

Konrad-Zuse-Zentrum für Informationstechnik Berlin
Takustraße 7, D-14195 Berlin

Andreas Brandt
Manfred Brandt

**A Note on the Stability of the Many-Queue Head-of-the-Line
Processor-Sharing System with Permanent Customers**

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Andreas Brandt

*Wirtschaftswissenschaftliche Fakultät, Humboldt-Universität Berlin,
Spandauer Str. 1, D-10178 Berlin, Germany*

Manfred Brandt

*Konrad-Zuse-Zentrum für Informationstechnik Berlin (ZIB),
Takustr. 7, D-14195 Berlin, Germany*

Abstract

We consider a single server system consisting of n queues with different types of customers and k permanent customers. The permanent customers and those at the head of the queues are served in processor-sharing by the service facility (head-of-the-line processor-sharing). By means of Loynes' monotonicity method a stationary work load process is constructed and using sample path analysis general stability conditions are derived. They allow to decide which queues are stable and moreover to compute the fraction of processor capacity devoted to the permanent customers. In case of a stable system the constructed stationary state process is the only one and for any initial state the system converges pathwise to the steady state.

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1 Introduction

In this note we consider a single server system consisting of n queues and k permanent customers. At the system there arrives a stream of n types of customers. We assume that the input is given by a stationary ergodic marked point process $\Phi = \{[T_\ell, I_\ell, S_\ell]\}_{\ell=-\infty}^{\infty}$ on the real line with the mark space $\mathcal{K} = \{1, \dots, n\} \times \mathbb{R}_+$ and $\dots < T_0 \leq 0 < T_1 < \dots$, where T_ℓ are the arrival instants of the customers, $I_\ell \in \{1, \dots, n\}$ indicates the type, i.e. the queue where the customer goes to, and S_ℓ denotes the service time of the customer arriving at T_ℓ . Note, that there are no independence assumptions. The input at queue i , i.e. the stream of type i -customers including their required service times, is given by the stationary ergodic marked point process

$$\Phi_i = \sum_{\ell} \delta_{[T_\ell, S_\ell]} \mathbb{I}\{I_\ell = i\} = \{[T_{i,\ell}, S_{i,\ell}]\}_{\ell=-\infty}^{\infty}, \quad (1.1)$$

where $\dots < T_{i,0} \leq 0 < T_{i,1} < \dots$ and $\mathbb{I}\{\cdot\}$ denotes the indicator function.

The permanent customers and those at the head of the n queues are served in processor-sharing (PS) by the service facility. This means if there are b ($\in \{0, \dots, n\}$) types of customers present in the system (i.e. b queues are not empty) then the permanent customers and each of the b customers at the head of the queues get a fraction of $1/(b+k)$ of the processor capacity, cf. Fig. 1. Note, that the fraction of the processor capacity devoted to the permanent customers changes randomly.

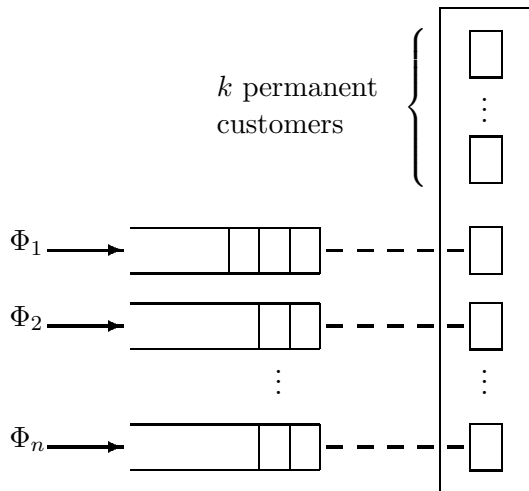


Figure 1: Many-queue processor-sharing system with k permanent customers and n queues of different customer types. \square corresponds to a customer.

The aim of the paper is to construct a stationary work load process by means of Loynes' monotonicity method and to derive general stability conditions for the separate queues and for the whole system. The stability conditions allow to decide which queues are stable and moreover to compute the fraction of processor capacity devoted to the permanent customers. Further it will be shown that for the stable system the constructed stationary state process is the only one and that under any initial state the system converges pathwise to the steady state. Concerning the extensive literature for processor-sharing systems we refer to the survey papers [Y3]–[Y5]. In [BB] the above model is analysed for Poisson arrival streams in detail: for independent identically distributed service times of the different types of customers a pseudo conservation law for the mean work load is given. In case of exponentially distributed service times a pseudo conservation law and a heuristic approximation for the mean sojourn times is derived. Further a numerical iterative algorithm for computing the steady state distribution is presented and implemented for $n=2, 3$. For the two-queue model the generating function of the stationary distribution satisfies a functional equation being a Riemann-Hilbert problem which is reduced to a Dirichlet problem for a circle. Numerical and simulation results are presented demonstrating the approximations and algorithms.

2 Construction of the stationary work load process by Loynes' monotonicity method

Let

- $\hat{V}_i(t)$ – be the work load at queue i immediately before t , i.e. $\hat{V}_i(t-) = \hat{V}_i(t)$;
- $\hat{V}(t) := (\hat{V}_1(t), \dots, \hat{V}_n(t))$ – work load vector immediately before t ;
- $\hat{V}_\ell = (\hat{V}_{\ell,1}, \dots, \hat{V}_{\ell,n}) := \hat{V}(T_\ell)$ – work load immediately before the arrival of the ℓ -th customer.

The dynamics of the work load process can be described by means of a measurable function as follows. Let $f(v, a) = (f^{(1)}(v, a), \dots, f^{(n)}(v, a))$ be the work load vector at time $t + a$ if the system starts at t with the work load vector $v \in \mathbb{R}_+^n$ and if during the interval $[t, t + a]$ there is no customer arrival.

In view of the processor-sharing discipline the service rate for queue i is $1/(b + k)$ if there is work load in queue i and b queues are non-empty. Since during the time interval $[t, t + a]$ queues may become empty, the service rate changes dynamically. Hence it is convenient to define $f(v, a)$ recursively up to the first time instant where the service rate changes: For $v \in \mathbb{R}_+^n$ and $a \in \mathbb{R}_+$ $f(v, a)$ is given by

$$f(v, a) = \begin{cases} \left(v - \frac{a}{b(v)+k} \mathbf{1}\right)_+ & \text{for } 0 \leq a \leq m(v)(b(v) + k), \\ f((v - m(v)\mathbf{1})_+, a - m(v)(b(v) + k)) & \text{for } 0 < m(v)(b(v) + k) < a, \\ 0 & \text{for } 0 = m(v) < a, \end{cases} \quad (2.1)$$

where $\mathbf{1} = (1, \dots, 1)$, $b(v) = \#\{i : v_i > 0\}$, $m(v) = \mathbb{I}\{b(v) > 0\} \min\{v_i : v_i > 0, i = 1, \dots, n\}$ and $x_+ = (\max(x_1, 0), \dots, \max(x_n, 0))$. We also have

$$f(v, a) = f(f(v, s), a - s), \quad 0 \leq s \leq a, \quad v \in \mathbb{R}_+^n. \quad (2.2)$$

Without loss of generality let $v_1 \geq v_2 \geq \dots \geq v_b > v_{b+1} = \dots = v_n = 0$ with $b = b(v)$. From (2.1) we get immediately the explicit representation

$$f(v, a) = \sum_{i=0}^b \mathbb{I}\{v_{b+1} + v_b + \dots + v_{b-i+2} + (b + k - i + 1)v_{b-i+1} \leq a \\ \leq v_{b+1} + v_b + \dots + v_{b-i+1} + (b + k - i)v_{b-i}\} \\ \left(v - \left(v_{b-i+1} + \frac{a - v_{b+1} - v_b - \dots - v_{b-i+2} - (b + k - i + 1)v_{b-i+1}}{k + b - i}\right) \mathbf{1}\right)_+,$$

where $v_{n+1} := 0$ and $v_0 := \infty$.

If \hat{V}_ℓ is a feasible sequence of states for the input Φ at the times T_ℓ , then

$$\hat{V}_{\ell+1} = f(\hat{V}_\ell + S_\ell e(I_\ell), T_{\ell+1} - T_\ell), \quad \ell \in \mathbb{Z}, \quad (2.3)$$

where

$$e(I_\ell) = (\delta_{I_\ell, 1}, \dots, \delta_{I_\ell, n}) \quad (2.4)$$

is the I_ℓ -th unit vector and $\delta_{i,j} = \mathbb{I}\{i = j\}$ the Kronecker symbol. For $T_\ell < t \leq T_{\ell+1}$ it holds

$$\hat{V}(t) = f(\hat{V}_\ell + S_\ell e(I_\ell), t - T_\ell) \quad (2.5)$$

and for $T_\ell < s \leq t \leq T_{\ell+1}$

$$\hat{V}(t) = f(\hat{V}(s), t - s). \quad (2.6)$$

Now we will construct a stationary work load process by using Loynes' monotonicity method, cf. e.g. [Loy], [BFL]. The following lemma is the basis; its proof is immediate.

Lemma 2.1. For $a \in \mathbb{R}_+$ fixed, the function $f(v, a)$ is monotone non-decreasing and continuous in $v \in \mathbb{R}_+^n$.

For giving the backward construction, iterations of the dynamics are needed. Let

$$f_1(v, a, i, s) := f(v + se(i), a), \quad v \in \mathbb{R}_+^n, \quad a, s \in \mathbb{R}_+, \quad i \in \{1, \dots, n\}, \quad (2.7)$$

where $e(i) = (\delta_{i,1}, \dots, \delta_{i,n})$ is the i -th unit vector. The j -fold iteration of f_1 is defined by

$$\begin{aligned} f_j(v, a_1, i_1, s_1, a_2, i_2, s_2, \dots, a_j, i_j, s_j) \\ := f_1(f_{j-1}(v, a_1, i_1, s_1, \dots, a_{j-1}, i_{j-1}, s_{j-1}), a_j, i_j, s_j) \end{aligned} \quad (2.8)$$

for $j \in \{2, 3, \dots\}$, $v \in \mathbb{R}_+^n$, $a_1, \dots, a_j, s_1, \dots, s_j \in \mathbb{R}_+$ and $i_1, \dots, i_j \in \{1, \dots, n\}$.

Further let

$$\begin{aligned} M_K = \{ \varphi = \{ [t_\ell, i_\ell, s_\ell] \}_{\ell=-\infty}^\infty : \dots < t_0 \leq 0 < t_1 < \dots, \quad i_\ell \in \{1, \dots, n\}, \\ s_\ell \in \mathbb{R}_+, \quad \lim_{\ell \rightarrow \pm\infty} t_\ell = \pm\infty \} \end{aligned}$$

be the set of point process realizations, where Φ is concentrated on, i.e. $P(M_K) = 1$. For $\varphi \in M_K$ let $a_\ell := t_{\ell+1} - t_\ell$, $\ell^-(t) := \max\{\ell : t_\ell < t\}$. Let $v^{(\tau)}(t, \varphi) = (v_1^{(\tau)}(t, \varphi), \dots, v_n^{(\tau)}(t, \varphi))$ be the state of the system at time t if it was started at time $t - \tau$ with initial state $v = 0$, input realization $\varphi \in M_K$ and $\tau > 0$. Then $v^{(\tau)}(t, \varphi)$ is given by

$$\begin{aligned} v^{(\tau)}(t, \varphi) = \mathbb{I}\{\ell^-(t-\tau) < \ell^-(t)\} f_{\ell^-(t)-\ell^-(t-\tau)}(0, a_{\ell^-(t-\tau)+1}, i_{\ell^-(t-\tau)+1}, s_{\ell^-(t-\tau)+1}, \dots, \\ a_{\ell^-(t)-1}, i_{\ell^-(t)-1}, s_{\ell^-(t)-1}, t - t_{\ell^-(t)}, i_{\ell^-(t)}, s_{\ell^-(t)}), \\ \tau > 0, \quad t \in \mathbb{R}, \quad \varphi \in M_K. \end{aligned} \quad (2.9)$$

In view of Lemma 2.1 for fixed t and φ the work load vector $v^{(\tau)}(t, \varphi)$ is non-decreasing and hence the limit as $\tau \rightarrow \infty$ exists:

$$v(t, \varphi) := \lim_{\tau \rightarrow \infty} v^{(\tau)}(t, \varphi). \quad (2.10)$$

For $t = t_\ell$ let

$$v_\ell(\varphi) := v(t_\ell, \varphi). \quad (2.11)$$

The state $v(t, \varphi)$ resp. $v_\ell(\varphi)$ corresponds to the system state at t resp. t_ℓ if the processor-sharing system was started at $t = -\infty$ with the initial state 0. Clearly, components of $v(t, \varphi)$ may be infinite. The continuity of $f(\cdot, a)$ and $\lim_{n \rightarrow \pm\infty} t_n = \pm\infty$ imply that $v(t, \varphi)$, $v_\ell(\varphi)$ satisfy the system dynamics (cf. (2.3)–(2.6)):

$$v_{\ell+1}(\varphi) = f(v_\ell(\varphi) + s_\ell e(i_\ell), a_\ell) \quad \text{for } \ell \in \mathbb{Z}, \quad (2.12)$$

$$v(t, \varphi) = f(v_\ell(\varphi) + s_\ell e(i_\ell), t - t_\ell) \quad \text{for } t_\ell < t \leq t_{\ell+1}, \quad (2.13)$$

$$v(t, \varphi) = f(v(s, \varphi), t - s) \quad \text{for } t_\ell < s \leq t \leq t_{\ell+1}. \quad (2.14)$$

For $\varphi = \{[t_\ell, i_\ell, s_\ell]\}_{\ell=-\infty}^{\infty} \in M_K$ ($\dots < t_0 \leq 0 < t_1 < \dots$) the shifted sequence $\theta_\tau \varphi$ is defined by $\theta_\tau \varphi = \{[t_{\ell+\ell(\tau, \varphi)} - \tau, i_{\ell+\ell(\tau, \varphi)}, s_{\ell+\ell(\tau, \varphi)}]\}_{\ell=-\infty}^{\infty}$, where $\ell(\tau, \varphi) = \max\{\ell : t_\ell \leq \tau\}$. (Considering M_K as counting measures, i.e. $\varphi = \sum_{\ell} \delta_{[t_\ell, i_\ell, s_\ell]}$, then the shifted measure $\theta_\tau \varphi$ is defined by $\theta_\tau \varphi = \sum_{\ell} \delta_{[t_{\ell-\tau}, i_{\ell-\tau}, s_{\ell-\tau}]}$.) Then, by construction (cf. (2.9)) we have

$$v^{(\tau)}(t, \varphi) = v^{(\tau)}(0, \theta_t \varphi), \quad t \in \mathbb{R}, \quad \tau > 0, \quad \varphi \in M_K \quad (2.15)$$

and

$$v(t, \varphi) = v(0, \theta_t \varphi), \quad t \in \mathbb{R}, \quad \varphi \in M_K. \quad (2.16)$$

From now on let

$$\hat{V}(t) := v(t, \Phi), \quad t \in \mathbb{R} \quad \text{and} \quad \hat{V}_\ell := v_\ell(\Phi), \quad \ell \in \mathbb{Z}. \quad (2.17)$$

Then $\hat{V}(t)$, $t \in \mathbb{R}$ is a stationary and ergodic process, which follows by the "shift" invariance construction (2.16) and the measurability properties of $v(\cdot, \cdot)$, note that $v(\cdot, \varphi)$ is left continuous for fixed $\varphi \in M_K$, for details see [BFL].

3 Stability condition

Definition 3.1 Queue $i \in \{1, \dots, n\}$ will be called *stable* if $P(\hat{V}_i(0) = 0) > 0$. The processor-sharing system will be called *stable* if $P(\hat{V}_i(0) = 0) > 0$ for $i = 1, \dots, n$.

Corollary 3.2 Assume that queue i is stable and consider the point process $\Phi_i^* = \{T_{i,\ell}^*\}_{\ell=-\infty}^{\infty}$ of the time instants where the i -th queue becomes empty for a positive duration of time:

$$\begin{aligned} T_{i,0}^* &= \sup\{t \leq 0 : \hat{V}_i(t-0) > 0, \hat{V}_i(t+0) = 0\}, \\ T_{i,\ell-1}^* &= \sup\{t < T_{i,\ell}^* : \hat{V}_i(t-0) > 0, \hat{V}_i(t+0) = 0\}, \quad \ell = 0, -1, -2, \dots, \\ T_{i,\ell+1}^* &= \inf\{t > T_{i,\ell}^* : \hat{V}_i(t-0) > 0, \hat{V}_i(t+0) = 0\}, \quad \ell = 0, 1, 2, \dots \end{aligned}$$

From the stationarity, ergodicity and construction of $\hat{V}_i(t)$ and from the stability condition $P(\hat{V}_i(0) = 0) > 0$ we conclude that Φ_i^* is a stationary ergodic point process, in particular that $\lim_{\ell \rightarrow \pm\infty} T_{i,\ell}^* = \pm\infty$ a.s. Since during the interval $[T_{i,\ell}^*, T_{i,\ell+1}^*]$ only a finite amount of work arrives at queue i , we conclude

$$P(\hat{V}_i(s) < \infty, s \in \mathbb{R}) = 1, \quad (3.1)$$

in particular

$$P(\hat{V}_i(t) < \infty) = 1, \quad t \in \mathbb{R}. \quad (3.2)$$

From (3.1) finally for a stable system we have

$$P(\hat{V}(s) < \infty, s \in \mathbb{R}) = P(\hat{V}(t) < \infty) = 1, \quad t \in \mathbb{R}, \quad (3.3)$$

where $\hat{V}(t) < \infty$ means $\hat{V}_i(t) < \infty$ for $i = 1, \dots, n$.

Remark 3.3. Another definition of stability for a system which often is used is condition (3.3). However, in general (3.3) does not imply stability of a system in sense of our Definition 3.1, which can be seen by the following example: Let $n = 1$, $k = 1$, $S_\ell \equiv 1/2$, T_1 be uniformly distributed on $[0, 1]$ and $T_{\ell+1} - T_\ell \equiv 1$, $\ell \in \mathbb{Z}$. (Clearly, $I_\ell \equiv 1$ in view of $n = 1$). Then $\{[T_\ell, S_\ell]\}_{\ell=-\infty}^\infty$ is a stationary and ergodic input. Loynes' backward construction (2.10), (2.11) yields for $t_\ell < t \leq t_{\ell+1}$ that $v(t, \varphi) = \frac{1-(t-t_\ell)}{2}$ and $v_\ell(\varphi) = 0$. But this implies $P(\hat{V}(t) > 0) = 1$, i.e. this queue is not stable in our sense.

Remark 3.4. The stability of the system (in the sense of our Definition 3.1) does not imply in general that the system has empty points (idle periods), i.e.

$$P(\hat{V}_i(0) = 0, i = 1, \dots, n) > 0. \quad (3.4)$$

This phenomenon is well known from the many server $G/G/m/\infty$ system. The following example illustrates this for our processor-sharing model. Let $k = 1$, $n = 2$, T_1 be uniformly distributed on $[0, 1]$, $T_{\ell+1} - T_\ell = 1$ and

$$\begin{aligned} P(S_{2j} = 2/3, S_{2j+1} = 1/2, I_{2j} = 1, I_{2j+1} = 2, j \in \mathbb{Z}) = \\ P(S_{2j+1} = 2/3, S_{2j} = 1/2, I_{2j+1} = 1, I_{2j} = 2, j \in \mathbb{Z}) = 1/2. \end{aligned}$$

Then $\Phi = \{[T_\ell, I_\ell, S_\ell]\}_{\ell=-\infty}^\infty$ is a stationary ergodic marked point process and the states $v_\ell(\Phi)$ at the arrival instants T_ℓ are given by

$$v_\ell(\Phi) = \begin{cases} (2/9, 0) & \text{if } I_\ell = 2, S_\ell = 1/2, \\ (0, 1/9) & \text{if } I_\ell = 1, S_\ell = 2/3. \end{cases}$$

An inspection of the dynamics between customer arrivals shows that both queues never become empty together, but each queue has idle periods, i.e. the system is stable (in the sense of Definition 3.1) but it has no empty points.

For stating the stability results we need some notations and considerations. The traffic intensity ϱ_i of the type i -customers (given by the point process Φ_i , cf. (1.1)) is

$$\varrho_i = E \sum_{\ell=1}^{\infty} S_{i,\ell} \mathbb{I}\{0 < T_{i,\ell} \leq 1\} = E \int_{(0,1] \times \mathbb{R}_+} s \Phi_i(d(t, s)) = \lambda_i m_{B_i^0},$$

where $\lambda_i = E \Phi_i((0, 1] \times \mathbb{R}_+)$ is the intensity of Φ_i and $m_{B_i^0} = ES_i^0$ is the expectation of the service time S_i^0 of a typical type i -customer, which is given by the Palm distribution of $P(\Phi_i \in (\cdot))$, cf. e.g. [FKAS] formula (1.2.8). The total arrival intensity λ and the total traffic intensity $\bar{\varrho}$ are given by $\lambda := E \Phi((0, 1] \times \{1, \dots, n\} \times \mathbb{R}_+) = \lambda_1 + \dots + \lambda_n$ resp.

$$\bar{\varrho} := E \sum_{\ell=1}^{\infty} S_\ell \mathbb{I}\{0 < T_\ell \leq 1\} = E \int_{(0,1] \times \{1, \dots, n\} \times \mathbb{R}_+} s \Phi(d(t, i, s)) = \lambda m_{B^0},$$

where $m_{B^0} = ES^0$ is the expectation of the service time of a typical customer. Clearly, it holds $\bar{\varrho} = \varrho_1 + \dots + \varrho_n$.

Let $\hat{V}(t)$ be given by (2.17). Then at time t queue i gets a random fraction $C_i(t)$ of the processor capacity, which is given by

$$C_i(t) = \frac{\mathbb{I}\{\hat{V}_i(t) > 0\}}{b(\hat{V}(t)) + k}, \quad i = 1, \dots, n. \quad (3.5)$$

The k permanent customers get the fractions

$$C_i(t) = \frac{1}{b(\hat{V}(t)) + k}, \quad i = n + 1, \dots, n + k, \quad (3.6)$$

where the index i corresponds to the $(i-n)$ -th permanent customer. The mean fractions of the processor capacity are given by

$$EC_i(0) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t C_i(\tau) d\tau, \quad i = 1, \dots, n + k \quad P - a.s. \quad (3.7)$$

The mean fractions of the permanent customers are the same:

$$\varrho^* := EC_i(0), \quad i = n + 1, \dots, n + k, \quad (3.8)$$

cf. (3.6). In view of (3.5)–(3.8) we further conclude that

$$EC_i(0) \leq \varrho^*, \quad i = 1, \dots, n \quad (3.9)$$

and

$$EC_i(0) = \varrho^* \quad \text{iff} \quad P(\hat{V}_i(0) > 0) = 1. \quad (3.10)$$

Let

$$p_i := P(\hat{V}_i(0) > 0) = E\mathbb{I}\{\hat{V}_i(0) > 0\} = E\mathbb{I}\{C_i(0) > 0\}, \quad i = 1, \dots, n \quad (3.11)$$

be the probability that at time $t = 0$ a type i -customer will be served. By Definition 3.1 queue i is stable iff $p_i < 1$.

Theorem 3.5. Let $\hat{V}(t)$ be given by (2.10), (2.17). Then

$$(i) \quad p_i \leq k\varrho_i + \sum_{j=1}^n \min(\varrho_j, \varrho_i), \quad i = 1, \dots, n. \quad (3.12)$$

(ii) Queue $i \in \{1, \dots, n\}$ is stable, i.e. $p_i < 1$, iff $\varrho_i < \varrho^*$.

$$(iii) \quad k\varrho^* + \sum_{j=1}^n \min(\varrho_j, \varrho^*) = 1. \quad (3.13)$$

(iv) The processor-sharing system is stable iff

$$\bar{\varrho} + k\varrho_{\max} < 1, \quad (3.14)$$

where $\varrho_{\max} = \max\{\varrho_1, \dots, \varrho_n\}$.

Proof. (i) We start with Loynes' construction (2.9), (2.10). Let $\tau > 0$ and $T > 0$ be fixed. Starting the dynamics of the processor-sharing system at time $-\tau$ with the work load vector $v = 0$ (empty system) then

$$w_i(\tau, T) := v_i^{(\tau)}(0, \Phi) - v_i^{(\tau+T)}(T, \Phi) + \sum_{\ell=1}^{\infty} S_{i,\ell} \mathbb{I}\{0 < T_{i,\ell} \leq T\}, \quad i = 1, \dots, n \quad (3.15)$$

is just the amount of service that receive the type i -customers during the interval $[0, T]$ by the processor. By an appropriate numbering we may assume without loss of generality that

$$w_1(\tau, T) \leq w_2(\tau, T) \leq \dots \leq w_n(\tau, T). \quad (3.16)$$

Since queue 1 receives at least the fraction $1/(n+k)$ of the processor capacity if it is non-empty, during the interval $[0, T]$ there is at most the duration of

$$D_1 = (n+k)w_1(\tau, T)$$

necessary for serving the work load $w_1(\tau, T)$. If queue 2 is non-empty it receives at least the fraction $1/(n+k)$ of the capacity and if no type 1-customers are present in the system then even at least the fraction $1/(n+k-1)$ of the processor capacity. Thus during $[0, T]$ there is at most the duration of

$$\begin{aligned} D_2 &= D_1 + (n+k-1)(w_2(\tau, T) - w_1(\tau, T)) \\ &= (n+k-1)w_2(\tau, T) + w_1(\tau, T) \end{aligned}$$

necessary for providing queue 2 an amount of service of $w_2(\tau, T)$. By induction we conclude that during $[0, T]$ there is at most a duration of

$$\begin{aligned} D_i &= (n+k+1-i)w_i(\tau, T) + \sum_{j=1}^{i-1} w_j(\tau, T) \\ &= kw_i(\tau, T) + \sum_{j=1}^n \min(w_j(\tau, T), w_i(\tau, T)), \quad i = 1, \dots, n \end{aligned} \quad (3.17)$$

necessary for providing queue i an amount of service of $w_i(\tau, T)$. The representation (3.17) of D_i allows to drop the assumption (3.16). Thus we get

$$\begin{aligned} &\int_0^T \mathbb{I}\{v_i^{(\tau+s)}(s, \Phi) > 0\} ds \\ &\leq D_i = kw_i(\tau, T) + \sum_{j=1}^n \min(w_j(\tau, T), w_i(\tau, T)), \quad i = 1, \dots, n. \end{aligned} \quad (3.18)$$

From $v_i^{(\tau)}(0, \Phi) > T/(k+1)$ it follows $v_i^{(\tau+s)}(s, \Phi) > 0$ for $s \in [0, T]$ since queue i receives at most the fraction $1/(k+1)$ of the processor capacity if type i -customers are present. Thus from (3.15) and (3.18) we conclude for $i = 1, \dots, n$

$$\begin{aligned}
\mathbb{I} \left\{ v_i^{(\tau)}(0, \Phi) > T/(k+1) \right\} T &\leq \int_0^T \mathbb{I} \left\{ v_i^{(\tau+s)}(s, \Phi) > 0 \right\} ds \\
&\leq kw_i(\tau, T) + \sum_{j=1}^n \min(w_j(\tau, T), w_i(\tau, T)) \\
&= k(v_i^{(\tau)}(0, \Phi) - v_i^{(\tau+T)}(T, \Phi) + \sum_{\ell=1}^{\infty} S_{i,\ell} \mathbb{I} \{0 < T_{i,\ell} \leq T\}) \\
&\quad + \sum_{j=1}^n \min \left(v_j^{(\tau)}(0, \Phi) - v_j^{(\tau+T)}(T, \Phi) + \sum_{\ell=1}^{\infty} S_{j,\ell} \mathbb{I} \{0 < T_{j,\ell} \leq T\}, \right. \\
&\quad \left. v_i^{(\tau)}(0, \Phi) - v_i^{(\tau+T)}(T, \Phi) + \sum_{\ell=1}^{\infty} S_{i,\ell} \mathbb{I} \{0 < T_{i,\ell} \leq T\} \right).
\end{aligned}$$

Taking expectations and using the fact that $E \min(X, Y) \leq \min(EX, EY)$ for two r.v.'s X and Y we get for $i = 1, \dots, n$

$$\begin{aligned}
E \mathbb{I} \left\{ v_i^{(\tau)}(0, \Phi) > T/(k+1) \right\} T &\leq k \left(E v_i^{(\tau)}(0, \Phi) - E v_i^{(\tau+T)}(T, \Phi) + \varrho_i T \right) \\
&\quad + \sum_{j=1}^n \min \left(E v_j^{(\tau)}(0, \Phi) - E v_j^{(\tau+T)}(T, \Phi) + \varrho_j T, \right. \\
&\quad \left. E v_i^{(\tau)}(0, \Phi) - E v_i^{(\tau+T)}(T, \Phi) + \varrho_i T \right). \tag{3.19}
\end{aligned}$$

By (2.15), the stationarity of Φ and since $v_i^{(\tau)}(0, \Phi)$ is non-decreasing with respect to τ it holds

$$E v_i^{(\tau+T)}(T, \Phi) = E v_i^{(\tau+T)}(0, \Phi) \geq E v_i^{(\tau)}(0, \Phi), \quad i = 1, \dots, n.$$

Thus (3.19) yields

$$E \mathbb{I} \left\{ v_i^{(\tau)}(0, \Phi) > T/(k+1) \right\} \leq k \varrho_i + \sum_{j=1}^n \min(\varrho_j, \varrho_i), \quad i = 1, \dots, n.$$

Taking now the limit as $T \rightarrow 0+0$ we get

$$E \mathbb{I} \{ v_i^{(\tau)}(0, \Phi) > 0 \} \leq k \varrho_i + \sum_{j=1}^n \min(\varrho_j, \varrho_i), \quad i = 1, \dots, n.$$

In view of the monotonicity of $v_i^{(\tau)}(0, \Phi)$ with respect to τ we can take the limit as $\tau \rightarrow \infty$ and obtain the assertion (i):

$$p_i = E \mathbb{I} \{ v_i(0, \Phi) > 0 \} \leq k \varrho_i + \sum_{j=1}^n \min(\varrho_j, \varrho_i), \quad i = 1, \dots, n.$$

(ii) Remember, the mean fraction ϱ^* of the processor capacity received by each permanent customer is given by (cf. (3.6), (3.7), (3.8)):

$$\varrho^* = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{1}{b(\hat{V}(\tau) + k)} d\tau. \quad (3.20)$$

If $p_i < 1$ then we conclude by the stationarity and ergodicity of $\hat{V}(t) = v(t, \Phi)$ and in view of $\lim_{\ell \rightarrow \pm\infty} T_{i,\ell} = \pm\infty$ that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{I}\{\hat{V}_i(\tau) = 0\} d\tau = E\mathbb{I}\{\hat{V}_i(0) = 0\} = 1 - p_i > 0 \quad P - a.s.,$$

i.e. queue i possesses infinitely many idle periods. Thus, in view of the system dynamics all customers arriving at queue i will be served. Since $\int_0^t C_i(\tau) d\tau$ is the amount of service received by queue i during the interval $[0, t]$ (cf. (3.5)) we conclude

$$\varrho_i = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{(0,t] \times \mathbb{R}_+} s \Phi_i(d(\tau, s)) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t C_i(\tau) d\tau = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{\mathbb{I}\{\hat{V}_i(\tau) > 0\}}{b(\hat{V}(\tau) + k)} d\tau, \quad (3.21)$$

i.e. ϱ_i is just the mean fraction of the processor capacity received by queue i provided $p_i < 1$.

From (3.10), (3.11), (3.20) and (3.21) we get

$$\begin{aligned} k\varrho^* + \sum_{j=1}^n \left(\mathbb{I}\{p_j < 1\} \varrho_j + \mathbb{I}\{p_j = 1\} \varrho^* \right) \\ = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left(k + \sum_{j=1}^n \left(\mathbb{I}\{p_j < 1\} \mathbb{I}\{\hat{V}_j(\tau) > 0\} + \mathbb{I}\{p_j = 1\} \right) \right) / \left(b(\hat{V}(\tau)) + k \right) d\tau \\ = 1. \end{aligned} \quad (3.22)$$

Assume now that $\varrho_i < \varrho^*$. From (i) and (3.22) we find

$$\begin{aligned} p_i &\leq k\varrho_i + \sum_{j=1}^n \min(\varrho_j, \varrho_i) < k\varrho^* + \sum_{j=1}^n \min(\varrho_j, \varrho^*) \\ &\leq k\varrho^* + \sum_{j=1}^n (\mathbb{I}\{p_j < 1\} \varrho_j + \mathbb{I}\{p_j = 1\} \varrho^*) = 1, \end{aligned}$$

i.e. $\varrho_i < \varrho^*$ implies $p_i < 1$. Conversely, if $p_i < 1$, then we obtain from (3.20) and (3.21) that

$$\varrho^* - \varrho_i = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \frac{\mathbb{I}\{\hat{V}_i(\tau) = 0\}}{b(\hat{V}(\tau)) + k} d\tau$$

providing the following boundaries for $1 - p_i$:

$$k(\varrho^* - \varrho_i) \leq 1 - p_i \leq (n + k - 1)(\varrho^* - \varrho_i). \quad (3.23)$$

In particular we get that $p_i < 1$ implies $\varrho_i < \varrho^*$. Thus we have shown that $p_i < 1$ iff $\varrho_i < \varrho^*$.

(iii) Using (ii), equation (3.22) reduces to (3.13).

(iv) From Definition 3.1 and assertion (ii) we have that the processor-sharing system is stable iff $\varrho_i < \varrho^*$ for $i = 1, \dots, n$. But in view of (3.13) this is equivalent to (3.14).

Remark 3.6. Since the left-hand side of (3.13) is strictly monotone increasing with respect to ϱ^* , it follows that ϱ^* is uniquely determined by (3.13) and can be computed numerically e.g. by an intersection procedure. The inequality (3.23) implies the estimate

$$k(\varrho^* - \varrho_i)_+ \leq 1 - p_i \leq (n + k - 1)(\varrho^* - \varrho_i)_+, \quad i = 1, \dots, n \quad (3.24)$$

and hence a lower and upper bound for $1 - p_i$ can be computed numerically. If some queues are instable then these types of customers may be considered as permanent customers, too. Hence it is sufficient to investigate stable processor-sharing systems.

Remark 3.7. The theorem justifies that it is useful to define the stability of the system accordingly to Definition 3.1; the property $P(\hat{V}(t) < \infty, t \in \mathbb{R}) = 1$ is not equivalent to the stability of the system in sense of Definition 3.1 as the example given in Remark 3.3 shows: There it holds $P(\hat{V}(t) < \infty) = 1$ but $\varrho_1 + 1\varrho_1 = 1$, i.e. the stability condition (3.14) fails.

The stationary ergodic state process $\hat{V}(t) = v(t, \Phi)$ constructed via Loynes' monotonicity method (2.10) is minimal, namely if $\bar{V}(t)$ is a state process – not necessarily stationary – satisfying the dynamics (2.3)–(2.6) for the input Φ , then $\hat{V}(t) \leq \bar{V}(t)$, $t \in \mathbb{R}$ P -a.s. This follows by the monotonicity of $f(\cdot, a)$, the construction (2.10), (2.17) and the fact that 0 is the minimal element in the state space \mathbb{R}_+^n ; for details we refer to [BFL], Sec. 1.3. Therefore in the following we refer to $\hat{V}(t)$ also as the *minimal state process*. What can be said about the uniqueness of the state process if $\hat{V}(t) < \infty$? Clearly, in the class of all state processes there is no hope for giving conditions ensuring uniqueness, because a feasible state process can evolve from left from "infinity". Thus it is only reasonable to look for uniqueness in the class of all stationary state processes (possibly) "belonging" to Φ . For the example given in Remark 3.3 the state process is not unique. Namely, it is easy to check that for any $c > 0$

$$\bar{v}^c(t, \varphi) := c + \frac{1 - (t - t_\ell)}{2}, \quad t_\ell < t \leq t_{\ell+1}, \quad v_\ell^c(\varphi) := c$$

defines a stationary state process $\bar{v}^c(t, \Phi)$ "belonging" to Φ . However, if the stability condition (3.14) is satisfied, then each queue has idle periods and as a consequence the minimal state process $\hat{V}(t)$ is the only stationary state process and for any initial state the system evolves toward $\hat{V}(t)$. Before stating and proving the corresponding results we need some further notations. Denote by $v(u, t, \varphi) = (v_1(u, t, \varphi), \dots, v_n(u, t, \varphi))$ the state of the system at time $t > 0$ if it was started at time zero with state u and input realization φ . Taking into account the system dynamics it holds

$$v(u, t, \varphi) = v(v(u, \tau, \varphi), t - \tau, \theta_\tau \varphi), \quad 0 \leq \tau \leq t, \quad u \in \mathbb{R}_+^n, \quad \varphi \in M_K. \quad (3.25)$$

From Lemma 2.1 and taking into account the system dynamics again it follows that for $t \in \mathbb{R}_+$, $\varphi \in M_K$

$$v(u, t, \varphi) \quad \text{is monotone non-decreasing in } u. \quad (3.26)$$

Further, for $u \in \mathbb{R}_+^n$ let $I_u := \{i \in \{1, \dots, n\} : u_i > 0\}$ be the set indicating the positive components of the vector $u \in \mathbb{R}_+^n$ and $I(u, t, \varphi) := I_{v(u, t, \varphi)}$ the corresponding set at time t with initial state u and input φ . For the minimal state process $v(t, \varphi) = (v_1(t, \varphi), \dots, v_n(t, \varphi))$ given by (2.10) analogously we define $I(t, \varphi) := I_{v(t, \varphi)}$. In accordance to Corollary 3.2, for $\varphi \in M_K$ let $t_{i, \ell}^* (\dots < t_{i, 0}^* \leq 0 < t_{i, 1}^* < \dots)$ be the time instants where the i -th queue becomes idle with respect to the minimal state process $v(t, \varphi)$ for a positive duration of time denoted by $d_{i, \ell}$, i.e. $v_i(t, \varphi) = 0$ iff $t_{i, \ell}^* \leq t \leq t_{i, \ell}^* + d_{i, \ell}$ for some ℓ . Denoting by $|A|$ the number of elements of a set A we recall that $b(v) = |I_v|$.

Lemma 3.8.

(i) The function $f(v, a)$ is semi contractive in v with respect to the l_1 -norm in \mathbb{R}_+^n , i.e.

$$\|f(v, a) - f(w, a)\|_1 \leq \|v - w\|_1, \quad v, w \in \mathbb{R}_+^n, \quad a \in \mathbb{R}_+. \quad (3.27)$$

(iia) If $I_v \neq I_w$ for $v, w \in \mathbb{R}_+^n$, then

$$\|f(v, a) - f(w, a)\|_1 \leq \|v - w\|_1 - \frac{k}{(k+n)^2} \min(a^*, a), \quad (3.28)$$

where

$$a^* = \begin{cases} \min(m(v)(b(v) + k), m(w)(b(w) + k)) & \text{for } b(v)b(w) > 0, \\ \max(m(v)(b(v) + k), m(w)(b(w) + k)) & \text{for } b(v)b(w) = 0. \end{cases} \quad (3.29)$$

(iib) If for $v, w \in \mathbb{R}_+^n$, $a \in \mathbb{R}_+$ it holds $I_{f(v, s)} \neq I_{f(w, s)}$ for $0 \leq s \leq a$, then

$$\|f(v, a) - f(w, a)\|_1 \leq \|v - w\|_1 - \frac{k}{(k+n)^2} a.$$

$$(iii) \quad \|v(u, t, \varphi) - v(w, t, \varphi)\|_1 \leq \|u - w\|_1, \quad u, w \in \mathbb{R}_+^n, \quad t \in \mathbb{R}_+, \quad \varphi \in M_K. \quad (3.30)$$

(iv) $\|v(u, t, \varphi) - v(w, t, \varphi)\|_1$ is non-increasing in $t \in \mathbb{R}_+$ for $u, w \in \mathbb{R}_+^n$, $\varphi \in M_K$.

(v) If $I(u, s, \varphi) \neq I(w, s, \varphi)$ for $s \in [0, t]$, then

$$\|v(u, t, \varphi) - v(w, t, \varphi)\|_1 \leq \|u - w\|_1 - \frac{k}{(k+n)^2} t. \quad (3.31)$$

(vi) If $u \leq w$ and $v_i(u, s, \varphi) > 0$ for $s \in [0, t]$ for some i , then

$$v_i(w, t, \varphi) - v_i(u, t, \varphi) \geq w_i - u_i. \quad (3.32)$$

(vii) If $I(u, s, \varphi) = I(w, s, \varphi)$ for $s \in [0, t]$, then

$$v(u, s, \varphi) - v(w, s, \varphi) = u - w \quad \text{for } s \in [0, t]. \quad (3.33)$$

Proof. (i),(ia) Let $v, w \in \mathbb{R}_+^n$. Without loss of generality we assume for $b_v := b(v)$ and $b_w := b(w)$ that

$$b_v \geq b_w. \quad (3.34)$$

Let a^* be defined as in statement (ia). Assume that $a^* > 0$, i.e. $I_v \cup I_w \neq \emptyset$ and $b_v + b_w \geq 1$. In the following let $0 \leq a \leq a^*$. Then the components of the work load vectors $f(v, a)$ and $f(w, a)$ are given by

$$f^{(i)}(v, a) = \begin{cases} v_i - \frac{a}{k+b_v} & \text{for } i \in I_v, \\ 0 & \text{for } i \in \bar{I}_v \end{cases}, \quad f^{(i)}(w, a) = \begin{cases} w_i - \frac{a}{k+b_w} & \text{for } i \in I_w, \\ 0 & \text{for } i \in \bar{I}_w \end{cases}$$

for $i = 1, \dots, n$, where $\bar{I}_v = \{1, \dots, n\} \setminus I_v$ and $\bar{I}_w = \{1, \dots, n\} \setminus I_w$. Using this explicit expressions and taking into account (3.34) we get

$$\begin{aligned} & \|f(v, a) - f(w, a)\|_1 \\ &= \sum_{i \in I_v \cap I_w} \left| v_i - \frac{a}{k+b_v} - w_i + \frac{a}{k+b_w} \right| + \sum_{i \in I_v \cap \bar{I}_w} \left(v_i - \frac{a}{k+b_v} \right) + \sum_{i \in \bar{I}_v \cap I_w} \left(w_i - \frac{a}{k+b_w} \right) \\ &\leq \|v - w\|_1 + a \sum_{i \in I_v \cap I_w} \left(\frac{1}{k+b_w} - \frac{1}{k+b_v} \right) - a \sum_{i \in I_v \cap \bar{I}_w} \frac{1}{k+b_v} - a \sum_{i \in \bar{I}_v \cap I_w} \frac{1}{k+b_w} \\ &= \|v - w\|_1 + \frac{a}{(k+b_v)(k+b_w)} ((k+b_v)|I_v \cap I_w| - (k+b_w)b_v - (k+b_v)|\bar{I}_v \cap I_w|) \\ &\leq \|v - w\|_1 + \frac{a}{(k+b_v)(k+b_w)} ((k+b_v)b_w - (k+b_w)b_v - (k+b_v)|\bar{I}_v \cap I_w|) \\ &= \|v - w\|_1 - \frac{a}{(k+b_v)(k+b_w)} (k(b_v - b_w) + (k+b_v)|\bar{I}_v \cap I_w|). \end{aligned} \quad (3.35)$$

Thus we have (3.27) for $0 \leq a \leq a^*$. Since $I_v \neq I_w$ and $b_v = b_w$ implies $\bar{I}_v \cap I_w \neq \emptyset$ we get for $I_v \neq I_w$ the inequality $k(b_v - b_w) + (k+b_v)|\bar{I}_v \cap I_w| \geq k$, and hence (3.35) implies (3.28) for $0 \leq a \leq a^*$. Looking at the derivation of (3.35) and taking into account the considerations above we conclude that equality holds in (3.27) iff $I_v = I_w$, i.e. for $0 < a \leq a^*$ we have

$$\|f(v, a) - f(w, a)\|_1 = \|v - w\|_1 \quad \text{iff } I_v = I_w. \quad (3.36)$$

For $a > a^*$ we prove (3.27) and (3.28) by iterating the above result. Let $a^{(1)} := a^*$, $v^{(1)} := v$, $w^{(1)} := w$. Using the recursion (2.2) we get for $a > a^{(1)}$

$$f(v^{(1)}, a) = f(v^{(2)}, a - a^{(1)}), \quad f(w^{(1)}, a) = f(w^{(2)}, a - a^{(1)}),$$

where

$$v^{(2)} := f(v^{(1)}, a^{(1)}), \quad w^{(2)} := f(w^{(1)}, a^{(1)}).$$

The same arguments as given above applied to $v^{(2)}$ and $w^{(2)}$ yield an $a^{(2)}$ such that (3.27) holds with $v^{(2)}$, $w^{(2)}$ and $a \leq a^{(2)}$. For $a^{(1)} < a \leq a^{(1)} + a^{(2)}$ thus we have

$$\|f(v^{(1)}, a) - f(w^{(1)}, a)\|_1 = \|f(v^{(2)}, a - a^{(1)}) - f(w^{(2)}, a - a^{(1)})\|_1 \leq \|v^{(2)} - w^{(2)}\|_1. \quad (3.37)$$

Since (3.27) and (3.28) hold for $v^{(1)}$, $w^{(1)}$ and $a = a^{(1)}$ we thus conclude from (3.37) the validity for $a^{(1)} < a \leq a^{(1)} + a^{(2)}$. Proceeding in the same way we obtain a sequence $a^{(i)}$ of at most $2n$ numbers and iterating we find that (3.27) and (3.28) are true for all $a > 0$.

(iib) This is an immediate consequence of (iia) and the iterative construction just given and of (3.37).

(iii) The dynamics $v(u, t, \varphi)$ satisfies

$$v(u, t, \varphi) = f(v(u, t_\ell, \varphi) + s_\ell e(i_\ell), t - t_\ell) \quad \text{for } t_\ell < t \leq t_{\ell+1}, \quad \ell \geq 1, \quad (3.38)$$

$$v(u, t, \varphi) = f(u, t) \quad \text{for } t_0 < 0 \leq t \leq t_1, \quad (3.39)$$

$$v(u, t, \varphi) = f(u + s_0 e(i_0), t) \quad \text{for } t_0 = 0 \leq t \leq t_1, \quad (3.40)$$

where $\varphi = \{[t_\ell, s_\ell, i_\ell]\}_{\ell=-\infty}^{\infty} \in M_K$. In view of these dynamics we can apply (3.27) successively from t over the arrival instants t_ℓ down to 0 ($\dots < t_0 \leq 0 < t_1 < \dots < t_\ell < t \leq t_{\ell+1}$) and thus conclude (iii).

(iv) From (iii) and (3.25) we get

$$\begin{aligned} \|v(u, t, \varphi) - v(w, t, \varphi)\|_1 &= \|v(v(u, s, \varphi), t - s, \theta_s \varphi) - v(v(w, s, \varphi), t - s, \theta_s \varphi)\| \\ &\leq \|v(u, s, \varphi) - v(w, s, \varphi)\|_1 \quad \text{for } 0 \leq s \leq t, \end{aligned}$$

which implies (iv).

(v) Assume that $I(u, s, \varphi) \neq I(w, s, \varphi)$ for $0 \leq s \leq t$. Since $\|v(u, s, \varphi) - v(w, s, \varphi)\|_1$ does not jump at the arrival instants t_ℓ , we conclude by taking into account the dynamics (3.38)–(3.40) and (iib), that during $[0, t]$ the distance $\|v(u, s, \varphi) - v(w, s, \varphi)\|_1$ decreases in s at least with rate $k/(k+n)^2$, which gives (3.31).

(vi) Since $v(\cdot, s, \varphi)$ is non-decreasing, cf. (3.26), $u \leq w$ implies that $v(u, s, \varphi) \leq v(w, s, \varphi)$ and $b(v(u, s, \varphi)) = |I(u, s, \varphi)| \leq |I(w, s, \varphi)| = b(v(w, s, \varphi))$. Now, if $v_i(u, s, \varphi) > 0$, $s \in [0, t]$ for some i then it holds $v_i(w, s, \varphi) > 0$, $s \in [0, t]$, too, and during $[0, t]$ both components $v_i(u, s, \varphi)$ and $v_i(w, s, \varphi)$ get continuously service with rate $1/(|I(u, s, \varphi)| + k) \geq 1/(|I(w, s, \varphi)| + k)$. At the arrival instants $t_{i,\ell}$ of type i -customers both components jump with the same height $s_{i,\ell}$ and in summary we must have $w_i - u_i \leq v_i(w, t, \varphi) - v_i(u, t, \varphi)$.

(vii) If $I(u, s, \varphi) = I(w, s, \varphi)$ for $s \in [0, t]$ then in view of the dynamics (3.38)–(3.40) and (3.36) we conclude that $v(u, s, \varphi)$ and $v(w, s, \varphi)$ get the same service rate at the queues and since the input process φ is the same, (3.33) follows directly.

Theorem 3.9. If the processor-sharing system is stable, i.e. if (3.14) is satisfied, then

(i) $\hat{V}(t) = v(t, \Phi)$ is the only stationary state process.

(ii) For any random initial state W it holds

$$\|\hat{V}(t) - v(W, t, \Phi)\|_1 \xrightarrow[t \rightarrow \infty]{} 0 \quad P - a.s.$$

Proof. (i) The assertion (i) is an easy consequence of (ii) and the semi-contractivity property (iii) of Lemma 3.8 as follows: Assume $\bar{V}(t)$, $t \in \mathbb{R}$ is a stationary state process for the input Φ , i.e.

(2.3)–(2.6) hold with $\bar{V}_\ell = \bar{V}(T_\ell - 0)$ and $\Psi = \{[T_\ell, I_\ell, S_\ell, \bar{V}_\ell]\}_{\ell=-\infty}^\infty$ is a stationary marked point process. Then we get from Lemma 3.8, (iv) that $H(t) = \|\hat{V}(t) - \bar{V}(t)\|_1$ is $P - a.s.$ monotone non-increasing in t and hence by the Birkhoff-Chintchin Ergodic-Theorem we conclude

$$H(0) = \|\hat{V}(t) - \bar{V}(t)\|_1, \quad t \in \mathbb{R} \quad P - a.s.$$

Applying now (ii) for $W = \bar{V}(0)$, we get $H(0) = \|\hat{V}(t) - v(\bar{V}(0), t, \Phi)\|_1 \xrightarrow[t \rightarrow \infty]{} 0$ $P - a.s.$, i.e. $H(0) = 0$ $P - a.s.$ and thus $\bar{V}(t) = \hat{V}(t)$, $t \in \mathbb{R}$ $P - a.s.$

(ii) The proof is divided into several steps.

a) It holds

$$\begin{aligned} P(\lim_{t \rightarrow \infty} \|\hat{V}(t) - v(W, t, \Phi)\|_1 = 0) \\ = \lim_{m \rightarrow \infty} P(\lim_{t \rightarrow \infty} \|\hat{V}(t) - v(W, t, \Phi)\|_1 = 0, W \leq \hat{V}(0) + m\mathbf{1}), \end{aligned}$$

where $\mathbf{1} = (1, \dots, 1)$. Consider a fixed $w \in \mathbb{R}_+^n$ and $\varphi \in M_K$ such that $v(t, \varphi) < \infty$ for $t \in \mathbb{R}$. Then there is an $m \in \mathbb{Z}_+$ such that

$$0 \leq w \leq v(0, \varphi) + m\mathbf{1}, \quad (3.41)$$

Using the triangle inequality for $\|\cdot\|_1$, the monotonicity of $v(\cdot, t, \varphi)$ (cf. (3.26)) and (3.41) we get

$$\begin{aligned} \|v(t, \varphi) - v(w, t, \varphi)\|_1 &\leq \|v(t, \varphi) - v(0, t, \varphi)\|_1 + \|v(w, t, \varphi) - v(0, t, \varphi)\|_1 \\ &\leq \|v(t, \varphi) - v(0, t, \varphi)\|_1 + \|v(v(0, \varphi) + m\mathbf{1}, t, \varphi) - v(0, t, \varphi)\|_1 \\ &\leq 2\|v(t, \varphi) - v(0, t, \varphi)\|_1 + \|v(v(0, \varphi) + m\mathbf{1}, t, \varphi) - v(t, \varphi)\|_1. \end{aligned}$$

b) Using the semi-contractivity of $v(\cdot, t, \Phi)$, the construction of $\hat{V}(t)$, cf. (2.10), (2.15), (2.16), (2.17) and the stationarity of Φ we get for $j = 1, 2, \dots$

$$\begin{aligned} P(\lim_{t \rightarrow \infty} \|v(0, t, \Phi) - \hat{V}(t)\|_1 \leq 1/j) &\geq \lim_{t \rightarrow \infty} P(\|v(0, t, \Phi) - v(t, \Phi)\|_1 \leq 1/j) \\ &= \lim_{t \rightarrow \infty} P(\|v^{(t)}(0, \theta_t \Phi) - v(0, \theta_t \Phi)\|_1 \leq 1/j) \\ &= \lim_{t \rightarrow \infty} P(\|v^{(t)}(0, \Phi) - v(0, \Phi)\|_1 \leq 1/j) = 1. \quad (3.42) \end{aligned}$$

From (3.42) we obtain

$$P(\lim_{t \rightarrow \infty} \|v(0, t, \Phi) - \hat{V}(t)\|_1 = 0) = \lim_{j \rightarrow \infty} P(\lim_{t \rightarrow \infty} \|v(0, t, \Phi) - \hat{V}(t)\|_1 \leq 1/j) = 1.$$

c) In view of a) and b) it remains to prove:

$$\|v(v(0, \varphi) + m\mathbf{1}, t, \varphi) - v(t, \varphi)\|_1 \longrightarrow 0 \quad \text{for } P - a.e. \varphi. \quad (3.43)$$

Now let $\varphi \in M_K$ and $t_{i,\ell}^*$ the time instants where queue $i \in \{1, \dots, n\}$ becomes idle for a positive duration $d_{i,\ell}$. By the assumed stability of the system we may assume that

$$\lim_{\ell \rightarrow \pm\infty} t_{i,\ell}^* = \pm\infty \quad \text{and} \quad \sum_{\ell=-\infty}^0 d_{i,\ell} = \sum_{\ell=1}^{\infty} d_{i,\ell} = \infty. \quad (3.44)$$

Further let $w = v(0, \varphi) + m\mathbf{1}$ for some $m \in \mathbb{Z}_+$. For $t \in \mathbb{R}_+$ then

$$h_i(t) := v_i(w, t, \varphi) - v_i(v(0, \varphi), t, \varphi) = v_i(w, t, \varphi) - v_i(t, \varphi) \geq 0, \quad i = 1, \dots, n$$

and $h(t) = h_1(t) + \dots + h_n(t)$ gives the distance between $v(w, t, \varphi)$ and $v(t, \varphi)$ with respect to the l_1 -norm. From Lemma 3.8, (iv) it follows that $h(t)$ is non-increasing in t and hence the limit

$$\eta := \lim_{t \rightarrow \infty} h(t)$$

exists. Assume now that $\eta > 0$. Let $\varepsilon := \eta k^2 / (2(k+n)^3)$. Then there is a $t^* \in \mathbb{R}_+$ such that

$$\eta \leq h(t) \leq \eta + \varepsilon \quad \text{for } t \geq t^*. \quad (3.45)$$

Let $j = \arg \max_i h_i(t^*)$. Then from (3.45) we have

$$h_j(t^*) \geq \frac{\eta}{n}. \quad (3.46)$$

The following observations are crucial:

1. The j -th component $v_j(t, \varphi)$ becomes infinitely times idle during the interval (t^*, ∞) and the cumulative idle time is infinite, i.e.

$$\int_{t^*}^{\infty} \mathbb{I}\{v_j(s, \varphi) = 0\} ds = \infty.$$

2. If $v_j(w, t_{j,\ell}^*, \varphi) > 0$ then $v_j(w, s, \varphi)$ decreases with respect to s in $[t_{j,\ell}^*, t_{j,\ell}^* + d_{j,\ell}]$ with a rate bounded by $1/(k+1)$ to zero. (Note, $v_j(s, \varphi) = 0$ for $s \in [t_{j,\ell}^*, t_{j,\ell}^* + d_{j,\ell}]$). If $v_j(w, s, \varphi)$ has reached zero, then $v_j(w, s, \varphi)$ remains zero up to $t_{j,\ell}^* + d_{j,\ell}$. Thus we conclude $v_j(w, s, \varphi) > 0 = v_j(s, \varphi)$ on $s \in [t_{j,\ell}^*, t_{j,\ell}^* + d]$, where $d = \min(v_j(w, t_{j,\ell}^*, \varphi)(k+1), d_{j,\ell})$.
3. If $v_j(w, t_{j,\ell}^* + d_{j,\ell}, \varphi) > 0$ then $h_j(t_{j,\ell}^* + d_{j,\ell}) > 0$. Since $v_j(s, \varphi) > 0$ for $s \in (t_{j,\ell}^* + d_{j,\ell}, t_{j,\ell+1}^*)$ we conclude from Lemma 3.8, (vi) that $h_j(s)$ is non-decreasing for $s \in (t_{j,\ell}^* + d_{j,\ell}, t_{j,\ell+1}^*)$. Now, by iterating the arguments in step 1 and 2 we conclude that on the interval $[t^*, \infty)$ the j -th components $v_j(w, t, \varphi)$ and $v_j(t, \varphi)$ differ on the idle periods of $v_j(t, \varphi)$ at least for $\eta(k+1)/n$ time units, cf. (3.46), hence there is a t^{**} such that

$$\int_{t^*}^{t^{**}} \mathbb{I}\{v_j(w, s, \varphi) > 0 = v_j(s, \varphi)\} ds \geq \frac{\eta(k+1)}{2n}. \quad (3.47)$$

Since $v_j(w, s, \varphi) > 0 = v_j(s, \varphi)$ implies $I(w, s, \varphi) \neq I(s, \varphi)$, we conclude by applying Lemma 3.8, (v), (iii), (3.44), (3.47), (3.45) and using the dynamics (3.25)

$$\begin{aligned} h(t^{**}) &= \|v(w, t^{**}, \varphi) - v(t^{**}, \varphi)\|_1 \leq \|v(w, t^*, \varphi) - v(t^*, \varphi)\|_1 - \frac{k}{(k+n)^2} \frac{\eta(k+1)}{2n} \\ &< h(t^*) - \frac{\eta k^2}{2(n+k)^3} \leq (\eta + \varepsilon) - \varepsilon = \eta, \end{aligned}$$

contradicting the left-hand side of (3.45). Thus we conclude $\eta = 0$ and hence (3.43) is proved.

4 Conclusions

The results given in this paper remain valid for $k \in (0, \infty)$. The k permanent customers can be considered as one permanent customer getting the k -fold portion of the processor capacity obtained by the customers at the head of the queues. The proofs remain valid; only a minor change is necessary: instead of ϱ^* one considers $k\varrho^*$ as the fraction of processor capacity that the permanent customer gets.

Also, the results of the paper remain valid if one considers arrival processes where at the T_n simultaneously at each of the n queues a customer arrives. The input is given by the marked point process $\Phi = \{[T_\ell, \underline{S}_\ell]\}_{\ell=-\infty}^\infty$, where $\underline{S}_\ell = [S_{1,\ell}, \dots, S_{n,\ell}]$ is the vector of service times; $S_{i,\ell}$ is the service time of the customer arriving at the i -th queue at time T_ℓ .

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