid Techniques.*
SC 89-11. B. Einfeldt. *A Conception for the Design of Difference Methods for Nonlinear Conservation Laws: I. The Geometric Approach.*

SC 90-1. M. Wulkow; P. Deuflhard. *Towards an Efficient Computational Treatment of Heterogeneous Polymer Reactions.*

Veröffentlichungen des Konrad-Zuse-Zentrum für Informationstechnik Berlin
Technical Reports December 1989

TR 86-1. H. J. Schuster. *Tätigkeitsbericht (vergriffen)*

TR 87-1. Hubert Busch; Uwe Pöhle; Wolfgang Stech. *CRAY-Handbuch. - Einführung in die Benutzung der CRAY.*
TR 87-2. Herbert Melenk; Winfried Neun. *Portable Standard LISP Implementation for CRAY X–MP Computers. Release of PSL 3.4 for COS.*
TR 87-3. Herbert Melenk; Winfried Neun. *Portable Common LISP Subset Implementation for CRAY X–MP Computers.*
TR 87-4. Herbert Melenk; Winfried Neun. *REDUCE Installation Guide for CRAY 1 / X-MP Systems Running COS Version 3.3*
TR 87-5. Herbert Melenk; Winfried Neun. *REDUCE Users Guide for the CRAY 1 / X-MP Series Running COS. Version 3.3*
TR 87-6. Rainer Buhtz; Jens Langendorf; Olaf Paetsch; Danuta Anna Buhtz. *ZUGRIFF - Eine vereinheitlichte Datenspezifikation für graphische Darstellungen und ihre graphische Aufbereitung.*
TR 87-7. J. Langendorf; O. Paetsch. *GRAZIL (Graphical ZIB Language).*

TR 88-1. Rainer Buhtz; Danuta Anna Buhtz. *TDLG 3.1 - Ein interaktives Programm zur Darstellung dreidimensionaler Modelle auf Rastergraphikgeräten.*
TR 88-2. Herbert Melenk; Winfried Neun. *REDUCE User's Guide for the CRAY 1 / CRAY X-MP Series Running UNICOS. Version 3.3.*
TR 88-3. Herbert Melenk; Winfried Neun. *REDUCE Installation Guide for CRAY 1 / CRAY X-MP Systems Running UNICOS. Version 3.3.*
TR 88-4. Danuta Anna Buhtz; Jens Langendorf; Olaf Paetsch. *GRAZIL-3D. Ein graphisches Anwendungsprogramm zur Darstellung von Kurven- und Funktionsverläufen im räumlichen Koordinatensystem.*
TR 88-5. Gerhard Maierhöfer; Georg Skorobohatyj. *Parallel-TRAPEX. Ein paralleler, adaptiver Algorithmus zur numerischen Integration ; seine Implementierung für SUPRENUM-artige Architekturen mit SUSI.*

TR 89-1. *CRAY-HANDBUCH. Einführung in die Benutzung der CRAY X-MP unter UNICOS.*
TR 89-2. Peter Deuflhard. *Numerik von Anfangswertmethoden für gewöhnliche Differential-*gleichungen.
TR 89-3. Artur Rudolf Walter. *Ein Finite-Element-Verfahren zur numerischen Lösung von Erhaltungsgleichungen.*
TR 89-4. Rainer Roitzsch. *KASKADE User's Manual.*
TR 89-5. Rainer Roitzsch. *KASKADE Programmer's Manual.*
TR 89-6. Herbert Melenk; Winfried Neun. *Implementation of Portable Standard LISP for the SPARC Processor.*
TR 89-7. Folkmar A. Bornemann. *Adaptive multilevel discretization in time and space for parabolic partial differential equations.*
TR 89-8. Gerhard Maierhöfer; Georg Skorobohatyj. *Implementierung des parallelen TRAPEX auf Transputern.*