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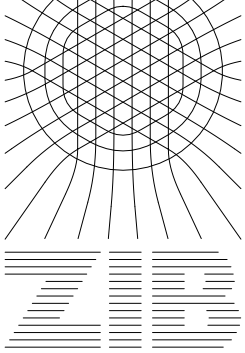

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An Adaptive Hierarchical Approximation Method on the Sphere Using Axisymmetric Locally Supported Basis Functions

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Abstract

The paper discusses the approximation of scattered data on the sphere which is one of the major tasks in geomathematics. Starting from the discretization of singular integrals on the sphere the authors devise a simple approximation method that employs locally supported spherical polynomials and does not require equidistributed grids. It is the basis for a hierarchical approximation algorithm using differently scaled basis functions, adaptivity and error control. The method is applied to two examples one of which is a digital terrain model of Australia.

Keywords : functional approximation, computational methods in geophysics, adaptive hierarchical discretization

Subject classification (AMS) : 65D17, 86-08

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1 Introduction

It is well known that approximation in the topology of a Hilbert space is a powerful tool, in particular if the family of trial functions constitutes a complete orthonormal system. For a function given on the (unit) sphere Ω the classical Fourier series expansion in terms of spherical harmonics is such a method. It gives satisfactory results in many cases, but it also has some severe drawbacks that restrict its applicability. One is that the orthonormalization process and the recovery of the function is in many instances of very undesirable computational complexity. Second, local changes of the function affect the whole table of Fourier coefficients. This shortcoming was already observed by Gabor (1946) who introduced the Gaussian kernel to window the Fourier integral, a technique which recently has been extended to spherical geometry by Freeden, Schreiner (1993, 1994). A different approach to localization is based on a generalized Fourier series in terms of localizing kernels such as the Gauss-Weierstrass kernel, the Abel-Poisson kernel, or compactly supported polynomial kernels. Starting from these, even orthonormal bases with controled localization properties can be derived (cf. Freeden (1990)). Both constructions map local changes of the function being represented to local changes of the coefficients in the expansion and thereby also reduce the computational complexity. However, there is still a defect in reconstructing a function using a sole, fixed "window parameter". It poorly resolves phenomena shorter than the window which leads to non-optimal computational cost in many circumstances. This can be remedied by kernels with decreasing window diameters exhibiting the so-called "zooming-in" property.

Apart from localization, orthogonality of the employed basis is an important feature. For the spherical case it has been shown in recent years that dropping this requirement can lead to substantial improvements (cf. Schaffeld (1988), Cui, Freeden, Witte, (1992), Freeden, Schreiner (1993,1994)). In these references, the approximation theory of some "overcomplete" systems of mutually non-orthogonal functions has been developed which are piecewise polynomial, compactly supported and axisymmetric. Non-orthogonal bases give a lot more freedom than orthogonal ones so that they can be generated in relatively simple ways. On the sphere, the basis is constructed using rotation and dilation of the window function. In fact, the latter can be carried over quite naturally to the sperical case when using a parametized kernel representation of singular spherical integrals.

Wedding non-orthogonal systems and different window sizes results in a spherical Toeplitz transform investigated in Freeden, Schreiner (1993,1994) and Brand (1994). In this way information of a function is displayed on various levels of resolution (mulilevel approximation). Of course, the number of levels needed in a particular numerical computation depends on the "phenotype" of the function under consideration. In most cases, in fact, a "one-level computation" using an adapted window parameter is of sufficient accuracy when the function is well-behaved. Multilevel computation, however, is indispensable when "high frequency phenomena" must be modeled. The automatic "zoom-in process" is stopped, if the resolution of the a priori given data is reached and no further refinement of the window parameter is reasonable. Toeplitz summation is intended to improve convergence compared to the original sequence of singular integral values which has been verified by the numerical experiments in Brand (1994) for the present case. Nevertheless, this summation procedure is still not satisfactory as it suffers from unde-

sirable redundancy. In the present paper we therefore devise a method which keeps the conceptual advantages of the Toeplitz transform but which is much more flexible in modelling local information of a function, eliminating redundancy as far as possible. This can be guaranteed in form of an adaptive hierarchical approximation method.

The outline of the paper is a follows: Section 2 recapitulates the necessary material about singular integrals on the sphere. The "non-orthogonal Fourier type expansion" in terms of axisymmetric locally supported basis functions is discussed in Section 3. The adaptive hierarchical approximation method is proposed in Section 4 and subsequently applied to some testcalculations (Section 5).

2 Theory of Singular Integrals on the Sphere

Singular integrals are an old technique in approximation theory. In this section we first recall how singular integrals on the sphere can be defined by means of the convolution on the sphere. Next we state related convergence theorems before discussing a particular class of locally supported kernels that will be employed in the sequel. (For more details on the employed definitions the reader is referred e.g. to Berens et al. (1968) and the references therein.)

We start with some notations. For all $\xi \in \mathbb{R}^3, x = (x_1, x_2, x_3)^T$, different from the origin, we have $x = r\xi, r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, where $\xi = (\xi_1, \xi_2, \xi_3)^T$ is the uniquely determined directional unit vector of $x \in \mathbb{R}^3$. The unit sphere in $\mathbb{R}^3$ is denoted by Ω. If the vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$ form the canonical orthonormal basis in $\mathbb{R}^3$, we may represent the points $\xi \in \Omega$ by

$$\xi = t\,\varepsilon^3 + \sqrt{1-t^2}(\cos\varphi\,\varepsilon^1 + \sin\varphi\,\varepsilon^2), \tag{1}$$

$$t = \cos\vartheta,\ \vartheta \in [0,\pi], \varphi \in [0, 2\pi). \tag{2}$$

The class of continuous functions on the sphere are denoted by $\mathcal{C}(\Omega)$. Equipped with the norm $||F||_{\mathcal{C}(\Omega)} = \sup_{\xi\in\Omega} |F(\xi)|$, $\mathcal{C}(\Omega)$ is a Banach space.
$\mathcal{L}^1[-1,1]$ is the space of absolutely integrable functions on the intervall $[-1,1]$ equipped with the norm $||F||_{\mathcal{L}^1[-1,1]} = (2\pi \int_{-1}^{1} |F(t)|\ dt)$. A close connection between integrals over the sphere Ω and the Legendre polynomials of degree n, P_n, is given by the Funk–Hecke formula

$$\int_\Omega G(\xi\zeta)\ P_n(\eta\zeta)\ d\omega(\zeta) = G^\wedge(n)\ P_n(\xi\eta), \quad G \in \mathcal{L}^1[-1,1] \tag{3}$$

($d\omega$ is the surface element on Ω) in which the "Legendre transform" $G^\wedge(n)$ for $G \in \mathcal{L}^1[-1,1]$ is defined by

$$G^\wedge(n) := 2\pi \int_{-1}^{1} G(t)\ P_n(t)\ dt \tag{4}$$

and where $\xi\eta$ denotes the inner product of $\xi, \eta \in \Omega$. Now the convolution on the sphere can be defined (see also Calderon, Zygmund (1955)).

Assume that $G \in \mathcal{L}^1[-1,1]$ and $F \in \mathcal{C}(\Omega)$. Then $G * F$ defined by

$$(G * F)(\xi) := \int_\Omega G(\xi\eta)\ F(\eta)\ d\omega(\eta) \tag{5}$$

is called the convolution of G and F.

Analogously the q–th iterated convolution of G and F is defined by $(G^{(q)} * F) = G^{(q-1)} * F$ with $G^{(1)} = G$ and $q \in \mathbb{N}$. Obviously one has

$$G^{(q)}(\xi\eta) = \int_\Omega G^{(q-1)}(\xi\zeta)\ G(\zeta\eta)\ d\omega(\zeta)\ ,\ q = 2,3,\dots \tag{6}$$

and

$$\left(G^{(q)}\right)^\wedge (n) = (G^\wedge(n))^q\ ,\ n = 0,1,\dots\ ,\ q = 1,2,\dots \tag{7}$$

We continue with the definition of singular integrals on the sphere (Berens et al. (1968)).

Definition 2.1 *Let $\{K_\rho\}, \rho \in (-1,1)$, be a subfamily of $\mathcal{L}^1[-1,1]$ satisfying the condition $K_\rho^\wedge(0) = 1$. Then $\{I_\rho\}, \rho \in (-1,1)$, defined by the convolution $I_\rho(F) := K_\rho * F\ ,\ F \in \mathcal{C}(\Omega)$ is called (spherical) singular integral corresponding to the integral kernel $\{K_\rho\}$. A singular integral $\{I_\rho\}$ is called an approximate identity in $\mathcal{C}(\Omega)$ if*

$$\lim_{\substack{\rho\to 1\\ \rho<1}} ||F - I_\rho(F)||_{\mathcal{C}(\Omega)} = 0 \tag{8}$$

for all $F \in \mathcal{C}(\Omega)$.

In addition, if

$$||K_\rho||_{\mathcal{L}^1[-1,1]} = 2\pi \int_{-1}^{1} |K_\rho(t)|\ dt \le M \tag{9}$$

is uniformly bounded for all $\rho \in (-1,1)$, it follows from the properties of the convolution that

$$||I_\rho(F)||_{\mathcal{C}(\Omega)} \le M\ ||F||_{\mathcal{C}(\Omega)}. \tag{10}$$

Furthermore, since the singular integrals fulfil the condition $K_\rho^\wedge(0) = 1$ they have the integral–preserving property

$$\int_\Omega I_\rho(F)(\xi)\ d\omega(\xi) = \int_\Omega F(\xi)\ d\omega(\xi) \tag{11}$$

which characterizes the approximation. The first convergence theorem (Freeden, Schreiner (1993)) for uniformly bounded kernels reads

Theorem 2.1 *Assume that the kernel $\{K_\rho\}, \rho \in (-1,1)$, is uniformly bounded. Therefore the corresponding singular integrals $\{I_\rho\}, \rho \in (-1,1)$, satisfy the above uniform boundedness criterion. Then the following statements are equivalent :*

(*i*) *For* $n = 0, 1, \ldots$

$$\lim_{\substack{\rho \to 1 \\ \rho < 1}} K_\rho^\wedge(n) = 1 \tag{12}$$

(*ii*) *For all spherical harmonics* $Y_n \in Harm_n(\Omega)$, $n = 0, 1, \ldots$

$$\lim_{\substack{\rho \to 1 \\ \rho < 1}} ||I_\rho(Y_n) - Y_n||_{\mathcal{C}(\Omega)} = 0 \tag{13}$$

(*iii*) *For all* $F \in \mathcal{C}(\Omega)$

$$\lim_{\substack{\rho \to 1 \\ \rho < 1}} ||I_\rho(F) - F||_{\mathcal{C}(\Omega)} = 0. \tag{14}$$

These results can be improved for non–negative kernels (cf. Freeden, Schreiner (1993)).

Theorem 2.2 *Suppose that* $\{K_\rho\}$, $\rho \in (-1, 1)$, *is a non–negative kernel . Then the following statements are equivalent :*

(*i*) *For* $n = 0, 1, \ldots$

$$\lim_{\substack{\rho \to 1 \\ \rho < 1}} K_\rho^\wedge(n) = 1 \tag{15}$$

(*ii*)

$$\lim_{\substack{\rho \to 1 \\ \rho < 1}} K_\rho^\wedge(1) = 1 \tag{16}$$

(*iii*)

$$\lim_{\substack{\rho \to 1 \\ \rho < 1}} \int_{-1}^{\delta} K_\rho(t)\, dt = 0, \quad \delta \in (-1, 1), \quad \textit{("localization property")} \tag{17}$$

(*iv*) I_ρ *is an approximate identity.*

For the q–th iterated convolution $K_\rho^{(q)}$ of a non–negative kernel K_ρ holds the following

Theorem 2.3 *Assume that* $\{K_\rho\}$, $K_\rho \in \mathcal{L}^1[-1, 1]$, $\rho \in (-1, 1)$, *is a non–negative kernel. Then* $\{I_\rho^{(q)}\}$, $q \in \mathbb{N}$, *defined by*

$$I_\rho^{(q)}(F) := K_\rho^{(q)} * F \ , \ F \in \mathcal{C}(\Omega) \tag{18}$$

is an approximate identity in $\mathcal{C}(\Omega)$ *if and only if* $\{I_\rho\}$, $\rho \in (-1, 1)$, *is an approximate identity in* $\mathcal{C}(\Omega)$.

This leads to the convergence statements for the first and second iteration of I_ρ, to which our considerations will be restricted.

Corollary 2.1 *Assume that* $\{K_\rho\}$, $K_\rho \in \mathcal{L}^1[-1,1]$, $\rho \in (-1,1)$, *is a family of non–negative kernels satisfying the localization property. Then for all* $F \in \mathcal{C}(\Omega)$ *the limit relations*

$$\lim_{\substack{\rho\to 1\\ \rho<1}} ||I_\rho^{(1)}(F) - F||_{\mathcal{C}(\Omega)} = 0 \tag{19}$$

and

$$\lim_{\substack{\rho\to 1\\ \rho<1}} ||I_\rho^{(2)}(F) - F||_{\mathcal{C}(\Omega)} = 0 \tag{20}$$

hold.

In the present work we will exclusively consider singular integrals with locally supported kernels ("finite–elements") due to their practical relevance. Moreover, we consider axi-symmetric kernels that are defined via a one-dimensional envelope function and set, as in Freeden, Schreiner (1993), $K_\rho(\xi\eta) = \tilde{B}_\rho(\xi\eta)$ with

$$\tilde{B}_\rho(\xi\eta) = \frac{B_\rho^{(k)}(\xi\eta)}{(B_\rho^{(k)})\hat{}(0)}, \quad \xi,\eta \in \Omega \tag{21}$$

The function $B_\rho^{(k)} : [-1,1] \to \mathbb{R}$, $k = 0,1,2,\ldots$ is merely the k–th Bernstein–polynomial of degree k over the interval $(\rho,1]$ and defined as

$$B_\rho^{(k)}(t) := \begin{cases} 0 & \text{for } -1 \le t \le \rho \\ (\frac{t-\rho}{1-\rho})^k & \text{for } \rho < t \le 1 \end{cases}, \tag{22}$$

see Figure 1. The regularity of $\tilde{B}_\rho$ is not decisive for most considerations, so that the index (k) is deliberately suppressed in (21) for clarity.

Obviously, $\tilde{B}_\rho(\cdot\eta) : \Omega \to \mathbb{R}$ has symmetry about the axis through the point $\eta \in \Omega$ since $\tilde{B}_\rho(\xi\eta)$ depends only on the spherical distance of the arguments $\xi,\eta \in \Omega$. Furthermore $\tilde{B}_\rho(\cdot\eta)$ has compact support

$$\Omega_{\rho,\eta} = \overline{\text{supp } \tilde{B}_\rho(\cdot\eta)} = \{\xi \in \Omega |\ \rho \le \xi\eta \le 1\}. \tag{23}$$

which is a spherical cap of radius ρ around η.

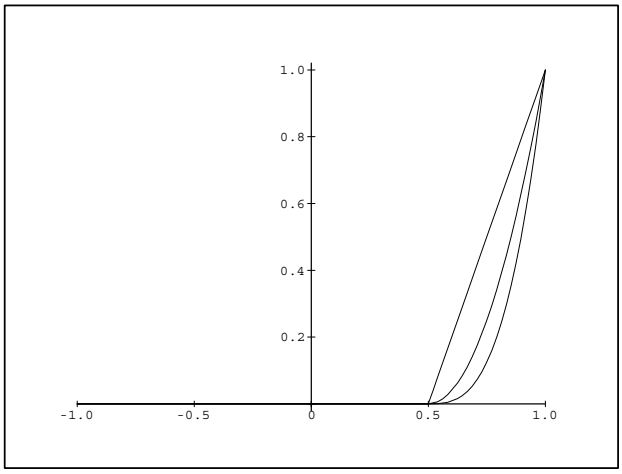
The singular integrals corresponding to the locally supported kernels are constructed using (21). They are called "integral mean" for $k = 0$ and "weighted integral mean" for $k = 1,2,\ldots$, respectively. In view of Corollary 2.1 their approximation behaviour is characterized by the following two estimations. Using the modulus of continuity

$$\mu(F, 1-\delta) = \max_{1-\xi\eta\le 1-\delta} |F(\eta) - F(\xi)|\,, \tag{24}$$

we obtain

$$\sup_{\xi\in\Omega} \left| \int_\Omega F(\eta)\ \tilde{B}_\rho(\xi\eta)\ d\omega - F(\xi) \right| \le \mu(F, 1-\rho) \tag{25}$$

Figure 1 : $B_\rho^{(k)}(t)$ for $k = 1, 2, 3$ and $\rho = 0.5$.



and

$$\sup_{\xi\in\Omega}\left|\int_\Omega\int_\Omega F(\eta)\ \tilde{B}_\rho(\eta\zeta)\ \tilde{B}_\rho(\xi\zeta)\ d\omega(\eta)\ d\omega(\zeta) - F(\xi)\ \right| \le \mu(F, 2(1-\rho^2)). \tag{26}$$

Estimating the distance of two points in a spherical cap we get the following results for Lipschitz–continuous functions F with Lipschitz-constant C_F

$$\sup_{\xi\in\Omega}\left|\int_\Omega F(\eta)\ \tilde{B}_\rho(\xi\eta)\ d\omega - F(\xi)\ \right| \le C_F\ \sqrt{2}\ \sqrt{1-\rho} \tag{27}$$

$$\sup_{\xi\in\Omega}\left|\int_\Omega\int_\Omega F(\eta)\ \tilde{B}_\rho(\eta\zeta)\ \tilde{B}_\rho(\xi\zeta)\ d\omega(\eta)\ d\omega(\zeta) - F(\xi)\ \right| \le C_F\ 2\ \sqrt{2}\ \sqrt{1-\rho}. \tag{28}$$

3 Discretization

The purpose of this work is the approximation of functions on the sphere from (measurements or observations at) certain points. From Section 2 we know that a function $F \in \mathcal{C}(\Omega)$ can be approximated very well by its singular integral $I_\rho^{(q)}(F)$ provided that the parameter ρ is chosen near 1. Hence it can be used to develop a simply structured and very economical quadrature formula for singular integrals on the sphere. In former articles (Schaffeld (1988), Cui, Freeden, Witte (1992), Freeden, Schreiner (1993)) an integration rule employing equidistributed pointsets (Weyl (1916)) is used. The idea of this method is the following : approximate the integral of a function $F : \Omega \to \mathbb{R}$ by an arithmetical mean of the function's values at prescribed points. An introduction to this "Low Discrepancy Method" as well as the derivation of error–estimations for Lipschitz–continuous functions and for functions of class $\mathcal{C}^{(2s)}$ are given in Freeden, Schreiner (1993). One deficiency of such a method is that it can only be used with restrictions for

local problems. Our aim in this section is the development of a method, which can also be used for local problems and which is not connected to equidistributed point sets.

The following theory holds for non-negative localizing kernels with or without compact support of this type, provided that technical details are thoroughly accounted for in the latter case. Since the hierarchical approximation technique to be presented employs compactly supported kernels for reasons of computational efficiency, we restict ourselves to this case here and employ $\tilde{B}_\rho$ from (21) for ease of presentation.

Let us consider a quadrature formula of the following type :

$$\int_\Omega F(\eta)\ \tilde{B}_\rho(\eta\xi)\ d\omega(\eta) \approx d_\Gamma(\xi)\ \sum_{\zeta\in\Gamma} F(\zeta)\ \tilde{B}_\rho(\zeta\xi), \quad \xi \in \Omega \tag{29}$$

where $\Gamma \subset \Omega$ is an arbitrary pointset on Ω. The factor $d_\Gamma(\xi)$ in (29) can be chosen in such a way as to get a quadrature formula which is exact for constant functions. Due to the normalization in (21) $\int_\Omega \tilde{B}_\rho(\eta\xi)\ d\omega(\eta) = \tilde{B}_\rho^\wedge(0) = 1,\ \xi \in \Omega$, this is achieved with

$$d_\Gamma(\xi) = \frac{1}{\sum_{\zeta\in\Gamma} \tilde{B}_\rho(\zeta\xi)}, \quad \xi \in \Omega. \tag{30}$$

so that the singular integrals are discretized in the following way

$$\int_\Omega F(\eta)\ \tilde{B}_\rho(\eta\xi)\ d\omega(\eta) \approx \frac{\sum_{\zeta\in\Gamma} F(\zeta)\ \tilde{B}_\rho(\zeta\xi)}{\sum_{\zeta\in\Gamma} \tilde{B}_\rho(\zeta\xi)}, \quad \xi \in \Omega. \tag{31}$$

Note that with $k = 0$ in (21) we get the arithmetical mean of $F(\zeta),\ \zeta \in \Omega_{\rho,\xi}$, as approximation of F at $\xi \in \Omega$ with $\Omega_{\rho,\xi}$ from (23).
For non–negative locally supported kernel functions we can calculate the following error–estimation

Theorem 3.1 *(Linear case) Let $\Gamma \subset \Omega$ be a given grid and $\rho \in (-1, 1)$ chosen such that for this grid $\sum_{\zeta'\in\Gamma} \tilde{B}_\rho(\xi\zeta') \neq 0$, with $\tilde{B}_\rho$ from (21) for all $\xi \in \Omega$. Then for $F \in \mathcal{C}(\Omega)$*

$$\left| \int_{\Omega_{\rho,\xi}} F(\eta)\ \tilde{B}_\rho(\xi\eta)\ d\omega(\eta) - \frac{\sum_{\zeta\in\Gamma} F(\zeta)\ \tilde{B}_\rho(\zeta\xi)}{\sum_{\zeta'\in\Gamma} \tilde{B}_\rho(\zeta'\xi)} \right| \leq \max_{\substack{\eta\in\Omega_{\rho,\xi} \\ \zeta\in\Gamma\cap\Omega_{\rho,\xi}}} |F(\eta) - F(\zeta)| \tag{32}$$

with $\Omega_{\rho,\xi}$ from (23)*. In particular, for Lipschitz–continuous F on Ω*

$$\sup_{\xi\in\Omega} \left| F(\xi) - \frac{\sum_{\zeta\in\Gamma} F(\zeta)\ \tilde{B}_\rho(\zeta\xi)}{\sum_{\zeta'\in\Gamma} \tilde{B}_\rho(\zeta'\xi)} \right| \leq C_F\ 3\ \sqrt{2}\ \sqrt{1-\rho}. \tag{33}$$

Proof

$$\left| \int_{\Omega_{\rho,\xi}} F(\eta)\ \tilde{B}_\rho(\xi\eta)\ d\omega(\eta) - \frac{1}{\sum\limits_{\zeta'\in\Gamma} \tilde{B}_\rho(\xi\zeta')} \sum_{\zeta\in\Gamma} F(\zeta)\ \tilde{B}_\rho(\xi\zeta) \right| =$$

$$\left| \frac{1}{\sum\limits_{\zeta' \in \Gamma} \tilde{B}_\rho(\xi\zeta')} \int_{\Omega_{\rho,\xi}} \Big\{ \sum_{\zeta \in \Gamma} \tilde{B}_\rho(\xi\zeta) \big(F(\eta) - F(\zeta)\big) \Big\} \tilde{B}_\rho(\xi\eta)\ d\omega(\eta) \right| \leq$$

$$\max_{\substack{\eta \in \Omega_{\rho,\xi} \\ \zeta \in \Gamma \cap \Omega_{\rho,\xi}}} |F(\eta) - F(\zeta)| \,.$$

Combining the last result with (27) the second part of the Theorem is proved. □

This first result remains valid for iterated kernel functions.

The above approximation procedure can be described in an alternative way as a "Fourier expansion method" (non–orthogonal series expansion). It can be seen as an expansion into a series of normalized basis functions $\overline{B}_\rho(\cdot\ \zeta) : \Omega \to \mathbb{R},\ \zeta \in \Gamma$

$$F(\xi) \approx f(\xi) = \sum_{\zeta \in \Gamma} F(\zeta)\ \overline{B}_\rho(\xi\zeta),\ \xi \in \Omega \tag{34}$$

with

$$\overline{B}_\rho(\xi\zeta) = \frac{1}{\sum_{\zeta' \in \Gamma} \tilde{B}_\rho(\xi\zeta')}\ \tilde{B}_\rho(\xi\zeta),\ \xi \in \Omega. \tag{35}$$

To assess the behaviour of the approximating function $f(\xi)$ we investigate the normalized basis functions $\overline{B}_\rho(\cdot\ \zeta) : \Omega \to \mathbb{R}$ for an arbitrary grid $\Gamma \subset \Omega$.

Lemma 3.1 *Given a grid* $\Gamma \subset \Omega$ *and* $\rho \in (-1, 1)$ *such that for this grid* $\sum\limits_{\zeta' \in \Gamma} \tilde{B}_\rho(\xi\zeta') \neq 0$, $\xi \in \Omega$, *the normalized basis functions* $\overline{B}_\rho(\cdot\ \zeta),\ \zeta \in \Gamma$ *have the properties :*

(P1) $\overline{B}_\rho(\cdot\ \zeta) \in \mathcal{C}(\Omega)$ *continuity*

(P2) $\overline{B}_\rho(\xi\zeta) \geq 0,\ \xi \in \Omega$ *positivity*

(P3) $\sum_{\zeta \in \Gamma} \overline{B}_\rho(\xi\zeta) = 1,\ \xi \in \Omega$ *normalization.*

The following two properties of the approximation function $f(\xi)$ defined in (34) can be shown

1. non–negativity : if $F(\zeta) \geq 0$, for all $\zeta \in \Gamma$ then

$$0 \leq f(\xi), \quad \xi \in \Omega \tag{36}$$

2. minimum–maximum principle :

$$\min_{\zeta \in \Gamma} F(\zeta) \leq f(\xi) \leq \max_{\zeta \in \Gamma} F(\zeta), \quad \xi \in \Omega. \tag{37}$$

The first one is a consequence of the construction principle (34), (35), namely that $f(\xi)$ is a convex combination of the data $F(\zeta),\ \zeta \in \Gamma$. The minimum–maximum principle, (37), is reminiscent of the same familiar property of harmonic functions.

The reader acquainted with scattered data methods will recognize that the above method is similar to the Shepard method (Shepard (1968)) used in meteorology and geology, described e.g. in Hoschek, Lasser (1992). In contrast to the Shepard method the above method is developed on the sphere and does not aim at interpolating the data but at approximation. The former is obtained for general data only if ρ is chosen in such a way that the spherical cap around each point $\zeta \in \Gamma$ does not contain any further grid point $\zeta' \in \Gamma$. This generally leads to a very wiggled approximation $f(\xi)$.

Let us describe in detail the approximation using the second iterated convolution $\tilde{B}_\rho^{(2)}$ of a non–negative kernel for later use in Section 4. The iterated integral is defined by

$$\begin{aligned} I_\rho^{(2)}(F)(\xi) &= (\tilde{B}_\rho^{(2)} * F)(\xi) && (38)\\ &= \int_\Omega F(\eta) \int_\Omega \tilde{B}_\rho(\xi\zeta)\ \tilde{B}_\rho(\zeta\eta)\ d\omega(\zeta)\ d\omega(\eta) && (39)\\ &= \int_\Omega \tilde{B}_\rho(\xi\zeta) \int_\Omega F(\eta)\ \tilde{B}_\rho(\zeta\eta)\ d\omega(\eta)\ d\omega(\zeta) && (40)\\ &= (\tilde{B}_\rho * I_\rho(F))(\xi) && (41) \end{aligned}$$

and can also be interpreted as a convolution between the kernel $\tilde{B}_\rho$ and the singular integral $I_\rho(F)$. Since generally the iterated convolution cannot be calculated explicitly (especially for the locally supported kernels defined in Section 2), we need two discretizations for the determination of $I_\rho^{(2)}(F)$ (bilinear case). On one hand we have to discretize $\tilde{B}_\rho^{(2)}$ using an artificial grid $\Gamma \subset \Omega$ and on the other hand we calculate $I_\rho^{(2)}(F)$ using the grid of measurements $\Xi \subset \Omega$. According to the second interpretation of the iterated singular integral we get a "non–orthogonal Fourier series type expansion"

$$F(\xi) \approx f(\xi) = \sum_{\eta\in\Gamma} c(\eta)\ \overline{B}_\rho(\xi\eta), \quad \xi \in \Omega \tag{42}$$

$$c(\eta) = I_\rho(F)(\eta) = \sum_{\zeta\in\Xi} F(\zeta)\ \overline{B}_\rho(\eta\zeta), \quad \eta \in \Gamma. \tag{43}$$

Now the two points of view for describing the approximation can be formulated clearly : On one side it is an approximation by integral means with smoothing kernel functions, on the other side the method can be seen as expansion in terms of basis functions in which the basis coefficients are calculated as integral means.
Especially for the locally supported basis functions introduced in Section 2 an error estimation can be derived. Using the abbreviations

$$I_\rho^{(2)}(F)(\xi) = \int_\Omega \int_\Omega F(\eta)\ \tilde{B}_\rho(\eta\zeta)\ \tilde{B}_\rho(\xi\zeta)\ d\omega(\eta)\ d\omega(\zeta) \tag{44}$$

and

$$f(\xi) = \sum_{\eta\in\Gamma} \left(\sum_{\zeta\in\Xi} \frac{F(\zeta)\ \tilde{B}_\rho(\eta\zeta)}{\sum_{\zeta'\in\Xi} \tilde{B}_\rho(\zeta'\eta)} \right) \frac{\tilde{B}_\rho(\xi\eta)}{\sum_{\eta'\in\Gamma} \tilde{B}_\rho(\xi\eta')} \tag{45}$$

we get

Theorem 3.2 *(Bilinear case) Let $\Gamma \subset \Omega$ and $\Xi \subset \Omega$ be two grids with $\sharp\Gamma = N$ and $\sharp\Xi = M$. Assume further that $\rho \in (-1,1)$ is chosen in such a way that $\sum_{\zeta'\in\Xi} \tilde{B}_\rho(\zeta'\eta) \neq 0$, for all $\eta \in \Gamma$ and $\sum_{\eta'\in\Gamma} \tilde{B}_\rho(\xi\eta') \neq 0$, for all $\xi \in \Omega$. Then, for any Lipschitz–continuous function $F : \Omega \to \mathbb{R}$ with Lipschitz–constant C_F*

$$||F - f||_{\mathcal{C}(\Omega)} \leq C_F\ 6\ \sqrt{2}\ \sqrt{1-\rho}. \tag{46}$$

Proof From eq. (26) the following estimation is valid for the iterated weighted integral mean

$$||I_\rho^{(2)}(F) - F||_{\mathcal{C}(\Omega)} \leq \max_{-1+2\rho^2\leq\xi\eta\leq 1} |F(\xi) - F(\eta)|. \tag{47}$$

This can be reformulated for Lipschitz–continuous functions F

$$||I_\rho^{(2)}(F) - F||_{\mathcal{C}(\Omega)} \leq C_F\ 2\ \sqrt{2}\ \sqrt{1-\rho}. \tag{48}$$

Using Theorem 3.1 we get for all $\xi \in \Omega$

$$\left| I_\rho^{(2)}(F)(\xi) - \int_{\Omega_{\rho,\xi}} \left(\sum_{i=0}^{M} \frac{F(\zeta_i)\ \tilde{B}_\rho(\zeta_i\eta)}{\sum_{j=0}^{M} \tilde{B}_\rho(\zeta_j\eta)} \right) \tilde{B}_\rho(\xi\eta)\ d\omega(\eta) \right|$$

$$\leq C_F\ 2\ \sqrt{2}\ \sqrt{1-\rho}. \tag{49}$$

Setting

$$G(\eta) := \sum_{i=0}^{M} \frac{F(\zeta_i)\ \tilde{B}_\rho(\zeta_i\eta)}{\sum_{j=0}^{M} \tilde{B}_\rho(\zeta_j\eta)}, \quad \eta \in \Omega_{\rho,\xi} \tag{50}$$

we obtain for a second discretization

$$\left| \int_{\Omega_{\rho,\xi}} G(\eta)\tilde{B}_\rho(\xi\eta)\ d\omega(\eta) - \sum_{i=0}^{N} G(\eta_i)\ \frac{\tilde{B}_\rho(\eta_i\xi)}{\sum_{j=0}^{N} \tilde{B}_\rho(\eta_j\xi)} \right|$$

$$\leq \max_{\substack{\eta\in\Omega_{\rho,\xi} \\ \eta_i\in\Omega_{\rho,\xi}\cap\Gamma}} |G(\eta) - G(\eta_i)| \tag{51}$$

and since $G(\eta) \leq \max\limits_{\eta\in\Omega_{\rho,\xi}} F(\eta)$ and $G(\eta_i) \geq \min\limits_{i=0,1,\ldots,N} \{F(\eta_i)\}$ it follows for Lipschitz–continuous F

$$\max_{\substack{\eta\in\Omega_{\rho,\xi} \\ \eta_i\in\Omega_{\rho,\xi}\cap\Gamma}} |G(\eta) - G(\eta_i)| \leq C_F\ 2\ \sqrt{2}\ \sqrt{1-\rho}. \tag{52}$$

Combining the last three results the theorem is proved. □

This rough estimate helps us to get some insight from a practical point of view. It demonstrates the approximability of F by a bilinear discretization of its singular integral provided that $1 - \rho$ is sufficiently small. It is also the basis of the multilevel approach described in the next section. Formula (45) may be understood as discretized non–orthogonal sum expansion of Fourier series type in terms of the normalized basis functions $\overline{B}_\rho$ and results in the following one–scale method.

Algorithm 3.1 *Given a grid* $\Xi \subset \Omega$ *and the data set* $(\zeta, F(\zeta)) \in \Xi \times \mathbb{R}$.

(*i*) *Choose a grid* Γ.

(*ii*) *Choose suitable parameters* ρ *and* k *defining* $\tilde{B}_\rho$ *in* (21).

(*iii*) *Compute the "discrete Fourier coefficients"* $c(\eta), \eta \in \Gamma$, *from equation* (43).

(*iv*) *Then the "discrete Fourier expansion"* f *from equation* (42) *is an approximation to the function* F.

4 The Hierarchical Method

Algorithm 3.1 is tested on several artificial and practical examples in Brand (1994) for different grids Γ and Ξ. It works well but has the following inherent drawbacks :

- A suitable value for the scale parameter ρ is not known in advance. It can be guessed according to the smallest spatial feature that has to be represented. This may require several runs for optimization and may be inappropriate for a given function.
- When the characteristic scale of the data to be represented varies in space, one has to choose the value of ρ according to the smallest feature of the scale size in the domain. If not doing so, essential information may be smoothed out. Consequently, there is a requirement of narrow spaced data for the whole domain which leads to a waste in computational effort due to the calculations of lots of fine scale amplitudes even in parts where they are not required. A spatially variable value of ρ would be appropriate in that case. However the practical specification would be highly problem dependent and rather delicate.
- The error can only be calculated at the end. An improvement is based on a completely new application with modified ρ and/or modification of the grid Γ, i.e. the number and location of basis functions and can only be controled with difficulties.
- A spatial variation of the tolerance of the measurements on the grid Ξ cannot be accounted for a straightforward manner.

These drawbacks can be remedied by a hierarchical algorithm which is described now. In fact, the one–scale algorithm of the previous section has mainly been set up in this paper to allow for this generalization. In particular the requirement of an equidistributed

grid Γ for former methods is not compatible with the local approach neccessarily to be adopted in the hierarchical construction. Let us first describe the algorithm in words.

Algorithm 4.1

(*i*) *Choose a relatively small scale parameter* ρ_0 *(large spherical caps) and relatively coarse basis grid* Γ_0.

(*ii*) *Compute an approximation* f_0 *from Algorithm 3.1.*

(*iii*) *Compute the error* E_1 *at the points of* Ξ.

(*iv*) *Decide wether* E_1 *is sufficiently small in all parts of the domain. If true stop.*

(*v*) *Increase the scale parameter to* ρ_1 *(smaller spherical caps) and refine the basis grid (grid of coefficients) to* Γ_1

(*vi*) *Compute the approximation of the discrete error* E_1 *in those parts of the domain where* E_1 *is above the threshold in* (*iv*). *Add this contribution to the approximation obtained in* (*ii*).

(*vii*) *Iterate steps* (*iii*) *to* (*vi*) *up to the situation where*

- (*a*) *The error* E_j *is sufficiently small in the whole domain or*
- (*b*) *The refinement of the grid* Γ_j *approaches a cell size where the number of points from* Ξ *that serve to determine a particular coefficient decreases below a threshold which ensures sufficient averaging*

Let us make some comments to detail the method. There are two different ways of increasing the quality of the approximations from given ρ and Γ. The first is to refine Γ keeping ρ constant. This leads to better results, if being done with the right parameter, because the iterations are the first members of a Neumann–series for the inversion of the corresponding integral operator (cf. Brand (1994)). On the other hand, increasing ρ without modification of Γ leads in its extreme to non–overlapping supports (considering locally supported kernel functions) and inadequate resolution of the corresponding scale. Therefore, in step (*v*) it is reasonable to refine $1-\rho_j$ and Γ_j at the same time. One arrives in some sense at contracted versions (ρ_j, Γ_j) of the first choice (ρ_0, Γ_0) (see also remarks at the end of this section). The initial ratio between the grid spacing of Γ_0 and the scale parameter ρ_0 has to be determined from experience but is rather uncritical since deficiencies will be corrected in the subsequent steps.

Now consider step (*vii*)(*b*) in more detail. To be applicable, the sums (42), (43) have to contain at least a certain number of points to ensure correct averaging. The support of the basis functions $\overline{B}_\rho(\cdot\eta)$ decreases with $1-\rho$, so that (since Ξ is fixed) the number of entries in the sum (43) diminishes. When a given minimal number is reached the algorithm might react in different ways :

(α) a warning is issued and the refinement is simply not continued in this region

(β) a warning is issued and only the grid Γ_j is refined to Γ_{j+1} keeping the scale parameter ρ_j constant. This is to be understood as an improvement for the approximation of $\tilde{B}_\rho^{(2)}$ (cf. (38)).

The source of the behaviour in $(vii)(b)$ may originate from two similar events :

(α) a (perhaps single) measurement in Ξ has been erroneous. If this is be verified, the corresponding point could simply be discarded from later processing.

(β) the measurements in Ξ have been too coarse to adequately describe a particular fine scale behaviour with sufficient reliability. The warning in this case indicates the need of additional points in Ξ if this is to be done.

Supplementary points forming a new set Ξ' can be added easily when the grid Ξ has been found insufficient in a particular region. The approximation has to be evaluated at these new points to get E_j. The procedure may in most cases be continued without the need of recalculating previous coefficients, since contributions to large scales will generally cancel out. Moreover, the employed bases (ρ_j, Γ_j) are not orthogonal. The union contains by construction a certain amount of redundancy, so that supplementary data points can also be represented by the fine scale basis functions.

We now define the hierarchical approximation in a formal way by means of the methodology developed above. It reads

$$F(\xi) \approx f_J(\xi) = (EI)_J(F)(\xi), \quad \xi \in \Omega \tag{53}$$

with the J–th discrete hierarchical error iteration $(EI)_J(F)$ of F given by the following

Definition 4.1 *The expression*

$$(EI)_J(F)(\xi) = \sum_{j=0}^{J} \sum_{i=1}^{N_j} c_{ji}(\eta_{ji})\ \overline{B}_{\rho_j}(\xi\eta_{ji}) \tag{54}$$

with the basis coefficients

$$c_{ji}(\eta_{ji}) = \sum_{\zeta \in \Xi} E_j(\zeta)\ \overline{B}_{\rho_j}(\xi\eta_{ji}) \tag{55}$$

for $\eta_{ji} \in \Gamma_j, j = 0, 1, \ldots, J, i = 1, 2, \ldots, N_j$, *with*

$$E_j(\zeta) = F(\zeta) - (EI)_{j-1}(F)(\zeta), \quad j \geq 1 \tag{56}$$

$$E_0(\zeta) = F(\zeta) \tag{57}$$

for $\zeta \in \Xi$ *is called J–th discrete hierarchical error iteration.*

The above method generalizes Algorithm 3.1 in a substantial way. It is triggered by the multilevel approaches that have been developed in the last years (multigrid, hierarchical

basis, wavelets). However, only the idea of using different scales is applied here without bothering about other properties, such as e.g. imbedding or orthogonality of subspaces which are possible in the continuous setting. Such a construction has recently been obtained in Freeden, Windheuser (1994), Freeden, Schreiner (1994), for the Abel–Poisson kernel. It is not known if a similar decomposition can also be obtained for the present kernel (21). Most of all, in spherical geometry it is impossible to introduce a regular grid which would permit to define a discrete counterpart to the continuous decomposition such as obtainable e.g. for periodic wavelet decomposition in $\mathbb{R}^n$, i.e. a discrete orthonormal basis of shifted and scaled versions of one function. This immediately rules out the possibility to develop a fast recursive algorithm as for the classical multiresolutions.

We therefore resorted to a different recursive approach by calculating the approximation error for each level of iteration as being done in Fröhlich, Schneider (1994) for the adaptive wavelet–discretization of a PDE. Note that the order of the theoretical asymptotic operation count is not bad. Assume e.g. that, as a worst case, the error E_j has a similar size all over the sphere so that refinement and correction is required everywhere. If then reduction of the cap size and refinement of Γ_j equilibrate, the number of operations to compute the required amplitudes and the corresponding errors on Ξ is roughly the same for each level j, $O(\sharp\Xi)$, giving an overall count of $O(J\ \sharp\Xi)$. For different settings, when refinement is local, the estimation of the computational cost is difficult since it depends on the actual data, the precise refinement strategy for (ρ_j, Γ_j), and the required tolerances. In any case calculations are only executed where this is demanded by a persisting error. A further advantage of the above procedure is that it deliberately allows the use of highly non–regular and non–equidistributed grids Ξ. They appear through a variable distance of the data points Ξ right from the beginning when it is known in advance that this will be necessary. Second, replacing a point of measurement by another one nearby may be necessary for pratical purposes when at the former point a value cannot be obtained (presence of an obstacle, etc.). Both occasions do frequently arise in geophysical applications. Finally, redundancy and non–orthogonality of the basis can be beneficious when additional data becomes available. Note that allthough an equidistributed grid Ξ is no longer employed, the construction requires a rule to define parametrize equidistributed grids Γ_j for each level j (see Section 5 for example). However, these are not used entirely but just in some part of the domain "switching on" only the required contributions.

An important point to be stressed is that the above is an approximation procedure which is inadequate if interpolation is required. On the other hand it is extremely simple in its principle and efficiently implementable. It is also very flexible and by its adaptivity may significantly reduce the computational cost with respect to other approximation methods.

5 Test calculations

The hierarchical approximation method is now applied in two cases. The first is an artificial example, the second is a digital terrain model of Australia. As basis grids, in (43), we use hierarchical grid sequences (see Freedeen, Schreiner (1993), Brand (1994)), which

have the basic properties that $\Gamma_j \subset \Gamma_{j+1}$ and that the knot width $\sigma(\Gamma_j)$ tends to zero if j tends to infinity. In the grid with nearly equidistributed grid points the points are arranged along latitudes in such a way that the spherical distance $t = \eta\eta'$ between two neighbouring grid points η, η' is equal. These grids are therefore parametrized by the number of latitudes γ (for the whole sphere). In contrast to this grid the grid points in a Corput–Halton grid are arranged irregularly. The idea is to use a sequence in a rectangle which is uniformly distributed and then transform it via spherical coordinates onto the unit sphere Ω (cf. van der Corput (1935), Kuipers, Niederreiter (1974)).

In every calculation we use the "weighted integral means" with $k = 3$ in (21) (twice continuously differentiable functions) as singular integral and consider the errors

$$maxerror := \max_{\zeta \in \Xi} |F(\zeta) - f(\zeta)| \tag{58}$$

$$mean - error := \frac{1}{\sharp\Xi} \sum_{\zeta \in \Xi} |F(\zeta) - f(\zeta)|. \tag{59}$$

The first example is set up by the test function $F : \Omega \to \mathbb{R}$

$$F(x_1, x_2, x_3) = \frac{1}{\sqrt{x_1^2 + (x_2 - 0.9)^2 + x_3^2}} \tag{60}$$

which is plotted for $(\varphi, \vartheta) \in [\pi/2, 3\pi/2] \times [\pi/4, 3\pi/4]$ in Figure 2. We assume that the function F is given on some non–equidistributed grid Ξ (see Figure 3) with 923 points of the following properties :

$$\begin{aligned} \pi/2 < 1.5707 \leq \quad & \varphi \quad && \leq 4.6796 < 3\pi/2 \\ \pi/4 < 0.8796 \leq \quad & \vartheta \quad && \leq 2.2619 < 3\pi/4 \\ 0.5264 \leq \quad & F(\zeta) \quad && \leq 9.1139, \quad \zeta \in \Xi \end{aligned}$$

The mean of the values of F on Ξ is $\overline{F} = 1.1835$.

In order to assess the quality of the hierarchical method, it has to be compared with a one–scale approximation. Two computations have been carried out, one with $\rho = 0.5$, one with $\rho = 0.9965$, both using the same nearly equidistributed grid Γ with 1601 points in the test area.
The results are reported in Table 1 and Figure 4, showing cuts along the equator. Figure 4 depicts the data points as circles, while the continuous line corresponds to the computed approximation. It is generated by basis functions of which the centers lying on the equators are marked by crosses on the abcissa. The 5th and 6th column in Table 1 give an idea of the mutual influence of points in Γ and Ξ.

It is apparent that if the size of the basis functions is too large the one–scale approximation is poor due to excessive smoothing even with $\sharp\Gamma$ large . The approximation with smaller caps is almost satisfactory. We will see, however, that the hierarchical method gives a better result with even less basis functions. We remark that the present integration method leads to acceptable results near boundaries of local domains since it takes into

Table 1 : Synthetic example, one–scale approximation.

One–scale method						
γ	ρ	maxerror	mean–error	$\sharp(\Xi \cap \Omega_{\rho,\eta})$	$\sharp(\Gamma \cap \Omega_{\rho,\xi})$	coefficients
64	0.5	6.496911	0.248608	195 – 599	3 – 5	1601
64	0.9965	0.675085	0.007291	2 – 9	3 – 10	1601

Figure 2 : Function $F(x_1, x_2, x_3)$.

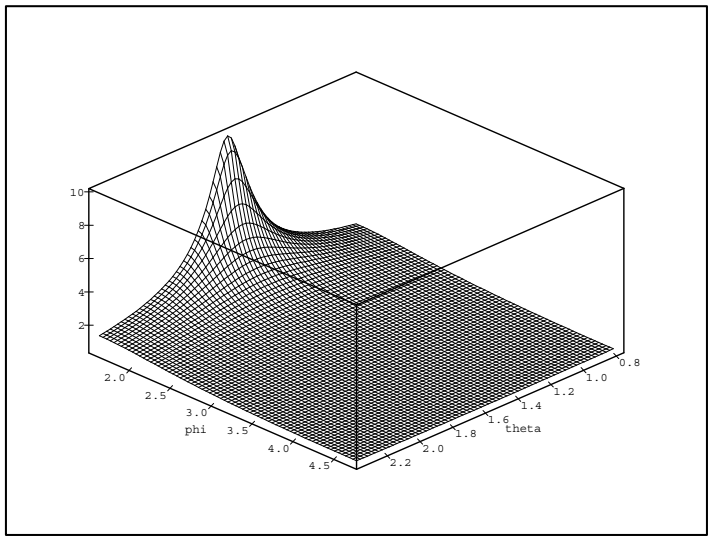


Figure 3 : Data distribution, i.e. grid Ξ, for the first example.

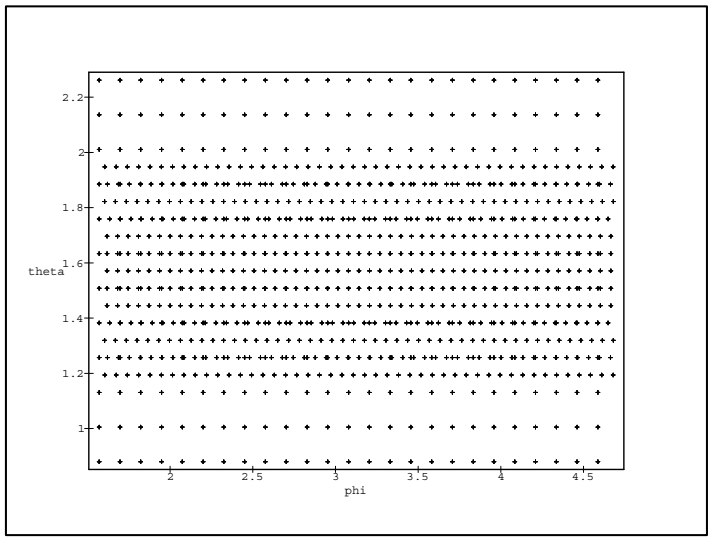


Figure 4 : Synthetic example, cuts through the one–scale approximation along the equator for $\rho = 0.5$ (left) and $\rho = 0.9965$ (right).

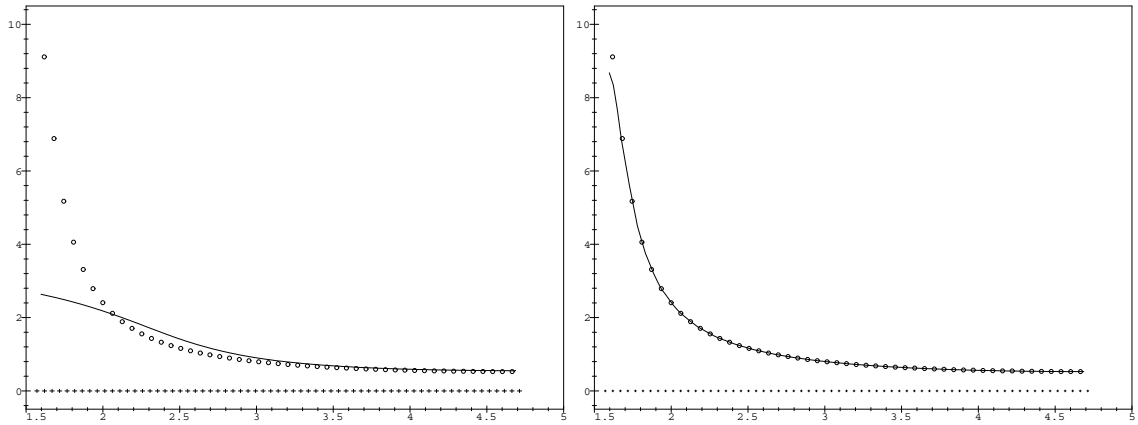


Table 2 : Performance of the hierarchical method for the synthetic example.

Hierarchical method						
γ	ρ_j	maxerror	mean–error	$\sharp(\Xi \cap \Omega_{\rho_j,\eta})$	$\sharp(\Gamma_j \cap \Omega_{\rho_j,\xi})$	coefficients $\lvert c_{ji}\rvert > 10^{-3}$
4	0.5	6.048130	0.232746	195 – 599	3 – 5	11 of 11
8	0.85	4.605950	0.121893	31 – 266	3 – 7	33 of 33
16	0.96	2.713190	0.046071	3 – 81	3 – 8	106 of 113
32	0.99	1.106561	0.012208	2 – 23	3 – 8	336 of 417
64	0.9965	0.333455	0.002862	2 – 9	3 – 10	685 of 1601

account a variable number of points for the quadrature (55). Methods based on an equidistributed grid fail near boundaries as they assume a constant number of entries in the sum, so that this part is generally masked. Let us now apply the hierarchical algorithm using the following rules. The refinement of (ρ_j, Γ_j) is controled by two characteristic values : $\sharp(\Gamma_j \cap \Omega_{\rho_j,\xi})$, $\xi \in \Omega$ monitors the number of overlapping basis functions, while $\sharp(\Xi \cap \Omega_{\rho_j,\eta})$, $\eta \in \Gamma$ indicates the number of points used to determine c_{ji} in (55). We experienced that a value of two or more for both quantities is appropriate and set up the refinement roughly fulfil this rule. Finally an amplitude of $|c_{ji}| > 10^{-3}$ was required, if not, the contribution has been left unconsidered.

Cuts along the equator of the resulting succesive approximations are reported in Figure 5. Figure 6 and 7 display the final approximation and its error in a perspective plot. The plot shows that in most of the computational domain the error is less than 1 % of the maximal value of F. Only at the boundary of the considered domain, where less measurements are available, the errors grow up to 5 % . Furthermore, the peak of the test function is approximated better than with a one–scale approximation method. This is a general observation backed also by other tests. Table 2 gives detailed quantitative information of the single steps. Note that with respect to the one–scale case a better result could be obtained with less than half of the number of basis functions. The adaptive choice of the basis points of Γ_j, used for the calculation, is plotted in Figure 8. Since we use hierarchical grids ($\Gamma_j \subset \Gamma_{j+1}$) the number j of the step, in which the basis point $\eta \in \Gamma_j$ is used for the calculation the last time, is assigned a gray scale (square grid for implementational reasons). In the second example we compute a digital terrain model of Australia observing the sperical figure of the earth. The elevation data used are derived from $0.5^\circ \times 0.5^\circ$ measurements of mean elevation. It represents a spacing of at least 1 point per 55 km^2. The data is given on latitude–longitude lattice with 5381 points has the following properties :

$$\begin{aligned} 1.9285\ (110.5^\circ) \leq \quad & \varphi \quad \leq 2.7139\ (155.5^\circ) \\ 1.7366\ (-9.5^\circ) \leq \quad & \vartheta \quad \leq 2.4521\ (-50.5^\circ) \\ -6000\ m \leq \quad & F \quad \leq 1344\ m. \end{aligned}$$

The mean value of the values is $\overline{F} = -755.8305\ m$.

Figure 5 : Synthetic example, hierarchical method, cuts along the equator for different levels of approximation $f_j,\ j = 0, \ldots, 4$.

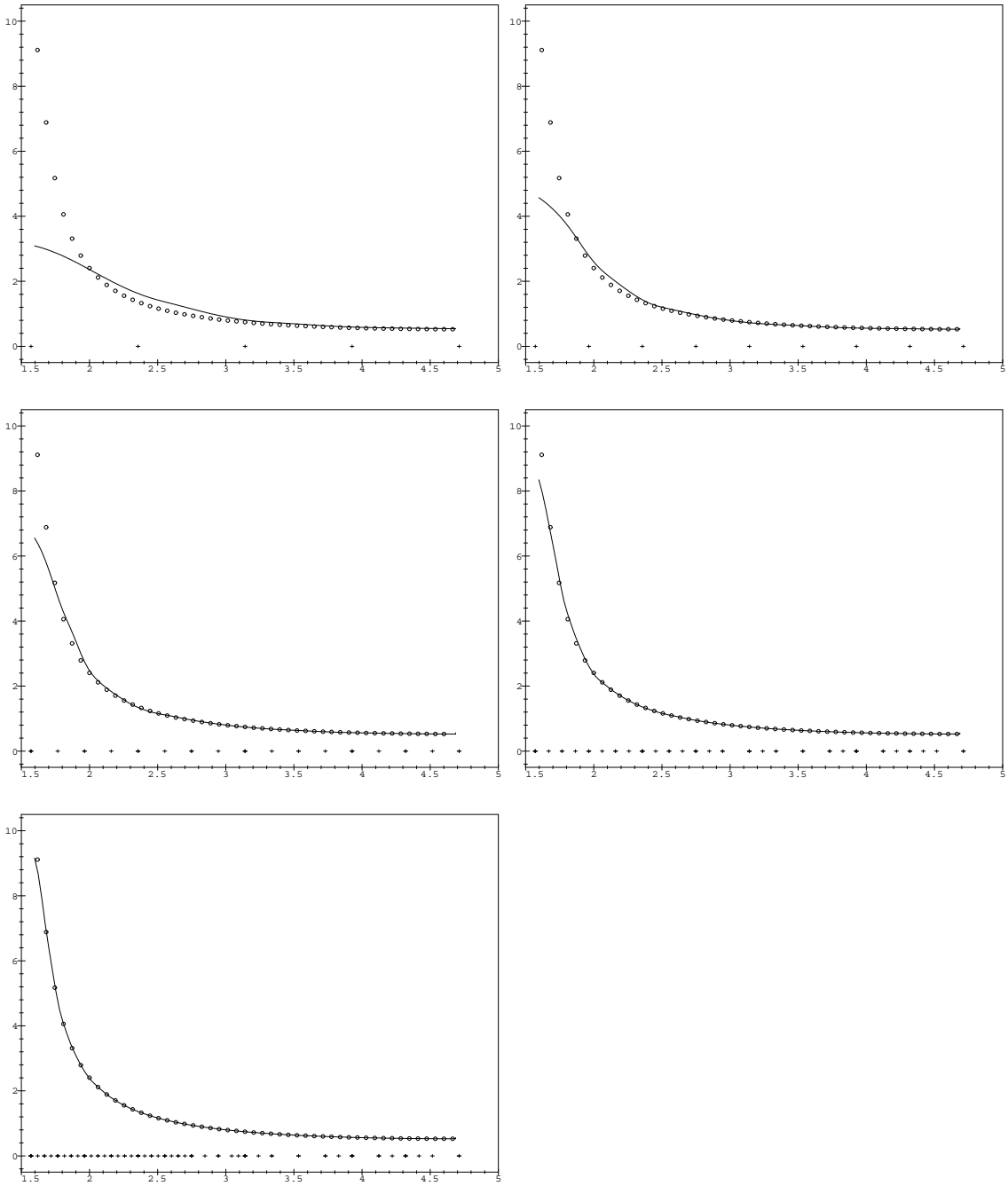


Figure 6 : Synthetic example, approximation after step 5
using the hierarchical method.

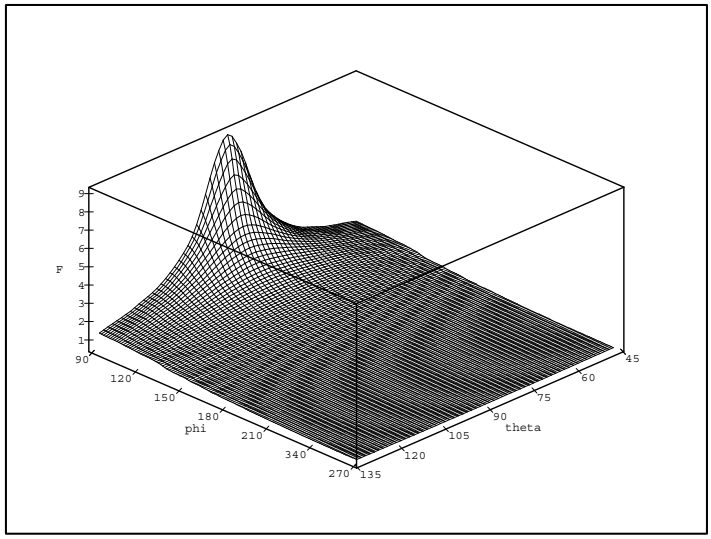


Figure 7 : Synthetic example, error plot of the approximation after step 5
using the hierarchical method.

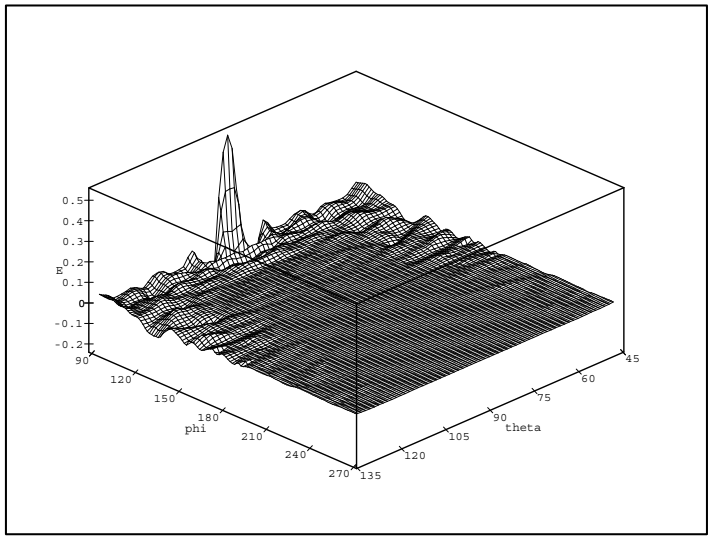
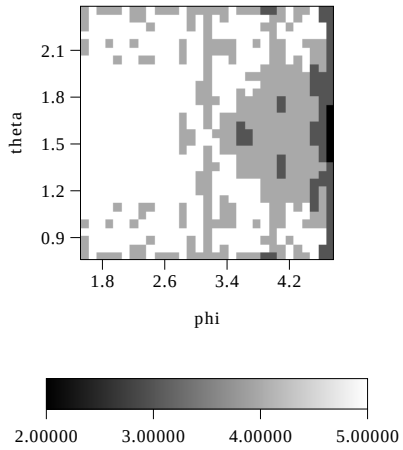


Figure 8 : Adaption of basis grid Γ_j (see text).



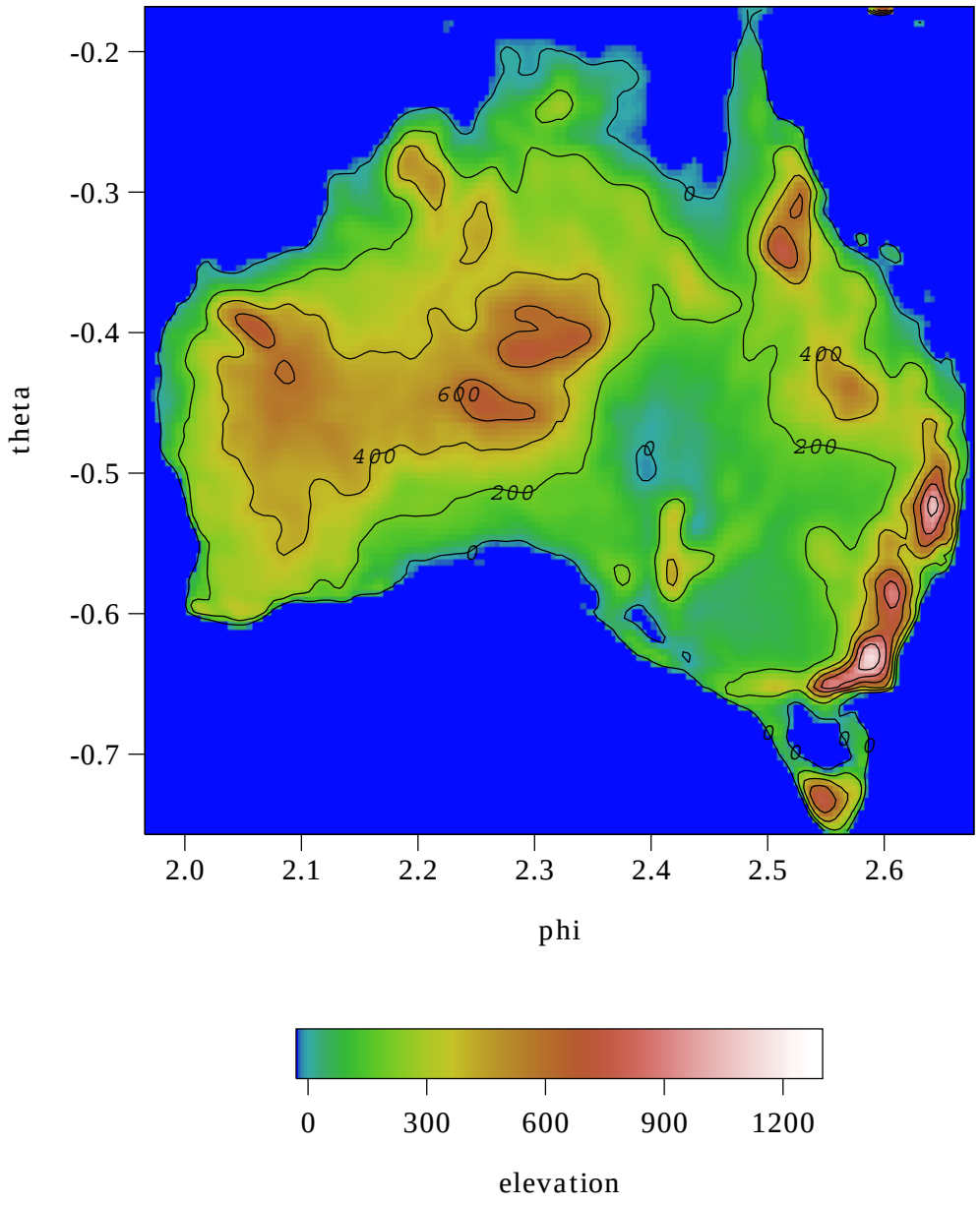
The hierarchical error iteration is calculated with a hierarchical sequence of Corput–Halton basis grids Γ_j (doubling of the number of basis points in each step), starting with $512 = 2^9$ basis functions, using the sequence of scale parameters $\rho_1 = 1 - 2^{-9}$, $\rho_2 = 1 - 2^{-10}, \ldots, \rho_6 = 1 - 2^{-14}$, that roughly correspond to the mesh width (normalization of the earth's radius to 1). The results are shown in Figure 9. It is the first time that this kind of approximation method is tested for large data sets for which interpolation methods can only be used after domain decomposition or data reduction. The difficulty of this example resides in the fact that at the East coast of Australia the values vary extremly (mountains and sea depths). Since the approximation method (approximation by singular integrals) is a smoothing method (building of integral means) the mountains and the coast line cannot be approximated efficiently using a one–scale method. In Figure 9 only positive values of the approximation are plotted and it is apparent that the mountains and coast line are approximated well. The deficiencies in the region between Australia and Tasmania are caused by too few measurements and our method recognizes this problem.

6 Conclusion

In this paper we have presented a simple and efficient hierarchical method for approximation of data given at a set of arbitrary points on the sphere. It is based on the principle of non–orthogonal series expansions and employs locally supported axisymmetric polynomial basis functions. The crucial point is the explicit computation of the error on each level which allows arbitrary size and distribution of these functions and easy adaption.

First applications of the new method to an artificial case and a large data set, a digital terrain model of Australia, have demonstrated the power and flexibility of the approach.

Figure 9 : Digital terrain model of Australia
obtained by the hierarchical method.



Immediate extensions concern the basis functions which need not be polynomials. Other locally supported functions may be used as well. Furthermore, a spatially varying tolerance of geodetic measurements can easily be accounted for by a variable threshold for the computed error at the data points. It is immediate to assign each point a tolerance and to require the approximation to lie within this range. It is also possible to introduce different weights for each measurement and use the following discretization

$$F(\xi) \approx \frac{\sum_{i=0}^{M} \beta_i \; F(\zeta_i) \; \tilde{B}_\rho(\xi\zeta_i)}{\sum_{i=0}^{M} \beta_i \; \tilde{B}_\rho(\xi\zeta_i)} \quad \zeta_i \in \Gamma, \sharp\Gamma = M, \; \xi \in \Omega. \tag{61}$$

In connection with regular grids on the sphere (cf. Freeden, Schreiner (1993)) this method can also used for modelling purposes.

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