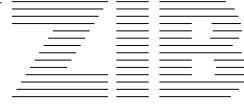


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**Approximations for the Distribution Function
of the Sum of iid Random Variables
with Compact Support in \mathbb{R}_+**

Approximations for the Distribution Function of the Sum of iid Random Variables with Compact Support in \mathbb{R}_+ ¹

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Abstract

In this paper a unified approach to central and decentral approximations of the distribution function $F(x, n)$ of the sum of n iid random variables with compact support in \mathbb{R}_+ is given. This approach yields direct Edgeworth expansion (especially the Central limit theorem) and indirect Edgeworth expansion (Theorem of Bahadur-Rao, large deviation results) within a unified framework. An approximative inversion of the LST of $F(x, n)$ (approximation of the complex inversion integral over a line by an integral over a proper bounded arc with a proper integrand) allows to get these approximations and moreover explicit error bounds.

1 Introduction

The distribution of the number $N(t)$ of packets arriving during an interval $[0, t]$ and generated by independent identically distributed sources within an ATM network, cf. [BB94], led to the problem of approximating the distribution function of the sum of iid random variables with compact support in \mathbb{R}_+ . Also in risk theory often appears the problem of approximating the distribution function of the sum of iid random variables. The distribution function of the sum of n real-valued iid random variables can be approximated by the Central limit theorem or, with more accuracy, by a direct Edgeworth expansion. But in case of very rare events large deviation techniques yield better approximations.

Applying complex analytic methods, these approximations and explicit error bounds are obtained within a unified framework. An investigation of the LST of a distribution function with compact support in \mathbb{R}_+ and an approximative inversion of the n -th power of the LST yield these results. The approximative inversion of the n -th power of the LST is based on an approximation of the complex inversion integral for Laplace transforms over a line by an integral over a proper bounded arc and a proper integrand. Depending on the choice of the arc in the complex plane we obtain a central approximation (Central limit theorem or direct Edgeworth expansion, cf. e.g. [F71] p. 541, depending on the choice of the approximating integrand) or a decentral approximation (large deviation results, Theorem of Cramér, indirect Edgeworth expansion, Theorem of Bahadur-Rao, cf. e.g. [BR60], [BC79], [B90]). In particular, firstly this approach yields a unified

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framework for the approximation of the distribution function of the sum of iid random variables in the different regions, secondly we obtain new error bounds for these approximations.

The paper is organized as follows. In Section 2 we investigate the LST of the distribution function of a random variable with compact support in \mathbb{R}_+ . The LST is an entire holomorphic function of first order. We obtain bounds for the LST and for the semi-invariants of the distribution function. Moreover we show that the inversion integral for Laplace transforms yields an approximation of the distribution function by substituting the path of integration over a line in the right half plane by a proper bounded arc in the complex plane. In Section 3 we give central approximations (Central limit theorem and direct Edgeworth expansion) of the distribution function of the sum of iid random variables with compact support in \mathbb{R}_+ and explicit error bounds. In Section 4 in an analogous way we obtain a decentral approximation (indirect Edgeworth expansion resp. Theorem of Bahadur-Rao) of the distribution function of the sum of iid random variables with compact support in \mathbb{R}_+ and also explicit error bounds. As an application in Section 5 we investigate the superposition of iid two-state sources (e.g. voice sources) within an ATM network, cf. [BB94].

2 The LST of the distribution function of a real-valued random variable with compact support

Let $F(x) := P(X \leq x)$ be the distribution function of a real-valued random variable X . We assume that $F(x)$ has the following property:

P1: The support of $F(x)$ is a compact subset of \mathbb{R}_+ and contains at least two points.

Later also we will make use of the following property of $F(x)$ which especially excludes the case of a lattice distribution:

P2: There exist an interval I of positive length ℓ and $\delta > 0$ such that

$$|F(x_2) - F(x_1)| \geq \delta |x_2 - x_1| \quad \forall (x_1, x_2) \in I^2. \quad (2.1)$$

Let us denote by $[a, b]$ the shortest interval containing the support of $F(x)$ and by $L := b - a$ its length. Let $g(s) := Ee^{-sX}$ be the LST of the distribution function $F(x)$. Since the support of $F(x)$ is compact in \mathbb{R}_+ and contains at least two points $g(s)$ is an entire holomorphic function of first order with an infinite number of zeros. Choosing that branch of $\sigma_0(s) := \log g(s)$ vanishing at $s = 0$ we obtain a function which is holomorphic in a neighborhood of $s = 0$. Finally, let

$$\sigma_\nu(s) := (-1)^\nu \frac{d^\nu}{ds^\nu} \sigma_0(s), \quad \nu \in \mathbb{N}. \quad (2.2)$$

$\sigma_\nu(s)$ is a meromorphic function for $\nu \in \mathbb{N}$. $\sigma_1(0)$ and $\sigma_2(0)$ are the expectation resp. the variance of the random variable X , in general $\sigma_\nu(0)$ is the semi-invariant of order ν of the distribution function $F(x)$, cf. [F71].

The following Lemma 2.1 gives bounds for $\sigma_\nu(u)$, $u \in \mathbb{R}$, $\nu = 2, 3, \dots$. The distribution functions

$$F_2(x) := \frac{1}{2} \mathbb{I}\{x \geq a\} + \frac{1}{2} \mathbb{I}\{x \geq b\},$$

$$F_3(x) := \left(\frac{1}{2} \pm \frac{1}{\sqrt{12}}\right) \mathbb{I}\{x \geq a\} + \left(\frac{1}{2} \mp \frac{1}{\sqrt{12}}\right) \mathbb{I}\{x \geq b\}$$

show that these bounds are sharp in case of $\nu = 2$ and $\nu = 3$.

Lemma 2.1 *Let $F(x)$ be a distribution function satisfying (P1). Then we have:*

i) $\sigma_0(s)$ is holomorphic in $G := \{s \in \mathcal{C} : |\operatorname{Im} s| < \pi/L\}$ satisfying the estimate

$$\left| \operatorname{Im} \left(\sigma_0(s) + \frac{a+b}{2}s \right) \right| \leq \frac{L}{2} |\operatorname{Im} s| \quad \forall s \in G. \quad (2.3)$$

ii) $\sigma_1(u)$ is strictly decreasing on \mathbb{R} and maps \mathbb{R} onto the interval (a, b) .

iii) For $u \in \mathbb{R}$ it holds

$$0 < \sigma_2(u) \leq \frac{1}{4}L^2, \quad (2.4)$$

$$|\sigma_3(u)| \leq \frac{\sqrt{3}}{18}L^3, \quad (2.5)$$

$$\left| \frac{\sigma_\nu(u)}{\nu!} \right| \leq \frac{3\sqrt{3}}{4} \left(\frac{L}{\pi} \right)^\nu, \quad \nu = 2, 3, \dots \quad (2.6)$$

Proof:

i) For $s \in G$, $s = u + iv$ it follows

$$\operatorname{Re} \left(e^{\frac{a+b}{2}s} g(s) \right) = \int_a^b e^{(\frac{a+b}{2}-x)u} \cos \left(\left(\frac{a+b}{2} - x \right) v \right) dF(x) \geq e^{-\frac{L}{2}|u|} \cos \left(\frac{Lv}{2} \right) > 0.$$

Therefore, $g(s)$ has no zeros in G and $\sigma_0(s) = \log g(s)$ is holomorphic in G . Furthermore, for $s \in G$, $\operatorname{Im} s > 0$ it follows

$$\operatorname{Im} \left(e^{i\frac{Lv}{2}} e^{\frac{a+b}{2}s} g(s) \right) = \int_a^b e^{(\frac{a+b}{2}-x)u} \sin((b-x)v) dF(x) \geq 0,$$

$$\operatorname{Im} \left(e^{-i\frac{Lv}{2}} e^{\frac{a+b}{2}s} g(s) \right) = \int_a^b e^{(\frac{a+b}{2}-x)u} \sin((a-x)v) dF(x) \leq 0$$

hence

$$|\operatorname{Im} \log(e^{\frac{a+b}{2}s} g(s))| = |\arg(e^{\frac{a+b}{2}s} g(s))| \leq \frac{L}{2} |\operatorname{Im} s| \quad \forall s \in G.$$

iii) Let

$$F_u(x) := \frac{1}{g(u)} \int_a^x e^{-u\xi} dF(\xi)$$

denote the twisted distribution function, cf. [B90]. For fixed $c \in \mathbb{R}$ the expansions

$$\frac{e^{ch}g(u+h)}{g(u)} = \int_a^b e^{(c-x)h} dF_u(x) = 1 + \sum_{k=1}^{\infty} \frac{h^k}{k!} \int_a^b (c-x)^k dF_u(x)$$

and

$$\begin{aligned} \log \frac{e^{ch}g(u+h)}{g(u)} &= h \int_a^b (c-x) dF_u(x) + \frac{h^2}{2!} \left[\int_a^b (c-x)^2 dF_u(x) - \left(\int_a^b (c-x) dF_u(x) \right)^2 \right] \\ &+ \frac{h^3}{3!} \left[\int_a^b (c-x)^3 dF_u(x) - 3 \int_a^b (c-x) dF_u(x) \int_a^b (c-x)^2 dF_u(x) \right. \\ &\quad \left. + 2 \left(\int_a^b (c-x) dF_u(x) \right)^3 \right] + \dots \end{aligned}$$

converge for $|h| < \pi/L$. On the other hand, for $|h| < \pi/L$ we have the Taylor-expansion

$$\log \frac{e^{ch}g(u+h)}{g(u)} = ch + \sigma_0(u+h) - \sigma_0(u) = h(c - \sigma_1(u)) + \sum_{k=2}^{\infty} \frac{h^k}{k!} (-1)^k \sigma_k(u).$$

Comparing the coefficients we obtain representations of $\sigma_k(u)$ for $k \in \mathbb{N}$. Choosing $c = \sigma_1(u)$ resp. $c = (a+b)/2$ the representation

$$\sigma_2(u) = \int_a^b (c-x)^2 dF_u(x) - \left(\int_a^b (c-x) dF_u(x) \right)^2 \quad (2.7)$$

yields

$$\sigma_2(u) = \int_a^b (\sigma_1(u) - x)^2 dF_u(x) > 0, \quad (2.8)$$

$$\sigma_2(u) \leq \int_a^b \left(\frac{a+b}{2} - x \right)^2 dF_u(x) \leq \frac{1}{4} L^2.$$

Choosing $c = b$ the representation of $\sigma_3(u)$ yields in view of Cauchy-Schwarz's inequality

$$\begin{aligned} \sigma_3(u) &= - \int_a^b (b-x)^3 dF_u(x) + 3 \int_a^b (b-x) dF_u(x) \int_a^b (b-x)^2 dF_u(x) - 2 \left(\int_a^b (b-x) dF_u(x) \right)^3 \\ &\leq - \int_a^b (b-x)^3 dF_u(x) + 3 \left(\int_a^b (b-x) dF_u(x) \right)^{3/2} \left(\int_a^b (b-x)^3 dF_u(x) \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
& -2 \left(\int_a^b (b-x) dF_u(x) \right)^3 \\
&= \left(\frac{\int_a^b (b-x)^3 dF_u(x)}{\int_a^b (b-x) dF_u(x)} \right)^{3/2} (-\xi + 3\xi^2 - 2\xi^3),
\end{aligned}$$

where

$$\xi := \left(\int_a^b (b-x) dF_u(x) \right)^{3/2} \left(\int_a^b (b-x)^3 dF_u(x) \right)^{-1/2}.$$

Since

$$-\xi + 3\xi^2 - 2\xi^3 \leq \frac{\sqrt{3}}{18}, \quad \xi \in [0, \infty)$$

we get

$$\sigma_3(u) \leq \frac{\sqrt{3}}{18} \left(\frac{\int_a^b (b-x)^3 dF_u(x)}{\int_a^b (b-x) dF_u(x)} \right)^{3/2} \leq \frac{\sqrt{3}}{18} L^3.$$

The corresponding estimate below for $\sigma_3(u)$ follows by choosing $c = a$.

Let $\varphi(s) := \sigma_\theta(s) + \frac{a+b}{2}s$, $s \in G$. Let $u \in \mathbb{R}$ and $r \in (0, \pi/L)$ be fixed. Then for $\nu = 1, 2, \dots$ it holds

$$\begin{aligned}
\int_0^\pi \operatorname{Im} \varphi(u + re^{i\alpha}) \sin(\nu\alpha) d\alpha &= \int_0^\pi \operatorname{Im} \left(\sum_{k=0}^{\infty} \frac{\varphi^{(k)}(u)}{k!} (re^{i\alpha})^k \right) \sin(\nu\alpha) d\alpha \\
&= \sum_{k=0}^{\infty} \frac{\varphi^{(k)}(u)}{k!} r^k \int_0^\pi \sin(k\alpha) \sin(\nu\alpha) d\alpha.
\end{aligned}$$

In view of

$$\int_0^\pi \sin(k\alpha) \sin(\nu\alpha) d\alpha = \frac{\pi}{2} \delta_{k\nu}$$

it follows

$$\frac{\varphi^{(\nu)}(u)}{\nu!} = \frac{2}{\pi r^\nu} \int_0^\pi \operatorname{Im} \varphi(u + re^{i\alpha}) \sin(\nu\alpha) d\alpha, \quad \nu = 1, 2, \dots$$

By the definitions of $\varphi(s)$ and $\sigma_\nu(s)$ therefore we have

$$\frac{\sigma_\nu(u)}{\nu!} = \frac{2(-1)^\nu}{\pi r^\nu} \int_0^\pi \operatorname{Im} \varphi(u + re^{i\alpha}) \sin(\nu\alpha) d\alpha, \quad \nu = 2, 3, \dots$$

Because of (2.3) we get the estimate

$$\left| \frac{\sigma_\nu(u)}{\nu!} \right| \leq \frac{2}{\pi r^\nu} \int_0^\pi \left| \frac{L}{2} r \sin(\alpha) \sin(\nu\alpha) \right| d\alpha = \frac{L}{\pi r^{\nu-1}} \frac{2\nu}{(\nu^2 - 1) \tan(\frac{\pi}{2\nu})}, \quad \nu = 2, 3, \dots$$

The limit $r \uparrow \pi/L$ yields

$$\left| \frac{\sigma_\nu(u)}{\nu!} \right| \leq \left(\frac{L}{\pi} \right)^\nu \frac{2\nu}{(\nu^2 - 1) \tan(\frac{\pi}{2\nu})}, \quad \nu = 2, 3, \dots$$

Elementary computation shows

$$\frac{2\nu}{(\nu^2 - 1) \tan(\frac{\pi}{2\nu})} \leq \frac{3\sqrt{3}}{4}, \quad \nu = 3, 4, \dots, 7,$$

and in case of $\nu \geq 8$ it follows

$$\frac{2\nu}{(\nu^2 - 1) \tan(\frac{\pi}{2\nu})} \leq \frac{2\nu}{(\nu^2 - 1) \frac{\pi}{2\nu}} = \frac{4/\pi}{1 - (1/\nu)^2} \leq \frac{4/\pi}{1 - (1/8)^2} < \frac{3\sqrt{3}}{4}.$$

In view of (2.4) therefore it holds (2.6).

ii) Because of $\sigma_1'(u) = -\sigma_2(u)$ and (2.4) the function $\sigma_1(u)$ is strictly decreasing on \mathbb{R} . Therefore only we have to prove

$$\lim_{u \rightarrow -\infty} \sigma_1(u) = b, \quad \lim_{u \rightarrow +\infty} \sigma_1(u) = a,$$

i.e.

$$\lim_{u \rightarrow -\infty} \frac{\int_a^b x e^{-xu} dF(x)}{\int_a^b e^{-xu} dF(x)} = b, \quad \lim_{u \rightarrow +\infty} \frac{\int_a^b x e^{-xu} dF(x)}{\int_a^b e^{-xu} dF(x)} = a.$$

From

$$\int_a^b (x - a) e^{-xu} dF(x) \geq 0$$

it follows

$$\frac{\int_a^b x e^{-xu} dF(x)}{\int_a^b e^{-xu} dF(x)} \geq a \quad \forall u \in \mathbb{R}. \tag{2.9}$$

For fixed $\varepsilon \in (0, L/2)$ and sufficiently large $u \in \mathbb{R}$ we have

$$\begin{aligned} \int_a^b (a + 2\varepsilon - x)e^{-xu} dF(x) &\geq \int_a^{a+\varepsilon} \varepsilon e^{-(a+\varepsilon)u} dF(x) - \int_{a+2\varepsilon}^b L e^{-(a+2\varepsilon)u} dF(x) \\ &\geq e^{-(a+\varepsilon)u} \left(\varepsilon \int_a^{a+\varepsilon} dF(x) - L e^{-\varepsilon u} \right) > 0, \end{aligned}$$

i.e.

$$\frac{\int_a^b x e^{-xu} dF(x)}{\int_a^b e^{-xu} dF(x)} < a + 2\varepsilon. \quad (2.10)$$

From (2.9), (2.10) we obtain the limit $u \rightarrow +\infty$. Analogously the limit $u \rightarrow -\infty$ follows.

The following Lemma 2.2 gives bounds for the quantity $|g(u+iv)/g(u)|$. Obviously, this quantity lies in the interval $[0, 1]$. In case of a distribution function $F(x)$ satisfying (P1) Lemma 2.2 gives nontrivial bounds only in a neighborhood of $v = 0$. If $F(x)$ satisfies (P1) and (P2) then Lemma 2.2 gives a global nontrivial upper bound, too.

Lemma 2.2 *Let $F(x)$ be a distribution function satisfying (P1). Then it holds*

$$\left| \frac{g(u+iv)}{g(u)} \right| \geq 1 - \frac{1}{2} \sigma_2(u) v^2 \quad \forall (u, v) \in \mathbb{R}^2, \quad (2.11)$$

$$\left| \frac{g(u+iv)}{g(u)} \right| \leq 1 - \frac{2}{L^2} \sigma_2(u) \sin^2 \left(\frac{Lv}{2} \right) \quad \forall (u, v) \in \mathbb{R} \times [-\pi/L, \pi/L]. \quad (2.12)$$

Let $F(x)$ be a distribution function satisfying (P1) and (P2). Then moreover we have

$$\left| \frac{g(u+iv)}{g(u)} \right| \leq 1 - \left(1 - \frac{2}{\pi} \right) \delta \ell e^{-L|u|} \min \left\{ 1, \left(\frac{\ell v}{\pi} \right)^2 \right\} \quad \forall (u, v) \in \mathbb{R}^2. \quad (2.13)$$

Proof: Let $F(x)$ be a distribution function satisfying (P1). Then for $(u, v) \in \mathbb{R}^2$ and for all $\varphi \in \mathbb{R}$ it holds

$$\begin{aligned} |g(u+iv)| &\geq \operatorname{Re} e^{i\varphi v} g(u+iv) = \int_a^b e^{-xu} \left(1 - 2 \sin^2 \left(\frac{\varphi - x}{2} v \right) \right) dF(x) \\ &\geq \int_a^b e^{-xu} \left(1 - \frac{v^2}{2} (\varphi - x)^2 \right) dF(x) = g(u) - \frac{v^2}{2} (g(u)\varphi^2 + 2g'(u)\varphi + g''(u)). \end{aligned}$$

Choosing now $\varphi := -g'(u)/g(u)$ we conclude (2.11).

Now, let $(u, v) \in \mathbb{R} \times (0, \pi/L)$. In view of (2.3) it holds

$$\left| \operatorname{Im}(\sigma_\theta(u + iv)) + \frac{a+b}{2}v \right| \leq \frac{b-a}{2}v,$$

i.e. $-\arg(g(u + iv)) \in [av, bv]$. Hence there exists $\varphi \in [a, b]$ such that

$$|g(u + iv)| = \operatorname{Re} e^{i\varphi v} g(u + iv) = \int_a^b e^{-xu} \left(1 - 2 \sin^2 \left(\frac{\varphi - x}{2} v \right) \right) dF(x).$$

As we have

$$\frac{\sin t}{t} \geq \frac{\sin(\frac{Lv}{2})}{\frac{Lv}{2}}, \quad |t| \leq \frac{Lv}{2} < \frac{\pi}{2}$$

furthermore it follows

$$\begin{aligned} |g(u + iv)| &\leq \int_a^b e^{-xu} \left(1 - \frac{2}{L^2} \sin^2 \left(\frac{Lv}{2} \right) (\varphi - x)^2 \right) dF(x) \\ &= g(u) - \frac{2}{L^2} \sin^2 \left(\frac{Lv}{2} \right) (g(u)\varphi^2 + 2g'(u)\varphi + g''(u)). \end{aligned}$$

Maximizing the right-hand side of this inequality wrt. $\varphi \in \mathbb{R}$ yields

$$\left| \frac{g(u + iv)}{g(u)} \right| \leq 1 - \frac{2}{L^2} \sigma_2(u) \sin^2 \left(\frac{Lv}{2} \right), \quad (u, v) \in \mathbb{R} \times (0, \pi/L).$$

Using continuity and symmetry wrt. real axis we get (2.12).

Now, let $F(x)$ be a distribution function satisfying (P1) and (P2). Then for $(u, v) \in \mathbb{R}^2$ and for proper $\varphi \in \mathbb{R}$ it holds

$$\begin{aligned} e^{\frac{a+b}{2}u} [g(u) - |g(u + iv)|] &= e^{\frac{a+b}{2}u} [g(u) - \operatorname{Re} e^{i\varphi v} g(u + iv)] \\ &= \int_a^b e^{(\frac{a+b}{2}-x)u} (1 - \cos((\varphi - x)v)) dF(x) \geq e^{-\frac{L}{2}|u|} \int_I (1 - \cos((\varphi - x)v)) dF(x) \\ &\geq \delta e^{-\frac{L}{2}|u|} \int_I (1 - \cos((\varphi - x)v)) dx = \delta e^{-\frac{L}{2}|u|} \left(\ell - \frac{2}{v} \sin\left(\frac{\ell v}{2}\right) \cos((\varphi - \theta)v) \right) \\ &\geq \delta \ell e^{-\frac{L}{2}|u|} \left(1 - \left| \frac{\sin(\frac{\ell v}{2})}{\frac{\ell v}{2}} \right| \right), \end{aligned}$$

where θ denotes the center of the interval I . Because of

$$e^{\frac{a+b}{2}u} g(u) = \int_a^b e^{(\frac{a+b}{2}-x)u} dF(x) \leq e^{\frac{L}{2}|u|}$$

we have

$$\left| \frac{g(u+iv)}{g(u)} \right| \leq 1 - \delta \ell e^{-L|u|} \left(1 - \left| \frac{\sin(\frac{\ell v}{2})}{\frac{\ell v}{2}} \right| \right). \quad (2.14)$$

Elementary analysis yields

$$1 - \left| \frac{\sin t}{t} \right| \geq \left(1 - \frac{2}{\pi} \right) \min \left\{ 1, \left(\frac{2t}{\pi} \right)^2 \right\} \quad \forall t \in \mathbb{R}. \quad (2.15)$$

From (2.14), (2.15) we obtain (2.13).

For $(u, v) \in \mathbb{R} \times (0, \infty)$ let $\gamma(u, v)$ be a rectifiable arc in $\mathcal{C} \setminus (-\infty, 0]$ from $u - iv$ to $u + iv$. The following Lemma 2.3 shows that for proper (u, v) the term

$$\frac{1}{2\pi i} \int_{\gamma(u, v)} g(s) e^{xs} \frac{ds}{s}$$

represents an approximation of the distribution function $F(x)$, i.e. taking the inversion formula for Laplace transforms and integrating over $\gamma(u, v)$ instead of a line in the right half plane we obtain for proper (u, v) an approximation of $F(x)$.

Lemma 2.3 *Let $F(x)$ be a distribution function satisfying (P1). Then for $(u, v) \in \mathbb{R} \times (0, \infty)$ it holds*

$$\begin{aligned} & \left| F(x) - \frac{1}{2\pi i} \int_{\gamma(u, v)} g(s) e^{xs} \frac{ds}{s} \right| \\ & < g(u) e^{xu+2|u|/v} \left(\sup_{t \geq v} \left| \frac{g(u+it)}{g(u)} \right| \right)^{5/6} \quad \forall x \in \mathbb{R}. \end{aligned} \quad (2.16)$$

Proof: From the inversion formula for Laplace transforms we obtain

$$\int_a^x F(\xi) d\xi = \frac{1}{2\pi i} \int_{\gamma} g(s) e^{xs} \frac{ds}{s^2},$$

where γ denotes a line in the right half plane. Because of Cauchy's integral theorem we may substitute the line γ by $\gamma(u, v) \cup \gamma^*(u, v)$, where $\gamma^*(u, v) := \{s \in \mathcal{C} : \operatorname{Re} s = u, |\operatorname{Im} s| \geq v\}$. Hence for $h \in \mathbb{R}$ with $0 < |h| \leq 2/v$ we get

$$\int_0^1 F(x + \tau h) d\tau = \frac{1}{h} \left(\int_a^{x+h} F(\xi) d\xi - \int_a^x F(\xi) d\xi \right) = \frac{1}{2\pi i} \int_{\gamma(u, v) \cup \gamma^*(u, v)} g(s) e^{xs} \frac{e^{hs} - 1}{hs} \frac{ds}{s}.$$

Cauchy's integral theorem yields

$$\int_0^1 F(x + \tau h) d\tau - \frac{1}{2\pi i} \int_{\gamma(u, v)} g(s) e^{xs} \frac{ds}{s}$$

$$= \frac{h}{2\pi i} \int_{u-iv}^{u+iv} g(s) e^{xs} \frac{e^{hs} - 1 - hs}{(hs)^2} ds + \frac{1}{2\pi i} \int_{\gamma^*(u,v)} g(s) e^{xs} \frac{e^{hs} - 1}{hs} \frac{ds}{s}, \quad (2.17)$$

where in case of the first integral on the right-hand side we may integrate along the line segment.

As for $z \in \mathcal{C} \setminus \{0\}$ we have

$$\left| \frac{e^z - 1}{z} \right| = \left| \int_0^1 e^{x_1 z} dx_1 \right| \leq e^{|Re z|} \min \left\{ 1, \frac{2}{|z|} \right\},$$

$$\left| \frac{e^z - 1 - z}{z^2} \right| = \left| \int_0^1 \int_0^1 e^{x_1 x_2 z} x_2 dx_1 dx_2 \right| \leq \frac{1}{2} e^{|Re z|}$$

from (2.17) it follows

$$\begin{aligned} & \left| \int_0^1 F(x + \tau h) d\tau - \frac{1}{2\pi i} \int_{\gamma(u,v)} g(s) e^{xs} \frac{ds}{s} \right| \\ & \leq \frac{1}{2\pi} g(u) e^{xu+|hu|} |hv| + \frac{1}{\pi} g(u) e^{xu+|hu|} \int_v^\infty \min \left\{ 1, \frac{2}{|ht|} \right\} \frac{dt}{t} e^{-\xi} \\ & \leq \frac{1}{\pi} g(u) e^{xu+2|u|/v} \left[\left| \frac{hv}{2} \right| + \left(1 + \log \left| \frac{2}{hv} \right| \right) e^{-\xi} \right], \end{aligned}$$

where

$$\xi := -\log \left(\sup_{t \geq v} \left| \frac{g(u+it)}{g(u)} \right| \right) \geq 0. \quad (2.18)$$

Consequently, choosing $h_0 := (2/v)e^{-\xi}$ it holds

$$\left| \int_0^1 F(x \pm \tau h_0) d\tau - \frac{1}{2\pi i} \int_{\gamma(u,v)} g(s) e^{xs} \frac{ds}{s} \right| \leq \frac{1}{\pi} g(u) e^{xu+2|u|/v} (2 + \xi) e^{-\xi}. \quad (2.19)$$

Since the distribution function $F(x)$ is increasing we have

$$\int_0^1 F(x - \tau h_0) d\tau \leq F(x) \leq \int_0^1 F(x + \tau h_0) d\tau,$$

and from (2.19) we get

$$\left| F(x) - \frac{1}{2\pi i} \int_{\gamma(u,v)} g(s) e^{xs} \frac{ds}{s} \right| \leq \frac{1}{\pi} g(u) e^{xu+2|u|/v} (2 + \xi) e^{-\xi}.$$

In view of

$$(2 + \xi)e^{-\xi/6} \leq 6e^{-2/3} < \pi, \quad \xi \geq 0$$

finally we obtain (2.16).

3 Central approximation of the sum of n iid random variables

Let $\{X_\nu\}_{\nu=1}^\infty$ be a sequence of iid real-valued random variables. We assume that the distribution function $F(x) = P(X_\nu \leq x)$ satisfies (P1). Let $F(x, n) := P(S_n \leq x)$ be the distribution function of the sum $S_n = X_1 + X_2 + \dots + X_n$ of the first n random variables. Obviously, $F(x, n)$ satisfies (P1) for $n \in \mathbb{N}$, too. We denote by $g(s)$ the LST of the distribution function $F(x)$, i.e. $g(s)^n$ is the LST of $F(x, n)$ for $n \in \mathbb{N}$.

The following Theorem 3.1 corresponds to a direct Edgeworth expansion. Especially in a neighbourhood of the expectation $n\sigma_1(0)$ of S_n we get a good approximation of its distribution function $F(x, n)$. This approximation makes use of the moments of $F(x)$ up to third order. The absolute error is uniformly bounded by $O(n^{-1})$. In case of the Central limit theorem, where only the first two moments of $F(x)$ are used, the absolute error is uniformly bounded by $O(n^{-1/2})$.

Theorem 3.1 *Let $F(x)$ be a distribution function satisfying (P1) and (P2).*

Then for $x \in \mathbb{R}$ and $n \in \mathbb{N}$ the approximation

$$F(x, n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{1}{2}\xi^2} d\xi + \frac{1}{6\sqrt{2\pi}} \sigma_3(0) \sigma_2(0)^{-3/2} n^{-1/2} (1 - y^2) e^{-\frac{1}{2}y^2} + r_n(y), \quad (3.1)$$

$$|r_n(y)| < \frac{1}{2} \left(\frac{L}{2\sqrt{\sigma_2(0)}} \right)^6 n^{-1} + e^{-n\kappa} \quad (3.2)$$

holds, where

$$y := \frac{x - n\sigma_1(0)}{\sqrt{n\sigma_2(0)}}, \quad (3.3)$$

$$\kappa := 0.3 \delta \ell^3 L^{-2}. \quad (3.4)$$

Moreover, for the Central limit approximation the following estimate holds:

$$\left| F(x, n) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{1}{2}\xi^2} d\xi \right| < \frac{1}{3} \left(\frac{L}{2\sqrt{\sigma_2(0)}} \right)^3 n^{-1/2} + e^{-n\kappa}. \quad (3.5)$$

Proof: For $(u, v) := (0, \pi/L)$ from Lemma 2.3 applied to the distribution function $F(x, n)$ and its LST $g(s)^n$ we get the estimate

$$|F(x, n) - J| < \left(\sup_{t \geq \pi/L} |g(it)| \right)^{\frac{5}{6}n}, \quad (3.6)$$

where

$$J := \frac{1}{2\pi i} \int_{\gamma(0, \pi/L)} g(s)^n e^{xs} \frac{ds}{s}. \quad (3.7)$$

In view of $\ell \leq L$ Lemma 2.2 yields

$$|g(it)| \leq 1 - \left(1 - \frac{2}{\pi}\right) \delta \ell^3 L^{-2} < \exp\left(-\left(1 - \frac{2}{\pi}\right) \delta \ell^3 L^{-2}\right), \quad t \geq \pi/L.$$

Therefore from (3.6) we get the estimate

$$|F(x, n) - J| < \exp\left(-\frac{5}{6}\left(1 - \frac{2}{\pi}\right) \delta \ell^3 L^{-2} n\right) < \exp(-\kappa n), \quad (3.8)$$

where κ is defined by (3.4).

Now, for fixed $\varepsilon \in (0, \pi/L)$ let $\gamma(0, \pi/L)$ be the union of the line segments from $-i\pi/L$ to $-i\varepsilon$ and from $i\varepsilon$ to $i\pi/L$ and of the circle arc from $-i\varepsilon$ over ε to $i\varepsilon$. Then it holds

$$J = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} g(\varepsilon e^{it})^n \exp(\varepsilon x e^{it}) dt + \frac{1}{\pi} \int_{\varepsilon}^{\pi/L} \operatorname{Im}(g(it)^n e^{ixt}) \frac{dt}{t}.$$

The limit $\varepsilon \downarrow 0$ yields

$$J = \frac{1}{2} + \frac{1}{\pi} \int_0^{\pi/L} \operatorname{Im}(g(it)^n e^{ixt}) \frac{dt}{t}, \quad (3.9)$$

and from (3.8) we obtain

$$\left|F(x, n) - \frac{1}{2} - \frac{1}{\pi} \int_0^{\pi/L} \operatorname{Im}(g(it)^n e^{ixt}) \frac{dt}{t}\right| < \exp(-\kappa n). \quad (3.10)$$

Defining

$$c_\nu := \frac{\sigma_\nu(0)}{\nu!} = \frac{(-1)^\nu}{\nu!} \frac{d^\nu}{ds^\nu} \log g(s) \Big|_{s=0}, \quad \nu = 1, 2, \dots, \quad (3.11)$$

$$z(t) := n \sum_{\nu=3}^{\infty} c_\nu (-it)^\nu \quad (3.12)$$

for any $\varrho \in (0, \pi)$ it follows

$$\begin{aligned} J &= \frac{1}{2} + \frac{1}{\pi} \int_0^{\varrho/L} \operatorname{Im} \exp(n\sigma_\varrho(it) + ixt) \frac{dt}{t} + \frac{1}{\pi} \int_{\varrho/L}^{\pi/L} \operatorname{Im}(g(it)^n e^{ixt}) \frac{dt}{t} \\ &= \frac{1}{2} + \frac{1}{\pi} \int_0^{\varrho/L} e^{-nc_2 t^2} \operatorname{Im}(e^{i(x-nc_1)t} e^{z(t)}) \frac{dt}{t} + \frac{1}{\pi} \int_{\varrho/L}^{\pi/L} \operatorname{Im}(g(it)^n e^{ixt}) \frac{dt}{t}. \end{aligned} \quad (3.13)$$

Because of the triangle inequality now we have the estimate

$$\Delta_3 := \left| J - \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} e^{-nc_2 t^2} \operatorname{Im}(e^{i(x-nc_1)t} (1 + inc_3 t^3)) \frac{dt}{t} \right| \leq \sum_{\nu=1}^4 J_\nu, \quad (3.14)$$

where

$$J_1 := \frac{1}{\pi} \int_{\varrho/L}^{\pi/L} |g(it)|^n \frac{dt}{t}, \quad (3.15)$$

$$J_2 := \frac{1}{\pi} \int_{\varrho/L}^{\infty} e^{-nc_2 t^2} |1 + inc_3 t^3| \frac{dt}{t}, \quad (3.16)$$

$$J_3 := \frac{1}{\pi} \int_0^{\varrho/L} e^{-nc_2 t^2} |e^{z(t)} - e^{inc_3 t^3}| \frac{dt}{t}, \quad (3.17)$$

$$J_4 := \frac{1}{\pi} \int_0^{\varrho/L} e^{-nc_2 t^2} |e^{inc_3 t^3} - (1 + inc_3 t^3)| \frac{dt}{t}. \quad (3.18)$$

In view of (3.11) and Lemma 2.1 it holds

$$0 < c_2 \leq \frac{1}{8} L^2, \quad (3.19)$$

$$|c_3| \leq \frac{\sqrt{3}}{108} L^3, \quad (3.20)$$

$$|c_\nu| \leq \frac{3\sqrt{3}}{4} \left(\frac{L}{\pi}\right)^\nu, \quad \nu = 2, 3, \dots \quad (3.21)$$

For $t \in [\varrho/L, \pi/L]$ from Lemma 2.2 it follows

$$|g(it)| \leq 1 - \frac{4c_2}{L^2} \sin^2 \frac{Lt}{2} < \exp\left(-\frac{4c_2}{L^2} \sin^2 \frac{Lt}{2}\right).$$

Defining

$$\xi := \frac{\sqrt{nc_2}}{L} \quad (3.22)$$

and taking into account (3.15), (3.19) we obtain

$$J_1 \leq \frac{1}{\pi} \int_{\varrho/L}^{\pi/L} \exp\left(-\frac{4nc_2}{L^2} \sin^2 \frac{Lt}{2}\right) \frac{dt}{t} \quad (3.23)$$

$$\begin{aligned} &= \frac{1}{\pi} \left(\frac{L}{\sqrt{nc_2}}\right)^2 \xi^2 \int_{\varrho}^{\pi} \exp(-4\xi^2 \sin^2 \frac{\tau}{2}) \frac{d\tau}{\tau} \\ &\leq \left(\frac{L}{\sqrt{8c_2}}\right)^6 \frac{1}{n} \frac{8}{\pi} \xi^2 \int_{\varrho}^{\pi} \exp(-4\xi^2 \sin^2 \frac{\tau}{2}) \frac{d\tau}{\tau}. \end{aligned} \quad (3.24)$$

Because of (3.16), (3.19), (3.20), (3.22) it holds

$$\begin{aligned}
J_2 &\leq \frac{1}{\pi} \int_{\varrho/L}^{\infty} e^{-nc_2t^2} \left(1 + \frac{n^2}{3888}(Lt)^6\right)^{1/2} \frac{dt}{t} \\
&= \frac{1}{\pi} \int_{\varrho}^{\infty} e^{-\xi^2\tau^2} \left(\left(\frac{L}{\sqrt{8c_2}}\right)^4 \frac{64}{n^2}\xi^4 + \left(\frac{L}{\sqrt{8c_2}}\right)^8 \frac{256}{243n^2}\xi^8\tau^6\right)^{1/2} \frac{d\tau}{\tau} \\
&\leq \left(\frac{L}{\sqrt{8c_2}}\right)^6 \frac{1}{n} \frac{8}{\pi} \xi^2 \int_{\varrho}^{\infty} e^{-\xi^2\tau^2} \left(1 + \frac{4}{243}\xi^4\tau^6\right)^{1/2} \frac{d\tau}{\tau} \\
&\leq \left(\frac{L}{\sqrt{8c_2}}\right)^6 \frac{1}{n} \left(\frac{8}{\pi}\xi^2 \int_{\varrho}^{\infty} e^{-\xi^2\tau^2} \frac{d\tau}{\tau} + \frac{16}{243\pi}\xi^6 \int_{\varrho}^{\infty} e^{-\xi^2\tau^2} \tau^5 d\tau\right). \tag{3.25}
\end{aligned}$$

The convexity of the exponential function yields

$$\begin{aligned}
|e^z - 1| &= \left|z \int_0^1 e^{xz} dx\right| \leq |z| \int_0^1 e^{x\operatorname{Re}z} dx \leq |z| \int_0^1 (1 + x(e^{\operatorname{Re}z} - 1)) dx \\
&= \frac{1}{2}(1 + e^{\operatorname{Re}z})|z|. \tag{3.26}
\end{aligned}$$

Consequently, in view of (3.17) we get

$$J_3 \leq \frac{1}{2\pi} \int_0^{\varrho/L} e^{-nc_2t^2} (1 + e^{\operatorname{Re}z(t)}) |z(t) - inc_3t^3| \frac{dt}{t}. \tag{3.27}$$

For $t \in [0, \varrho/L]$ from Lemma 2.2 it follows

$$\begin{aligned}
\operatorname{Re} z(t) &= \operatorname{Re}\left(n\left(\sigma_0(it) + \frac{\sigma_2(0)}{2}t^2\right)\right) \leq nc_2t^2 + n \log\left(1 - \frac{4c_2}{L^2} \sin^2 \frac{Lt}{2}\right) \\
&\leq nc_2t^2 - \frac{4nc_2}{L^2} \sin^2 \frac{Lt}{2}. \tag{3.28}
\end{aligned}$$

Moreover for $t \in [0, \varrho/L]$ from (3.21) we obtain

$$|\operatorname{Re} z(t)| = \left|n \sum_{\nu=2}^{\infty} c_{2\nu}(-1)^\nu t^{2\nu}\right| \leq \frac{3\sqrt{3}}{4}n \sum_{\nu=2}^{\infty} \left(\frac{Lt}{\pi}\right)^{2\nu} = \frac{3\sqrt{3}}{4}n \left(\frac{Lt}{\pi}\right)^4 \frac{1}{1 - (Lt/\pi)^2}$$

and analogously

$$|\operatorname{Im}(z(t) - inc_3t^3)| = \left|n \sum_{\nu=2}^{\infty} c_{2\nu+1}(-1)^\nu t^{2\nu+1}\right| \leq \frac{3\sqrt{3}}{4}n \left(\frac{Lt}{\pi}\right)^5 \frac{1}{1 - (Lt/\pi)^2},$$

hence we have

$$|z(t) - inc_3t^3| \leq \frac{3\sqrt{3}}{4}n \frac{(1 + (Lt/\pi)^2)^{1/2}}{1 - (Lt/\pi)^2} \left(\frac{Lt}{\pi}\right)^4. \tag{3.29}$$

From (3.27), (3.28), (3.29) it follows

$$\begin{aligned}
J_3 &\leq \frac{1}{2\pi} \int_0^{\varrho/L} \left(\exp(-nc_2 t^2) + \exp\left(-\frac{4nc_2}{L^2} \sin^2 \frac{Lt}{2}\right) \right) \frac{3\sqrt{3}}{4} n \frac{(1 + (Lt/\pi)^2)^{1/2}}{1 - (Lt/\pi)^2} \left(\frac{Lt}{\pi}\right)^4 \frac{dt}{t} \\
&= \frac{3\sqrt{3}n}{8\pi} \left(\frac{L}{\sqrt{nc_2}}\right)^4 \xi^4 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \right) \frac{(1 + (\tau/\pi)^2)^{1/2}}{1 - (\tau/\pi)^2} \left(\frac{\tau}{\pi}\right)^4 \frac{d\tau}{\tau} \\
&\leq \left(\frac{L}{\sqrt{8c_2}}\right)^6 \frac{1}{n} \frac{24\sqrt{3}}{\pi^4} \xi^4 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \right) \frac{(\pi^2 + \tau^2)^{1/2}}{\pi^2 - \tau^2} \tau^3 d\tau.
\end{aligned} \tag{3.30}$$

The convexity of the exponential function also yields

$$\begin{aligned}
|e^z - (1+z)| &= \left| z^2 \int_0^1 \int_0^1 e^{xyz} y dx dy \right| \leq |z|^2 \int_0^1 \int_0^1 e^{xy \operatorname{Re} z} y dx dy \\
&\leq |z|^2 \int_0^1 \int_0^1 (1 + xy(e^{\operatorname{Re} z} - 1)) y dx dy = \frac{1}{6} (2 + e^{\operatorname{Re} z}) |z|^2.
\end{aligned} \tag{3.31}$$

Therefore from (3.18), (3.20) we get

$$\begin{aligned}
J_4 &\leq \frac{1}{\pi} \int_0^{\varrho/L} e^{-nc_2 t^2} \frac{1}{2} n^2 \frac{(Lt)^6}{3888} \frac{dt}{t} = \frac{1}{\pi} \left(\frac{L}{\sqrt{nc_2}}\right)^6 \xi^6 \int_0^{\varrho} e^{-\xi^2 \tau^2} \frac{n^2}{7776} \tau^5 d\tau \\
&= \left(\frac{L}{\sqrt{8c_2}}\right)^6 \frac{1}{n} \frac{16}{243\pi} \xi^6 \int_0^{\varrho} e^{-\xi^2 \tau^2} \tau^5 d\tau.
\end{aligned} \tag{3.32}$$

From (3.14), (3.24), (3.25), (3.30), (3.32) for any $\varrho \in (0, \pi)$ it follows

$$\Delta_3 \leq \left(\frac{L}{\sqrt{8c_2}}\right)^6 \frac{1}{n} \varphi_3(\xi, \varrho), \tag{3.33}$$

where

$$\begin{aligned}
\varphi_3(\xi, \varrho) &:= \frac{8}{\pi} \xi^2 \int_{\varrho}^{\pi} \exp(-4\xi^2 \sin^2 \frac{\tau}{2}) \frac{d\tau}{\tau} + \frac{8}{\pi} \xi^2 \int_{\varrho}^{\infty} \exp(-\xi^2 \tau^2) \frac{d\tau}{\tau} \\
&\quad + \frac{24\sqrt{3}}{\pi^4} \xi^4 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \right) \frac{(\pi^2 + \tau^2)^{1/2}}{\pi^2 - \tau^2} \tau^3 d\tau \\
&\quad + \frac{16}{243\pi} \xi^6 \int_0^{\infty} \exp(-\xi^2 \tau^2) \tau^5 d\tau.
\end{aligned}$$

Because of $\sin(\tau/2) \geq (\tau/\varrho) \sin(\varrho/2)$, $\tau \in [0, \varrho]$ we get

$$\begin{aligned}
\varphi_3(\xi, \varrho) &\leq \frac{8}{\pi} \xi^2 \exp(-4\xi^2 \sin^2 \frac{\varrho}{2}) \log \frac{\pi}{\varrho} + \frac{8}{\pi} \xi^2 \int_{\varrho}^{\infty} \exp(-\xi^2 \tau^2) \frac{d\tau}{\tau} + \frac{16}{243\pi} \\
&\quad + \frac{24\sqrt{3}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \xi^4 \int_0^{\varrho} (\exp(-\xi^2 \tau^2) + \exp(-4\xi^2 \tau^2 \varrho^{-2} \sin^2(\varrho/2))) \tau^3 d\tau \\
&= \frac{2 \log(\pi/\varrho)}{\pi \sin^2(\varrho/2)} t e^{-t} + \frac{4}{\pi \varrho^2} \xi^2 \varrho^2 \int_{\xi^2 \varrho^2}^{\infty} e^{-\sigma} \frac{d\sigma}{\sigma} + \frac{16}{243\pi} \\
&\quad + \frac{12\sqrt{3}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \left(\int_0^{\xi^2 \varrho^2} e^{-\sigma} \sigma d\sigma + \left(\frac{\varrho/2}{\sin(\varrho/2)} \right)^4 \int_0^t e^{-\sigma} \sigma d\sigma \right), \tag{3.34}
\end{aligned}$$

where $t := 4\xi^2 \sin^2(\varrho/2)$. Cauchy-Schwarz's inequality yields

$$\begin{aligned}
\xi^2 \varrho^2 \int_{\xi^2 \varrho^2}^{\infty} e^{-\sigma} \frac{d\sigma}{\sigma} &\leq \xi^2 \varrho^2 \left(\int_{\xi^2 \varrho^2}^{\infty} e^{-2\sigma} d\sigma \right)^{1/2} \left(\int_{\xi^2 \varrho^2}^{\infty} \frac{d\sigma}{\sigma^2} \right)^{1/2} \\
&= \frac{\xi \varrho}{\sqrt{2}} e^{-\xi^2 \varrho^2} \leq \frac{1}{2\sqrt{2}} (1 + \xi^2 \varrho^2) e^{-\xi^2 \varrho^2} = \frac{1}{2\sqrt{2}} \int_{\xi^2 \varrho^2}^{\infty} e^{-\sigma} \sigma d\sigma.
\end{aligned}$$

Hence from (3.34) it follows

$$\begin{aligned}
\varphi_3(\xi, \varrho) &\leq \frac{2 \log(\pi/\varrho)}{\pi \sin^2(\varrho/2)} t e^{-t} + \frac{12\sqrt{3}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \left(\frac{\varrho/2}{\sin(\varrho/2)} \right)^4 (1 - (1+t)e^{-t}) \\
&\quad + \max \left\{ \frac{\sqrt{2}}{\pi \varrho^2}, \frac{12\sqrt{3}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \right\} + \frac{16}{243\pi}. \tag{3.35}
\end{aligned}$$

Choosing $\varrho := 1.9$ we obtain

$$\varphi_3(\xi, \varrho) < 0.5 t e^{-t} + 0.25 (1 - (1+t)e^{-t}) + 0.15.$$

Since the maximum of the right-hand side is attained at $t = 2$ we get the estimate

$$\varphi_3(\xi, \varrho) < 0.4 + 0.25 e^{-2} = 0.4338 \dots < 1/2.$$

Consequently, in view of (3.8), (3.14), (3.33) it holds

$$\begin{aligned}
&\left| F(x, n) - \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} e^{-nc_2 t^2} \operatorname{Im}(e^{i(x-nc_1)t} (1 + inc_3 t^3)) \frac{dt}{t} \right| \\
&\leq \Delta_3 + |F(x, n) - J| < \frac{1}{2} \left(\frac{L}{2\sqrt{\sigma_2(0)}} \right)^6 n^{-1} + e^{-\kappa n}. \tag{3.36}
\end{aligned}$$

Let

$$\psi(y) := \frac{1}{\pi} \int_0^{\infty} e^{-\frac{1}{2}\tau^2} \sin(y\tau) \frac{d\tau}{\tau}, \quad y \in \mathbb{R}. \quad (3.37)$$

Integration by parts yields

$$\psi''(y) = -\frac{1}{\pi} \int_0^{\infty} \tau e^{-\frac{1}{2}\tau^2} \sin(y\tau) d\tau = -\frac{y}{\pi} \int_0^{\infty} e^{-\frac{1}{2}\tau^2} \cos(y\tau) d\tau = -y\psi'(y), \quad y \in \mathbb{R}.$$

Therefore we have

$$\psi'(y) = \psi'(0) e^{-\frac{1}{2}y^2}, \quad y \in \mathbb{R}. \quad (3.38)$$

Because of

$$\psi'(0) = \frac{1}{\pi} \int_0^{\infty} e^{-\frac{1}{2}\tau^2} d\tau = \frac{1}{\sqrt{2\pi}}$$

and $\psi(0) = 0$ from (3.38) it results

$$\psi(y) = \frac{1}{\sqrt{2\pi}} \int_0^y e^{-\frac{1}{2}\tau^2} d\tau, \quad y \in \mathbb{R}. \quad (3.39)$$

Moreover from (3.37), (3.39) it follows

$$\frac{1}{\pi} \int_0^{\infty} e^{-\frac{1}{2}\tau^2} \cos(y\tau) \tau^2 d\tau = -\psi'''(y) = \frac{1}{\sqrt{2\pi}} (1 - y^2) e^{-\frac{1}{2}y^2}, \quad y \in \mathbb{R}. \quad (3.40)$$

Choosing y corresponding to (3.3) in view of (3.11) from (3.37), (3.39) and (3.40) we obtain

$$\begin{aligned} \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} e^{-nc_2 t^2} \operatorname{Im} e^{i(x-nc_1)t} \frac{dt}{t} &= \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} e^{-\frac{1}{2}\tau^2} \sin\left(\frac{x-nc_1}{\sqrt{2nc_2}}\tau\right) \frac{d\tau}{\tau} \\ &= \frac{1}{2} + \psi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{1}{2}\tau^2} d\tau, \end{aligned} \quad (3.41)$$

$$\begin{aligned} \frac{1}{\pi} \int_0^{\infty} e^{-nc_2 t^2} \operatorname{Im}(e^{i(x-nc_1)t} i nc_3 t^3) \frac{dt}{t} &= \frac{nc_3}{(2nc_2)^{3/2}} \frac{1}{\pi} \int_0^{\infty} e^{-\frac{1}{2}\tau^2} \cos\left(\frac{x-nc_1}{\sqrt{2nc_2}}\tau\right) \tau^2 d\tau \\ &= \frac{1}{6\sqrt{2\pi}} \sigma_3(0) \sigma_2(0)^{-3/2} n^{-1/2} (1 - y^2) e^{-\frac{1}{2}y^2}. \end{aligned} \quad (3.42)$$

Finally from (3.36), (3.41), (3.42) the estimate (3.2) follows.

For proving estimate (3.5) we start at (3.8). In view of (3.41) it holds

$$\begin{aligned} \left| F(x, n) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{1}{2}\xi^2} d\xi \right| &\leq |F(x, n) - J| + \left| J - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{1}{2}\xi^2} d\xi \right| \\ &< e^{-\kappa n} + \Delta_2, \end{aligned} \quad (3.43)$$

where

$$\Delta_2 := \left| J - \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} e^{-nc_2 t^2} \operatorname{Im} e^{i(x-nc_1)t} \frac{dt}{t} \right|.$$

Because of (3.13), (3.15) for any $\varrho \in (0, \pi)$ it follows

$$\Delta_2 \leq J_1 + J_5 + J_6, \quad (3.44)$$

where

$$J_5 := \frac{1}{\pi} \int_{\varrho/L}^{\infty} e^{-nc_2 t^2} \frac{dt}{t}, \quad (3.45)$$

$$J_6 := \frac{1}{\pi} \int_0^{\varrho/L} e^{-nc_2 t^2} |e^{z(t)} - 1| \frac{dt}{t}. \quad (3.46)$$

Analogously to (3.24) we obtain

$$J_1 \leq \left(\frac{L}{\sqrt{8c_2}} \right)^3 n^{-1/2} \frac{\sqrt{8}}{\pi} \xi \int_{\varrho}^{\pi} \exp(-4\xi^2 \sin^2 \frac{\tau}{2}) \frac{d\tau}{\tau}. \quad (3.47)$$

In view of (3.19) it follows

$$J_5 = \frac{1}{\pi} \left(\frac{L}{\sqrt{nc_2}} \right) \xi \int_{\varrho}^{\infty} e^{-\xi^2 \tau^2} \frac{d\tau}{\tau} \leq \left(\frac{L}{\sqrt{8c_2}} \right)^3 n^{-1/2} \frac{\sqrt{8}}{\pi} \xi \int_{\varrho}^{\infty} e^{-\xi^2 \tau^2} \frac{d\tau}{\tau}. \quad (3.48)$$

Taking into account (3.22), (3.26), (3.28) from (3.46) we get

$$\begin{aligned} J_6 &\leq \frac{1}{2\pi} \int_0^{\varrho/L} e^{-nc_2 t^2} (1 + e^{\operatorname{Re} z(t)}) |z(t)| \frac{dt}{t} \\ &\leq \frac{1}{2\pi} \int_0^{\varrho/L} \left(\exp(-nc_2 t^2) + \exp\left(-\frac{4nc_2}{L^2} \sin^2 \frac{Lt}{2}\right) \right) |z(t)| \frac{dt}{t} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \right) \left| z\left(\frac{\tau}{L}\right) \right| \frac{d\tau}{\tau} \\
&= \left(\frac{L}{\sqrt{8c_2}} \right)^3 n^{-1/2} \frac{8\sqrt{2}}{\pi} \xi^3 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \right) \left| \frac{1}{n} z\left(\frac{\tau}{L}\right) \right| \frac{d\tau}{\tau}. \tag{3.49}
\end{aligned}$$

For $\tau \in [0, \varrho]$ from (3.20), (3.29) it results

$$\left| \frac{1}{n} z\left(\frac{\tau}{L}\right) \right| \leq \frac{3\sqrt{3}}{4} \frac{(1 + (\tau/\pi)^2)^{1/2}}{1 - (\tau/\pi)^2} \left(\frac{\tau}{\pi}\right)^4 + \frac{\sqrt{3}}{108} \tau^3 \leq \left(\frac{3\sqrt{3}}{4\pi^3} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \varrho + \frac{\sqrt{3}}{108} \right) \tau^3.$$

Therefore from (3.49) we obtain

$$J_6 \leq \left(\frac{L}{\sqrt{8c_2}} \right)^3 n^{-1/2} C(\varrho) \xi^3 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \right) \tau^2 d\tau, \tag{3.50}$$

where

$$C(\varrho) := \frac{6\sqrt{6}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \varrho + \frac{2\sqrt{6}}{27\pi}. \tag{3.51}$$

In view of (3.44), (3.47), (3.48), (3.50) for any $\varrho \in (0, \pi)$ it holds

$$\Delta_2 \leq \left(\frac{L}{\sqrt{8c_2}} \right)^3 n^{-1/2} \varphi_2(\xi, \varrho), \tag{3.52}$$

where

$$\begin{aligned}
\varphi_2(\xi, \varrho) &:= \frac{\sqrt{8}}{\pi} \xi \int_{\varrho}^{\pi} \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \frac{d\tau}{\tau} + \frac{\sqrt{8}}{\pi} \xi \int_{\varrho}^{\infty} \exp(-\xi^2 \tau^2) \frac{d\tau}{\tau} \\
&\quad + C(\varrho) \xi^3 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \right) \tau^2 d\tau.
\end{aligned}$$

Because of $\sin(\tau/2) \geq (\tau/\varrho) \sin(\varrho/2)$, $\tau \in [0, \varrho]$ we get

$$\begin{aligned}
\varphi_2(\xi, \varrho) &\leq \frac{\sqrt{8}}{\pi} \xi \exp\left(-4\xi^2 \sin^2 \frac{\varrho}{2}\right) \log \frac{\pi}{\varrho} + \frac{\sqrt{8}}{\pi} \xi \int_{\varrho}^{\infty} \exp(-\xi^2 \tau^2) \frac{d\tau}{\tau} \\
&\quad + C(\varrho) \xi^3 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp(-4\xi^2 \tau^2 \varrho^{-2} \sin^2(\varrho/2)) \right) \tau^2 d\tau \\
&= \frac{\sqrt{2} \log(\pi/\varrho)}{\pi \sin(\varrho/2)} t^{1/2} e^{-t} + \frac{\sqrt{2}}{\pi \varrho} \xi \varrho \int_{\xi^2 \varrho^2}^{\infty} e^{-\sigma} \frac{d\sigma}{\sigma} \\
&\quad + \frac{1}{2} C(\varrho) \int_0^{\xi^2 \varrho^2} e^{-\sigma} \sigma^{1/2} d\sigma + \frac{1}{2} C(\varrho) \left(\frac{\varrho/2}{\sin(\varrho/2)} \right)^3 \int_0^t e^{-\sigma} \sigma^{1/2} d\sigma, \tag{3.53}
\end{aligned}$$

where $t := 4\xi^2 \sin^2(\varrho/2)$. As for $\eta > 0$ we have

$$\begin{aligned} \eta \int_{\eta^2}^{\infty} e^{-\sigma} \frac{d\sigma}{\sigma} &\leq \eta \left(\int_{\eta^2}^{\infty} e^{-2\sigma} d\sigma \right)^{1/2} \left(\int_{\eta^2}^{\infty} \frac{d\sigma}{\sigma^2} \right)^{1/2} = \frac{e^{-\eta^2}}{\sqrt{2}} \\ &< \frac{1}{2\sqrt{2}} (2 + \eta^2) e^{-\eta^2} = \frac{1}{2\sqrt{2}} \int_{\eta^2}^{\infty} e^{-\sigma} (1 + \sigma) d\sigma, \\ \int_0^{\eta^2} e^{-\sigma} \sigma^{1/2} d\sigma &< \frac{1}{2} \int_0^{\eta^2} e^{-\sigma} (1 + \sigma) d\sigma \end{aligned}$$

from (3.53) it follows

$$\begin{aligned} \varphi_{\varrho}(\xi, \varrho) &< \frac{\sqrt{2} \log(\pi/\varrho)}{2\pi \sin(\varrho/2)} (1+t)e^{-t} + \frac{1}{2\pi\varrho} \int_{\xi^2\varrho^2}^{\infty} e^{-\sigma} (1 + \sigma) d\sigma \\ &\quad + \frac{1}{4} C(\varrho) \int_0^{\xi^2\varrho^2} e^{-\sigma} (1 + \sigma) d\sigma + \frac{1}{4} C(\varrho) \left(\frac{\varrho/2}{\sin(\varrho/2)} \right)^3 \int_0^t e^{-\sigma} (1 + \sigma) d\sigma. \end{aligned}$$

Choosing $\varrho := 1.9$ and taking into account $\varrho C(\varrho) < 2/\pi$ and $t < \xi^2\varrho^2$ we obtain

$$\begin{aligned} \varphi_{\varrho}(\xi, \varrho) &< \frac{\sqrt{2} \log(\pi/\varrho)}{2\pi \sin(\varrho/2)} (1+t)e^{-t} + \frac{1}{\pi\varrho} \\ &\quad + \left(\frac{1}{4} C(\varrho) \left(\frac{\varrho/2}{\sin(\varrho/2)} \right)^3 + \frac{1}{4} C(\varrho) - \frac{1}{2\pi\varrho} \right) \int_0^t e^{-\sigma} (1 + \sigma) d\sigma \\ &< 0.14 (1+t)e^{-t} + 0.16754 + 0.07 \int_0^t e^{-\sigma} (1 + \sigma) d\sigma \\ &= 0.07 t e^{-t} + 0.30754. \end{aligned}$$

Since the maximum of the right-hand side is attained at $t = 1$ we get the estimate

$$\varphi_{\varrho}(\xi, \varrho) < 0.30754 + 0.07e^{-1} = 0.3332\dots < 1/3.$$

Therefore in view of (3.43) and (3.52) also the estimate (3.5) holds.

4 Decentral approximation of the sum of n iid random variables

In case of very rare events in general a decentral approximation (large deviation results resp. indirect Edgeworth expansion) of the distribution function $F(x, n)$ of the sum of n iid random variables is better than a central one. We assume again that the distribution function $F(x)$ of the generic random variable satisfies (P1) and (P2). As in case of the central approximation the starting point is Lemma 2.3, but now we choose $u \in \mathbb{R} \setminus \{0\}$.

Theorem 4.1 Let $F(x)$ be a distribution function satisfying (P1) and (P2).

Then the equation

$$\sigma_1(u) = x \quad (4.1)$$

has a unique solution $u \in \mathbb{R}$ for any $x \in (a, b)$. For $u \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$ the approximation

$$F(n\sigma_1(u), n) = \mathbb{I}\{u < 0\} + \exp(n(\sigma_0(u) + u\sigma_1(u))) \left(\frac{1}{u\sqrt{2\pi\sigma_2(u)n}} + R_n(u) \right), \quad (4.2)$$

$$\begin{aligned} |R_n(u)| &< \left(3 \left(\frac{L}{2\sqrt{\sigma_2(u)}} \right)^\gamma \frac{1}{L|u|} + 4 \left(\frac{L}{2\sqrt{\sigma_2(u)}} \right)^3 \frac{1}{(L|u|)^3} \right) n^{-3/2} \\ &\quad + \exp\left(\frac{2L}{\pi}|u| - n\kappa e^{-L|u|} \right) \end{aligned} \quad (4.3)$$

holds, where

$$\kappa := 0.3 \delta \ell^3 L^{-2}. \quad (4.4)$$

Proof: By the second part of Lemma 2.1 it follows that equation (4.1) has a unique solution for any $x \in (a, b)$.

Applying Lemma 2.3 with $v := \pi/L$ to the distribution function $F(x, n)$ and its LST $g(s)^n$ we get the estimate

$$|F(n\sigma_1(u), n) - J| < \exp(n(\sigma_0(u) + u\sigma_1(u))) + (2L/\pi)|u| \left(\sup_{t \geq \pi/L} \left| \frac{g(u+it)}{g(u)} \right| \right)^{\frac{5}{6}n}, \quad (4.5)$$

where

$$J := \frac{1}{2\pi i} \int_{\gamma(u, \pi/L)} g(s)^n e^{n\sigma_1(u)s} \frac{ds}{s}. \quad (4.6)$$

Since in view of $\ell \leq L$ Lemma 2.2 yields

$$\left| \frac{g(u+it)}{g(u)} \right| \leq 1 - \left(1 - \frac{2}{\pi} \right) \delta \ell^3 L^{-2} e^{-L|u|} < \exp\left(- \left(1 - \frac{2}{\pi} \right) \delta \ell^3 L^{-2} e^{-L|u|} \right), \quad t \geq \pi/L$$

from (4.5) the estimate

$$|F(n\sigma_1(u), n) - J| < \exp\left(n(\sigma_0(u) + u\sigma_1(u)) \right) \exp\left(\frac{2L}{\pi}|u| - n\kappa e^{-L|u|} \right) \quad (4.7)$$

follows, where κ is defined by (4.4).

Choosing the line segment as path of integration the Residue theorem yields

$$\begin{aligned}
J &= \mathbb{I}\{u < 0\} + \frac{1}{2\pi i} \int_{u-i\pi/L}^{u+i\pi/L} g(s)^n e^{n\sigma_1(u)s} \frac{ds}{s} \\
&= \mathbb{I}\{u < 0\} + \exp(n(\sigma_0(u) + u\sigma_1(u))) \frac{1}{\pi} \int_0^{\pi/L} \operatorname{Re} \left(\frac{1}{u+it} \left(\frac{g(u+it)}{g(u)} \right)^n e^{in\sigma_1(u)t} \right) dt.
\end{aligned} \tag{4.8}$$

Defining

$$c_\nu := \frac{\sigma_\nu(u)}{\nu!} = \frac{(-1)^\nu}{\nu!} \frac{d^\nu}{ds^\nu} \log g(s) \Big|_{s=u}, \quad \nu = 1, 2, \dots, \tag{4.9}$$

$$z(t) := n \sum_{\nu=3}^{\infty} c_\nu (-it)^\nu \tag{4.10}$$

for any $\varrho \in (0, \pi)$ it follows

$$\begin{aligned}
J &= \mathbb{I}\{u < 0\} + \exp(n(\sigma_0(u) + u\sigma_1(u))) \left[\frac{1}{\pi u} \int_0^{\varrho/L} e^{-nc_2 t^2} \operatorname{Re} \left(\frac{1}{1+it/u} e^{z(t)} \right) dt \right. \\
&\quad \left. + \frac{1}{\pi u} \int_{\varrho/L}^{\pi/L} \operatorname{Re} \left(\frac{1}{1+it/u} \left(\frac{g(u+it)}{g(u)} \right)^n e^{in\sigma_1(u)t} \right) dt \right].
\end{aligned} \tag{4.11}$$

Because of

$$\frac{1}{\pi u} \int_0^{\infty} e^{-nc_2 t^2} dt = \frac{1}{u \sqrt{2\pi\sigma_2(u)n}}$$

and of the triangle inequality from (4.11) we get the estimate

$$\begin{aligned}
\Delta &:= \left| J - \mathbb{I}\{u < 0\} - \frac{\exp(n(\sigma_0(u) + u\sigma_1(u)))}{u \sqrt{2\pi\sigma_2(u)n}} \right| \\
&\leq \exp(n(\sigma_0(u) + u\sigma_1(u))) \sum_{\nu=1}^4 J_\nu,
\end{aligned} \tag{4.12}$$

where

$$J_1 := \frac{1}{\pi|u|} \int_{\varrho/L}^{\pi/L} \left| \frac{g(u+it)}{g(u)} \right|^n dt, \tag{4.13}$$

$$J_2 := \frac{1}{\pi|u|} \int_{\varrho/L}^{\infty} e^{-nc_2 t^2} dt, \tag{4.14}$$

$$J_3 := \frac{1}{\pi|u|} \int_0^{\varrho/L} e^{-nc_2 t^2} |e^{z(t)} - e^{inc_3 t^3}| dt, \quad (4.15)$$

$$J_4 := \frac{1}{\pi|u|} \int_0^{\varrho/L} e^{-nc_2 t^2} \left| \operatorname{Re} \frac{e^{inc_3 t^3}}{1 + it/u} - 1 \right| dt. \quad (4.16)$$

Applying (4.9) and Lemma 2.1 it results

$$0 < c_2 \leq \frac{1}{8} L^2, \quad (4.17)$$

$$|c_3| \leq \frac{\sqrt{3}}{108} L^3, \quad (4.18)$$

$$|c_\nu| \leq \frac{3\sqrt{3}}{4} \left(\frac{L}{\pi}\right)^\nu, \quad \nu = 2, 3, \dots \quad (4.19)$$

For $t \in [\varrho/L, \pi/L]$ from Lemma 2.2 it follows

$$\left| \frac{g(u + it)}{g(u)} \right| \leq 1 - \frac{4c_2}{L^2} \sin^2 \frac{Lt}{2} < \exp \left(- \frac{4c_2}{L^2} \sin^2 \frac{Lt}{2} \right).$$

Defining

$$\xi := \frac{\sqrt{nc_2}}{L} \quad (4.20)$$

and taking into account (4.13), (4.17) we obtain

$$\begin{aligned} J_1 &\leq \frac{1}{\pi|u|} \int_{\varrho/L}^{\pi/L} \exp \left(- \frac{4nc_2}{L^2} \sin^2 \frac{Lt}{2} \right) dt = \frac{1}{\pi L|u|} \left(\frac{L}{\sqrt{nc_2}} \right)^3 \xi^3 \int_{\varrho}^{\pi} \exp \left(- 4\xi^2 \sin^2 \frac{\tau}{2} \right) d\tau \\ &\leq \left(\frac{L}{\sqrt{8c_2}} \right)^\gamma n^{-3/2} \frac{16\sqrt{2}}{\pi L|u|} \xi^3 \int_{\varrho}^{\pi} \exp \left(- 4\xi^2 \sin^2 \frac{\tau}{2} \right) d\tau. \end{aligned} \quad (4.21)$$

Because of (4.14), (4.17) it holds

$$\begin{aligned} J_2 &= \frac{1}{\pi|u|} \int_{\varrho/L}^{\infty} e^{-\xi^2 (Lt)^2} dt = \frac{1}{\pi L|u|} \left(\frac{L}{\sqrt{nc_2}} \right)^3 \xi^3 \int_{\varrho}^{\infty} e^{-\xi^2 \tau^2} d\tau \\ &\leq \left(\frac{L}{\sqrt{8c_2}} \right)^\gamma n^{-3/2} \frac{16\sqrt{2}}{\pi L|u|} \xi^3 \int_{\varrho}^{\infty} e^{-\xi^2 \tau^2} d\tau. \end{aligned} \quad (4.22)$$

Taking into account (4.15), (3.26) it follows

$$J_3 \leq \frac{1}{2\pi|u|} \int_0^{\varrho/L} e^{-nc_2 t^2} (1 + e^{\operatorname{Re} z(t)}) |z(t) - inc_3 t^3| dt. \quad (4.23)$$

For $t \in [0, \varrho/L]$, from Lemma 2.2 analogously to (3.28) we get

$$\operatorname{Re} z(t) \leq nc_2 t^2 - \frac{4nc_2}{L^2} \sin^2 \frac{Lt}{2}, \quad (4.24)$$

and from (4.19) analogously to (3.29) it follows

$$|z(t) - inc_3 t^3| \leq \frac{3\sqrt{3}}{4} n \frac{(1 + (Lt/\pi)^2)^{1/2}}{1 - (Lt/\pi)^2} \left(\frac{Lt}{\pi}\right)^4. \quad (4.25)$$

From (4.17), (4.23), (4.24), (4.25) it results

$$\begin{aligned} J_3 &\leq \frac{1}{2\pi|u|} \int_0^{\varrho/L} \left(\exp(-nc_2 t^2) + \exp\left(-\frac{4nc_2}{L^2} \sin^2 \frac{Lt}{2}\right) \right) \frac{3\sqrt{3}}{4} n \frac{(1 + (Lt/\pi)^2)^{1/2}}{1 - (Lt/\pi)^2} \left(\frac{Lt}{\pi}\right)^4 dt \\ &\leq \frac{3\sqrt{3}n}{8\pi^4 L|u|} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \left(\frac{L}{\sqrt{nc_2}}\right)^5 \xi^5 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \right) \tau^4 d\tau \\ &\leq \left(\frac{L}{\sqrt{8c_2}}\right)^7 n^{-3/2} \frac{48\sqrt{6}}{\pi^4 L|u|} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \xi^5 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) \right) \tau^4 d\tau. \end{aligned} \quad (4.26)$$

In view of $|\sin \tau| \leq |\tau|$ for $\tau \in \mathbb{R}$ and (4.18) we get

$$\begin{aligned} \left| \operatorname{Re} \frac{e^{inc_3 t^3}}{1 + it/u} - 1 \right| &= \frac{1}{1 + (t/u)^2} \left| 2 \sin^2 \frac{nc_3 t^3}{2} - \frac{t}{u} \sin(nc_3 t^3) + \frac{t^2}{u^2} \right| \\ &\leq \frac{1}{2} n^2 c_3^2 t^6 + |nc_3 t^3| \left| \frac{t}{u} \right| + \frac{t^2}{u^2} \leq \frac{3}{2} n^2 c_3^2 t^6 + \frac{5}{4} \frac{t^2}{u^2} \\ &\leq \frac{1}{2592} n^2 (Lt)^6 + \frac{5}{4} \frac{t^2}{u^2}. \end{aligned}$$

Therefore from (4.16) we obtain

$$\begin{aligned} J_4 &\leq \frac{n^2}{2592\pi|u|} \int_0^{\varrho/L} e^{-nc_2 t^2} (Lt)^6 dt + \frac{5}{4\pi|u|^3} \int_0^{\varrho/L} e^{-nc_2 t^2} t^2 dt \\ &\leq \frac{n^2}{2592\pi L|u|} \left(\frac{L}{\sqrt{nc_2}}\right)^7 \xi^7 \int_0^{\varrho} e^{-\xi^2 \tau^2} \tau^6 d\tau + \frac{5}{4\pi(L|u|)^3} \left(\frac{L}{\sqrt{nc_2}}\right)^3 \xi^3 \int_0^{\infty} e^{-\xi^2 \tau^2} \tau^2 d\tau \\ &= \left(\frac{L}{\sqrt{8c_2}}\right)^7 n^{-3/2} \frac{32\sqrt{2}}{81\pi L|u|} \xi^7 \int_0^{\varrho} e^{-\xi^2 \tau^2} \tau^6 d\tau + \frac{10}{\sqrt{2\pi}} \left(\frac{L}{\sqrt{8c_2}}\right)^3 \frac{n^{-3/2}}{(L|u|)^3}. \end{aligned} \quad (4.27)$$

From (4.12), (4.21), (4.22), (4.26), (4.27) for any $\varrho \in (0, \pi)$ it follows

$$\Delta \leq \exp(n(\sigma_0(u) + u\sigma_1(u))) \left(\left(\frac{L}{\sqrt{8c_2}}\right)^7 \frac{n^{-3/2}}{L|u|} \varphi(\xi, \varrho) + \frac{10}{\sqrt{2\pi}} \left(\frac{L}{\sqrt{8c_2}}\right)^3 \frac{n^{-3/2}}{(L|u|)^3} \right), \quad (4.28)$$

where

$$\begin{aligned}\varphi(\xi, \varrho) &:= \frac{16\sqrt{2}}{\pi}\xi^3 \int_{\varrho}^{\pi} \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right) d\tau + \frac{16\sqrt{2}}{\pi}\xi^3 \int_{\varrho}^{\infty} \exp(-\xi^2 \tau^2) d\tau \\ &+ \frac{48\sqrt{6}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \xi^5 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp\left(-4\xi^2 \sin^2 \frac{\tau}{2}\right)\right) \tau^4 d\tau \\ &+ \frac{32\sqrt{2}}{81\pi} \xi^7 \int_0^{\varrho} \exp(-\xi^2 \tau^2) \tau^6 d\tau.\end{aligned}$$

Because of $\sin(\tau/2) \geq (\tau/\varrho) \sin(\varrho/2)$ for $\tau \in [0, \varrho]$ we get

$$\begin{aligned}\varphi(\xi, \varrho) &\leq \frac{16\sqrt{2}}{\pi}\xi^3(\pi - \varrho) \exp(-4\xi^2 \sin^2 \frac{\varrho}{2}) + \frac{16\sqrt{2}}{\pi}\xi^3 \int_{\varrho}^{\infty} \exp(-\xi^2 \tau^2) d\tau \\ &+ \frac{48\sqrt{6}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \xi^5 \int_0^{\varrho} \left(\exp(-\xi^2 \tau^2) + \exp(-4\xi^2 \tau^2 \varrho^{-2} \sin^2(\varrho/2))\right) \tau^4 d\tau \\ &+ \frac{32\sqrt{2}}{81\pi} \xi^7 \int_0^{\varrho} \exp(-\xi^2 \tau^2) \tau^6 d\tau \\ &= \frac{2\sqrt{2}(\pi - \varrho)}{\pi \sin^3(\varrho/2)} t^{3/2} e^{-t} + \frac{8\sqrt{2}}{\pi \varrho^2} \xi^2 \varrho^2 \int_{\xi^2 \varrho^2}^{\infty} e^{-\sigma} \sigma^{-1/2} d\sigma + \frac{16\sqrt{2}}{81\pi} \int_0^{\xi^2 \varrho^2} e^{-\sigma} \sigma^{5/2} d\sigma \\ &+ \frac{24\sqrt{6}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \left(\int_0^{\xi^2 \varrho^2} e^{-\sigma} \sigma^{3/2} d\sigma + \left(\frac{\varrho/2}{\sin(\varrho/2)}\right)^5 \int_0^t e^{-\sigma} \sigma^{3/2} d\sigma \right), \quad (4.29)\end{aligned}$$

where $t := 4\xi^2 \sin^2(\varrho/2)$. As for $\eta > 0$ we have

$$\begin{aligned}\eta \int_{\eta}^{\infty} e^{-\sigma} \sigma^{-1/2} d\sigma &\leq \eta^{1/2} \int_{\eta}^{\infty} e^{-\sigma} d\sigma \leq \frac{1}{2} e^{-\eta} (1 + \eta) = \frac{1}{2} \int_{\eta}^{\infty} e^{-\sigma} \sigma d\sigma, \\ \int_0^{\eta} e^{-\sigma} \sigma^{3/2} d\sigma &\leq \frac{1}{2} \int_0^{\eta} e^{-\sigma} (\sigma + \sigma^2) d\sigma = \frac{3}{2} - \frac{1}{2} e^{-\eta} (3 + 3\eta + \eta^2) \\ &\leq \frac{3}{2} - \frac{3}{2} e^{-\eta} (1 + \eta) = \frac{3}{2} \int_0^{\eta} e^{-\sigma} \sigma d\sigma, \\ \int_0^{\eta} e^{-\sigma} \sigma^{5/2} d\sigma &\leq \frac{1}{2} \int_0^{\eta} e^{-\sigma} (\sigma^2 + \sigma^3) d\sigma = 4 - \frac{1}{2} e^{-\eta} (8 + 8\eta + 4\eta^2 + \eta^3) \\ &\leq 4 - 4e^{-\eta} (1 + \eta) = 4 \int_0^{\eta} e^{-\sigma} \sigma d\sigma\end{aligned}$$

from (4.29) it follows by integration by parts

$$\begin{aligned}
\varphi(\xi, \varrho) &< \frac{2\sqrt{2}(\pi - \varrho)}{\pi \sin^3(\varrho/2)} t^{3/2} e^{-t} + \frac{4\sqrt{2}}{\pi \varrho^2} \int_{\xi^2 \varrho^2}^{\infty} e^{-\sigma} \sigma d\sigma \\
&+ \left(\frac{36\sqrt{6}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} + \frac{64\sqrt{2}}{81\pi} \right) \int_0^{\xi^2 \varrho^2} e^{-\sigma} \sigma d\sigma \\
&+ \frac{24\sqrt{6}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \left(\frac{\varrho/2}{\sin(\varrho/2)} \right)^5 \int_0^t e^{-\sigma} \sigma^{3/2} d\sigma \\
&\leq \frac{2\sqrt{2}(\pi - \varrho)}{\pi \sin^3(\varrho/2)} t^{3/2} e^{-t} \\
&+ \frac{24\sqrt{6}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} \left(\frac{\varrho/2}{\sin(\varrho/2)} \right)^5 \left(\frac{3}{2} \int_0^t e^{-\sigma} \sigma^{1/2} d\sigma - t^{3/2} e^{-t} \right) \\
&+ \max \left\{ \frac{4\sqrt{2}}{\pi \varrho^2}, \frac{36\sqrt{6}}{\pi^4} \frac{(\pi^2 + \varrho^2)^{1/2}}{\pi^2 - \varrho^2} + \frac{64\sqrt{2}}{81\pi} \right\}.
\end{aligned}$$

Choosing $\varrho := \pi/2$ we obtain

$$\begin{aligned}
\varphi(\xi, \varrho) &< 4t^{3/2}e^{-t} + 0.5(1.5\Gamma(1.5) - t^{3/2}e^{-t}) + 0.8 \\
&= 3.5t^{3/2}e^{-t} + 0.375\sqrt{\pi} + 0.8.
\end{aligned}$$

Since the maximum of the right-hand side is attained at $t = 3/2$ we get the estimate

$$\varphi(\xi, \varrho) < 3.5 \left(\frac{3}{2e} \right)^{3/2} + 0.375\sqrt{\pi} + 0.8 = 2.899 \dots < 3. \quad (4.30)$$

In view of (4.2), (4.7), (4.12), (4.28) it holds

$$|R_n(u)| < \exp \left(\frac{2L}{\pi} |u| - n\kappa e^{-L|u|} \right) + \left(\frac{L}{\sqrt{8c_2}} \right)^\gamma \frac{n^{-3/2}}{L|u|} \varphi(\xi, \varrho) + \frac{10}{\sqrt{2\pi}} \left(\frac{L}{\sqrt{8c_2}} \right)^3 \frac{n^{-3/2}}{(L|u|)^3}.$$

Taking into account (4.9) and (4.30) we get (4.3). Therefore Theorem 4.1 is proven.

Because of Theorem 4.1 for any $x \in (a, b)$ equation $\sigma_I(u) = x$ has a unique solution $u \in \mathbb{R}$. Therefore

$$I(x) := -(\sigma_\theta(u) + u\sigma_I(u)), \quad \text{where } \sigma_I(u) := x, \quad (4.31)$$

is well defined for $x \in (a, b)$. $I(x)$ is known as the large deviation rate function, cf. e.g. [B90] p. 7.

From Theorem 4.1 for $x < EX_\nu$, i.e. for $u > 0$, it follows

$$\frac{1}{n} \log F(nx, n) = -I(x) + \frac{1}{n} \log \left(\frac{1}{u\sqrt{2\pi\sigma_2(u)n}} + R_n(u) \right),$$

hence in view of (4.3) for $x < EX_\nu$ it holds

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log F(nx, n) = -I(x). \quad (4.32)$$

In case of $x > EX_\nu$, i.e. for $u < 0$, it follows

$$\frac{1}{n} \log(1 - F(nx, n)) = -I(x) + \frac{1}{n} \log \left(\frac{-1}{u\sqrt{2\pi\sigma_2(u)n}} - R_n(u) \right),$$

therefore in view of (4.3) for $x > EX_\nu$ it holds

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(1 - F(nx, n)) = -I(x). \quad (4.33)$$

Equations (4.32) and (4.33) correspond to Cramér's theorem, cf. e.g. [B90] p. 9.

From estimate (4.3) obviously it follows

$$\lim_{n \rightarrow \infty} n^{1/2} R_n(u) = 0,$$

i.e. we have

$$\begin{aligned} \lim_{n \rightarrow \infty} n^{1/2} (F(n\sigma_1(u), n) - \mathbb{I}\{u < 0\}) \exp(-n(\sigma_0(u) + u\sigma_1(u))) \\ = \frac{1}{u\sqrt{2\pi\sigma_2(u)}}, \quad u \in \mathbb{R} \setminus \{0\}. \end{aligned} \quad (4.34)$$

This statement corresponds to the Theorem of Bahadur-Rao, cf. [BR60] Theorem 1, [B90] p. 121.

5 Numerical results

Fluid flow approximations are used for approximating models of communication systems where packet arrival streams are generated in a regular manner over certain intervals. The appropriate mathematical model for describing the bursty arrival streams in the fluid flow framework are the Markov modulated rate processes (MMRP's). For details see [BB94]. We consider the distribution of the number $N(t)$ of packets in an interval $[0, t]$ of a MMRP. In [BB94] for the distribution of $N(t)$ explicit formulas are given in case of two-state sources (e.g. voice sources).

The support of the distribution function $F(x) := P(N(t) \leq x)$ is the interval $[r_1 t, r_2 t]$, where $0 \leq r_1 < r_2$ and r_1, r_2 are the packet rates in the first resp. second state of the source. The LST $g(s)$ of the distribution function $F(x)$ has the following representation, cf. [BB94] p. 289:

$$g(s) = \sum_{k=0}^1 \frac{1}{2} \left[1 + (-1)^k \frac{1 + \frac{\lambda - \mu}{\lambda + \mu} \frac{r_2 - r_1}{\lambda + \mu} s}{\sqrt{D}/(\lambda + \mu)} \right] \exp \left(\frac{t}{2} ((-1)^k \sqrt{D} - (\lambda + \mu) - (r_2 + r_1)s) \right), \quad (5.1)$$

where $1/\mu$, $1/\lambda$ are the mean sojourn times in the first resp. second state of the source and

$$D := (\lambda - \mu + (r_2 - r_1)s)^2 + 4\lambda\mu. \quad (5.2)$$

By some algebra from (5.1), (5.2) it follows

$$\sigma_1(0) = \frac{\lambda r_1 + \mu r_2}{\lambda + \mu} t, \quad (5.3)$$

$$\sigma_2(0) = 2 \left(e^{-(\lambda + \mu)t} - 1 + (\lambda + \mu)t \right) \frac{\lambda\mu}{(\lambda + \mu)^2} \left(\frac{r_2 - r_1}{\lambda + \mu} \right)^2, \quad (5.4)$$

$$\sigma_3(0) = 6 \left((\lambda + \mu)t - 2 + ((\lambda + \mu)t + 2)e^{-(\lambda + \mu)t} \right) \frac{\lambda\mu(\lambda - \mu)}{(\lambda + \mu)^3} \left(\frac{r_2 - r_1}{\lambda + \mu} \right)^3. \quad (5.5)$$

The distribution function $F(x, n) = P(N_1(t) + N_2(t) + \dots + N_n(t) \leq x)$ of the superposition of n iid two-state sources can be approximated by means of Theorem 3.1 resp. Theorem 4.1. As an example we consider the following voice source choosing $t := 1000$ (in ms):

States	Packet rates (in ms^{-1})	Mean sojourn times (in ms)
1	$r_1 = 0$	$1/\mu = 650$
2	$r_2 = 1/16$	$1/\lambda = 352$

We have $a = r_1 t = 0$ and $b = r_2 t = 62.5$. From (5.3)–(5.5) we obtain

$$\sigma_1(0) = 21.95 \dots, \quad \sigma_2(0) = 314.86 \dots, \quad \sigma_3(0) = 2906.97 \dots$$

In the following three tables for $n = 50$, $n = 500$ resp. $n = 5000$ and some x values the exact values of the distribution function $F(x, n)$ as well as the values of the central approximations (the Central limit approximation and the approximation based on moments up to third order) and of the decentral approximation are given.

x	$F(x, n)$	Central Limit Theorem	Moments up to third Order	Decentral Approximation
560	0.000002	0.000009	0.000000	$2.0311 \cdot 10^{-6}$
740	0.001520	0.002175	0.001576	$1.6543 \cdot 10^{-3}$
920	0.076520	0.078231	0.076424	$9.9450 \cdot 10^{-2}$
1100	0.511860	0.506981	0.511871	$1 - 2.2795 \cdot 10^1$
1280	0.924773	0.926759	0.924869	$1 - 9.9251 \cdot 10^{-2}$
1460	0.997555	0.998053	0.997497	$1 - 2.7070 \cdot 10^{-3}$
1640	0.999985	0.999992	0.999985	$1 - 1.5781 \cdot 10^{-5}$

Table 5.1: $n = 50$, $nEN(t) = 1097.80 \dots$

x	$F(x, n)$	Central Limit Theorem	Moments up to third Order	Decentral Approximation
9200	0.000002	0.000004	0.000002	$2.5973 \cdot 10^{-6}$
9800	0.001342	0.001494	0.001347	$1.4663 \cdot 10^{-3}$
10400	0.071988	0.072581	0.071980	$9.3615 \cdot 10^{-2}$
11000	0.523598	0.522065	0.523605	$1 - 7.1985 \cdot 10^0$
11600	0.940840	0.941502	0.940842	$1 - 7.5598 \cdot 10^{-2}$
12200	0.998853	0.998964	0.998849	$1 - 1.2535 \cdot 10^{-3}$
12800	0.999997	0.999998	0.999997	$1 - 3.1802 \cdot 10^{-6}$

Table 5.2: $n = 500$, $nEN(t) = 10978.04 \dots$

x	$F(x, n)$	Central Limit Theorem	Moments up to third Order	Decentral Approximation
104000	0.000002	0.000002	0.000002	$1.8809 \cdot 10^{-6}$
106000	0.001251	0.001294	0.001251	$1.3669 \cdot 10^{-3}$
108000	0.077772	0.077952	0.077771	$1.0237 \cdot 10^{-1}$
110000	0.569920	0.569455	0.569922	$1 - 2.2452 \cdot 10^0$
112000	0.961332	0.961549	0.961331	$1 - 4.7492 \cdot 10^{-2}$
114000	0.999597	0.999614	0.999597	$1 - 4.3458 \cdot 10^{-4}$
116000	1.000000	1.000000	1.000000	$1 - 4.2860 \cdot 10^{-7}$

Table 5.3: $n = 5000$, $nEN(t) = 109780.43 \dots$

The examples show that the central approximation based on moments up to third order corresponding to the first part of Theorem 3.1 is very good in a neighbourhood of the expectation and better than the Central limit approximation, especially in case of a very large number n of sources. But in case of very rare events and only in this case a decentral approximation is much better than a central one, cf. Table 5.1 for $x = 560$ and $x = 1100$.

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