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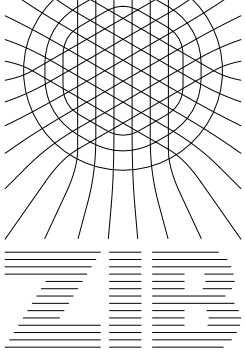

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# Spherically symmetric perfect fluids in shearfree motion - the symmetry approach.

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# Spherically symmetric perfect fluids in shearfree motion - the symmetry approach.

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**Abstract**

In General Relativity, the motion of expanding shearfree perfect fluids is governed by the ordinary differential equation $y'' = F(x)\, y^2$ , where $F$ is an arbitrary function from which the equation of state can be computed. A complete symmetry analysis of this differential equation is given; its solutions are classified according to this scheme, and in particular the relation to Wyman's Painlevé analysis is clarified.

Short title: Shearfree spherically symmetric perfect fluids

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## 1. The problem

Among the spherically symmetric perfect fluids those in shearfree motion play an outstanding rôle: they are physically preferred since for shearing motion friction should be

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taken into account and the model of a perfect fluid becomes obsolete, and they are mathematically interesting because of some properties which we intend to discuss in detail. Because of the high symmetry of the problem, the field equations become very simple: it was shown by Kustaanheimo and Qvist (1948)[1] that for the line element

$$\mathrm{d}s^2 = e^{2\lambda(r,t)}\left[\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right)\right] - e^{2\upsilon(r,t)}\mathrm{d}t^2 \tag{1.1}$$

they can be reduced to the ordinary differential equation

$$d^2y/dx^2 = y'' = F(x)\ y^2 \tag{1.2}$$

for the metric function

$$y\,(x,t) = \mathrm{e}^{-\lambda\,(r,t\,)}, \quad x \equiv r^2. \tag{1.3}$$

The second metric function is then

$$e^{\upsilon} = \lambda_{\,,t} e^{-f\,(t)}\ , \tag{1.4}$$

where $f(t)$ is an arbitrary function connected with the freedom of scaling $t$.
The function $F(x)$ depends on the equation of state of the fluid, and mass density $\mu$ and pressure $p$ can be computed from

$$\begin{aligned} \kappa\,\mu &= 3e^{2f} - \mathrm{e}^{-2\lambda}(2\lambda_{,rr} + \lambda_{,r}^{\,2} + 4\lambda_{,r}\,/r) \\ \kappa\,p\,\lambda_{,t} &= \mathrm{e}^{-3\lambda}\partial_{\,t}\left[\mathrm{e}^{\lambda}(\lambda_{,r}^{\,2} + 2\lambda_{,r}/r) - \mathrm{e}^{(3\lambda+2f)}\right] \end{aligned} \tag{1.5}$$

see Kramer et al. (1980) [2]. When deriving these equations the assumption

$$y_{,\,t} \not\equiv \ 0 \tag{1.6}$$

was made, i.e. the fields are not static (and the expansion of the fluid is not zero).
It may be a surprise that the time dependent motion of the fluid is described by a time-independent differential equation (1.2), but the condition (1.6) clarifies how the time enters: one has to find a solution of (1.2) which contains at least one arbitrary parameter (constant of integration), which then can be chosen to be a function of time; preferably we are interested in the general solution for a given function $F$, with two constants of integration.

The history of the spherically symmetric and shearfree solutions is long and rich in rediscoveries and detours. Although the central equation (1.2) was already found in 1948 by Kustaanheimo and Qvist [1], it remained hidden in the literature, and several papers published later contained only solutions which were just special cases of solutions already given in [1]. Kustaanheimo and Qvist constructed their solutions by means of the symmetry approach due to Lie, but this method also remained forgotten for a long time. Many new solutions found later could have been found using Lie symmetries.
In this paper we want to discuss this symmetry approach in some detail. But before doing this, we will give a short review of the different methods used so far in solving $y'' = F(x)\, y^{\,2}$. The trivial case $F = 0$ will always be excluded.

## 2. The three approaches for solving $y'' = F(x)\, y^{\,2}$.

### 2.1. The direct approach

In many papers either the full set of field equations (by making an ansatz for the metric functions $\lambda(r,t)$ and $v(r,t)$ ) or the central equation $y'' = F(x)\, y^{\,2}$ (by making an ansatz for $F(x)$ ) was solved directly, often in a rather ingenious way, and often leading to very complicated expressions for the metric. Since the two other approaches are superior in that they proved far more successful (all solutions found by any of the direct approaches are contained as subcases) and moreover admit an invariant classification of the different cases, we refer the reader to the literature [2], [3],[10], where most of these solutions are given (or quoted) and their interrelation is discussed.

### 2.2. Wyman's Painlevé approach

In his search for functions $F(x)$ for which the solutions of $y'' = F(x)\, y^{\,2}$ have fixed critical points (branch points) and essential singularities, Wyman (1976) [4] found that solutions of $y'' = F(x)\, y^{\,2}$ can be constructed from solutions of

$$\frac{\mathrm{d}^2\,\tilde{y}}{\mathrm{d}\,\tilde{x}^2} \;=\; \tilde{F}(\tilde{x})\,\tilde{y}^{\,2} + p\tilde{x} + q \tag{2.1}$$

via a mapping

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,\tilde{x}} = \frac{1}{\tilde{\psi}(\,\tilde{x}\,)^2}\,, \quad y(x) \;=\; \frac{1}{\tilde{\psi}(\,\tilde{x}\,)}\,[\;\tilde{y}\,(\,\tilde{x}\,) - \tilde{H}(\tilde{x}\,)], \quad F(x) = \tilde{\psi}^5(\tilde{x}\,)\;\tilde{F}(\tilde{x})\,, \tag{2.2}$$

if the functions $\tilde{\psi}$ and $\tilde{H}$ satisfy

$$\frac{\mathrm{d}^2\tilde{H}}{\mathrm{d}\;\tilde{x}^{\,2}} = \tilde{F}(\tilde{x}\,)\,\tilde{H}^2(\tilde{x}\,) + p\tilde{x}\; + q\;, \quad \frac{\mathrm{d}^2\tilde{\psi}(\,\tilde{x}\,)}{\mathrm{d}\,\tilde{x}^{\,2}} = \;2\,\tilde{H}\,(\tilde{x}\,)\;\tilde{F}\,(\tilde{x})\,\tilde{\psi}(\,\tilde{x}\,) \tag{2.3}$$

(the group-theoretical background of this transformation has been discussed by Herlt and Stephani (1992) [5] ).

For $p = 0 = q$, these mappings leave the *form* of the differential equation invariant, as $\mathrm{d}^2y/\mathrm{d}x^2 = F(x)y^2$ is transformed (mapped) into $\mathrm{d}^2\tilde{y}/\mathrm{d}\tilde{x}^2 = \tilde{F}(\tilde{x})\,\tilde{y}^{\,2}$. We therefore refer to these transformations as gauge transformations of the function $F$, as opposed to the symmetry transformations to be discussed later which leave *the function* $F$ fixed.

The general solution of the differential equation (2.1) is known in the following three cases (normal forms): For

$$\frac{\mathrm{d}^2\,\tilde{y}}{\mathrm{d}\,\hat{x}^2} = \;\tilde{y}^{\,2} + p\tilde{x}\; + q, \tag{2.4}$$

where it defines a Painlevé transcendent, and for its subcases

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}\,\tilde{x}^2} = \tilde{y}^2 + q \tag{2.5}$$

and

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}\,\tilde{x}^2} = \tilde{y}^2, \tag{2.6}$$

where it leads to elliptic integrals. Because of the existence of the gauge transformations (2.1)-(2.3), the solutions are also known if $\mathrm{d}^2y/\mathrm{d}x^2 = F(x)y^2$ can be mapped onto one of these three cases.

At first glance, this mapping procedure looks very complicated, and indeed it is. But for the most interesting case (2.6), the equations (2.1) - (2.3) give a straightforward scheme for finding solutions: take any two solutions of the simple differential equation (2.6) as $\tilde{y}$ and $\tilde{H}$, and $\tilde{\psi}$ as $\tilde{\psi} = c_1\mathrm{d}\tilde{H}\,/\mathrm{d}\tilde{x} + c_2(2\tilde{H} + \tilde{x}\mathrm{d}\tilde{H}\,/\mathrm{d}\tilde{x})$, (having thus solved (2.1) and (2.3) with $\tilde{F} = 1$ and $p = 0 = q$). Then $y(x)$ constructed according to (2.2) is a solution of $\;y'' = F(x)\;y^{\,2}$ with $F(x) = \tilde{\psi}^5$. As was shown by Srivastava [3], many of the known solutions can be mapped onto (2.6) and and therefore be generated this way.

When can a given $F(x)$ be mapped onto one of the three cases (2.4)-(2.6)? It is easy to use Wyman's procedure to answer this question. From the desired result $\tilde{F} = 1$ and the transformation law (2.2) we see that we have to define a function $\tilde{\psi}$ by $\tilde{\psi}(\tilde{x}) = F(x)^{1/5}$, then to differentiate it twice with respect to $\tilde{x}$ (using $\mathrm{d}\,x/\mathrm{d}\,\tilde{x} = \tilde{\psi}^{-2}$ when differentiating the right hand side) and to define a function $\tilde{H}(\tilde{x})$ by $H(x) = (\mathrm{d}^2\tilde{\psi}(\tilde{x})/\mathrm{d}\,\tilde{x}^2)\,/\,2\tilde{\psi}$. This function $\tilde{H}(\tilde{x})$ then has to satisfy (2.3) with $\tilde{F} = 1$. For the three cases (2.4)-(2.6) the resulting conditions for $F(x)$ read

$$F^{-2/5}\left\{2F^{-2/5}\left[F^{-2/5}\left(F^{-3/5}(F^{-1/5})''\right)'\right]' + F^{-6/5}\left((F^{-1/5})''\right)^2\right\}' = -4p \tag{2.7}$$

$$2F^{-2/5}\left[F^{-2/5}\left(F^{-3/5}(F^{-1/5})''\right)'\right]' + F^{-6/5}\left((F^{-1/5})''\right)^2 = -4q \tag{2.8}$$

$$2F^{-2/5}\left[F^{-2/5}\left(F^{-3/5}(F^{-1/5})''\right)'\right]' + F^{-6/5}\left((F^{-1/5})''\right)^2 = 0, \tag{2.9}$$

respectively. Equations (2.7)-(2.9) are the necessary and sufficient conditions that $y'' = F(x)\,y^2$ can be transformed into (2.4)-(2.6).

The last condition (2.9) can also be written as

$$2[F^{-4/5}(F^{-1/5})'']'' = F^{-3/5}[(F^{-1/5})'']^2 \tag{2.10}$$

compare [5]. Because of its three Lie point symmetries, this condition can be reduced to a first order differential equation: with

$$F^{-2/5} = \mathrm{e}^S \tag{2.11}$$

one obtains

$$2(S'^2 + S'')'' = (S'^2 + S'')^2 \tag{2.12}$$

i.e. $S'^2 + S''$ is an elliptic function $K(x)$, and

$$S'^2 + S'' = K(x) \tag{2.13}$$

is a first order differential equation for $S'(x)$. Functions $F(x)$ which lead to integrable cases can be constructed this way. Wyman (1976) [4] also gave explicit constructions of functions $F$ satisfying (2.8) or (2.9).

We will show in Section 3 how the three cases (2.7)-(2.9) can be characterized in terms of the Lie symmetries of $\mathrm{d}^2y/\mathrm{d}x^2 = F(x)y^2$.

### 2.3. The symmetry approach

It is well known from the theory of differential equations that a second order equation like $y'' = F(x)\, y^{\,2}$can be reduced to a first order equation if it admits *one* Lie point symmetry, and that its general solution can be given in terms of quadratures if it admits *two* (or more) Lie point symmetries. Using standard techniques (see e.g Stephani (1989) [6] or Olver (1986) [8]), one finds that for a given $F(x)$ the equation $y'' = F(x)\, y^{\,2}$ admits a symmetry if a function $B(x)$ exists which satisfies

$$F(\frac{5}{2}B' + c) + BF' = 0 \tag{2.14}$$

and

$$B''' = 4(d\,x + e)\,F\,, \tag{2.15}$$

where $c$ , $d$ and $e$ are (arbitrary) constants. The generator of this symmetry then reads

$$X = B(x)\,\partial_x + [(c + B'/2)\,y + d\,x + e]\partial_{\,y}\ . \tag{2.16}$$

The condition (2.14) can be integrated to give $B(x)$ in terms of $F(x)$:

$$B = a\,F^{-2/5} - \frac{2}{5}cF^{-2/5}\int F^{2/5}d\,x. \tag{2.17}$$

If one inserts this result into the second condition (2.15), the result is the equation

$$4(d\,x\ + e)F = [a\,F^{-2/5} - \frac{2}{5}cF^{-2/5}\int F^{2/5}\mathrm{d}\,x]''' \tag{2.18}$$

(Kustaanheimo and Qvist [1]) which $F(x)$ has to fulfill to give rise to a symmetry. If this condition is satisfied, then $B(x)$ can be computed from (2.17); because of the two constants $a$ and $c$ appearing in (2.17), there are at most two symmetries for a given $F(x)$. The symmetry approach opens up a systematic way to search for classes of functions $F(x)$ which give rise to symmetries, and to construct solutions of $y'' = F(x)\,y^{\,2}$ for these cases. We will discuss this in some detail in the following sections.

## 3. Solutions of $y'' = F(x)\ y^{\,2}$ with one Lie point symmetry

### 3.1. Equivalent classes of functions $F(x)$

The existence of a symmetry does not depend on the particular form in which the differential equation under consideration is given; any point transformation

$$x = x(\tilde{x}, \tilde{y})\ ,\ \ y = y(\tilde{x}, \tilde{y}) \tag{3.1}$$

which leaves $y'' = F(x)\ y^{\,2}$ invariant only changes the explicit form of the symmetry. Transformations of this type are often used to simplify calculations or to get rid of apparently different cases of functions $F(x)$ which are rather trivially related to each other. Simple examples of such transformations are

$$\begin{aligned}
&\tilde{x} = x + \lambda,\ \tilde{y} = y\,, \quad \tilde{F}(\tilde{x}) = F(\tilde{x} - \lambda), \qquad \tilde{e} = e + d\lambda \\
&\tilde{x} = \lambda x\,,\ \ \tilde{y} = y\,, \qquad \tilde{F}(\tilde{x}) = F(\tilde{x}/\lambda)/\lambda^2, \quad \tilde{d} = d/\lambda \\
&\tilde{x} = x,\ \tilde{y} = \lambda y, \qquad \tilde{F}(\tilde{x}) = F(\tilde{x})/\lambda \qquad \tilde{d} = \lambda\, d,\ \tilde{e} = \lambda\, e
\end{aligned} \tag{3.2}$$

(where the last column shows the effect on the constants appearing in the symmetry (2.16) and in the condition (2.18)). A more general invariance (or gauge) transformation of $y'' = F(x)\ y^{\,2}$ is provided by the transformation (2.2) if restricted to $p = q = 0$.
Whereas all these transformations are members of a continuous group, there is also a discrete transformation which has its origin in the invariance of the metric under the reflection

$$\tilde{r} = 1/r. \tag{3.3}$$

It is the transformation

$$\begin{aligned}
&\tilde{x} = 1/x\ ,\ \ \tilde{y}\,(\,\tilde{x}\,) = y\,(x)\,/x\ ,\ \ \tilde{F}(\,\tilde{x}\,) = x^5 F(x)\,, \\
&\qquad \tilde{B}(\tilde{x}) = -B(x)\,/\,x^2\,, \quad \tilde{c} = c\,,\ \ \tilde{d} = e\,,\ \ \tilde{e} = d\,.
\end{aligned} \tag{3.4}$$

Together with (3.2) it can be used to achieve e.g. $d = 0$.
If two functions $F(x)$ and $\tilde{F}(\tilde{x})$ are equivalent in the sense discussed above, their related differential equations $y'' = F(x)\ y^{\,2}$ and $\tilde{y}'' = \tilde{F}(\tilde{x})\ \tilde{y}^{\,2}$ are equivalent, too - although the physical interpretation of the relevant metrics may be quite different.

### 3.2. Functions $F(x)$ for which $y'' = F(x)\, y^{\,2}$ admits one symmetry

To find functions $F(x)$ which give rise to a symmetry one has to solve the system (2.14)-(2.15), i.e.

$$F(\frac{5}{2}B' + c) + BF' = 0 \tag{3.5}$$

and

$$B''' = 4(d\,x + e)\,F\,. \tag{3.6}$$

For c=0, dx+e≠0 , the general solution to these equations can be constructed as follows [5]: equation (3.5) is integrated by $F = B^{-5/2}$, and inserting this into (3.6) and taking the gauge $d = 0,\ e = 1$ one gets

$$B''' = B^{-5/2}. \tag{3.7}$$

To integrate this equation, write $B' = p(B)$ and make the substitutions $\xi = 1/B, \quad p^2(B)/B = f(\xi)$. The result is the differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} = \frac{2}{\sqrt{f}}\ , \tag{3.8}$$

which can be solved by standard methods.

For c≠0, dx+e=0 , the general solution is also known: one has to solve $B''' = 0$ and to insert the result

$$B = \alpha x^2 + 2\beta x + \gamma = \alpha(x - x_1)(x - x_2) \tag{3.9}$$

into the representation

$$F = \ B^{-5/2} \exp(-c \int \frac{\mathrm{d}x}{B}) \tag{3.10}$$

of $F$.

In the general case c≠0, dx+e≠0 , one can simplify the task by realizing that the system (3.5)-(3.6) of differential equations admits two symmetries, which in the gauge $d = 0$ have the form

$$Y_1 = \partial_x, \quad Y_2 = x\ \partial_x + B\,\partial_B - 2\,F\,\partial_F. \tag{3.11}$$

By introducing the variables $\lambda = -\ln\sqrt{F}$ and $z = B\sqrt{F}$ and making the substitution $w = \mathrm{d}\,z\,/\,\mathrm{d}\,\lambda = w(z)$, one can use these symmetries to reduce the system to the second

order equation

$$4e = -\frac{8\,c^3}{z+5w}\left[\frac{\mathrm{d}}{\mathrm{d}z}\frac{w}{z+5w}\frac{\mathrm{d}}{\mathrm{d}z}\frac{z+w}{z+5w} - \frac{w}{z+5w}\frac{\mathrm{d}}{\mathrm{d}z}\frac{z+w}{z+5w}\right] \tag{3.12}$$

which admits no further symmetries. No explicit solution of (2.9) is known, although in principle a class of solutions could be constructed: For the two- symmetry case (4.7) there always is one symmetry with $c \neq 0$, $dx + e \neq 0$.

For illustration, we give a few examples of functions $F(x)$ - taken from the literature ([1], [3], [4], [9]) together with the corresponding symmetries:

$$\begin{array}{ll}
F = \mathrm{e}^x & B = 1,\ c = 1,\ d = 0 = e \\[2ex]
F = x^n & B = x,\ c = -(n+5/2),\ d = 0 = e \\[2ex]
F = (x-x_1)^n\,(x-x_2)^{-n-5} & B = (x-x_1)(x-x_2), \\
 & c = (2n+5)(x_2-x_1)/2;\ d = 0 = e \\[2ex]
F = x^{-15/7} & B = x^{6/7}\,,\ c = 0 = d,\ e = 12/7^3.
\end{array} \tag{3.13}$$

### 3.3. Invariant solutions

If there is a symmetry, it may be worthwhile to ask whether interesting solutions exist which are fixed points of the symmetry [6], i.e. solutions of $y'' = F(x)\,y^2$ which lie on an invariant surface $y - y(x) = 0$ and therefore satisfy

$$X\ \ [\ y - y(x)] = \ (c + B'/2)y + d\,x + e - B\,y' = 0. \tag{3.14}$$

In our context, the interesting solutions are those which contain an arbitrary parameter *not* occurring in $F$ (which then could be taken as a function of time). Since the general solution of the inhomogeneous linear differential equation (3.14) has the form $y = \alpha\,y_{(\mathrm{hom})} + y_{(\mathrm{inh})}$, the parameter $\alpha$ does not drop out when inserting this into $y'' = F(x)\,y^2$ and cannot be taken as a function of time; the invariant solutions are not of interest.

### 3.4. Normal form of the differential equation $y'' = F(x)\,y^2$ and reduction of order (first integrals)

If a symmetry is known, one can use it to simplify the differential equation $y'' = F(x)\,y^2$ by bringing it into a normal form; this is possible without specifying the function $F$ and the symmetry. To do this, we introduce new variables $Y(y,x)$ and $s(x)$ adopted to the symmetry

$$X = B(x)\,\partial_x + [(c + B'/2)\,y + d\,x + e]\partial_{\,y} \tag{3.15}$$

by $XY = 0,\ Xs = 1$ by defining

$$Y = y\ u(x) + v(x), \qquad X\,Y = 0, \qquad u(x) = \tfrac{1}{\sqrt{B}} \exp(-\textstyle\int \frac{c}{B}\mathrm{d}x),$$

$$v(x) = -\textstyle\int u\frac{dx+e}{B}\mathrm{d}x = -\frac{1}{4}\int BB'''\mathrm{d}x = -\frac{1}{4}BB'' + \frac{1}{8}(B')^2 + v_0 \tag{3.16}$$

and

$$s = \int \frac{\mathrm{d}x}{B}, \qquad X\,s = 1. \tag{3.17}$$

Using the relation

$$F = B^{-5/2} \exp(-\int \frac{c}{B}\mathrm{d}x) = \frac{u}{B^2}\,, \tag{3.18}$$

which follows from the definition (3.16) of $u$ and from the representation ( 3.10) of $F$, we obtain from $y'' = F(x)y^2$ the differential equation for $Y(s)$

$$\ddot{Y} = -2c\dot{Y} + Y^{\,2} + b_1 Y + b_2, \tag{3.19}$$

where the two constants $b_1$ and $b_2$ are given by

$$b_1 = \frac{B'^{\,2}}{4} - \frac{BB''}{2} - c^2 - 2v \tag{3.20}$$

and

$$b_2 = -v\,(v + b_1) + B^2\,u^2\,(v'\,/\,u^2)'. \tag{3.21}$$

It follows from the general theory that $b_1$ and $b_2$ are really constants: since $X$ is a symmetry and $Xs = 1$ holds, the differential equation (3.19) must not depend on $s$. Of course it

is possible to check this directly by taking the derivative with respect to $s$ and using the symmetry conditions (3.5)-(3.6).The values of the two constants depend on the symmetry; for $B = \alpha x^2 + 2\beta x + \gamma$ we have e.g. $b_1 = \beta^2 - \alpha\gamma - c^2,\ b_2 = 0$. Since $v$ is defined only up to an additive constant, $b_1 = 0$ can always be achieved.

The result of these calculations is that the existence of a symmetry implies that $y'' = F(x)y^2$ can be transformed into the simple looking equation $\ddot{Y} = -2c\dot{Y} + Y^{\ 2} + b_1 Y + b_2$. Because of its symmetry, which now reads $X = \partial_s$, the substitution $\dot{Y} = p(Y), \ddot{Y} = pp_Y$ is possible; it leads to the first order differential equation (Abelian differential equation for $p(Y)$)

$$pp_Y = -2cp + Y^2 + b_1 Y + b_2. \tag{3.22}$$

Since a transformation $Y \to Y + const$ is always possible, the type and the integrability of these differential equations is characterized by

$$\triangle = b_2 - b_1^2/4. \tag{3.23}$$

For c=0, the equation (3.19) simplifies to $\ddot{Y} = Y^2 + b_1 Y + b_2$ , which can obviously be solved by quadratures even if there is no second symmetry (remember that for $c = 0$ the general $F$ admitting a symmetry is known). Thus we can state: *If $y'' = F(x)\, y^2$ admits a symmetry with $B = F^{-2/5}$, then it can be solved by quadratures.* Comparing the above form $\ddot{Y} = Y^2 + \ b_1 Y + b_2$ with Wyman's classification, we can also state: *If $y'' = F(x)\, y^2$ admits a symmetry with $B = F^{-2/5}$, then it can be transformed into the Wyman cases (2.5) or (2.6).* This result agrees with that of Srivastava [3] that if there is a symmetry with $c = 0$, then there is either a second symmetry or a mapping to the Wyman class $y'' = y^2 + q$ is possible.

For c$\neq$0, one can gauge $c = 1$ and $b_1 = 0$; then the differential equation (3.19) reads

$$\ddot{Y} = -2\dot{Y} + Y^2 + b_2. \tag{3.24}$$

This is a one-parameter family of differential equations, and correspondingly the functions $F(x)$ with $c \neq 0$ form (up to mappings $F \to \tilde{F}$ !) a one-parameter set.

Are there cases where $\ddot{Y} = -2\dot{Y} + Y^2 + b_2$ can be solved? We will come back to this question when discussing the two-symmetries case in the following section, and show that

in the above mentioned set the point $\triangle = b_2 = -(24/25)^2$ plays an exceptional rôle: only for this value of $\triangle$ solutions are known.

## 4. Solutions of $y'' = F(x)\, y^{\,2}$ with two Lie point symmetries

### 4.1. The conditions for the existence of two symmetries

As already stated in Section 1, there are at most two symmetries for a given $F(x)$. Because of (2.17), we can choose them to be

$$\begin{aligned} B_1 = F^{-2/5}, \qquad & X_1 = B_1\partial_x + [B_1'/2)\, y + d_1\, x + e_1]\partial_{\,y}\,, \\ & a = 1,\, c = 0 \\ B_2 = F^{-2/5}\int F^{2/5}\mathrm{d}x, \quad & X_2 = B_2\partial_x + [(B_2' - 5)\, y/2 + d_2\, x + e_2]\partial_{\,y}\,, \\ & a = 0,\, c = -5/2. \end{aligned} \tag{4.1}$$

The commutator of the two symmetries is

$$[X_1\,,\, X_{\,2}\,] = X_{\,1}\,; \tag{4.2}$$

we have a non-abelian group. These two symmetries have to obey the symmetry condition (2.18) *with the same* $F(x)$. Introducing the notation

$$\phi = \int F^{2/5}\mathrm{d}x \;\;,\;\; F = (\phi')^{5/2} \;\;,\;\; B_1 = \frac{1}{\phi'}\;,\; B_2 = \frac{\phi}{\phi'}\;\;, \tag{4.3}$$

this gives rise to the two conditions

$$\begin{aligned} 4(d_{\,1}\, x + e_1)(\phi')^{5/2} &= (1/\phi')''' \\ 4(d_{\,2}\, x + e_2)(\phi')^{5/2} &= (\phi/\phi')''' \end{aligned} \tag{4.4}$$

which have to be satisfied simultaneously.

### 4.2. Functions $F(x)$ for which $y'' = F(x)\, y^2$ admits two symmetries

For the general case $(d_1\; x + e_1)(d_2\; x + e_2) \neq 0$, and in the gauge $d_1 = 0$, $e_1 = 1$, the general solution of the system (4.4) could be found (Wolf 1996 [8]) by means of the substitution

$$W(t) \equiv k(\phi - d_2\, x - e_2)/d_2\,,\; t = kx,\;\; k = d_2^{\,7/6} \tag{4.5}$$

and some computer algebra; one obtains

$$W^3\,(\mathrm{d}W/\mathrm{d}\,t+1)^{3/2}-12(\mathrm{d}W/\mathrm{d}\,t+1)+4=0\ , \tag{4.6}$$

which is $t-$independent and can be solved by a quadrature. Equation (4.6) is a first order differential equation for $\phi$, and if we insert the corresponding condition on $F$ into (2.9), we find that (2.9) is satisfied: *All differential equations* $y''=F(x)\,y^2$ *which admit two symmetries with* $(d_1\ x+e_1)(d_2\ x+e_2)\neq 0$ *can be transformed into* $y''=y^2$.

We will not pursue this line of thought further (and not consider the other cases with $(d_1\ x+e_1)(d_2\ x+e_2)=0$) since we can take a more immediate way by starting from the normal form

$$\ddot{Y}=-2c\dot{Y}+Y^{\,2}+b_1Y+b_2 \tag{4.7}$$

of the differential equation $y''=F(x)\,y^2$ for the case that it admits *one* symmetry. If there are *two* symmetries, then (4.7) has also to admit two symmetries. If we perform a symmetry analysis of (4.7), the result is that two symmetries can occur if and only if

$$\triangle=b_2-b_1^2/4=-(24\,c^2/\,25)^2 \tag{4.8}$$

holds. If there are two symmetries, then - as shown in (4.1) - there exists always one with $c=0$; starting with that symmetry and taking the gauge $b_1=0$ , we have $\ddot{Y}=Y^{\,2}+b_2$ instead of (4.7), and the condition (4.8) for the existence of two symmetries gives $b_2=0$ : the equation $y''=F(x)y^2$ admits two symmetries exactly if its normal form is $\ddot{Y}=Y^2$. Of course, if there are two symmetries we need not transform the given $F(x)$ into $F=1$; we would rather follow the usual procedure [6] to solve $y''=F(x)y^2$ directly by means of quadratures.

We now come back to the question raised at the end of the preceding Section: Are there cases with *one* symmetry where $\ddot{Y}=-2\dot{Y}+Y^2+b_2$ (or equivalently $pp_Y=-2p+Y^2+b$) can be solved in terms of known functions? The answer is: No; the solution is known for $b_2=-(24/25)^2$, but in this case a second symmetry exists. In all other cases no solution has been found so far.

We close this section with a few examples of functions $F(x)$ which admit two symmetries;

these are

$$
\begin{aligned}
&F = 1 \qquad && X_1 = \partial_x\,, \quad X_2 = x\partial_x - 2y\partial_y \\
&F = x^{-5} \qquad && X_1 = x^2\partial_x + xy\partial_y\,,\; X_2 = x\partial_x + 3y\partial_y \\
&F = x^{-15/7} \qquad && X_1 = x^{6/7}\partial_x + [(12/7^3) + (3/7)yx^{-1/7}]\partial_y \\
& && X_2 = 7x\partial_x + y\partial_y \\
&F = \frac{(x-x_1)^{-15/7}}{(x-x_2)^{20/7}} \qquad && X_1 = (x-x_1)(x-x_2)\partial_x \\
& && \quad +[(2x-x_1-x_2)/2 + 5(x_2-x_1)/14]y\partial_y \\
& && X_2 = (x-x_1)^{6/7}(x-x_2)^{8/7}\partial_x + \\
& && \quad +[\{(x-x_2)^{1/7}(x-x_1)^{-1/7}(7x-4x_1-3x_2)/7\}y \\
& && \quad +12(x-x_2)(x_1-x_2)^3/343]\partial_y
\end{aligned}
\tag{4.9, 4.10}
$$

(note that the symmetries are not given in the gauge (4.1) to avoid some irrelevant constants).

## 5. Summary

For the symmetry approach to the differential equation $y'' = F(x)y^2$ two different tasks can be imagined.

*The first task* is to determine all functions $F(x)$ for which $y'' = F(x)y^2$ admits a symmetry. Two cases have to be distinguished:

(1a) $y'' = F(x)y^2$ should admit one symmetry. In the notation of Section 3.2, the state of the art is: All functions $F$ belonging to symmetries with either $c = 0$ or $dx + e = 0$ are known and given by (3.7 - 3.8) for $c = 0$, and by (3.9 -3.10 ) for $dx + e = 0$; for the general case $c(dx + e) \neq 0$ one has to solve (3.12), but no solution is known so far which admits only one symmetry.

(1b) $y'' = F(x)y^2$ should admit two symmetries. All these functions $F$ are known; they satisfy $2[F^{-4/5}(F^{-1/5})'']'' = F^{-3/5}[(F^{-1/5})'']^2$. To construct them, one could go along the lines indicated in Section 2.2, or one could solve (4.7) for the general case $(d_1\ x+e_1)(d_2\ x+e_2) \neq 0$ and start from the one-symmetry case for the cases with $(d_1\ x+e_1)(d_2\ x+e_2) = 0$,

or one could follow Wyman's procedure [4] . It is important to note in this context that although all functions $F(x)$ belonging here can be mapped to $F = 1$, the (physical) meaning and interpretation of gravitational fields characterized by different $F$'s will be rather different.

*The second task* is that $F(x)$ is given and one intends to solve the differential equation. Then three main cases can occur:

(2a) $y'' = F(x)y^2$ does not admit a symmetry. Then the symmetry approach does not help. One may instead use Wyman's approach and test whether $F$ satisfies (2.7); if yes, then the solution can be given in terms of a Painlevé function. Otherwise no solution is known.

(2b) $y'' = F(x)y^2$ admits exactly one symmetry. Then $y'' = F(x)y^2$ can be transformed into $\ddot{Y} = -2c\dot{Y} + Y^2 + b_1 Y + b_2$ (or equivalently into the first order equation $pp_Y = -2cp + Y^2 + b_1 Y + b_2$) by means of the transformation (3.16)-(3.17). If the symmetry has $c = 0$, then this equation can be solved by quadratures (in that case $F$ also satisfies (2.8)). For $c \neq 0$, no solution of $y'' = F(x)y^2$ is known.

(2c) $y'' = F(x)y^2$ admits two symmetries. Then its general solution can be given in terms of quadratures by standard group theoretical methods [6], or by transforming $y'' = F(x)y^2$ into (4.8) using the symmetry with $c = 0$ and then performing some quadratures. In the two-symmetry case $F$ satisfies (2.9).

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