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Gröbner bases, invariant theory and equivariant dynamics

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Abstract

This paper is about algorithmic invariant theory as it is required within equivariant dynamical systems. The question of generic bifurcation equations requires the knowledge of fundamental invariants and equivariants. We discuss computations which are related to this for finite groups and semisimple Lie groups. We consider questions such as the completeness of invariants and equivariants. Efficient computations are gained by the Hilbert series driven Buchberger algorithm. Applications such as orbit space reduction are presented.

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1 Introduction

This paper deals with algorithmic invariant theory as it is relevant for the theory of equivariant dynamical systems. The emphasis is on efficiency of the involved algorithms.

The motivation for this paper and the associated software is given by the following tasks as the arise naturally in symmetric bifurcation theory, equivariant dynamic, exact solution of symmetric systems of equations, etc.

– A lot of papers in the area of equivariant dynamics start with setting up a *generic equivariant vector field* (the bifurcation equations) for a given group action. Then Hopf bifurcation with symmetry [21], [34] or Takens-Bogdanov points with symmetry [36] or heteroclinic bifurcation [10] or symmetry breaking bifurcation points of higher codimension [20] are investigated or direct bifurcating chaos [48] is demonstrated numerically. On the theoretical level the famous theorems by Schwarz and Poénaru are an essential ingregient, see [23, p. 46 and p. 51].

– A special method in equivariant dynamics is known as *orbit space reduction*. Where the solutions of a differential equation with symmetry are linked to the solutions of a system of differential equations for a set of fundamental invariants. By this method the group action is ruled out. Although the reduction leads to differential equations on an algebraic variety (restricted to a semialgebraic set) it has been applied successfully [11], [12], [33], [34] because the number of differential equations is often smaller than in the original system. For theoretical results see also [32] and [41]. The reduced phase space in Hamiltonian systems is similiar to the orbit space above.

– Systems of algebraic equations can be solved by the Gröbner basis method such that all solutions are found exactly. If symmetry is present the use of invariant theory is helpful, see [44, p. 58]. In case one is interested in real solutions only, one might as well consider a result involving the fundamental equivariants, see [18], [28], [48].
All three methods have in common that for a given group action one needs to know the fundamental invariants and fundamental equivariants. For another interesting application of invariant theory see [1]. The following questions often need to be solved:

A. Given a set of homogeneous invariants.

   1.) What are the relation in the invariants? (This knowledge is needed for the subsequent questions)
   2.) Does the set of invariants generate the invariant ring?
   2b) Do the homogeneous invariants generate the ring up to a certain degree?
   3.) How is a given invariant represented in terms of fundamental invariants?
   3b.) How is the given invariant represented if the fundamental invariants form a Hironaka decomposition?

B. Given a set of homogeneous invariants $\pi_1, \dots, \pi_r$ and a set of homogeneous equivariants $b_1, \dots, b_s$.

   1.) What are the relations?

2.) Do the given equivariants generate the module of equivariants over the ring generated by the given invariants $\pi_1, \ldots, \pi_r$?

2b) Is the module generated up to a certain degree?

3.) How does one represent a given equivariant in terms of fundamental invariants and equivariants?

C. Given a representation of a finite group.

1.) How can one determine the fundamental invariants algorithmically such that they form a Hironaka decomposition? (This means that for each invariant there is a unique representation in terms of fundamental invariants).

2.) How does one determine the fundamental equivariants such that they form a free module over the ring generated by the primary invariants?

D. Given a representation of a compact Lie group. Since we are interested in equivariant dynamics we restrict to the case of linear reductive groups which include all finite-dimensional representations of compact Lie groups. Then the invariant ring is Cohen-Macaulay, see [7], [25].

1.) How can one compute a homogeneous invariant for a given degree? (Observe that the Reynold operator for finite groups (projection onto invariants) does not exist or can not be evaluated for continuous groups).

2.) How can one compute a fundamental set of invariants?

3.) How can one modify a fundamental set of invariants into a Hironaka decomposition? Then each invariant can be written in terms of the fundamental invariants in a unique way.

4.) How does one determine a homogeneous equivariant of a given degree?

5.) How can one compute a generating set of equivariants over a ring generated by a given set of invariants?

6.) Given a set of fundamental invariants and equivariants, how is a Stanley decomposition determined?

E. Given an equivariant vector field and a set of fundamental invariants, how are the equations on the orbit space computed?

There is a lot of information in the literature about these questions. However this is not well recognized by the dynamical systems community. Table 1 summarizises the relevant books and articles so far they contain efficient algorithms and gives an overview of implementations. *Invar* [29] and *Symmetry* [19] are implemented in Maple while an implementation for C.1.) in Singular [24] exists as well.

The aim of this paper is to communicate this existing knowledge to the community of researchers in equivariant dynamics. Secondly, we want to demonstrate that the algorithms work for realistic examples from the literature in finite time (which is not obvious

	A					B			
	1.)	2.)	2b.)	3.)	3b.)	1.)	2.)	2b.)	3.)
literature	[5] p.269	[44] p.32		[5] p.269	[44] p.52				[18] p.115
Invar	x	x	x	x	x				
Symmetry	x	x	x	x	x	x	x	x	x

	C		D					
	1.)	2.)	1.)	2.)	3.)	4.)	5.)	6.)
literature	[44, 30] p.57 –	[48, 18]			[15]			[46]
Invar	x							
Symmetry	x	x	x	x	x	x	x	

Table 1: Literature containing algorithms answering questions in invariant theory and their implementations.

for this class of algorithms). New contributions are the use of the *multigraded Hilbert series driven Buchberger algorithm* [9, 22] for computation of relations, completeness of equivariants (B. 2.)), restriction of completeness to certain degree (A.2b.), B. 2b.)), membership of free module. Restriction with respect to various gradings are the key to the partial completeness questions. This is illustrated by examples which have been computed on a Dec Alpha workstation. These new ideas mainly improve the efficiency of existing algorithms. The algorithmic treatment for continuous groups has been implemented and tested for the first time.

2 Preliminaries: computations with ideals and modules

This section gives an overview of the computations including Gröbner bases. They are necessary in order to understand the sections on algorithmic invariant theory. Moreover we are using a special variant which proofs to be very efficient. For those who want to learn more about it we refer to [5], [9], [13], [37], and the references therein.

2.1 Gröbner bases and syzygies

We are dealing with ideals in a polynomial ring $K[x_1, \ldots, x_n]$ where the field K is in most practical computations $\mathbf{Q}$. A Gröbner basis is a special ideal basis depending on an order of the monomials.

Definition 2.1 *([5] p. 189) $\leq$ is called a term order, if* $\forall\, x^\alpha, x^\beta, x^\gamma \in K[x_1, \ldots, x_n]$

$$\begin{array}{lll} x^\alpha \leq x^\alpha & & \\ x^\alpha \leq x^\beta \text{ and } x^\beta \leq x^\gamma & \Rightarrow & x^\alpha \leq x^\gamma \\ x^\alpha \leq x^\beta \text{ and } x^\beta \leq x^\alpha & \Rightarrow & x^\alpha = x^\beta \\ x^\alpha \leq x^\beta \text{ or } x^\beta \leq x^\alpha & & \\ 1 \leq x^\alpha & & \\ x^\alpha \leq x^\beta \quad \Rightarrow \quad x^\alpha x^\gamma \leq x^\beta x^\gamma & & \end{array}$$

Example 2.2 *: Let the matrix $M \in \mathbf{Z}^{n,n}$ have the following properties:*

i.) for each column j the first nonzero entry is positive.

$$\forall j\ \exists k \text{ with } m_{ij} = 0 \quad \forall i < k \text{ and } m_{kj} > 0$$

ii.) M has full rank.

By $M\alpha \leq M\beta$ ($\Leftrightarrow \exists k$ with $(M\alpha)_i = (M\beta)_i\ \forall i < k, (M\alpha)_k \leq (M\beta)_k$) is a term order defined. Although term orders might be defined in a different way they are all equivalent to such a matrix order ([39], [47]).

For a polynomial $f \in K[x]$ we denote by $ht(f)$ its leading term, i.e. the monomial with highest order and nonvanishing coefficient $hc(f)$ in f:

$$f = hc(f) \cdot ht(f) + \text{lower order terms.}$$

Definition 2.3 *An ideal basis $\{f_1, \ldots, f_m\}$ of $I = (f_1, \ldots, f_m)$ is called a Gröbner basis, if the ideal of the leading terms equals the ideal generated by all leading terms of elements of I:*

$$(ht(f_1), \ldots, ht(f_m)) = (ht(f))_{f \in I}$$

So the theory of Gröbner bases is closely related to the theory of monomial ideals. From a political point of view the most important thing about Gröbner bases is that they are used to solve systems of equations. From a mathematical point of view the most important point is that the quotient ring $K[x]/I$ has unique representatives which are computable by a division algorithm modulo a Gröbner basis. The division algorithm is defined for a set of polynomials F. normalf(f, F) or short $\bar{f}^F$ denotes the result of the division applied to f and is called *normal form*. The result may still depend on how the division is done. It is an important fact that the result of the division algorithm modulo a Gröbner basis is unique.

Definition 2.4 *A Gröbner basis $\{f_1, \ldots, f_m\}$ such that $\{ht(f_1), \ldots, ht(f_m)\}$ form a minimal basis of the monomial ideal and the f_i are interreduced is called a* reduced Gröbner basis.

A reduced Gröbner basis is unique. Although there exist infinitely many term orders for each ideal only a finite number of reduced Gröbner basis exists ([3, 38, 45]).

Definition 2.5 *Given a set $F = \{f_1, \ldots, f_m\}$ with leading terms $ht(f_i)$ a* syzygie *is a tuple $(s_1, \ldots, s_m) \in K[x]^m$ such that*

$$\sum_{i=1}^{m} s_i \cdot hc(f_i)ht(f_i) = 0.$$

The set of all syzygies form a $K[x]$-module denoted by $S(F)$. Each syzygie s corresponds to a polynomial in the ideal generated by F by defining $s \cdot F = \sum_{i=1}^{m} s_i f_i$. The Buchberger algorithm is based on the fact that special sparse syzygies

$$S^{ij} \in S(F), \quad 1 \leq i < j \leq m, \tag{1}$$

form a module basis, where $S_k^{ij} = 0$, $\forall k \neq i, k \neq j$, and

$$S_i^{ij} = \frac{lcm(ht(f_i), ht(f_j))}{ht(f_i)} lc(f_j), \quad S_j^{ij} = \frac{lcm(ht(f_i), ht(f_j))}{ht(f_j)} lc(f_i).$$

$S(f_i, f_j) := S^{ij} \cdot F$ is called *S-polynomial.*
F is a Gröbner basis if

$$normalf(s \cdot f, F) = 0, \quad \forall s \in S(F).$$

This condition is satisfied if it holds for a module basis of $S(F)$, e.g. for the basis $S^{ij}, 1 \leq i < j \leq m$. It turns out that S^{ij} form not a minimal basis. Some can be dropped by the Buchberger criteria.
1. Buchberger criterion: If $ht(f_i), ht(f_j)$ are disjoint then S^{ij} is superfluous.
2. Buchberger criterion: If $ht(f_j) | lcm(ht(f_i), ht(f_k))$ and S^{ij}, S^{jk} are considered then S^{ik} is superfluous.

2.2 Graded rings and graded modules

Definition 2.6 *([14] p. 29) A ring is called* graded*, if a direct sum decomposition $R = \bigoplus_{i=0}^{\infty} R_i$ exists such that*

$$R_i \cdot R_j \subseteq R_{i+j}$$

holds for all $i, j \in \mathbf{N}$.

Example 2.7 *: The polynomial ring $K[x]$ with the usual degree is a graded ring. Besides this natural grading there are other gradings: Let $w_1, \ldots, w_n \in \mathbf{N}$ be weights on the variables $x_1, \ldots, x_n$. ($W : \{x_1, \ldots, x_n\} \to \mathbf{N}, W(x_i) = w_i$). The* weighted degree *is defined by*

$$\deg_W(x^\alpha) = \sum_{i=1}^{n} w_i \alpha_i.$$

*Polynomials $f = \sum_{\alpha \in A} a_\alpha x^\alpha$ with $deg_W(x^\alpha)$ equal for all $\alpha \in A$ are called W-*homogeneous*. Homogeneous polynomials of a certain degree form vector spaces yielding the graded structure*

$$K[x] = \bigoplus_{i=0}^{\infty} H_i^W(K[x]).$$

The weights $1, \ldots, 1$ refer to the natural grading. All gradings of $K[x]$ are given in this way ([5] p. 467). If all $w_i > 0$ then $H_0^W(K[x]) = K$ and all K-vector spaces $H_i^W(K[x])$ have finite dimension.

Of course a ring may be graded several times.

Definition 2.8 *([14] p.42) Let R be a graded ring. A module M is called graded, if it is the direct sum $M = \bigoplus_{j=0}^{\infty} M_j$ such that $R_i M_j \subset M_{i+j} \, \forall \, i, j \in N$.*

Example 2.9 *: i.) Let $K[x_1, \ldots, x_n]$ be graded with weights $w_1, \ldots, w_n$. If $z_1, \ldots, z_m$ are additional variables then $\Gamma : \{x_1, \ldots, x_n, z_1, \ldots, z_m\} \to \mathbf{N}$*

$$\Gamma(x_i) = 0, \, i = 1, \ldots, n, \quad \Gamma(z_j) = 1, \, j = 1, \ldots, m,$$

defines a grading on $K[x_1, \ldots, x_n, z_1, \ldots, z_m]$ yielding

$$K[x, z] = \bigoplus_{i=0}^{\infty} H_i^{\Gamma}(K[x, z]).$$

Each $H_i^{\Gamma}(K[x, z])$ is a $K[x]$-module. $H_1^{\Gamma}(K[x, z])$ is especially interesting because each finitely generated, free $K[x]$-module is isomorphic to a $H_1^{\Gamma}(K[x, z])$.
The grading W on $K[x]$ may be extended to $K[x, z]$ by weights on z_j:

$$\Theta : \{x_1, \ldots, z_m\} \to N, \Theta_{|K[x]} = W.$$

A restriction is a grading of the module $H_1^{\Gamma}(K[x, z])$:

$$K[x, z] = \bigoplus_{k=0}^{\infty} H_k^{\Theta}(K[x, z]) \qquad (2)$$

$$H_1^{\Gamma}(K[x, z]) = \bigoplus_{k=0}^{\infty} H_{1,k}^{\Gamma,\Theta}(K[x, z]) \text{ with} \qquad (3)$$

$$H_{1,k}^{\Gamma,\Theta}(K[x, z]) = H_1^{\Gamma}(K[x, z]) \cap H_k^{\Theta}(K[x, z]) \qquad (4)$$

ii.) A second example for a graded module is an ideal I of a Θ-graded ring $K[x]$ which is generated by Θ-homogeneous polynomials.

Definition 2.10 *A* Θ-homogeneous ideal *I of $K[x]$ where $K[x]$ is Θ-graded, is an ideal which respects the grading, i.e. is a Θ-graded module.*

For each Θ-homogeneous ideal I the quotient ring $K[x]/I$ is a graded module over $K[x]$ as well.

Definition 2.11 *([2] p.116) Let M be a finitely generated, graded module ($M = \bigoplus_{i=0}^{\infty} M_i$) over a Noetherian graded ring $R = \bigoplus_{i=0}^{\infty} R_i$ such that $R_0 = K$ is a field. Then*

$$\mathcal{HP}(\lambda) = \sum_{i=0}^{\infty} \dim(M_i)\lambda^i$$

is called the Hilbert-Poincaré series of M. Here $\dim(M_i)$ denotes the dimension of the K-vector spaces M_i.

If M is multigraded the multiple Hilbert series is defined in a similiar way.

Definition 2.12 *([9]) A tupel of gradings $(W_1, \ldots, W_r)$ of $K[x]$ is a* weight system *if for all $i = 1, \ldots, n$ exists $j \in \{1, \ldots, r\}$ with $W_j(x_i) > 0$.*

Example 2.13 *If $K[x]$ is graded by W such that all values $W(x_i) = w_i$ are positive then this grading is a weight system. Then all K-vector spaces $H_i(K[x]/I)$ have finite dimension which equals the codimension of $H_i(I)$ in $H_i(K[x])$. Thus the Hilbert series $\mathcal{HP}^W_{K[x]/I}$ is well-defined.*

For monomial ideals the corresponding Hilbert series (single graded or multigraded) may be computed by an algorithm described in [4]. It is implemented in Macaulay and in Maple in the moregroebner package [17]. The input consists of a finite set of monomials and a list of gradings forming a weight system.

It is an important and well-known fact about Gröbner bases that they enable the computation of Hilbert series of homogeneous ideals as was already pointed out in [8].

Lemma 2.14 *[35] Let $(W_1, \ldots, W_r)$ be a weight system for $K[x]$ and I a W-homogeneous ideal. Let $LT(I)$ be the monomial ideal generated by all leading terms $ht(f)$ of $f \in I$ with respect to a term order of $K[x]$. Then the Hilbert series of I and $LT(I)$ are equal.*

Since the leading terms $\{ht(f), f \in \mathcal{GB}\}$ of a Gröbner basis $\mathcal{GB}$ generate the monomial ideal $LT(I)$ the series $\mathcal{HP}_{K[x]/I}$ is easily computed.

2.3 Hilbert series driven Buchberger algorithm

Definition 2.15 *Let $U = \{U_1, \ldots, U_r\}$ be a set of gradings of $K[x]$ and I a U-homogeneous ideal. Let $d \in \mathbf{N}^r$ be a fixed degree. A finite set of U-homogeneous polynomials $F \subset I$ is called a d-truncated Gröbner basis of I with respect to U and denoted by $\mathcal{GB}(\bigoplus_{j \le d} H^U_j(I))$, if*

$$\{ht(f)| \quad f \in F \text{ and } deg_{U_i}(f) \le d_i, i = 1, \ldots, r\}$$

generates

$$\bigcap_{i=1}^{r} \bigoplus_{j_i=0}^{d_i} H^{U_i}_{j_i}(LT(I)) = \bigoplus_{j \le d} H^U_j(LT(I)).$$

Observe that this definition only makes sense for homogeneous ideals and that there is no restriction on the term order.
This definition is useful in at least two ways.

Definition 2.16 *([5]) Consider $K[x,z]$ with the grading $\Gamma(x_i)=0, \Gamma(z_j)=1$. A* module Gröbner basis *of a submodule of $H_1^\Gamma(K[x,z])$ is a truncated Gröbner basis of degree 1 with respect to Γ.*

Lemma 2.17 *Let $U=\{U_1,\ldots,U_r\}$ be a set of gradings of $K[x]$ and $d\in\mathbf{N}^r$ a fixed degree. Assume $\mathcal{GB}\subset K[x]$ is a d-truncated Gröbner basis of a U-homogeneous ideal I wrt U. Let $f\in K[x]$ be a polynomial with $deg_{U_i}(f)\le d_i$, $i=1,\ldots,r$. Then*

$$f\in I \Leftrightarrow normalf(f,\mathcal{GB})=0.$$

This is the generalization of Thm. 10.39, p. 471 in [5] from one grading to multiple grading.

The gradings $U=\{U_1,\ldots,U_r\}$ inherit to the module of syzygies $S(F)$ for a set $F=\{f_1,\ldots,f_m\}$ in the following way:

$$S\in H^{U_1\ldots U_r}_{j_1\ldots j_r}(S(F)) \quad :\Leftrightarrow S_k\cdot ht(f_k)\in H^{U_1\ldots U_r}_{j_1\ldots j_r}(K[x]), \quad k=1,\ldots,m.$$

Especially the syzygies S^{kl} as defined in (1) are homogeneous of degree

$$\deg_U(lcm(ht(f_k),ht(f_l))).$$

Theorem 2.18 *Let $U=\{U_1,\ldots,U_r\}$ be a set of gradings of $K[x]$ and $W=(W_1,\ldots,W_s)$ a weight system of $K[x]$. Let $I\subset K[x]$ be an ideal which is U-homogeneous and W-homogeneous. Let $d\in\mathbf{N}^r$ be a degree, $F=\{f_1,\ldots,f_m\}\subset I$ a set of U-homogeneous, W-homogeneous polynomials. Assume the Hilbert series $\mathcal{HP}^W_{K[x]/I}(\lambda)=\sum_{i\in\mathbf{N}^s}a^i\lambda^i$ is given. Let the tentative Hilbert series denote by $\mathcal{HP}^W_{K[x]/(LT(F))}(\lambda)=\sum_{i\in\mathbf{N}^s}b^i\lambda^i$. Assume for all degree $i\in N^s$*

$$\begin{array}{lll} a^i=b^i \quad \textbf{or} & \textit{for all syzygies } S:=S^{kl} \textit{ with} & \deg_{U_j}(S)\le d_j, j=1,\ldots,r, \textit{ and} \\ & & \deg_{W_\nu}(S)\le i_\nu, \nu=1,\ldots,s \\ & normalf(S\cdot F,F)=0. & \end{array}$$

Then F is a d-truncated Gröbner basis of I with respect to U.

Proof: We need to show that for all degrees $j\le d$ for all syzygies $S\in H_j^U(S(F))$ normalf$(s\cdot F,F)=0$. The weight system is exploited: The condition is equivalent to

$$\forall\, j\le d \quad \forall\, i\in\mathbf{N}^s \quad \forall S\in H^{W,U}_{i,j}(S(F)) \quad \text{normalf}(S\cdot F,F)=0. \tag{5}$$

The graded structure garantees that

$$\dim(H^{W,U}_{i,j}(I))=\dim(H^{W,U}_{i,j}(LT(F))\Leftrightarrow \text{normalf}(S\cdot F,F)=0 \text{ for all } S\in H^{W,U}_{i,j}(S(F)).$$

The dimensions are given by the Hilbert series

$$\mathcal{HP}^{W,U}_{K[x]/I}(\lambda,\mu) = \sum_{i\in\mathbf{N}^s, j\in\mathbf{N}^r} c^{ij}\lambda^i\mu^j$$

$$\mathcal{HP}^{W,U}_{K[x]/(LT(F))}(\lambda,\mu) = \sum_{i\in\mathbf{N}^s, j\in\mathbf{N}^r} e^{ij}\lambda^i\mu^j$$

where $a^i = \sum_{j\in\mathbf{N}^r} c^{ij}$ and $b^i = \sum_{j\in\mathbf{N}^r} e^{ij}$. Thus $a^i = b^i$ for some $i\in\mathbf{N}^s$ implies condition (5) for this $i\in\mathbf{N}^s$ and all $j\le d$.

One may attack condition (5) in another way. For $i\in\mathbf{N}^s, j\in\mathbf{N}^r, j\le d$ one needs to show (5) for a vector space basis of $H^{W,U}_{i,j}(S(F))$. Since the syzygies S^{kl} are (W,U)-homogeneous the syzygies $x^\alpha\cdot S^{kl}$ with all $k,l=1,\ldots,m, k\neq l$ and all $\alpha\in\mathbf{N}^n$ such that $\deg_{W,U}(x^\alpha\cdot lcm(ht(f_k),ht(f_l))=(i,j)$ form a set of generators for $H^{W,U}_{i,j}(S(F))$ as a vector space. This completes the proof. □

An implementation using this theorem motivated by the non-truncated version of the multi-graded Hilbert series driven Buchberger algorithm in [9] is available by www, [17]. The basic idea is to climb up by degree and use the information on the Hilbert series for cancelation of S-polynomials.

Observe that the exploitation of the Buchberger criteria are not affected by the truncation because the gradings inherit onto the module of syzygies.

2.4 Elimination

Let $I\subset K[x_1,\ldots,x_n,y_1,\ldots,y_m]$ be an ideal.

Definition 2.19 *$\tilde{I} = I\cap K[y]$ is called the* elimination ideal *of I in $K[y]$.*

Definition 2.20 *A monomial order $\le$ of $K[x,y]$ is called an elimination order, if*

$$x^\alpha y^\gamma \ge y^\delta$$

for all cases with $x^\alpha\neq 1$.

By [5] p. 257 Lemma 6.14 it is sufficient to demand

$$x^\alpha \ge y^\delta \quad \forall\, x^\alpha\neq 1, y^\delta.$$

Example 2.21 *Elimination orders on $K[x_1,\ldots,x_n,y_1,\ldots,y_m]$ include*

a.) the lexicographical order.

b.) all matrix orders with matrix $M=(M_{ij})$ with first column $M_{1i}=1, i=1,\ldots,n, M_{1i}=0, i=n+1,\ldots,n+m$. This includes the elimination order by Bayer and Stillman.

c.) block orders $\ge$ consisting of orders $\ge_x$ and $\ge_y$ on $K[x]$ and $K[y]$, respectively. ($x^\alpha y^\gamma > x^\beta y^\delta \Leftrightarrow x^\alpha >_x x^\beta$ or $x^\alpha = x^\beta$ and $y^\gamma >_y y^\delta$).

Lemma 2.22 *([5] p.269, [13] p. 329) Let* $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ *and* $\leq$ *be an elimination order for* $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$. *Let* $\mathcal{GB}$ *be a Gröbner basis of*

$$I = (f_1(x) - y_1, \ldots, f_m(x) - y_m).$$

Then $\mathcal{GB} \cap K[y]$ *is a Gröbner basis of the elimination ideal with respect to* $\leq_{|K[y]}$.

(i) $f(x) \in K[f_1, \ldots, f_m] \Leftrightarrow normalf(f, \mathcal{GB})(x, y) \in K[y]$.

(ii) if $f \in K[f_1, \ldots, f_m]$ *then* $g := normalf(f, \mathcal{GB}) \in K[y]$ *gives a rewriting* $f(x) = g(f_1(x), \ldots, f_m(x))$.

Observe that $normalf(f, \mathcal{GB})$ depends in general on the term order. The relevance of Lemma 2.22 is that $K[f_1(x), \ldots, f_m(x)]$ and $K[y]/\tilde{I}$ are isomorphic as rings.

3 Gröbner bases for invariants and equivariants

In equivariant dynamics we are concerned with the invariant ring and the module of equivariants. Assume G is a group and $\vartheta : G \to GL(K^n)$ its representation. Let $\rho : G \to GL(K^m)$ be another representation. (Of course the field K is $\mathbf{R}$ in theory, but for practical computations it often will be $\mathbf{Q}$ or an extension of it.) Moreover we assume that ϑ, ρ are orthogonal.
$p \in K[x]$ is called *invariant*, if

$$p(\vartheta(g)x) = p(x) \quad \forall g \in G.$$

By $K[x]_\vartheta$ we denote the invariant ring. $f \in K[x]^m$ is called *ϑ-ρ-equivariant*, if

$$f(\vartheta(g)x) = \rho(g)f(x) \quad \forall g \in G.$$

The $K[x]_\vartheta$-module is denoted by $K[x]_\vartheta^\rho$.

In equivariant dynamical systems one studies

$$\dot{x} = f(x)$$

where f is a generic equivariant vector field. f is polynomial if a truncation of the Taylor expansion has been performed. So the knowledge of generators of the invariant ring and the module are the key for the investigation of generic equivariant dynamical systems.

3.1 Relations

Let $\{p_1, \ldots, p_k\}$ denote the Hilbert basis of the invariant ring $K[x]_\vartheta$. The relations among the basis elements are easily determined with the help of Gröbner bases. Recall that the relations are polynomials $r \in K[y_1, \ldots, y_k]$ with $r(p_1(x), \ldots, p_k(x)) \equiv 0$. They form an ideal.

Algorithm 3.1 *(Relations in fundamental invariants up to degree d)*
Input: *Invariant polynomials* $p_1(x), \ldots, p_k(x) \in K[x]$, *homogeneous with respect to the natural grading* N *on* $K[x]$
Output: *relations* $r_1, \ldots, r_l \in K[y_1, \ldots, y_k]$
i.) Compute the Hilbert series $\mathcal{HP}_{K[x,y]/(y_1,\ldots,y_k)}$ *with respect to the induced grading* $W : \{x, y\} \to \mathbf{N}, W_{|\{x\}} = N,\ W(y_i) = \deg_N(p_i(x)), i = 1, \ldots, k.$
ii.) Choose an elimination order $\leq$ *on* $K[x, y]$ *which eliminates* x.
iii.) Compute a Gröbner basis $\mathcal{GB}(\bigoplus_{i\leq d} H_i^W(I))$ *of the ideal*

$$I := (p_1(x) - y_1, \ldots, p_k(x) - y_k) \subset K[x, y],$$

with respect to the elimination order, using the Hilbert series driven Buchberger algorithm with the Hilbert series under i.), truncate with respect to the grading W *up to degree* d.
The polynomials $\mathcal{GB} \cap K[y]$ *are* $W_{|\{y\}}$*-homogeneous and form a truncated Gröbner basis of degree* d *with respect to* W *of the ideal of relations* $\tilde{I} \subset K[y]$.

Proof: Without exploiting homogenicity the correctness of the algorithm is given by Lemma 2.22. Since the polynomials $p_i(x) - y_i, i = 1, \ldots, k$ are homogeneous with respect to the induced grading W, it makes sense to use truncation and the Hilbert series driven Buchberger algorithm. By the 1. Buchberger criterion the polynomials $p_i(x) - y_i$ form a Gröbner basis of I with respect to a term order which select y_i as leading terms. The Hilbert series $\mathcal{HP}^W_{K[x,y]/I}(\lambda)$ is given by the series of the monomial ideal $(y_1, \ldots, y_k)$ by Lemma 2.14. The restriction $W_{|\{y\}}$ is a weighted grading on $K[y]$. Since $p_i - y_i$ are homogeneous all other polynomials in the Buchberger algorithm are homogeneous, especially $\mathcal{GB}(\tilde{I})$. Thus the ideal of relations is W-homogenous and it makes sense to consider the part up to degree d. □

Remark 3.2 *i.) Different choices of elimination term orders in ii.) give different sets of output polynomials. Nevertheless the same space* $\bigoplus_{i=0}^d H_i^W(\tilde{I})$ *is generated.*
ii.) In case no Hilbert series driven Buchberger algorithm is available, special choices of an elimination order influence the cpu time a lot. Good experience has been gained with a matrix order with first row given by W. *Since* $p_i - y_i$ *are* W*-homogeneous this does not affect the final Gröbner basis, but gives good exploitation of the Buchberger criteria because the S-polynomials are ordered with respect to sugar and then ties are broken by the term order.*

Given some homogeneous invariant polynomials $p_1(x), \ldots, p_k(x)$ and some homogeneous equivariants $f_1(x), \ldots, f_l(x) \in K[x]^\rho_\vartheta$ one denotes by relations the set of polynomials

$$\mathcal{R} = \{r \in K[y, u] | r(p(x), f(x)) \equiv 0, r \in H_1^\Gamma(K[y, u])\}$$

where $\Gamma : \{y, u\} \to \mathbf{N}, \Gamma_{|\{y\}} = 0, \Gamma_{|\{u\}} = N$ is a Kronecker grading.

It is interesting to know a generating set of $\mathcal{R}$ since the $K[p(x)]$-module generated by $f_1, \ldots, f_l$ is isomorphic to the $K[y]/\tilde{I}$-module $H_1^\Gamma(K[y, u]/\mathcal{R}$, where $\tilde{I}$ is the ideal of relations in the invariants.

A Gröbner basis $\mathcal{GB}(\tilde{I})$ gives some generators $u_i \cdot g, g \in \mathcal{GB}(\tilde{I}), i = 1, \ldots, l$, of $\mathcal{R}$. By the following algorithm these are completed to a generating set of $\mathcal{R}$.

Algorithm 3.3 *(Relations in fundamental invariants and equivariants up to degree d)*
Input: *homogeneous invariants* $p_1(x), \ldots, p_k(x)$
homogeneous equivariants $f_1(x), \ldots, f_l(x) \in K[x]^m$
degree d
Output: *relations* $r_1, \ldots, r_s \in H_1^\Gamma(K[y, u])$
i.) Choose slack variables $z_1, \ldots, z_m$.
ii.) Choose a termorder $\leq$ *on* $K[x, z, y, u]$ *which eliminates* x *and* z.
iii.) $W : \{x, z, y, u\} \to \mathbf{N}, W_{|\{x\}} = N, W_{|\{z\}} = 0, W(y_i) = \deg_N(p_i(x)), i = 1, \ldots, k,$
$W(u_i) = \deg_N(f_i(x)), i = 1, \ldots, l.$
iv.) Compute $\mathcal{GB}(\bigoplus_{i \leq d} H_i^W(I))$ *with respect to* $\leq_{|K[x,y]}$ *of*
$I = (p_1(x) - y_1, \ldots, p_k(x) - y_k) \subset K[x, y]$ *by Algorithm 3.1.*
v.) Use grading $U : \{x, z, y, u\} \to \mathbf{N}, U_{|\{x,y\}} = 0, U_{|\{z,u\}} = N.$
vi.) $g_i(x, z) := \sum_{j=1}^m (f_i(x))_j z_j, i = 1, \ldots, l.$ *Define*

$$J = (p_1(x) - y_1, \ldots, p_k(x) - y_k, g_1(x, z) - u_1, \ldots, g_l(x, z) - u_l) \subset K[x, z, y, u].$$

vii.) Compute the Hilbert series $\mathcal{HP}^{W+U,W}_{K[x,z,y,u]/J}$ *equal to the Hilbert series given by* $(y_1, \ldots, y_k, u_1, \ldots, u_l)$.
viii.) Compute $\mathcal{GB}(\bigoplus_{(i,j) \leq (d,1)} H_{i,j}^{W,U}(J))$ *with respect to* $\leq$ *of* J. *Use the Hilbert series driven Buchberger algorithm with Hilbert series* $\mathcal{HP}^{W+U,W}_{K[x,z,y,u]/J}$ *as under vii.) and as input use*

$$\mathcal{GB}(\bigoplus_{i \leq d} H_i^W(I)) \text{ and } \mathrm{normalf}(g_i(x, z) - u_i, \mathcal{GB}(\bigoplus_{i \leq d} H_i^W(I))),\ i = 1, \ldots, l.$$

ix.) The relations are $\mathcal{GB} \cap H_1^U(K[y, u])$.

Proof: By using the slack variables the relations in p_i and g_j of degree 1 equals the set $\mathcal{R}$, the relations in p_i and f_j. The grading U enables the restriction to the module. The Gröbner basis of J gives a Gröbner basis of $\tilde{J} = J \cap K[y, u]$ by Lemma 2.22. With U one obtains a Gröbner basis of $\bigoplus_{i=0}^1 H_i^U(\tilde{J})$ where $H_0(\tilde{J}) = \tilde{I}$. Since $I \subset J$ one can first compute a Gröbner basis of I. Because J is homogeneous with respect to W and the Buchberger algorithm preserves homogenecity one can restrict to degree d. $\square$

Remark 3.4 *i.) If no Hilbert series driven Buchberger algorithm is available Remark 3.2 is valid analogously.*
ii.) If the representation $\rho : G \to GL(K^m)$ *contains no trivial irreducible representation in its isotypic decomposition the efficiency can be improved significantly by using a termorder which eliminates* x *only. If* ρ *contains not the trivial representation the slack variables are eliminated automatically.* ρ *induces an action on* z *as* $\rho^t : G \to GL(K^m), g \mapsto \rho(g)^t$. *The polynomials* $g_i(x, z)$ *are* $(\rho + \rho^t)$*-invariant.* $J \cap H_1^U(K[z, y])$ *is generated over* $K[y]$

	relations in invs.		relations in equivariants		
	no HP	with HP	1 GB no HP	1 GB with HP	GB Alg. 3.3
total nr of pairs	3003	1743	26478	16775	15419
elim. pairs by trunc.			39952	21148	23362
elim. pairs by criteria	2502	1276	23126	13558	12681
elim. pairs by HP		322		2469	2290
treated pairs	501	145	3352	748	448
cpu	807 sec	239 sec	10.5 h	6.2 h	4.9 h

Table 2: Computation of relations in invariants and equivariants for $D_3 + D_3$ with and without use of Hilbert series and as in Algorithm 3.3.

by the ρ^t-invariant in $K[z]$ of degree 1. Since ρ has no trivial irreducible representation there exists no ρ^t-invariant of degree 1 and thus $J \cap H_1^U(K[z,y])$ is empty. Because of the symmetry this is equivalent to $J \cap H_1^U(K[z,y,u]) = J \cap H_1^U(K[y,u])$.
iii.) If ρ contains the trivial representation then one should use an elimination order which first eliminates x and in a second step z.
iv.) It is prefarable to use the weight system $[U+W,W]$ instead of $[U,W]$ since $U+W$ is already a weight system. This is used in the Hilbert series driven version of the Buchberger algorithm.

Example 3.5 *In order to investigate a Takens-Bogdanov point with D_3-symmetry in [36] a generic equivariant vector field is investigated for the action of D_3 generated by*

$$flip(v,w) = (\bar{v},\bar{w}), \;\; and \;\; rotation(v,w) = (e^{i\frac{2\pi}{3}}v, e^{i\frac{2\pi}{3}}w)$$

which decomposes as two times the natural two-dimensional action. The invariants and equivariants in these coordinates are suggested as

$$s_1 = v\bar{v}, s_2 = w\bar{w}, t_0 = w^3+\bar{w}^3, t_3 = v^3+\bar{v}^3, s_3 = v\bar{w}+\bar{v}\,w, t_1 = vw^2+\bar{v}\,\bar{w}^2, t_2 = v^2w+\bar{v}^2\,\bar{w}.$$

$$g_0 = \begin{pmatrix} v \\ 0 \end{pmatrix}, g_1 = \begin{pmatrix} 0 \\ v \end{pmatrix}, g_2 = \begin{pmatrix} w \\ 0 \end{pmatrix}, g_3 = \begin{pmatrix} 0 \\ w \end{pmatrix}, \;\; and$$

$$f_j = \begin{pmatrix} \bar{v}\bar{w}^{3-j-1} \\ 0 \end{pmatrix}, h_j = \begin{pmatrix} 0 \\ \bar{v}\bar{w}^{3-j-1} \end{pmatrix}, \quad j = 0,1,2.$$

The complex notation is chosen since this is common in analysis and is appropriate for hand calculations. For computation in a Computer Algebra Package one chooses different coordinates: $v = v_r + i\cdot v_i, w = w_r + i\cdot w_i$. Then the invariants and equivariants are

$$s_1 = {v_r}^2 + {v_i}^2, s_2 = {w_r}^2 + {w_i}^2, t_0 = 2\,{w_r}^3 - 6\,w_r\,{w_i}^2, t_3 = 2\,{v_r}^3 - 6\,v_r\,{v_i}^2,$$

$$s_3 = 2\,v_r\,w_r + 2\,v_i\,w_i, t_1 = 2\,v_r\,{w_r}^2 - 2\,v_r\,{w_i}^2 - 4\,v_i\,w_r\,w_i, t_2 = 2\,{v_r}^2 w_r - 4\,v_r\,v_i\,w_i - 2\,{v_i}^2 w_r,$$

$$g_0 = [v_r, v_i, 0, 0], g_1 = [0, 0, v_r, v_i], g_2 = [w_r, w_i, 0, 0], g_3 = [0, 0, w_r, w_i],$$
$$f_0 = [{w_r}^2 - {w_i}^2, -2\, w_r\, w_i, 0, 0], f_1 = [v_r\, w_r - v_i\, w_i, -v_r\, w_i - v_i\, w_r, 0, 0],$$
$$f_2 = [{v_r}^2 - {v_i}^2, -2\, v_r\, v_i, 0, 0],$$
$$h_0 = [0, 0, {w_r}^2 - {w_i}^2, -2\, w_r\, w_i], h_1 = [0, 0, v_r\, w_r - v_i\, w_i, -v_r\, w_i - v_i\, w_r],$$
$$h_2 = [0, 0, {v_r}^2 - {v_i}^2, -2\, v_r\, v_i]$$

Table 2 shows some statistics of the computations of the relations. With the algorithms in subsection 3.2 it could be shown that the invariant ring and the module of equivariants are generated completely.

3.2 Completeness

Given some homogeneous invariants it is easily checked whether the invariant ring $K[x]_\vartheta$ is generated up to degree d by Gröbner bases. Without restriction in degree this is Algorithm 2.2.5 p. 32 in [44].

Algorithm 3.6 *(completeness of invariants up to degree d)*
Input: *Molien series of invariant ring* $\mathcal{HP}_{K[x]_\vartheta}(z)$*,*
homogeneous invariants $p_1(x), \ldots, p_k(x)$*,*
degree d
Output: *true or minimal degree of missing invariant*
i.) Compute relations $\mathcal{GB}(\bigoplus_{i\leq d} H_i^W(\tilde{I}))$ *by Algorithm 3.1.*
ii.) $LT := \{ht(f) | \quad f \in \mathcal{GB}\}$
iii.) Compute $\mathcal{HP}^W_{K[y]/(LT)}(z)$
iv.) If $\mathcal{HP}^W_{K[y]/(LT)}(z) \equiv \mathcal{HP}^W_{K[x]_\vartheta}(z)$ *then* "invariants generate invariant ring completely"
If series($\mathcal{HP}^W_{K[y]/(LT)}(z), d) \equiv$ *series(*$\mathcal{HP}^W_{K[x]_\vartheta}(z), d)$ *then* true
else mindeg(series($\mathcal{HP}^W_{K[x]_\vartheta}(z) - \mathcal{HP}^W_{K[y]/(LT)}(z), d)$

Proof: Since $p_1, \ldots, p_k$ are invariants we have $K[p(x)] \subset K[x]_\vartheta$. Since p_i are homogeneous in the natural grading N we have moreover

$$H_i^N(K[p]) \subset H_i^N(K[x]_\vartheta), \quad i = 0, \ldots$$

and thus

$$\dim H_i^N(K[p]) \leq \dim H_i^N(K[x]_\vartheta), \quad i = 0, \ldots,$$

Thus one only needs to compare $\mathcal{HP}^N_{K[p]}(z)$ and $\mathcal{HP}^N_{K[x]_\vartheta}(z)$ (up to degree d). Since the ideal of relations $\tilde{I} \subset K[y]$ is homogeneous with respect to the induced grading W the residue ring $K[y]/\tilde{I}$ is graded by $W_{|\{y\}}$ as well. Moreover $K[y]/\tilde{I}$ with respect to W and $K[p(x)]$ with respect to the natural grading are isomorphic as graded rings. Thus $\mathcal{HP}^W_{K[p]}(z) = \mathcal{HP}^W_{K[y]/\tilde{I}}(z)$. By Lemma 2.14 the Gröbner basis $\mathcal{GB}$ of $\tilde{I}$ yields $\mathcal{HP}^N_{K[p]}(z) = \mathcal{HP}^W_{K[y]/LT(\tilde{I})}(z)$. If one truncates the Gröbner basis to degree d the last equality is only valid for degrees $0, 1, \ldots, d$. $\square$

For the equivariants the algorithmic treatment is analogous.

Algorithm 3.7 *(completeness of fundamental equivariants up to degree d)*
Input: *Hilbert-Poincaré series* $\mathcal{HP}^N_{K[x]^\rho_\vartheta}(z)$ *of* $K[x]_\vartheta$*-module of equivariants* $K[x]^\rho_\vartheta$,
homogeneous invariants $p_1(x),\ldots,p_k(x)$,
homogeneous equivariants $f_1(x),\ldots,f_l(x)$,
degree d
Output: *true, if* $K[x]^\rho_\vartheta$ *is generated by* $f_1,\ldots,f_l$ *as* $K[p]$*-module up to degree d else minimal degree of missing part.*
i.) Compute $\mathcal{GB}(\bigoplus_{(i,j)\le(d,1)} H^{W,U}_{i,j}(\tilde J)$ *as in Algorithm 3.3.*
ii.) Define $LT := \{ht(f)| \quad f\in\mathcal{GB}\}$.
iii.) Compute the Hilbert series of the $K[p]$*-module* M *generated by* $f_1,\ldots,f_l$ *by*
hp:=0
for j from 1 to l do
$T_i = \{y^\alpha \in K[y] | y^\alpha \in LT \text{ or } u_i y^\alpha \in LT\}$
$hp := hp + z^{\deg(f_i)} \cdot \mathcal{HP}^W_{K[y]/(T_i)}(z)$
$\mathcal{HP}^N_M(z)_{|0..d} = \mathcal{HP}^W_{H^U_1(K[y,u])/\mathcal{R}}(z)_{|0..d} := hp(z)_{|0..d}$
iv.) If $\mathcal{HP}^N_M(z) = \mathcal{HP}^N_{K[x]^\rho_\vartheta}(z)$ *then* "module is generated completely".
If series($\mathcal{HP}^N_M(z),d$*) = series(*$\mathcal{HP}^N_{K[x]^\rho_\vartheta}(z),d$*) then* true
else mindeg(series($\mathcal{HP}^N_M(z) - \mathcal{HP}^N_{K[x]^\rho_\vartheta}(z),d$*)).*

Proof: As in the proof of correctness of Algorithm 3.6 we use that $K[p]$ and $K[y]/\tilde I$ are isomorphic as graded rings with respect to the natural grading N and $W_{|\{y\}}$ respectively. For $\tilde J := J\cap K[y,u]$ the submodule $H^U_1(\tilde J) = \mathcal{R}$ is a module over $K[y]$ and since $H^U_0(\tilde J) = \tilde I$ and $\bigoplus_{i=0}^l u_i \tilde I \subset H^U_1(\tilde J)$ the module is a $K[y]/\tilde I$-module. Since $H^U_1(\tilde J)$ is generated by $W_{|\{y,u\}}$-homogeneous elements the quotient $H^U_1(K[y,u]/H^U_1(\tilde J)$ is graded by the quotient grading of W over the graded ring $K[y]/\tilde I$.

In fact the K-vector spaces

$$H^W_i\left(H^U_1(K[y,u])/H^U_1(\tilde J)\right) \text{ and } H^N_i(M)$$

are isomorphic for all i. Together U, W form a weight system. Thus the spaces have the same dimension and the Hilbert series are equal.

In generalization of Lemma 2.14 the $K[y]/\tilde I$-module $H^U_1(K[y,u])/H^U_1(\tilde J)$ and the $K[y]/LT(\tilde I)$-module $H^U_1(K[y,u])/LT(H^U_1(\tilde J))$ are isomorphic as W-graded modules. The monomial module $LT(H^U_1(\tilde J))$ is easily determined from the Gröbner basis $\mathcal{GB}(H^U_0(\tilde J) + H^U_1(\tilde J))$. The monomials $\bigcup_{i=1}^l u_i LT(\mathcal{GB}(H^U_0))$ and

$$\{m \in LT(\mathcal{GB}(H^U_0(\tilde J) + H^U_1(\tilde J))| \quad \deg_U(m) = 1\}$$

generate $LT(H^U_1(\tilde J))$.

By exploiting $LT(H^U_1(\tilde J)) = \bigoplus_{i=1}^l u_i \cdot (T_i)$ the modules $H^U_1(K[y,u])/H^U_1(\tilde J)$ and $\bigoplus_{i=1}^l K[y]/(T_i)$ are isomorphic. As graded modules one needs to take into account the shifts in the gradings by $\deg_W(u_i) = \deg_N(f_i)$.

In case the Gröbner basis is only computed up to degree d the computed Hilbert series is of course only valid up to degree d. □

degree	invariants		equivariants	
	cpu	complete up to	cpu	complete up to
5	9 sec	5	18 min	5
6	20 sec	6	53 min	6
7	28 sec	7	2.0 h	8
8	47 sec	∞	4.6 h	10
11			21.9 h	∞
∞	265 sec	∞		

Table 3: Timings for completeness of invariants and equivariants for $O(2) \times S^1$.

Example 3.8 *In [34] a Hopf bifurcation with $O(3)$ symmetry is investigated. After reduction to a fixed point space one is left with the action*

$$\begin{aligned} r_\theta(z_{-2}, z_0, z_2) &= (e^{-i\theta} z_{-2}, z_0, e^{i\theta} z_2), \\ \kappa(z_{-2}, z_0, z_2) &= (z_2, z_0, z_{-2}), \\ \phi(z_{-2}, z_0, z_2) &= (e^{i\phi} z_{-2}, e^{i\phi} z_0, e^{i\phi} z_2) \end{aligned}$$

of $O(2) \times S^1$ where S^1 comes in due to the Hopf bifurcation. The invariants are suggested as

$$\begin{aligned} &\pi_1 = |z_0|^2, \quad \pi_2 = |z_{-2}|^2 + |z_2|^2, \quad \pi_3 = |z_{-2}|^2 \cdot |z_2|^2, \\ &\pi_4 = \tfrac{1}{2}(\overline{z_0}^2 z_{-2} z_2 + z_0^2 \overline{z_{-2}}\, \overline{z_2}) \\ &\pi_5 = \tfrac{i}{2}(\overline{z_0}^2 z_{-2} z_2 - z_0^2 \overline{z_{-2}}\, \overline{z_2}) \end{aligned}$$

For the equivariants Leis has given

$$e_1 = \begin{pmatrix} 0 \\ z_0 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} z_{-2} \\ 0 \\ z_2 \end{pmatrix}, \; e_3 = \begin{pmatrix} z_{-2}|z_2|^2 \\ 0 \\ z_2|z_{-2}|^2 \end{pmatrix},$$

$$e_4 = \frac{1}{2} \begin{pmatrix} z_0^2 \bar{z}_2 \\ 2 z_{-2} z_2 \overline{z_0} \\ z_0^2 \overline{z_{-2}} \end{pmatrix}, \; e_5 = -\frac{i}{2} \begin{pmatrix} z_0^2 \bar{z}_2 \\ -2 z_{-2} z_2 \overline{z_0} \\ z_0^2 \overline{z_{-2}} \end{pmatrix}.$$

such that the equivariants are $E_i = (e_i, \bar{e}_i)$. These coordinates are very common in dynamical systems. Then the multiplication for a complex number c is meant to be $c \cdot E_i = (c \cdot e_i, \bar{c} \cdot \bar{e}_i)$. In order to be able to compute one needs to change coordinates: $z_{-2} = \frac{1}{2}\sqrt{2}(x_1 + i \cdot y_1), z_0 = \frac{1}{2}\sqrt{2}(x_2 + i \cdot y_2), z_2 = \frac{1}{2}\sqrt{2}(x_3 + i \cdot y_3)$ or equivalently $(z, \bar{z}) = A \cdot (x, y)$. The transformed equivariants are

$$A^{-1} \cdot E_i(A \cdot (x, y)), \quad A^{-1} \cdot \begin{pmatrix} i \cdot e_i(A \cdot (x, y)) \\ -i \cdot e_i(A \cdot (x, y)) \end{pmatrix}, \; i = 1, \ldots, 5$$

which gives 10 equivariants. The Molien series and the equivariant Molien series have been computed in [34] with the Weyl integral formula and theorem of residues to be

$$\mathcal{HP}_{0(2)\times S^1}(\lambda) = \frac{1 + \lambda^4}{(1-\lambda^2)^2 \cdot (1-\lambda^4)^2}, \quad \mathcal{HP}^{0(2)\times S^1}_{0(2)\times S^1}(\lambda) = \frac{4\lambda + 6\lambda + 2\lambda^5}{(1-\lambda^2)^2 \cdot (1-\lambda^4)^2}.$$

Example	group	cpu time
Chossat [11]	O(2)	5 sec
Chossat, Dias [12]	$O(2)\times S^1$	72 sec
Leis [34]	$O(3)\times S^1$	160 sec

Table 4: Timings of Algorithm 3.9 (orbit space reduction) for examples from literature.

The completeness question for the equivariants turned out to be a hard problem. Table 3 gives some timings.

3.3 Orbit space reduction

In equivariant dynamics there is one method known as orbit space reduction, see [11, 12, 33, 32]. Assume an equivariant system

$$\dot{x} = f(x), \qquad x \in \mathbf{R}^n, \tag{6}$$

where $f \in \mathbf{R}[x]_\vartheta^\vartheta$ and ϑ is orthogonal. Let $p_1(x), \ldots, p_k(x)$ be the fundamental invariants which generate $\mathbf{R}[x]_\vartheta$. Then the dynamic of (6) is closely related to the dynamic of a system

$$\begin{array}{c} \dot{y} = g(y), \quad y \in \mathbf{R}^k \\ + \text{ algebraic relations} \end{array} \tag{7}$$

since

$$\frac{\partial}{\partial t} p_i(x) = \frac{dp_i}{dx}\dot{x} = \frac{dp_i}{dx} f(x) = h_i(x) = g_i(p_1(x), \ldots, p_k(x)),\ i = 1, \ldots, k,$$

where $g_i \in R[y]$ exist with this property since h_i is invariant. There is a lot to say about the topological structure but that is beyond the scope of this paper. Here we are only interested in those calculations which are often done by hand but can be done by computer.

The algebraic relations are computed by Algorithm 3.1. It is well-known in Computer Algebra but not in Analysis that the rewriting of h_i in $g_i(p(x))$ is done with Gröbner bases, see Lemma 2.22.

Algorithm 3.9 *(orbit space reduction)*

Input: *equivariant* $f(x) \in K[x]_\vartheta^\vartheta$,
homogeneous invariants $p_1(x), \ldots, p_k(x)$ *generating* $K[x]_\vartheta$

Output: $\dot{y} = g \in K[y_1, \ldots, y_k]^k$, *relations*

$d := \max(\deg(p_1), \ldots, \deg(p_k)) + \deg(f) - 1$
Compute the Gröbner basis $\mathcal{GB}(\bigoplus_{i=0}^d H_i^W(I))$ *as in Algorithm 3.1.*
Then compute the normalforms $g_i(y) = normalf(\frac{dp_i(x)}{dx} \cdot f(x), \mathcal{GB}), i = 1, \ldots, k.$

Remark 3.10 *i.) Observe that we do not need a full Gröbner basis in Algorithm 3.9.*
ii.) In case the invariants form a Hironaka decomposition (that means that the secondary invariants $p_{s+1}, \dots, p_k$ *form a free module over the ring* $\mathbf{R}[p_1, \dots, p_s]$ *and thus each invariant has a unique representation* $\sum_{i=s+1}^{k} a_i(p_1, \dots, p_s) \cdot p_i$*) one would like to compute the unique representations* $\sum a_i p_i$ *in Algorithm 3.9. This is done, if the eliminating term order* $\leq$ *on* $K[x, y]$ *is such that the matrix representating the order* $\leq_{|K[y]}$ *starts with first row with 0's for* $y_1, \dots, y_s$ *and 1's for* $y_{s+1}, \dots, y_k$ *([44], p. 52).*

3.4 Computing invariants

For finite groups there is one famous algorithm in [44] for the computation of invariants which appeared first in [46]. It makes use of the fact that the invariant ring $K[x]_\vartheta$ is Cohen-Macaulay, that means whenever we have polynomials $p_1, \dots, p_n$ such that $K[x]_\vartheta$ is a module over the subring $K[p(x)]$ then it is a free module.

The generators are called *secondary invariants* in contrast to the *primary invariants* p_i. The algorithm searches first primary invariants until the set of common zeros of $p_1, \dots, p_n$ is $\{0\}$ (the *nullcone*). The degrees of the secondaries are then known by the Molien series of $K[x]_\vartheta$.

Algorithm 3.11 *([44] p. 57)*
Input: *finite group* $\vartheta : G \to GL(K^n)$
i.) find primary invariants $p_1, \dots, p_n$
ii.) find secondary invariants: for degrees d_i *there are* c_i *many,* $i = 1, \dots, m$

$S = \{1\}$
for $i = 1, \dots, m$
  for $j = 1, \dots, c_i$
    $q :=$ *nextcandidate* $\in H^N_{d_i}(K[x]_\vartheta)$
    while $q(x)$ *is not an element of the free* $K[p]$*-module generated by* S
      $q :=$ *nexcandidate*
    $S := S \cup \{q\}$

There are at least four ways in performing the task

$$q(x) \in^? \bigoplus_{s \in S} s(x) K[p_1(x), \dots, p_n(x)]$$

whether q is a member of the free module generated by S over $K[p]$ where $S = \{1, s_2, \dots, s_l\}$. It is used that q_i is homogeneous of degree $d_i = \deg_N(q(x))$.

a.) Pure linear algebra:
Build all terms $sp_1^{j_1} \cdots p_n^{j_n}$ with $s \in S, j = (j_1, \dots, j_n)$ such that $\deg_N(sp^j) = d_i$ and determine linear dependence of $q(x)$ on these polynomials by comparing coefficients.

group	Ref.	degrees		up	invariants			
		primary	secondary	to	a	b	c	d
$D_3(\vartheta^2+\vartheta^3)$	[21]	2,2,3	4	all	6 sec	9 sec	9 sec	6 sec
D_3+D_3	[36]	2,2,3,3	2,3,3,4,5	all	23 sec	243 sec	343 sec	19 sec
$D_4(\vartheta^2+\vartheta^5)$	[10]	2,2,4	3	all	4 sec	5 sec	5 sec	4 sec
$Z_4\times Z_2$	[48]	2,4,4,8	6,6,6,8,8,8	8	108 sec	466 sec	529 sec	107 sec
			14	all	213 sec	$>1\ h$	$>1\ h$	179 sec

Table 5: Timings for Algorithm 3.11 for various variants.

b.) Normalform with respect to Hironaka decomposition:
Compute a Gröbner basis $\mathcal{GB}(\bigoplus_{j=1}^{d_i} H_j^N(I))$ of

$$I=(p_1(x)-y_1,\ldots,p_n(x)-y_n,\eta_2-s_2(x),\ldots)$$

with respect to a termorder which eliminates first x and on $K[y,\eta]$ eliminates η, starting with natural degree on η, see [44], p. 52. (Use the Hilbert series driven Buchberger algorithm with $\mathcal{HP}^W_{K[x,y,\eta]/I}$ given by $(y_1,\ldots,y_n,\eta_2,\ldots,\eta_l)$.) If normalf$(q,\mathcal{GB})\in K[y,\eta]$ and is linear in η then $q\in\bigoplus_{s\in S}s\cdot K[p]$. Once a new secondary invariant is found the Gröbner basis needs to be updated.

c.) Restriction to module with slack variable:
Choose a variable z and consider the gradings $W:\{x,y,\eta,z\}\to\mathbf{N}$ $W_{|\{x\}}=N, W(y_i)=\deg_N(p_i(x)), i=1,\ldots,n, W(\eta_i)=\deg_N(s_i(x)), i=2,\ldots,l, W(z)=0$ for restriction in degree and $U:\{x,y,\eta,z\}\to\mathbf{N}$ $U_{|\{x,y\}}=0, U_{|\{\eta\}}=N, U(z)=1$ for restriction to the module. For

$$I:=(y_1-p_1(x),\ldots,y_n-p_n(x),\eta_2-z\cdot s_2,\ldots,\eta_l-z\cdot s_l),$$

compute $\mathcal{GB}=\mathcal{GB}(\bigoplus_{k\le d_i,j=0,1}H^{W,U}_{k,j}(I))$. (Use the Hilbert series $\mathcal{HP}^{W,U}_{K[x,y,\eta,z]/I}$ given by $(y_1,\ldots,y_n,\eta_2,\eta_l)$.) If $g(x,y,\eta,z)=$ normalf$(z\cdot q,\mathcal{GB})$ does not depend on x then q is a member of the module and $g(p(x),s(x),1)=q(x)$. Like method b.) the Gröbner basis needs update after a new secondary invariant was found.

d.) Using one Gröbner basis:
The Gröbner basis $\mathcal{GB}(\bigoplus_{j=0}^{max(d_i)}H_j^N((p_1(x),\ldots,p_n(x)))$ with respect to any termorder is computed once in the beginning. Consider the set of normal forms $\tilde{S}=\{\text{normalf}(s,\mathcal{GB})|s\in S\}$ and $\tilde{q}:=\text{normalf}(q,\mathcal{GB})\in K[x]$.
If $\tilde{q}$ is K-linear dependent on $\tilde{S}$ (on those of degree d_i) then $q(x)=\bigoplus_{s\in S}s(x)\cdot B^s(x)$ with $B^s(x)\in(p_1(x),\ldots,p_n(x))$ members of the ideal. Because of the invariance $B^s(x)$ can be assumed to be invariant and thus to be members of $K[p]$. Here one can use restriction to actual degree.

In *Invar* [29] method d.) is used. We tested these methods for various examples. Timings are given in Table 5. Method b.) and c.) need more time because more information than

group	Ref.	degrees of equivariants	up to	equivariants a	c	d
$D_3(\vartheta^2+\vartheta^3)$	[21]	1,1,2,2,3,3	all	37 sec	85 sec	35 sec
D_3+D_3	[36]	1,1,1,1,2,2,2,2,2,2,	2	45 sec	185 sec	46 sec
		3,3,3,3,4,4,4,4,4,4,5,5,5,5	all	324 sec	$>1\ h$	229 sec
$D_4(\vartheta^2+\vartheta^5)$	[10]	1,1,2,2,3,4	all	70 sec	129 sec	69 sec
$Z_4\times Z_2$	[48]	1,3,3,3,3,	3	158 sec	274 sec	164 sec
		(7*5), (8*7), (7*9), (4*11),13	all	4290 sec	$>3\ h$	2581 sec

Table 6: Timings for computation of equivariants for various variants.

needed (the representation in $p(x)$ and $s \in S$) is computed. Method d.) is best since it exploites the underlying structure as much as possible.

Equivariants

Since the module of equivariants is Cohen-Macaulay for finite groups the second part of the Algorithm 3.11 transfers immediately to the equivariant case see [18], [48].

Only the task $q(x) \in^? \bigoplus_{e\in E} e(x)K[p(x)]$ where $q(x)\in K[x]^r$ and $E\subset K[x]^r$ of order m needs to be discussed. The methods a.), c.) and d.) are valid analogously, but b.) has no generalisation. Timings are given in Table 6.

4 Invariants and equivariants for continuous groups

The algorithms given in section 3.4 can not be generalized for continuous groups. We will consider here a different method based on properties of the Lie algebra of the considered group. This is a direct extension of a method already used in [42] for the computation of the equivariants for SO(3) and O(3). It holds for the class of *semi-simple* Lie groups which includes most of the common classical groups (SO(n) and O(n) with $n>2$, SU(n) and U(n), ...). There is a huge amount of bibliography concerning Lie groups and Lie algebras. Let us cite for example [6, 26, 27] or [16].

From now on, the group G will be a connected Lie group, i.e. a connected subgroup of GL(n). Our approach can also be generalized to cases $G\times H$ where H is a finite group. We indicate at the end of this section how to generalise the algorithms of this section in some particular cases of a direct product of this form. In equivariant dynamics problems, we are often interested in subgroups of $\mathrm{GL}(n,\mathbb{R})$ acting on real vector spaces. However for most of the interesting groups (including SO(n), SL(n), Sp($2n$)) the real representations of subgroups of $\mathrm{GL}(n,\mathbb{R})$ can be obtained by restriction to the real part of complex representations of subgroups of $\mathrm{GL}(n,\mathbb{C})$.

4.1 Lie group - Lie algebra

For a connected compact Lie group G let $\vartheta : G \to GL(\mathbb{C}^n)$ and $\rho : G \to GL(\mathbb{C}^m)$ be two linear representations where ϑ is faithful. The aim of the following sections is the computation of a generic homogeneous equivariant of degree d, i.e. one computes a vector space basis of $H_d^N(\mathbb{C}[x]_\vartheta^\rho)$. Observe that for $m = 1, \rho(g) = 1$ the case of invariants is included.

The key for the algorithm is the use of a associated Lie algebra. Recall that a connected Lie group is a manifold and the tangent space at the identity T_eG has the special meaning of the Lie algebra $\mathfrak{g}$, i.e. T_eG is a vector space and additionally is provided with a Lie bracket.

Each $Y \in \mathfrak{g}$ is thus given by a path $\gamma : [-1,1] \to G, \gamma(0) = e, \frac{d}{ds}\gamma(s)_{|s=0} = Y$. The group action ϑ inherits via the representation $\Theta : G \to Aut(\mathbb{C}[x]^m)$,

$$\Theta(g)(f(x)) = f(\vartheta(g^{-1})x), \quad g \in G,$$

to an action of the Lie algebra on the polynomial vectors. Elements of the tangent space of $T_{\vartheta(e)}\vartheta(G)$ are given by paths

$$\gamma : [-1,1] \to G, \gamma(0) = e, \gamma'(0) = Y \in \mathfrak{g},$$

$$\zeta : \mathfrak{g} \to GL(\mathbb{C}^n), \zeta(Y) = \frac{d}{ds}\vartheta(\gamma(s))_{|s=0}.$$

Analogously, we have $T_{\Theta(e)}\Theta(G)$ by $\theta : \mathfrak{g} \to Aut(\mathbb{C}[x]^m)$

$$\begin{aligned} \theta(Y)(f(x)) &= \tfrac{d}{ds}\Theta(\gamma(s))(f(x))_{|s=0} = \tfrac{d}{ds}f(\vartheta(\gamma(s)^{-1})x)_{|s=0} \\ &= \tfrac{d}{dx}f(x) \cdot \tfrac{d}{ds}\vartheta(\gamma(s)^{-1})_{|s=0} \cdot x = \tfrac{d}{dx}f(x) \cdot \zeta(-Y) \cdot x \\ &= -\tfrac{d}{dx}f(x) \cdot \zeta(Y) \cdot x. \end{aligned}$$

Let the Lie algebra action associated to ρ be denoted by ϱ. The main ingredient in the algorithmic treatment is the following well-known lemma.

Lemma 4.1 *$f \in \mathbb{C}[x]^m$ is ϑ-ρ equivariant iff for all generators Y of $\mathfrak{g}$*

$$\theta(-Y)(f(x)) = \varrho(Y) \cdot f(x).$$

Special structure of the Lie algebra allows for more simplification as described in the following section. Besides this two simplifications are obvious. Assume ϑ decomposes into subrepresentations and the coordinates are such that the representations matrices have block diagonal form

$$\vartheta(g) = \begin{pmatrix} \vartheta^1(g) & & 0 \\ & \ddots & \\ 0 & & \vartheta^r(g) \end{pmatrix}$$

The set of variables then decomposes into r subsets $x = (x^1, \ldots, x^r)$. Then the Kronecker gradings $N_i : \{x_1, \ldots, x_n\} \to \mathbf{N}$,

$$N_i(x_j) = \begin{cases} 1, & if \quad x_j \in x^i \\ 0, & else \end{cases}$$

define a grading of $\mathbb{C}[x]$ such that the invariant ring is multigraded by $\underline{N} = (N_1, \ldots, N_r)$ as well.:

$$H_d(\mathbb{C}[x]_\vartheta) = \bigoplus_{k_1+\cdots+k_r=d} H_{\underline{k}}^{\underline{N}}(\mathbb{C}[x]_\vartheta).$$

The module of equivariants is multigraded analogously.

The second simplification is given by a decomposition of ρ in block diagonal form $\rho(g) = diag(\rho_1(g), \ldots, \rho_s(g))$. The ϑ-ρ-equivariants are given by the ϑ-ρ_i-equivariants $\tilde{f}$ since $(0, \ldots, \tilde{f}, 0, \ldots, 0)$ is a ϑ-ρ-equivariant.

Observe that the case $\vartheta = \rho = \vartheta^1 + \vartheta^2$ includes the case which in the theory of dynamical systems is known as mode interaction.

4.2 Properties of semi-simple Lie algebras

We consider now the case where the group G is a finite dimensional complex semi-simple Lie group G. Let us recall that a connected Lie group is called *semi-simple* if it does not contain any non trivial abelian normal subgroup. Similarly, a Lie algebra is called *semi-simple* if it contains no proper non trivial abelian ideal. Now a connected Lie group is semi-simple iff its Lie algebra $\mathfrak{g}$ is semi-simple ([27] p. 89). The classical complex Lie algebras $\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_n(\mathbb{C})(n > 2)$ and $\mathfrak{sp}_{2n}(\mathbb{C})$ are semi-simple. A maximal abelian subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ is called a *Cartan subalgebra.* Let us recall that the adjoint representation of $\mathfrak{g}$ is the action of $\mathfrak{g}$ on itself using the Lie bracket i.e

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{GL}(\mathfrak{g})$$

such that for $\mathrm{X}, \mathrm{Y} \in \mathfrak{g},\ \mathrm{ad}(X)(Y) = [X, Y]$.

The action of $\mathfrak{g}_0$ on $\mathfrak{g}$ can be thought of as diagonal (Theorem 9.20 [16]) i.e all matrices $ad(\mathrm{X})$ for $\mathrm{X} \in \mathfrak{g}_0$ can be simultaneously diagonalised. This property suggests a so-called *Cartan decomposition* of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus_\alpha \mathfrak{g}_\alpha \tag{8}$$

where each *root space* $\mathfrak{g}_\alpha$ is a one dimensional eigenspace for the elements of $\mathfrak{g}_0$ in the adjoint action of $\mathfrak{g}$. Each α occuring in the Cartan decomposition (8) is a linear form on $\mathfrak{g}_0$. We denote by $\mathcal{R}$ the set of these linear forms which are called *roots.* For any root space $\mathfrak{g}_\alpha$, the form $\alpha \in \mathfrak{g}_0^*$ is defined by

$$\alpha(X)Y = \mathrm{ad}(X)\,Y = [X, Y],\ X \in \mathfrak{g}_0, Y \in \mathfrak{g}_\alpha.$$

If $\mathfrak{g}_\alpha$ is a root space then $\mathfrak{g}_{-\alpha}$ is also a root space. Furthermore, there exists a particular basis of $\mathfrak{g}$ such that the elements of root spaces act in a nice way via the adjoint action.

Theorem 4.2 The Cartan-Weyl form

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra with the Cartan decomposition (8). There exists a basis $\{\mathrm{H}_i : i = 1 \dots r\}$ of $\mathfrak{g}_0$ and for each $\alpha \in \mathcal{R}$ some E_α generates $\mathfrak{g}_\alpha$ such that

- $\mathrm{ad}(\mathrm{H}_i)\,\mathrm{E}_\alpha = [\mathrm{H}_i, \mathrm{E}_\alpha] = \alpha(\mathrm{H}_i)\ \mathrm{E}_\alpha,$
- $\mathrm{ad}(\mathrm{E}_\alpha)\,\mathrm{E}_{-\alpha} = [\mathrm{E}_\alpha,\, \mathrm{E}_{-\alpha}] = \sum_{i=1}^{r} \alpha(\mathrm{H}_i)\ \mathrm{H}_i = \mathrm{H}_\alpha,$
- $\mathrm{ad}(\mathrm{E}_\alpha)\,\mathrm{E}_\beta = [\mathrm{E}_\alpha\, \mathrm{E}_\beta]\ = \mathrm{N}_{\alpha,\beta}\,\mathrm{E}_{\alpha+\beta}$ *with* $\mathrm{N}_{\alpha,\beta} = 0$ *unless* $\alpha + \beta \in \mathcal{R},$

with $\mathrm{N}_{\alpha,\beta} = -\mathrm{N}_{\beta,\alpha} = \mathrm{N}_{-\beta,-\alpha} = -\mathrm{N}_{-\alpha,-\beta}.$

Example 4.3 *: The Cartan subalgebra of $so(3)$ is generated by J^3 and there are two root spaces generated by J^+ and J^-. Then*

$$[J^3, J^\pm] = \pm J^\pm, \quad [J^+, J^-] = 2J^3.$$

Now consider an action $\varrho : \mathfrak{g} \to GL(V)$ where V is a complex vector space. V can be provided with a decomposition similar to (8). More precisely

$$\mathrm{V} = \bigoplus_{\beta \in \mathcal{W}} \mathrm{V}_\beta \tag{9}$$

where $\mathcal{W}$ is a subset of the linear forms on $\mathfrak{g}_0$. Each *weight space* V_β is characterised by

$$\forall\, v \in \mathrm{V}_\beta, \forall\, \mathrm{X} \in \mathfrak{g}_0,\ \ \varrho(X)(v) = \beta(X)v.$$

and the elements β in $\mathcal{W}$ are called the *weights* of the representation. The dimension of a weight space V_β is called its *multiplicity.* If $\dim \mathrm{V}_\beta = 1$ then the weight β is said *simple.* In some sense the adjoint action is a special case of this. So the roots α of the algebra are the weights of the adjoint representation.

The rest of the Lie algebra $\mathfrak{g}$ acts on V in the following way

- $\varrho(\mathfrak{g}_\alpha)(\mathrm{V}_\beta) \subset \mathrm{V}_{\beta+\alpha}$ if $\alpha + \beta \in \mathcal{W}$,
- $\varrho(\mathfrak{g}_\alpha)(\mathrm{V}_\beta) = 0$ if $\alpha + \beta \notin \mathcal{W}$.

Now, the set $\mathcal{R}$ can be splitted as $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$. The roots in $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) are called *positive* (resp. *negative*) roots. This decomposition can be done in such a way that there exists a weight space V_β called a *highest weight space* such that

$$\varrho(\mathrm{E}_\alpha)\mathrm{V}_\beta = 0 \ \text{ for all } \ \alpha \in \mathcal{R}_+ \tag{10}$$

Using this notation the key result is:

Theorem 4.4 ([16] prop. 14.13)
For any semisimple complex Lie algebra $\mathfrak{g}$ the following holds.

i) Every finite-dimensional representation on V of $\mathfrak{g}$ possesses a highest weight space. Take one vector v in this space.

ii) The subspace W of V generated by the images of the highest weight vector v under successive applications of generators E_α of root spaces $\mathfrak{g}_\alpha$ for $\alpha \in \mathcal{R}_-$ is an irreducible subrepresentation.

iii) An irreducible representation possesses a unique highest weight vector up to multiplication by scalar, i.e. the dimension of the highest weight space is one.

The negative roots allow to determine explicitely a basis of an irreducible representation once that a highest weight vector is determined. It is however possible to find a refinement of this method in order to obtain a more efficient way. Indeed, it can be shown that only a subset of $\mathcal{R}_-$ is necessary. More precisely, we call *primitive positive (resp. negative) roots* the subset of $\mathcal{R}_+$ (resp. $\mathcal{R}_-$) of roots that cannot be expressed by positive (resp. negative) roots. In the previous theorem, $\mathcal{R}_-$ can be replaced by the subset $\mathcal{R}_-^p$ of primitive negatives roots. More precisely

Proposition 4.5 ([16] Observation 14.16)
Any irreducible representation V is generated by the images of its highest weight vector v under successive applications of roots spaces $\mathfrak{g}_\alpha$ where α ranges over the primitive negative roots.

Furthermore, the highest weight space V_{β_0} is uniquely characterised by the following fact

$$\varrho(\mathrm{E}_\alpha)\mathrm{V}_{\beta_0} = 0 \text{ for all } \alpha \in \mathcal{R}_+^p. \tag{11}$$

Two other results are also usefull for our construction :

Proposition 4.6

- *If a representation is irreducible, its highest weight is simple.*
- *Two irreducible representations are equivalent if their highest weights are equal.*

The root systems are well known for the semi-simple complex Lie algebras. There are various equivalent ways to determine the sets of positive, negative and primitive roots (negative or positive) as well as classification of the corresponding highest weight space (see for instance [16]). Nevertheless, these informations can be found in the literature on Lie groups and Lie algebras.

Before closing this section, we need to introduce a bit more terminology. We assume that the representation space V is decomposed as in (9) and that V_{β_0} denotes the maximal weight space. Let $\mathcal{R}_-^p = \{\alpha_1, \ldots, \alpha_q\}$ be a set of negative primitive roots. For any weight space V_β, there is a word $P = \mathrm{E}_{p_1}\mathrm{E}_{p_2}\cdots\mathrm{E}_{p_n}$ with $p_i, i = 1\ldots n$ in $\mathcal{R}_-^p$ such that

$$\varrho(\mathrm{E}_{p_1}\mathrm{E}_{p_2}\cdots\mathrm{E}_{p_n})\mathrm{V}_{\beta_0} \subset \mathrm{V}_\beta.$$

For a given weight vector β the length of words $P' = E_{p'_1} \cdot E_{p'_2} \cdots E_{p'_m}$ such that $\varrho(P')(V_{\beta_0}) \subset V_\beta$ is unique (this lenght is the sum of the coordinates of the point β in the lattice in $\mathfrak{g}_0^*$ generated by $\alpha_1, \ldots, \alpha_q$ and with the origin in β_0). We collect all weight spaces of the same length n into a set, called the *n-th layer* Δ_n. With this definition, $\mathrm{V}_{\beta_0} \in \Delta_0$ as a corollary of proposition (4.5). The layer Δ_{n+1} can be generated by successive applications of elements E_α with α in $\mathcal{R}_-^p$ on the weight spaces of the layer Δ_n. Furthermore, since V is finite dimensional, the number of layers is finite.

4.3 Computing invariants and equivariants

We describe in this section an algorithm to determine a vector space basis of the equivariants $f : V \to W$, homogeneous of a certain degree of a connected semi-simple compact Lie group G, acting on finite-dimensional vector spaces V and W of dimensions n and m, respectively, This algorithm does not distinguish between primary and secondary invariants. Nevertheless Molien series and Gröbner basis may be usefull to recover some part of this information.

Let us choose a basis of V consisting of bases for each irreducible component which yields a decomposition of the variables into groups $x = (x^1, \ldots, x^r)$. So the action of G on V is given by $\vartheta : G \to (\mathbb{C}^n)$. Let Θ, ζ, θ be the associated actions as in section 4.1. On W the choice of the basis respects the irreducible components as well yielding a block diagonal representation $\rho : G \to GL(\mathbb{C}^m), \rho(g) = \mathrm{diag}(\rho^1(g), \ldots, \rho^s(g))$. The associated representation of the Lie algebra is denoted $\varrho : \mathfrak{g} \to GL(\mathbb{C}^m)$ and is diagonal as well. Additionally we assume that the basis is chosen such that they form a basis of the weight spaces, i.e. in $\mathbb{C}^m$ unit vectors correspond to the elements of the weight spaces. Indexing by the weights β we have the basis $w^i_{\beta,j}, i = 1, \ldots, s, j = 1, \ldots, \dim V_\beta$ where β is a weight of ϱ^i and

$$E_\alpha : w^i_{\beta,j} \to \sum_{k=1}^{\dim V_\gamma} \varrho^i(E_\alpha)_{\beta,j,\alpha+\beta,k}\; w^i_{\alpha+\beta,k}$$

for all generators E_α of root spaces $\mathfrak{g}_\alpha$. Analogously for the generators H_l of the Cartan subalgebra $\varrho^i(H_l)$ is a diagonal matrix with entries $\beta(H_l)$. For both ρ and ϱ we use the same coordiate system such that $\rho(exp(g)) = exp(\varrho(g))$. For ϑ and ζ the same is true. ϑ and ρ describe how in these coordinates $C[x]^\rho_\vartheta$ corresponds to the equivariant polynomial mappings $V \to W$. The following algorithm is a generalization of a description given in [43] for $SO(3)$.

Algorithm 4.7 *(Computation of generic equivariant)*
Input:

- *connected semi-simple Lie group given by associated Lie algebra represented by generators H_l of the Cartan subalgebra and primitive positive roots $E_\alpha \in \mathcal{R}_+^p$ and primitive negative roots $E_\alpha \in \mathcal{R}_-^p$,*
- *matrices $\zeta(E_\alpha), \zeta(H_l)$, matrices $\varrho(E_\alpha), \varrho(H_l)$*
- *degree $\underline{k} = (k_1, \ldots, k_r)$*

Output: *a vector space basis of $H_{\underline{k}}^N(\mathbb{C}[x]_\vartheta^\rho)$.*
Let $M \subset H_{\underline{k}}^N(\mathbb{C}[x])$ denote the set of monomials of multidegree $\underline{k}$. For $i = 1, \ldots, s$

i.) A generic homogeneous polynomial is of the form

$$P(x^1, \ldots, x^r) = \sum_{m \in M} a_m^i \cdot m(x)$$

where a_m^i are unknowns. This is an Ansatz for the component corresponding to the highest weight space $f_{\beta_0}^i(x) = P(x)$. The other components $f_{\beta,k}^i(x)$ remain undetermined until step iii.).

ii.) The unknowns a_m^i are determined by the following conditions

$$\begin{aligned} &\forall \text{ generators } H_l \qquad && \theta(-H_l)(f^i(x)) = \varrho(H_l) \cdot f^i(x), \\ &\forall \alpha \in \mathcal{R}_+^p && \theta(E_\alpha)(f^i(x)) = 0 \end{aligned}$$

By comparing coefficients in the component $f_{\beta_0}^i$ this gives linear equations in the unknowns a_m^i. Solve this system and substitute the solution into $f_{\beta_0}^i$ which then still depends on a subset of the a_m^i. This determines the layer $\triangle_0$.

iii.) Assume the components $f_{\beta,k}^i$ for all weights β in the layer $\triangle_n$ are known. Then the components $\alpha+\beta, j$ corresponding to the weight spaces in the following layer $\triangle_{n+1}$ are determined by $\alpha \in \mathcal{R}_-^p$ by

$$\theta(-E_\alpha) f^i(x) = \varrho^i(E_\alpha) f^i(x).$$

For $\alpha \in \mathcal{R}_-^p$ and components β, k this means

$$\frac{d}{dx} f_{\beta,k}^i(x) \cdot \zeta(E_\alpha) \cdot x = \sum_{\alpha+\beta, j} \varrho^i(E_\alpha)_{\beta,k,\alpha+\beta,j} f_{\alpha+\beta,j}^i,$$

where on the left hand side $f_{\beta,k}^i(x)$ are polynomials and $f_{\alpha+\beta,j}^i$ on the right hand side are unknown components. By linear manipulation this gives the components $f_{\alpha+\beta,j}^i$ of the layer $\triangle_{n+1}$.

Altogether $(f^1(x), \ldots, f^s(x))$ is a generic equivariant.

Proof: This algorithm is a direct application of the results of the previous section. The determination of the component $f^i_{\beta_0}(x)$ is provided by the fact that the maximal weight is in the kernel of all primitive positive roots and is an eigenspace for the elements in the Cartan subalgebra. Now, the successive applications of all primitive negative roots on the highest weight provides a set of basis of the full representation space W. Each iteration in the algorithm allows to determine a basis for the $n+1$-th layer from the knowledge of the n-th layer. Since the number of layers is finite, the algorithm reachs necessarely a layer $\Delta_{n'} = \{0\}$ with is empty. Since unit vectors correspond to weight spaces the result is an equivariant vector field in the orthonormal basis $\{w^i_{\beta_0}, w^i_{\beta_1,1}, \cdots, , w^i_{\beta_q,t_q}, i = 1, \ldots, s\}$. □

Remark 4.8 *i.) For real Lie groups such as $SO(3)$ the method is applicable as well. The real associated Lie algebra can be embedded into a complex one by complexification $\mathfrak{g}^c = \mathfrak{g} \times_R \mathbb{C}$. Now if $\mathfrak{g}$ is a simple Lie algebra (i.e. without proper non-trivial ideal) then its complexified $\mathfrak{g}^c$ is also simple (see [16]). Then the irreducible representations of $\mathfrak{g}$ are just obtained by restriction to the real part of the irreducible representations of $\mathfrak{g}^c$.*
ii.) For non-connected compact Lie groups one considers the connected component of the identity which is a normal subgroup. Then a finite group H exixts such that the group is the direct product $G \times H$. For example $O(3) = SO(3) \times Z_2$ is obtained this way. The generic $G \times H$-invariants and equivariants are determined form the G-invariants and G-equivariants by applying the Reynold projection of H and its generalization to equivariants. This is possible since G and H commute.
iii.) Concerning the multigrading observe that less blocks might exist in the diagonalized form of $\vartheta(G \times H)$ than of $\vartheta(G)$. Thus there may exist less grading of the invariant ring and the module of equivariants. iv.) Since G is normal in $G \times H$ the group H is operating on the vector space generated by the G-invariants and their conjugates. Thus the H-invariants of this action yield the $G \times H$-invariants, see [31], or more readable in [40].

Example 4.9 *For the representations $l = 1$ and $l = 2$ and degree $k = (2, 3)$ the computation of the generic invariant and generic equivariant takes* 148 *sec,* 298 *sec, respectively.*

Algorithm 4.7 together with the algorithms for completeness in section 3.2 yield an algorithmic determination of fundamental invariants and fundamental equivariants for special compact Lie groups. Details of these algorithms will appear elsewhere.

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