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## **On Critically Perfect Graphs**



# On Critically Perfect Graphs

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## Abstract

A perfect graph is critical if the deletion of any edge results in an imperfect graph. We give examples of such graphs and prove some basic properties. We relate critically perfect graphs to well-known classes of perfect graphs, investigate the structure of the class of critically perfect graphs, and study operations preserving critical perfectness.

## 1 Introduction

We assume familiarity with basic notions of graph theory and consider finite, undirected graphs without loops or multiple edges; subgraphs are node-induced.

To better understand a graph property  $\mathcal{P}$ , it is often useful to investigate extremal cases, i.e., graphs having  $\mathcal{P}$  but losing it by a small modification. We distinguish two types of such graphs: **minimal** graphs (they possess  $\mathcal{P}$  but lose it by the deletion of any *node*) and **critical** graphs (they have  $\mathcal{P}$  but lose it by the deletion of any *edge*). The subject of our investigation is a rich and well-studied graph property: perfectness.

BERGE proposed to call a graph **perfect** if, for each of its subgraphs  $G'$ , the chromatic number  $\chi(G')$  equals the clique number  $\omega(G')$ , otherwise the graph is **imperfect**. Chordless cycles of length at least four have been termed **holes** and their complements **antiholes**. Obviously, any graph that contains an odd hole or an odd antihole is imperfect. BERGE conjectured in [2] that a graph is perfect iff it contains neither odd holes nor odd antiholes as subgraphs, i.e., iff the graph is **Berge** (Strong Perfect Graph Conjecture, for short SPGC). PADBERG [16] introduced the notion of minimally imperfect graphs; in these terms, the SPGC states that the odd holes and the odd antiholes are the only minimally imperfect graphs. Therefore, minimally imperfect Berge graphs are called **monsters** (since the existence of this third type of minimally imperfect graphs would contradict the SPGC).

Although several, in general  $\mathcal{NP}$ -hard, combinatorial optimization problems can be solved in polynomial time for perfect graphs, see [8], the structure of perfect graphs is not well-understood. In particular, the SPGC still seems to be out of reach. On the other hand, the investigation of minimally imperfect graphs has revealed that these graphs have quite strong properties, see e.g. [5, 9, 13, 14, 15, 16, 18, 19]. That motivated us to introduce a new class of extremal cases with respect to perfectness: critically perfect graphs. We provide several examples and prove some basic properties in section 2.

We are interested in relating the class  $\mathcal{CP}$  of critically perfect graphs to well-known classes of perfect graphs. It turns out that many linegraphs of bipartite graphs as well as complements

of such graphs belong to  $\mathcal{CP}$ ; let us denote this subset of  $\mathcal{CP}$  by  $\mathcal{LG}$ . Since there is no class of perfect graphs known which contains both the linegraphs of bipartite graphs and their complements, it seems hopeless to find a class of perfect graphs containing all graphs of  $\mathcal{CP}$ .

In order to examine the structure of the class  $\mathcal{CP}$ , we first characterize the graphs belonging to  $\mathcal{LG}$  (see section 3). Then we study operations preserving critical perfectness in section 4. We obtain a subset of  $\mathcal{CP}$  which can be constructed from graphs in  $\mathcal{LG}$  by applying this kind of operations. Finally, we check when complementation preserves critical perfectness.

## 2 Critical Perfectness

We define an edge  $e \in E$  of a perfect graph  $G = (V, E)$  to be **critical** if  $G - e$  is imperfect. In particular, for every critical edge  $e$  of a perfect graph  $G$ , there is a subgraph  $G_e \subseteq G$  s.t.  $G_e - e$  is minimally imperfect.

According to the three different types of minimally imperfect graphs, we distinguish between three types of critical edges. We say that an edge  $e$  of a perfect graph  $G$  is **H-critical** (**A-critical**, **M-critical**, resp.) if  $G - e$  contains an odd hole (an odd antihole, a monster, resp.). Note that an H-critical edge is a single chord in an odd cycle of length  $\geq 5$  which forms a triangle with two edges of this cycle; further, if the SPGC is true, there are no M-critical edges.

We define a graph to be **critically perfect** if it is a perfect graph without isolated nodes and all of its edges are critical. Critical perfectness is a very strong property. Nevertheless, there are surprisingly many graphs having this property; a few examples are shown in Figure 1 (note that  $G_1, \dots, G_6$  are linegraphs of bipartite graphs<sup>†</sup> and  $G_1 = \overline{G_1} = G_4 = \overline{G_4}$  holds).

Some properties of minimally imperfect graphs immediately determine properties of a critically perfect graph  $G$ : if  $x$  and  $y$  are adjacent nodes of  $G$ , then neither  $x$  **dominates**  $y$ , i.e.,  $N(y) \subseteq N(x) \cup \{x\}$  (easy), nor do all induced paths connecting  $x$  and  $y$  in  $G - xy$  have even length (MEYNIEL [14]), nor are all nodes in  $G - \{x, y\}$  adjacent either to  $x$  or to  $y$  (OLARIU [15]). In addition, we state the following basic properties.

**Lemma 2.1** *For any critically perfect graph  $G$ , it holds that*

- (i) *every edge of  $G$  is contained in a triangle,*
- (ii) *there is no simplicial node in  $G$ , i.e., a node  $x$  s.t.  $N(x)$  is a clique,*
- (iii)  *$G$  has minimal degree  $\delta(G) \geq 4$  and maximal degree  $\Delta(G) \leq n - 3$ .*

**Proof.** Let  $G = (V, E)$  be a critically perfect graph.

(i) Suppose  $e \in E$  is not contained in a triangle. Then  $e$  is neither H-critical nor A-critical. Hence, there is  $G_e \subseteq G$  s.t.  $G_e - e$  is a monster. LOVÁSZ's characterization of perfect graphs [13] says  $\alpha(G_e)\omega(G_e) \geq n$  but  $\alpha(G_e - e)\omega(G_e - e) < n$ . Therefore,  $\omega(G_e - e) < \omega(G_e)$  (note that  $\omega(G_e) \geq \omega(G_e - e)$  and  $\alpha(G_e) \leq \alpha(G_e - e)$ ) holds and  $e$  is contained in the intersection of all maximum cliques of  $G_e$ . It follows  $\omega(G) = 2$ , since  $e$  is a maximal clique of  $G$ , a contradiction to  $\omega > 3$  for any monster by TUCKER [19].  $\diamond$

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<sup>†</sup>A well-known characterization of linegraphs of bipartite graphs is that one can colour their edges with two colours s.t. every maximal clique is monochromatic and no two maximal cliques of the same colour intersect.

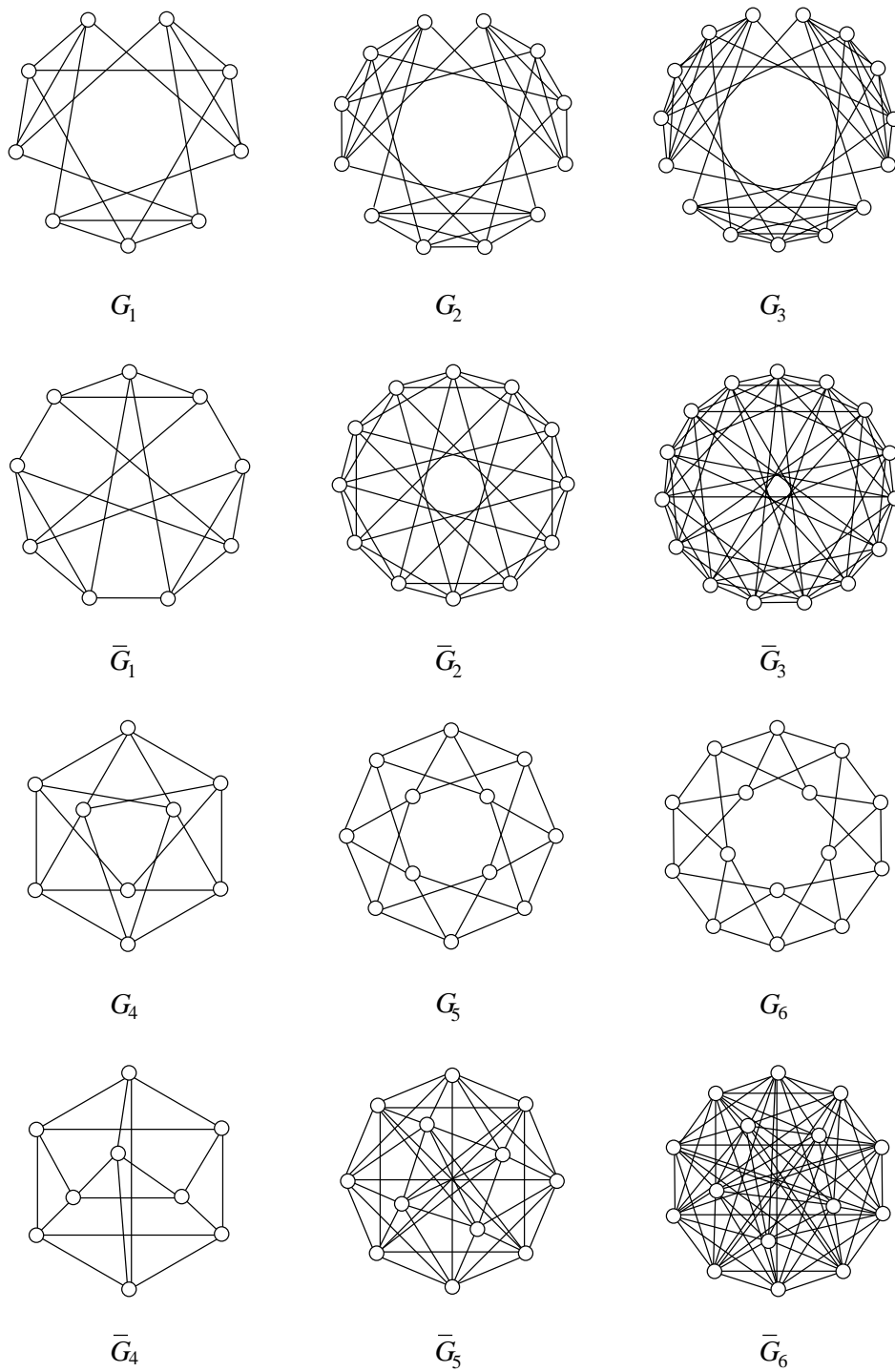


Figure 1: Examples for critically perfect graphs.

(ii) Since  $G$  is noncomplete, a node  $y$  in the clique  $N(x)$  would have a neighbour in  $G - (N(x) \cup \{x\})$ , i.e.,  $y$  would dominate  $x$ , a contradiction.  $\diamond$

(iii) We first show  $\delta(G) \geq 4$ . Edges incident to nodes with degree  $< 3$  are not critical. Assume that there is  $x \in V$  with  $N(x) = \{x_1, x_2, x_3\}$ . Since  $\delta(\overline{C_{2k+1}}) \geq 4$  for  $k > 2$  and  $\delta(M) \geq 6$  for any monster  $M$  holds by SEBÖ [18] and [19], the edges incident to  $x$  are H-critical. Consider the edge  $xx_1$ , it has to be contained in a triangle. Without loss of generality, let  $x_1x_2$  be an edge. Further,  $x_1, x, x_3$  are contained in an even hole  $C$  with  $x_2 \notin C$  and  $N_C(x_2) = \{x, x_1\}$ , i.e.,  $x_1x_3, x_2x_3 \notin E$ . But now,  $xx_3$  is not contained in a triangle, a contradiction to (i).

Now, we show  $\Delta(G) \leq n - 3$ . A node of degree  $n - 1$  would dominate all other nodes. Consider  $u \in V$  with  $d(u) = n - 2$  and  $w \notin N(u)$ . Since  $vw \notin E, v \in N(u)$  would imply that  $u$  dominates  $v$ , we have  $N(u) = N(w)$ . Let  $e$  be an edge incident to  $u$  and  $G_e - e$  the corresponding minimally imperfect subgraph. By MEYNIEL [14], it follows  $w \notin G_e - e$ , i.e.,  $G_e \subseteq N(u) \cup \{u\}$  and, in  $G_e - e$ ,  $u$  is contained in exactly one stable set of size 2, a contradiction to  $G_e - e$  minimally imperfect.  $\square$

Note that the bounds for  $\delta(G)$  and  $\Delta(G)$  in (iii) are sharp (this is shown by the graph  $G_1$  in Figure 1 and by the complement of the linegraph of the second graph in Figure 3, resp.).

### 3 Characterization of the Graphs in $\mathcal{LG}$

In order to characterize the graphs in  $\mathcal{LG}$ , we describe two structures in the underlying bipartite graphs  $F$  which guarantee that the edges of  $L(F)$  and  $\overline{L(F)}$ , respectively, are critical.

We say that two incident edges  $x$  and  $y$  form an **H-pair** if there is a  $K_{1,3}$  with edges  $x, y, z$  and an even cycle  $C$  that contains  $x$  and  $y$  but only one endnode of  $z$ . We call  $F$  an **H-graph** if it is connected and every two of its incident edges form an H-pair. Examples of H-graphs are the bipartite graphs shown in Figure 2 and all 3-connected bipartite graphs (easy).

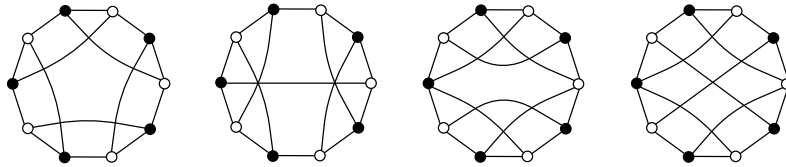


Figure 2: Examples for H-graphs.

We say that two nonincident edges form an **A-pair** if they are the end edges of an odd path with length at least five. We call  $F$  an **A-graph** if it is connected and every two of its nonincident edges form an A-pair. Examples for A-graphs are the bipartite graphs shown in Figure 3 and all 3-connected bipartite graphs (easy).

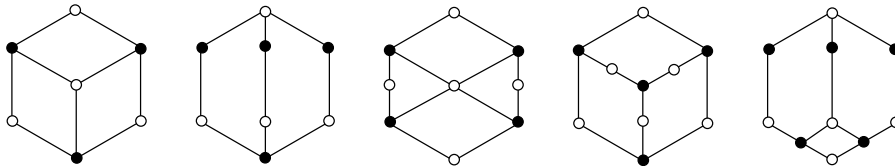


Figure 3: Examples for A-graphs.

Let us first investigate the relationship of the defined structure in H-graphs with critical edges in linegraphs.

**Theorem 3.1** *Let  $G$  be a perfect graph and the linegraph of a graph  $F$ . An edge  $e = xy$  of  $G$  is critical iff  $x$  and  $y$  form an H-pair in  $F$ .*

**Proof.** (If) Let  $x$  and  $y$  form an H-pair in  $F$ , then there is an edge  $z$  incident to both and an even cycle  $C$  containing  $x$  and  $y$  but only one endnode of  $z$ . In  $G = L(F)$ , the edge  $e = xy$  is contained in the even hole  $L(C)$  and it holds  $N_{L(C)}(z) = \{x, y\}$ , i.e., the edge  $e$  is H-critical.  $\diamond$

(Only if) Let  $G$  be perfect, the linegraph of a graph  $F$ , and  $e$  a critical edge of  $G$ . We show that the edges corresponding to the endnodes of  $e$  form an H-pair in  $F$ . According to the three types of critical edges, we distinguish between three cases.

*Case 1:  $e$  is H-critical.* Let  $e = xy$ , the nodes  $x$  and  $y$  are contained in an even hole  $C_{2k}$ ,  $k \geq 2$  and there is another node  $z$  with  $N_{C_{2k}}(z) = \{x, y\}$ . In  $F$ ,  $x$  and  $y$  are incident edges on the even cycle  $L^{-1}(C_{2k})$ ;  $z$  is incident to  $x$  and  $y$  and has exactly one endpoint on  $L^{-1}(C_{2k})$  (else  $z$  has four neighbours on  $C_{2k}$ ). Thus,  $x$  and  $y$  form an H-pair in  $F$ .

*Case 2:  $e$  is A-critical.* Let  $e = v_1v_{2k+1}$  and  $v_1, \dots, v_{2k+1}$  be the nodes of the corresponding odd antihole  $\overline{C_{2k+1}}$ ,  $k \geq 2$  in  $G - e$  with  $v_i v_{i+1} \notin E(G)$  for  $1 \leq i \leq 2k$ . For  $k \geq 3$ , the nodes  $v_1, v_2, v_3, v_{2k}, v_{2k+1}$  induce a subgraph of  $G$  which is forbidden according to BEINEKE's characterization of linegraphs [1]. Hence, it holds  $k = 2$  and the assertion follows by Case 1, since  $e$  is also H-critical (note  $\overline{C_5} + e \simeq C_5 + e$ ).

*Case 3:  $e$  is M-critical.* Let  $e = xy$  and  $M - e$  be a monster. By [1],  $M$  is  $K_{1,3}$ -free but  $M - e$  is not since the SPGC is true for  $K_{1,3}$ -free graphs by a result of PARTHASARATHY and RAVINDRA [17]. Let us now consider a  $K_{1,3} = \{w, x, y; z\}$  with center  $z$  in  $M - e$ . The nodes  $x, z, w$  and  $y, z, w$ , resp., induce  $P_3$ 's which are contained in the holes  $C_x$  and  $C_y$ , resp. (see HOÀNG [9]). Since  $M - e$  is Berge,  $C_x$  and  $C_y$  are even holes.  $L^{-1}(C_x)$  and  $L^{-1}(C_y)$  are even cycles in  $F$  and  $C = L^{-1}(C_x \cup C_y) - \{w, z\}$  is an even cycle containing the edges  $x, y$  and only one endnode of  $z$ . But now,  $L(C \cup \{z\}) - e$  is an odd hole in  $M - e$ , a contradiction to  $M - e$  Berge.  $\square$

We obtain the following corollary as an immediate consequence of the previous proof.

**Corollary 3.2** *Let  $G$  be perfect and a linegraph. An edge of  $G$  is critical iff it is H-critical.*

**Theorem 3.3** *Let  $G$  be the linegraph of  $F$ .  $G$  is critically perfect iff  $F$  is a bipartite H-graph.*

**Proof.** (If) Let  $F$  be a bipartite H-graph, then its linegraph  $G$  is perfect by KÖNIG's Edge-Colouring-Theorem [11]; every edge of  $G$  is critical by Theorem 3.1.  $\diamond$

(Only if) Let  $G$  be critically perfect and the linegraph of  $F$ . We know by Theorem 3.1 that the edges of  $F$  corresponding to adjacent nodes in  $G$  form an H-pair. Thus,  $F$  is an H-graph. We have to show that  $F$  is bipartite.

Assume  $F$  contains an odd cycle. Since  $G$  contains an odd hole if this cycle has a length  $\geq 5$ , it has to be a triangle with edges  $e_1, e_2, e_3$ . Since  $F$  is an H-graph,  $e_1$  and  $e_2$  appear on an even cycle  $C$  that has to be a  $C_4$  (else  $(C - \{e_1, e_2\}) \cup \{e_3\}$  would be an odd cycle of length  $\geq 5$ ). Let  $e_{13}$  be the edge of  $C$  incident to  $e_1$  and  $e_3$ , and  $e_{23}$  be the edge of  $C$  incident to  $e_2$  and  $e_3$ . But  $e_1$  and  $e_3$  also appear on an even cycle  $C'$  and we get an odd cycle of length  $\geq 5$  in each case:  $(C' - \{e_3\}) \cup \{e_{13}, e_{23}\}$  if  $e_{23} \notin C'$ , or  $(C' - \{e_{23}, e_3\}) \cup \{e_{13}\}$  if  $e_{23} \in C'$ . Since the existence of an odd cycle in  $F$  contradicts the perfectness of  $G$ ,  $F$  has to be a bipartite graph.  $\square$

Now, let us relate the structure in A-graphs to critical edges in complements of linegraphs in a similar way.

**Theorem 3.4** *Let  $G$  be perfect and the complement of the linegraph of a graph  $F$ . An edge  $e = xy$  of  $G$  is critical iff  $x$  and  $y$  form an A-pair in  $F$ .*

**Proof.** *(If)* Let  $x$  and  $y$  form an A-pair in  $F$ , then they are the end edges of an path  $P_{2k}$ ,  $k \geq 3$ . It holds  $\overline{L(P_{2k})} = \overline{C_{2k-1}} + e$  where  $e$  is an additional edge between these nodes of  $\overline{C_{2k-1}}$  which correspond to the end edges of the path  $P_{2k}$ , i.e., the edge  $e$  is A-critical.  $\diamond$

*(Only if)* Let  $G$  be perfect, the complement of the linegraph of  $F$ , and  $e$  a critical edge of  $G$ . We show that the edges corresponding to the endnodes of  $e$  form an A-pair in  $F$ . According to the three types of critical edges, we distinguish between three cases.

*Case 1:  $e$  is A-critical.* Let  $e = v_1v_{2k+1}$  and  $v_1, \dots, v_{2k+1}$  be the nodes of the corresponding odd antihole  $\overline{C_{2k+1}}$ ,  $k \geq 2$  in  $G - e$  with  $v_i v_{i+1} \notin E(G)$  for  $1 \leq i \leq 2k$ . It holds  $v_i v_{i+1} \in E(\overline{G})$  for  $1 \leq i \leq 2k$ , i.e.,  $v_1$  and  $v_{2k+1}$  are the endnodes of a path  $P_{2k+1} = \overline{C_{2k+1}} + e$  in  $\overline{G}$ , hence the end edges of a path with length  $2k + 1$  in  $F$ .

*Case 2:  $e$  is H-critical.* Let  $e = v_1v_{2k}$  and  $v_1, \dots, v_{2k+1}$  be the nodes of the corresponding odd hole  $C_{2k+1}$ ,  $k \geq 2$  in  $G - e$  with  $v_i v_{i+1} \in E(G)$  for  $1 \leq i \leq 2k + 1 \pmod{2k + 1}$ . For  $k \geq 3$ , the nodes  $v_1, v_3, v_{2k}, v_{2k+1}$  induce a  $\overline{K_{1,3}}$  in  $G$  which is forbidden by [1] (note that  $\overline{G}$  is  $K_{1,3}$ -free). Hence, it holds  $k = 2$  and the assertion follows by Case 1, since  $e$  is also A-critical.

*Case 3:  $e$  is M-critical.* Let  $e = xy$  and  $M - e$  be a monster. By [1],  $M$  is  $\overline{K_{1,3}}$ -free but  $M - e$  is not since  $\overline{M} + e$  is a monster as well by LOVÁSZ's Perfect Graph Theorem [13], and the SPGC is true for  $K_{1,3}$ -free graphs by [17]. Let us now consider a  $\overline{K_{1,3}} = \{w, x, z; y\}$  in  $M - e$ , i.e., a  $K_{1,3}$  with center  $y$  in  $\overline{M} + e$ . The nodes  $w, y, z$  and  $x, y, z$ , resp., induce  $P_3$ 's in  $\overline{M} + e$  which are contained in the holes  $C_w$  and  $C_x$ , resp. (see [9]). Since  $\overline{M} + e$  is Berge,  $C_w$  and  $C_x$  are even holes. In  $\overline{M}$ , we have  $C_w$  and an even path  $P_x$  connecting  $z$  and  $x$ . Finally,  $P = L^{-1}(C_w \cup P_x) - \{z\}$  is a path of odd length  $\geq 5$  with  $x$  and  $y$  as end edges; but  $\overline{L(P)}$  is an odd antihole in  $M - e$ , a contradiction to  $M - e$  Berge.  $\square$

**Corollary 3.5** *Let  $G$  be perfect and the complement of a linegraph. An edge of  $G$  is critical iff it is A-critical.*

**Theorem 3.6** *Let  $G$  be perfect and the complement of the linegraph of a graph  $F$ .  $G$  is critically perfect iff  $F$  is a bipartite A-graph.*

**Proof.** *(If)* Let  $F$  be a bipartite A-graph, then the complement  $G$  of its linegraph is perfect by KÖNIG's Matching-Theorem [12]; every edge of  $G$  is critical by Theorem 3.1.  $\diamond$

*(Only if)* Let  $G$  be critically perfect and the complement of the linegraph of  $F$ . We know by Theorem 3.4 that the edges of  $F$  corresponding to adjacent nodes in  $G$  form an A-pair. Thus,  $F$  is an A-graph. We have to show that  $F$  is bipartite. Assume  $F$  contains an odd cycle  $D$ . Since  $G$  contains an odd antihole if  $D$  has a length  $\geq 5$ ,  $D$  has to be a triangle with nodes  $x, y, z$ . Since  $F$  contains nonincident edges (note that  $G$  is not a stable set) and is edgeconnected, there is an edge incident to  $D$ , say  $x'$  is adjacent to  $x$ . The edges  $xx'$  and  $yz$  form an A-pair, i.e., they are the end edges of an odd path  $P$  with length  $\geq 5$  and we get an odd cycle  $C$  with length  $\geq 5$ :  $C = \{xx' \dots y\}$  if  $P = \{xx' \dots yz\}$ , or  $C = \{x \dots yz\}$  if  $P = \{x'x \dots yz\}$ , a contradiction.  $\square$



## 4 Operations Preserving Critical Perfectness

In this section, we study graph operations with the property that the class  $\mathcal{CP}$  is closed under applying these operations.

A graph operation transforms graphs  $G_1, \dots, G_i$  into a new graph  $G$ . If an operation transfers a common property  $\mathcal{P}$  of  $G_1, \dots, G_i$  to  $G$ , we say that this operation preserves the property  $\mathcal{P}$ . Let us call a perfection preserving operation a **P-operation**. We define a **P\*-operation** to be a P-operation that generates  $G$  from disjoint graphs  $G_1$  and  $G_2$  s.t.

- (i)  $G_1 \subseteq G$  and  $G_2 \subseteq G$
- (ii)  $e \in E(G_1)$  or  $e \in E(G_2) \forall e \in E(G)$ .

It follows from the definition that the class  $\mathcal{CP}$  is closed under applying P\*-operations.

**Lemma 4.1** *Every P\*-operation preserves critical perfectness.*

Let us give some examples of P\*-operations. Obviously, the union and all perfection preserving identification operations are P\*-operations: the clique identification; its generalization, the subgraph identification by HSU [10]; and the stable set identification by a result of CORNEIL and FONLUPT [6]. Furthermore, the multiplication and its generalization, the substitution by LOVÁSZ's Replacement Lemma [13], and the composition by BIXBY [3] are P\*-operations as well.

Let us mention that, if  $G = \overline{L(F)}$ , the multiplication of a node  $v \in V(G)$  is similar to the addition of parallel edges to  $v$  in the underlying graph  $F$ . Further, since no node of a critically perfect graph dominates one of its neighbours, no critically perfect graph satisfies the condition of the amalgam operation introduced by BURLET and FONLUPT [4], except for the special case of BIXBY's composition. The same holds for all further generalizations of the amalgam, e.g., the so-called 2-amalgam defined by CORNUÉJOLS and CUNNINGHAM [7].

Finally, note that a graph  $G$  generated from two graphs  $G_1, G_2 \in \mathcal{LG}$  by P\*-operations does not belong to  $\mathcal{LG}$  in general, as is shown by the two examples in Figure 4.

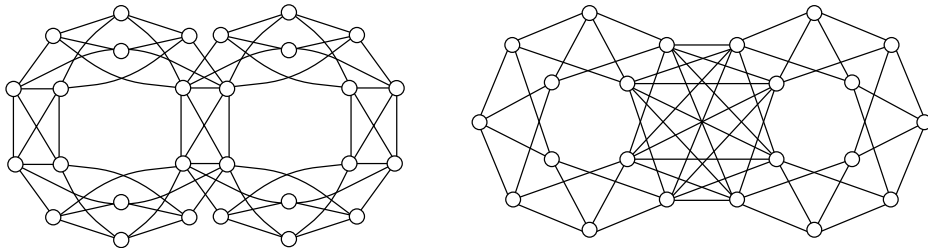


Figure 4: Examples for graphs generated by P\*-operations.

Let us now check, when complementation, which preserves perfectness by LOVÁSZ's Perfect Graph Theorem [13], preserves critical perfectness. Both a graph *and* its complement are critically perfect iff the deletion *and* the addition of any edge results in an imperfect graph. This requirement is very strong. Indeed, the elements of  $\mathcal{CP}$  have noncritical complements in general, i.e.,  $\mathcal{CP}$  is not closed under complementation. E.g., all linegraphs of bipartite A-graphs that are not also H-graphs (see the linegraphs of the A-graphs shown in Figure 3), are noncritical.

Obviously, the same holds for all graphs generated by P\*-operations from at least one graph with noncritical complement.

However, there are many examples of critically perfect graphs with critical complements, e.g., the graphs shown in Figure 1 and in Figure 4. We are able to prove some necessary conditions that the complement of a critically perfect graph is critical as well.

**Lemma 4.2** *If the complement of a critically perfect graph  $G$  is critical, then  $G$*

- (i) *is a connected graph,*
- (ii) *has no clique-cutset  $Q$  s.t. in at least two components of  $G - Q$ , there exists a node  $x$  with  $xv \in E(G) \forall v \in Q$ ,*
- (iii) *has no nonadjacent nodes with the same neighbourhood.*

**Proof.** Let  $\overline{G}$  be a critically perfect graph.

(i) Assume that  $G$  is disconnected with components  $G_1$  and  $G_2$ . In  $\overline{G}$ , consider an edge  $e = x_1x_2$  with  $x_1 \in V(G_1)$ ,  $x_2 \in V(G_2)$  and let  $\overline{G}_e - e$  be the corresponding minimally imperfect subgraph of  $\overline{G} - e$ . By LOVÁSZ [13],  $G_e + e \subseteq G + e$  is minimally imperfect as well, and  $e$  is the only edge connecting  $(G_e + e) \cap G_1$  and  $(G_e + e) \cap G_2$ . Therefore,  $G_e + e$  cannot be  $(2\omega - 2)$ -connected, a contradiction to SEBÖ [18].  $\diamond$

(ii) Assume that  $Q$  is a clique-cutset of  $G$ ,  $G_1$  and  $G_2$  are two components of  $G - Q$ , and  $x_1 \in G_1$ ,  $x_2 \in G_2$  are nodes with  $x_1v, x_2v \in E(G) \forall v \in Q$ . In  $\overline{G}$ , consider the edge  $e = x_1x_2$  and the minimally imperfect subgraph  $\overline{G}_e - e$ . By [13],  $G_e + e \subseteq G + e$  is minimally imperfect. But  $G_e + e \cap (Q \cup \{x_1, x_2\})$  is a clique-cutset of  $G_e + e$ , a contradiction to CHVÁTAL's Star-Cutset Lemma [5].  $\diamond$

(iii) Assume that  $x_1$  and  $x_2$  are nonadjacent nodes with the same neighbourhood; in  $\overline{G}$ , they are adjacent nodes and would dominate each other, a contradiction.  $\square$

Therefore, neither the complements of graphs generated by the union (see Lemma 4.2(i)), nor by the identification in a node or an edge (see Lemma 4.2(ii)), nor by the multiplication (see Lemma 4.2(iii)) are critically perfect. Furthermore, it follows by Lemma 4.2(i) that the well-known complete join does not preserve critical perfectness.

We are further interested in

- sufficient conditions that guarantee the critical perfectness of a graph and its complement,
- critically perfect graphs that cannot be constructed from the graphs in  $\mathcal{LG}$  by applying P\*-operations or complementation,
- additional examples for P\*-operations, and
- further operations preserving or constructing critical perfectness.

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