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Parameter Derivatives of the Jacobi Polynomials  
and the Gaussian Hypergeometric Function

# Parameter Derivatives of the Jacobi Polynomials and the Gaussian Hypergeometric Function

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## ABSTRACT

The paper's main result is a simple derivation rule for the Jacobi polynomials with respect to their parameters, i.e. for  $\partial_\alpha P_n^{\alpha,\beta}$ , and  $\partial_\beta P_n^{\alpha,\beta}$ . It is obtained via relations for the Gaussian hypergeometric function concerning parameter derivatives and integer shifts in the first two arguments. These have an interest on their own for further applications to continuous and discrete orthogonal polynomials. The study is motivated by a Galerkin method with moving weight, presents all proofs in detail, and terminates in a brief discussion of the generated polynomials.

KEY WORDS: special functions, Jacobi polynomials, classical hypergeometric functions, adaptive Galerkin methods

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# 1 INTRODUCTION

Orthogonal polynomials have wide applications in mathematics, mathematical physics, numerical analysis, and other fields. Consequently there exists an enormous variety of such polynomials and relations that describe their properties and interdependencies, see e.g. [Sze39], [AS64], [Chi78], [NU88], [PBM90]. Polynomials related to the interval such as the Legendre, Chebyshev, Jacobi, Gegenbauer, and others are intimately connected to the Gauss hypergeometric function  $F$  that will be discussed in an intermediate step. The same is true for polynomials that are orthogonal with respect to some discrete scalar product such as e.g. the discrete Laguerre and the Meixner polynomials [NU88]. Although the presented results therefore have an impact on discrete polynomials as well, we will focus the discussion on the Jacobi polynomials  $P_k^{\alpha,\beta}(x)$  for conciseness here. Further investigations will be reported in a forthcoming paper.

References as the above and the vast literature on the classical orthogonal polynomials give the impression of nearly every aspect already being investigated. The belowmentioned Galerkin method, however, triggered the interest in parameter derivatives which seem not having been discussed yet. Standard references treat the derivatives of the polynomials with respect to the spatial variable  $x$  in an exhaustive way (differential equations, recurrence relations for derivatives etc.). Concerning the parameters, however, the author could only find equations relating parameter values to their integer shifts, say  $\alpha \rightarrow \alpha + 1$  for example, as well as some involved relations such as [Sze39] (4.10.13):

$$(1-x)^{\alpha-\mu} P_n^{\alpha+\mu,\beta}(x) = (1+x)^{n+\alpha+1} P_n^{\alpha+\mu,\beta}(1) \times \frac{2^\mu \Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \int_x^1 \frac{(1-y)^\alpha (y-x)^{\mu-1}}{(1+y)^{n+\alpha+\mu+1}} \frac{P_n^{\alpha,\beta}(y)}{P_n^{\alpha,\beta}(1)} dy$$

which could in principle be used to get  $\partial_\alpha P_k^{\alpha,\beta} = \lim_{\mu \rightarrow 0} (P_n^{\alpha+\mu,\beta} - P_n^{\alpha,\beta})/\mu$ . A reference giving a simple expression for  $\partial_\alpha P_k^{\alpha,\beta}(x)$  and  $\partial_\beta P_k^{\alpha,\beta}(x)$  suitable for numerical evaluation could not be discerned.

To motivate the relevance of such expressions we briefly indicate an example of application to numerical analysis. Galerkin methods using orthogonal polynomials  $\{b_k\}$  (generally termed spectral methods) are based on a series

representation

$$u^N(x) = \sum_{k=0}^{N-1} \hat{u}_k b_k(x) \quad (1.1)$$

where  $u^N$  is an approximation to the exact solution  $u(x)$  of a differential equation. The basis  $\{b_k\}$  is orthogonal with respect to a weight  $g(x)$ , i.e.

$$\langle b_j, b_k \rangle_g = \int b_j(x) b_k(x) g(x) dx = \gamma_k \delta_{jk} \quad \gamma_k \in \mathbb{R}^{\neq 0} \quad (1.2)$$

The use in an evolutionary problem requires the expression  $\partial_t u(x, t)$  which, due to (1.1), is approximated by

$$\partial_t u^N(x) = \sum_{k=0}^{N-1} d_t \hat{u}_k b_k(x)$$

Classical methods employ trigonometric functions, Legendre polynomials or Chebyshev polynomials of the first kind (mostly due to the existence of fast transforms). The latter two are scaled Jacobi polynomials with  $\alpha = \beta = 0$ , and  $\alpha = \beta = -\frac{1}{2}$ , respectively.

Regardless of convergence properties one can use a different ansatz instead of (1.1) reading

$$u^N(x) = g(x) \sum_{k=0}^{N-1} \hat{u}_k b_k(x) \quad (1.3)$$

which then requires the test functions  $\{b_k/g\}$  to obtain orthogonality with respect to the scalar product (1.2). This "moving weight approach" was first introduced in [DW89] for  $b_k$  being the discrete Laguerre polynomials. The representation (1.3) is particularly advantageous if the weight function  $g(x)$  already is a good approximation to  $u(x)$ . If  $b_k$  and hence  $g$  depend on one or more parameters, the approximation (1.3) may be supplemented with an adaption of the parameters through additional requirements [DW89]. In the continuous case, using Jacobi polynomials, the solution of a space-time dependent problem would then be approximated by

$$u^N(x, t) = w^{\alpha(t), \beta(t)}(x) \sum_{k=0}^{N-1} \hat{u}_k(t) P_k^{\alpha(t), \beta(t)}(x) \quad (1.4)$$

where  $g = w^{\alpha, \beta}$  is the weight function for  $b_k = P_k^{\alpha, \beta}$  in (1.2). It is now obvious that a Galerkin method based on (1.4) requires the knowledge of  $\partial_\alpha P_k^{\alpha, \beta}(x)$  and  $\partial_\beta P_k^{\alpha, \beta}(x)$ , or more precisely of  $\langle \partial_\alpha P_j^{\alpha, \beta}(x), P_k^{\alpha, \beta}(x) \rangle_{w^{\alpha, \beta}}$  and  $\langle \partial_\beta P_j^{\alpha, \beta}(x), P_k^{\alpha, \beta}(x) \rangle_{w^{\alpha, \beta}}$ .

The paper is layed out as follows. In a first step the relation of the above scalar products to the Gaussian hypergeometric function  $F$  is recalled. Next, we determine the parameter derivatives of  $F$  which can also be useful in various other contexts as already mentioned. Finally, the derivation formulas for the Jacobi polynomials are given supplemented by a discussion of different properties. The calculations are elementary and all proofs have been carried out by hand. For the generation of the appropriate hypotheses the package REDUCE eased the tedious work.

## 2 REQUIRED PROPERTIES OF THE GAUSSIAN HYPERGEOMETRIC FUNCTION

In order to make the paper self-contained and to fix notation we start with recalling some definitions to be found in references such as [AS64], [PBM90]. A general hypergeometric function  ${}_pF_q$  is defined by

$${}_pF_q(a_1, a_2, \dots, a_p; c_1, c_2, \dots, c_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(c_1)_k (c_2)_k \dots (c_q)_k} \frac{z^k}{k!} \quad (2.1)$$

$$a_i, z \in \mathbb{R}, c_i \in \mathbb{R} \setminus \mathbb{Z}^{\leq 0}$$

where the Pochhammer symbol is

$$(a)_k = a(a+1)(a+2)\dots(a+k-1) \quad (a)_0 = 1 \quad a \in \mathbb{R}, k \in \mathbb{N}$$

The Gaussian hypergeometric function  $F$  is the important special case that we will be concerned with:

$$F(a, b, c, z) = {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k \quad (2.2)$$

If  $a$  or  $b$  are equal to a non-positive integer, the sum in (2.2) (analogously in (2.1)) becomes finite. Thus we have for later use

$$F(-n, b, c, z) = \sum_{k=0}^n \frac{(-n)_k (b)_k}{(c)_k k!} z^k \quad n \in \mathbb{N}_0, b \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}^{\leq 0} \quad (2.3)$$

The Jacobi polynomials can be related to the Gauss hypergeometric function in different ways. The most useful equations for our purpose are [NU88]

$$P_n^{\alpha, \beta}(x) = \frac{(-1)^n \Gamma(n + \beta + 1)}{n! \Gamma(\beta + 1)} F(-n, n + \alpha + \beta + 1, \beta + 1, \frac{1+x}{2}) \quad (2.4)$$

$$P_n^{\alpha,\beta}(x) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} F(-n, n + \alpha + \beta + 1, \alpha + 1, \frac{1-x}{2}) \quad (2.5)$$

with  $\alpha, \beta > -1, n \in \mathbb{N}_0$  and usually  $x \in [-1, 1]$ . The reason is that  $\alpha$  in (2.4) and  $\beta$  in (2.5) appear only in the second argument of  $F$ . Consequently, for the parameter derivatives of  $P_k^{\alpha,\beta}$  we are led to compute  $\partial_b F(-n, b, c, z)$  with the appropriate substitutions. Let us take (2.4) and (2.5) as definitions of  $P_k^{\alpha,\beta}$  here. It is observed immediately that

$$F(0, b, c, z) = 1 \quad \Rightarrow \quad \partial_b F(0, b, c, z) = \partial_c F(0, b, c, z) = 0$$

and

$$P_0^{\alpha,\beta}(x) = 1 \quad \Rightarrow \quad \partial_\alpha P_0^{\alpha,\beta}(x) = \partial_\beta P_0^{\alpha,\beta}(x) = 0$$

In the sequel we can therefore restrict ourselves to  $n \in \mathbb{N}$  which avoids distinguishing between different cases.

Before continuing it should be noted, that a relation that expresses  $\partial_\alpha P_n^{\alpha,\beta}(x)$  in terms of  $\{P_k^{\alpha,\beta}(x)\}_{k=0,\dots,n}$  must exist, indeed. The reason is that  $P_n^{\alpha,\beta}(x)$  is a polynomial of degree  $n$  in  $x$  of which the coefficients depend on  $\alpha$  and  $\beta$ . Deriving with respect to the latter still yields a polynomial of degree  $n$  in  $x$  which can always be written in the basis  $\{P_k^{\alpha,\beta}(x)\}_{k=0,\dots,n}$ .

### 3 THE DERIVATIVE OF $F$ WITH RESPECT TO THE SECOND AND THIRD ARGUMENT

For the following we assume in view of the application that the first argument of  $F$  is a negative integer. Deriving  $F$  as given by (2.3) with respect to  $b$  results in

$$\partial_b F(-n, b, c, z) = \left( \sum_{m=0}^{n-1} \frac{1}{b+m} \right) F(-n, b, c, z) - \sum_{m=0}^{n-1} \frac{1}{b+m} \sum_{k=0}^m \frac{(-n)_k (b)_k}{(c)_k k!} z^k \quad (3.1)$$

Having in mind  $b = n + \alpha + \beta + 1$  with  $n \geq 1$  from (2.4), (2.5) we assume  $b > 0$  for (3.1) and the sequel. The second term on the right hand side of (3.1) is a polynomial of degree  $n - 1$  in  $z$ , so that it can be expressed as a linear combination of hypergeometric functions of degree  $0, \dots, n - 1$ :

$$\sum_{m=0}^{n-1} \frac{1}{b+m} \sum_{k=0}^m \frac{(-n)_k (b)_k}{(c)_k k!} z^k = \sum_{m=0}^{n-1} A^{(n,m)} F(-m, b, c, z)$$

Equating coefficients for  $z$  leads to the relation

$$(-n)_k \sum_{m=k}^{n-1} \frac{1}{b+m} = \sum_{m=k}^{n-1} A^{(n,m)} (-q)_k \quad k = 0, \dots, n - 1 \quad (3.2)$$

which is equivalent to the recurrence

$$A^{(n,k)} = \binom{n}{k} \sum_{m=k}^{n-1} \frac{1}{b+m} - \sum_{q=k+1}^{n-1} A^{(n,q)} \binom{q}{k}$$

Using the relation

$$\sum_{p=0}^{j-m} (-1)^p \binom{j}{p} \binom{j-p-1}{m-1} = (-1)^{j-m} \quad j \in \mathbb{N}, m = 1, \dots, j$$

it can be shown that

$$A^{(n,k)} = (-1)^{n-k} \binom{n}{k} \sum_{p=1}^{n-k} (-1)^p \binom{n-k-1}{p-1} \frac{1}{b+n-p} \quad (3.3)$$

The identity

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{z-j} = (-1)^n \frac{n!}{(z-n)_{n+1}} \quad n \in \mathbb{N}_0, z \in \mathbb{R} \setminus \{0, 1, \dots, n\} \quad (3.4)$$

which is verified apart finally leads to the

**THEOREM 1.** For  $n \in \mathbb{N}$ ,  $b \in \mathbb{R}^{>0}$ ,  $c \in \mathbb{R} \setminus \mathbb{Z}^{\leq 0}$ ,  $z \in \mathbb{R}$  the derivative of  $F$  with respect to the second argument is given by

$$\begin{aligned} \partial_b F(-n, b, c, z) &= \left( \sum_{k=0}^{n-1} \frac{1}{b+k} \right) F(-n, b, c, z) \\ &\quad - \sum_{m=0}^{n-1} \frac{n!}{m!(n-m)} \frac{1}{(b+m)_{n-m}} F(-m, b, c, z) \end{aligned} \quad (3.5)$$

**Remark:** The assumption on  $b$  can immediately be relaxed to  $b \in \mathbb{R} \setminus \{0, -1, \dots, -n+1\}$ . For negative integers similar expressions can be developed considering  $F$  as a polynomial in  $z$  as above. For  $b = 0$  this is impossible as  $F(-n, 0, c, z) = 0$  whereas  $\partial_b F(-n, b, c, z)|_{b=0}$  is not, in general.

**Remark:** Formula (3.5) is different with respect to what is found in literature. For  $\partial_\alpha({}_pF_q)$ , [PBM90] give a relation which leads after the appropriate substitutions to

$$\partial_b F(-n, b, c, z) = \sum_{k=0}^n \frac{(-n)_k (b)_k}{(c)_k k!} (\psi(b+k) - \psi(b)) z^k$$

This is merely a restatement of (3.1) since

$$\psi(b+k) - \psi(b) = \sum_{j=0}^{k-1} \frac{1}{b+j}$$

For the sake of completeness we now turn to the expression  $\partial_c F(-n, b, c, z)$ . The equality

$$\partial_c ( (c)_n F(-n, b, c, z) ) = \sum_{k=0}^{n-1} \frac{(c)_k n!}{(n-k)k!} F(-k, b, c, z) \quad (3.6)$$

can be established similarly to the previous one by using the definition of  $F$ , equating coefficients, and applying (3.2), (3.3), and (3.4). This yields

**THEOREM 2.** For  $n \in \mathbb{N}$ ,  $b \in \mathbb{R}$ ,  $c \in \mathbb{R} \setminus \mathbb{Z}^{\leq 0}$ ,  $z \in \mathbb{R}$  the derivative of  $F$  with respect to the third argument is given by

$$\begin{aligned} \partial_c F(-n, b, c, z) = & - \left( \sum_{k=0}^{n-1} \frac{1}{c+k} \right) F(-n, b, c, z) \\ & + \sum_{m=0}^{n-1} \frac{n!}{m!(n-m)} \frac{1}{(c+m)_{n-m}} F(-m, b, c, z) \end{aligned} \quad (3.7)$$

Astonishing enough, the coefficients in (3.7) differ from (3.5) only by sign if  $c$  is substituted by  $b$ .

**Remark:** Let us indicate that (3.6) can also be obtained from the relation

$$\partial_\alpha l_n^{\alpha, \rho} = \sum_{\nu=0}^{n-1} \frac{\rho^{n-\nu}}{n-\nu} l_\nu^{\alpha, \rho} \quad (3.8)$$

in [Wul90]. The functions  $l_n^{\alpha, \rho}$  are modified discrete Laguerre polynomials defined by

$$l_n^{\alpha, \rho}(s) = \frac{\rho^n}{n!} m_n^{1+\alpha, \rho}(s-1) = \frac{\rho^n}{n!} (1+\alpha)_n F(-n, 1-s, 1+\alpha, 1-\frac{1}{\rho}) \quad (3.9)$$

$n \in \mathbb{N}_0, s \in \mathbb{N}, 0 < \rho < 1, \alpha > -1$

where  $m_n^{\gamma, \mu}$  are the Meixner polynomials. Inserting (3.9) in (3.8) yields (3.6) [Koe94]. This connection illustrates the potential applications of the parameter derivatives of  $F$  to an easier development of relations involving discrete orthogonal polynomials.



## 4 APPLICATION TO JACOBI POLYNOMIALS

Let us now set  $b = n + B$  with  $B = \alpha + \beta + 1 > -1$  and use (2.4) to determine  $\partial_\alpha P_k^{\alpha, \beta}(x)$  from (3.5) with  $c = \beta + 1, z = (1 + x)/2$ . It is obvious that in

$$\partial_\alpha P_n^{\alpha, \beta}(x) = \frac{(-1)^n \Gamma(n + c)}{n! \Gamma(c)} \left( \left( \sum_{k=0}^{n-1} \frac{1}{B + n + k} \right) F(-n, n + B, c, z) \right. \quad (4.1)$$

$$\left. - \sum_{m=0}^{n-1} A^{(n, m)} F(-m, n + B, c, z) \right)$$

we cannot convert  $F(-m, n + B, c, z)$  directly back to an expression in terms of  $P_k^{\alpha, \beta}$  by (2.4) in case of  $n \neq m$ . For this purpose we prove an important relation treating integer shifts in the first two arguments of  $F$ :

LEMMA 1. *For  $n \in \mathbb{N}$ ,  $B > -1$ ,  $c \in \mathbb{R}^{\neq 0}$ ,  $z \in \mathbb{R}$  the coefficients*

$$\rho^{(n, j, p)} = (-1)^{j-p} (n - j)_{j-p} \binom{j}{p} \frac{(B + n)_p (B + 2p)}{(B + p)_{j+1}} \quad (4.2)$$

allow the decomposition

$$F(-j, n + B, c, z) = \sum_{p=0}^j \rho^{(n, j, p)} F(-p, p + B, c, z) \quad j = 0, \dots, n - 1 \quad (4.3)$$

*Proof.* Inserting (4.2) in (4.3), equating coefficients, and setting  $Q = j - m$  leads to

$$1 = (-1)^Q \sum_{q=0}^Q \binom{Q}{q} (k)_{Q-q} \frac{(B + Q + k)_q (B + 2q)}{(B + q)_{Q+1}} \quad (4.4)$$

which can be proved by double induction. First, (4.4) is verified for  $k = 1$  by induction on  $Q$ . Next, we assume that (4.4) holds for  $k = 1, \dots, K$  and  $Q \in \mathbb{N}_0$  (since  $j = 0, \dots, n - 1$ ,  $m = 0, \dots, j$ , but  $n \in \mathbb{N}$ ). It has then to be shown that the same holds for  $k = K + 1$  and all  $Q$ . The case  $Q = 0$  being verified the step from  $Q$  to  $Q + 1$  requires to show that

$$1 = \frac{(B + Q + K + 2)_{Q+1}}{(B + Q + 1)_{Q+1}} - \frac{Q+1}{B + Q + K + 1} Q! (-1)^Q \sum_{q=0}^Q \frac{(-1)^q}{q} (B + 2q)$$

$$\left( \frac{(K+1)_{Q-q}}{(Q-q)!} \frac{(B + Q + K + 1)_q}{(B + q)_{Q+1}} \right) \left( \frac{K}{Q+1-q} + 1 \right) \left( \frac{K}{B + Q + 1 + q} + 1 \right) \quad (4.5)$$

Application of the assumption for  $K + 1$  and  $Q$  leads to the appearance of a similar sum as in (4.5), but with  $K$  replaced by  $K - 1$ . Repeating this  $K$  times yields with  $Z = B + Q + 1 \in \mathbb{R}^{>0}$  and  $p = Q + 1 > 1$

$$(Z + K)(Z)_p - (Z + K)_{p+1} = -p(Z)_p \sum_{j=0}^K \frac{(Z + p + 1 + K - j)_j}{(z + K - j)_j}$$

This is true by the subsequent relation.

For  $z \in \mathbb{R}^{>0}$ ,  $k \in \mathbb{N}_0$ ,  $p \in \mathbb{N}$

$$\sum_{j=0}^k \frac{1}{(z + k - j)_{p+1}} = \frac{1}{p} \left( \frac{1}{(z)_p} - \frac{1}{(z + k + 1)_p} \right)$$

which is verified by induction on  $k$ .  $\square$

Relation (4.3) is now inserted into (4.1). After changing the order of summation we obtain a sum on  $A^{(n,m)} \rho^{(n,m,p)}$  which can be simplified by the following statement.

LEMMA 2.

$$\sum_{m=p}^{n-1} A^{(n,m)} \rho^{(n,m,p)} = (-1)^{n-p-1} \frac{n!}{p!} \frac{1}{n-p} \frac{B+2p}{B+n+p} \frac{1}{(B+p)_{n-p}} \quad (4.6)$$

$$n \in \mathbb{N}, p = 0, \dots, n-1, b \in \mathbb{R}^{>0}$$

*Proof.* Straightforward simplifications lead, with  $J = n - 1 - p$ , to prove

$$\sum_{j=0}^J (-1)^j \frac{1}{J+1-j} \binom{J}{j} (z+j)_J = (-1)^J \frac{(z+J+1)_J}{J+1} \quad z \in \mathbb{R}, J \in \mathbb{N}_0$$

which is again done by induction.  $\square$

Applying the above relations to (4.1) using the same procedure for  $\partial_\beta P_k^{\alpha,\beta}(x)$  we are now in the position to state our main theorem.

THEOREM 3. For  $\alpha, \beta > -1$ ,  $n \in \mathbb{N}_0$ , and  $x \in [-1, 1]$

$$\partial_\alpha P_n^{\alpha,\beta}(x) = \sum_{k=0}^n d_{n,k}^{\alpha,\beta} \frac{\Gamma(\beta + 1 + n)}{\Gamma(\beta + 1 + k)} P_k^{\alpha,\beta}(x) \quad (4.7)$$

$$\partial_\beta P_n^{\alpha,\beta}(x) = \sum_{k=0}^n d_{n,k}^{\alpha,\beta} \frac{\Gamma(\alpha + 1 + n)}{\Gamma(\alpha + 1 + k)} (-1)^{n-k} P_k^{\alpha,\beta}(x)$$

where

$$d_{n,k}^{\alpha,\beta} = \begin{cases} \sum_{m=0}^{n-1} \frac{1}{\alpha+\beta+1+n+m} & n \geq 1, k = n \\ \frac{1}{n-k} \frac{\alpha+\beta+1+2k}{\alpha+\beta+1+k+n} \frac{1}{(\alpha+\beta+1+k)_{n-k}} & n \geq 1, k = 0, \dots, n-1 \\ 0 & n = k = 0 \end{cases}$$

The abbreviation

$$\begin{aligned} f_n^{\alpha,\beta}(x) &= \partial_\alpha P_n^{\alpha,\beta}(x) \\ g_n^{\alpha,\beta}(x) &= \partial_\beta P_n^{\alpha,\beta}(x) \end{aligned} \quad (4.8)$$

will be used for convenience.

## 5 PROPERTIES OF THE GENERATED POLYNOMIALS

It has been stated before, and is obvious from (4.7), that  $f_n^{\alpha,\beta}$  and  $g_n^{\alpha,\beta}$  are polynomials of degree  $n$  in  $x$ .

The symmetry of the Jacobi polynomials with respect to  $\alpha$  and  $\beta$  [AS64]

$$P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x)$$

and the symmetry of the coefficients in (4.7) yield

$$g_n^{\alpha,\beta}(x) = (-1)^n f_n^{\alpha,\beta}(-x) \quad (5.1)$$

Hence, for the remainder we concentrate on the function  $f_n^{\alpha,\beta}$ , as  $g_n^{\alpha,\beta}(x)$  behaves analogously.

For illustration let us explicitly write down the expression for low degrees:

$$\begin{aligned} f_0^{\alpha,\beta}(x) &= 0 \\ f_1^{\alpha,\beta}(x) &= \frac{1}{2}x + \frac{1}{2} \\ f_2^{\alpha,\beta}(x) &= \frac{2\alpha + 2\beta + 7}{8}x^2 + \frac{2\alpha + 3}{4}x + \frac{2\alpha - 2\beta - 1}{8} \\ f_3^{\alpha,\beta}(x) &= \frac{3\alpha^2 + 6\alpha\beta + 30\alpha + 3\beta^2 + 30\beta + 74}{48}x^3 + \frac{3\alpha^2 + 2\alpha\beta + 18\alpha - \beta^2 + 20}{16}x^2 \\ &\quad + \frac{3\alpha^2 - 2\alpha\beta + 6\alpha - \beta^2 - 10\beta - 10}{16}x + \frac{3\alpha^2 - 6\alpha\beta - 6\alpha + 3\beta^2 - 16}{48} \end{aligned} \quad (5.2)$$

Figure 1 shows plots of  $f_n^{\alpha,\beta}(x)$  for  $n = 1, \dots, 5$  and different parameter values.

From (4.7) the scalar products mentioned in the introduction can now be identified.

COROLLARY 1. For  $f_n^{\alpha,\beta}$ ,  $g_n^{\alpha,\beta}$  defined by (4.8),  $j, n \in \mathbb{N}_0$

$$\langle f_n^{\alpha,\beta}(x), P_j^{\alpha,\beta}(x) \rangle_{w^{\alpha,\beta}} = \begin{cases} 0 & n \geq 1, j > n \\ \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1+n)\Gamma(\beta+1+n)}{(2n+\alpha+\beta+1)n!\Gamma(\alpha+\beta+1+n)} \sum_{m=0}^{n-1} \frac{1}{\alpha+\beta+1+n+m} & n \geq 1, j = n \\ \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1+j)\Gamma(\beta+1+n)}{j!(n-j)(\alpha+\beta+1+j+n)\Gamma(\alpha+\beta+1+n)} & n \geq 1, j < n \\ 0 & n = 0, j \geq 0 \end{cases}$$

and

$$\langle g_n^{\alpha,\beta}(x), P_j^{\alpha,\beta}(x) \rangle_{w^{\alpha,\beta}} = (-1)^{n-k} \langle f_n^{\beta,\alpha}(x), P_j^{\beta,\alpha}(x) \rangle_{w^{\beta,\alpha}} \quad (5.3)$$

*Proof.* Recall that the Jacobi polynomials are related to the weight

$$w^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$$

leading to the constants  $\gamma_k = \gamma_k^{\alpha,\beta}$  in the orthogonality relation (1.2) [AS64]

$$\gamma_k^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}}{2k + \alpha + \beta + 1} \frac{\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)}{k! \Gamma(k + \alpha + \beta + 1)} \quad k \in \mathbb{N}_0$$

Equation (5.3) directly follows from (5.1).  $\square$

Remark: It is also obvious that neither the scalar products nor the functions themselves exhibit a singularity for  $\alpha=0$  or  $\beta=0$  as do  $\partial_\alpha w^{\alpha,\beta}$  and  $\partial_\beta w^{\alpha,\beta}$ , respectively.

Let us now discuss the numerical evaluation of  $f_n$  at a given point  $x$ . Even if this can be accomplished by (4.7) it is often computationally more efficient to employ a recursion. In our case such a relation follows immediately from the recursion for the Jacobi polynomials, e.g. [AS64]

$$a_{1,k} P_{k+1}(x) = (a_{2,k} + x a_{3,k}) P_k(x) - a_{4,k} P_{k-1}(x) \quad k \in \mathbb{N}_0 \quad (5.4)$$

with  $P_{-1}(x) = 0$ ,  $P_0(x) = 1$  and

$$\begin{aligned} a_{1,k} &= 2(k+1)(\alpha + \beta + k + 1)(\alpha + \beta + 2k) \\ a_{2,k} &= (\alpha + \beta + 2k + 1)(\alpha^2 - \beta^2) \\ a_{3,k} &= (\alpha + \beta + 2k)(\alpha + \beta + 2k + 1)(\alpha + \beta + 2k + 2) \\ a_{4,k} &= 2(\alpha + k)(\beta + k)(\alpha + \beta + 2k + 2) \end{aligned} \quad (5.5)$$

The superscripts are dropped here for better readability. Deriving (5.4) with respect to  $\alpha$  yields

$$a_{1,k} f_{k+1}(x) = -(\partial_\alpha a_{1,k})P_{k+1}(x) + (\partial_\alpha a_{2,k} + x \partial_\alpha a_{3,k})P_k(x) - (\partial_\alpha a_{4,k})P_{k-1}(x) + (a_{2,k} + x a_{3,k})f_k(x) - a_{4,k} f_{k-1}(x) \quad (5.6)$$

so that first  $P_{k+1}(x)$  is determined from (5.4), subsequently  $f_{k+1}(x)$  from (5.6).

As an application of the above we discuss the values of  $f_n$  at the boundaries  $x = 1, x = -1$ . Let us first recall that

$$P_n^{\alpha,\beta}(-1) = \frac{\Gamma(n + \beta + 1)}{n!\Gamma(\beta + 1)}(-1)^n \quad (5.7)$$

$$P_n^{\alpha,\beta}(1) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \quad (5.8)$$

which is nicely deduced from (2.4), and (2.5), respectively. The plots reveal that

$$f_n^{\alpha,\beta}(-1) = 0 \quad (5.9)$$

With the help of (5.6) this is readily verified by assuming that  $f_k(-1) = f_{k-1}(-1) = 0$  (true for  $k = 1, 2, 3$ , see (5.2)) and inserting (5.5) and (5.7). An interesting consequence for the coefficients in (4.7) is

$$\sum_{k=0}^n d_{n,k}^{\alpha,\beta} \frac{\Gamma(\beta + 1 + n)}{\Gamma(\beta + 1)} \frac{(-1)^k}{k!} = \sum_{k=0}^n d_{n,k}^{\alpha,\beta} \frac{\Gamma(\alpha + 1 + n)}{\Gamma(\alpha + 1)} \frac{(-1)^k}{k!} = 0$$

Catching the value of  $f_n(1)$  at the right boundary is less obvious. First of all  $g_n(1) = \partial_\beta P_n(1) = 0$  due to the symmetry (5.1), so that

$$\partial_\alpha(\partial_\beta P_n(1)) = \partial_\beta f_n(1) = 0$$

showing that  $f_n(1)$  does not depend on  $\beta$ . Inserting (5.8) and (5.5) into (5.6) we get a polynomial of second degree in  $\beta$ . Because of the independency from  $\beta$  each of its coefficients has to equate to zero leading to three individual recurrence relations. The simplest one reads

$$\tilde{f}_{k+1}(1) = (\alpha)_k + (\alpha + 2k + 1)\tilde{f}_k(1) - k(\alpha + k)\tilde{f}_{k-1}(1) \quad k \in \mathbb{N}$$

with  $\tilde{f}_k = k! f_k$  being started with  $\tilde{f}_0(1) = 0$  and  $\tilde{f}_1(1) = 1$ , (5.2).

We terminate the discussion here, conscious that properties of  $f_n$  and  $g_n$  such as range, distribution of zeros, possible orthogonality with respect to some weight and others still remain to be investigated.

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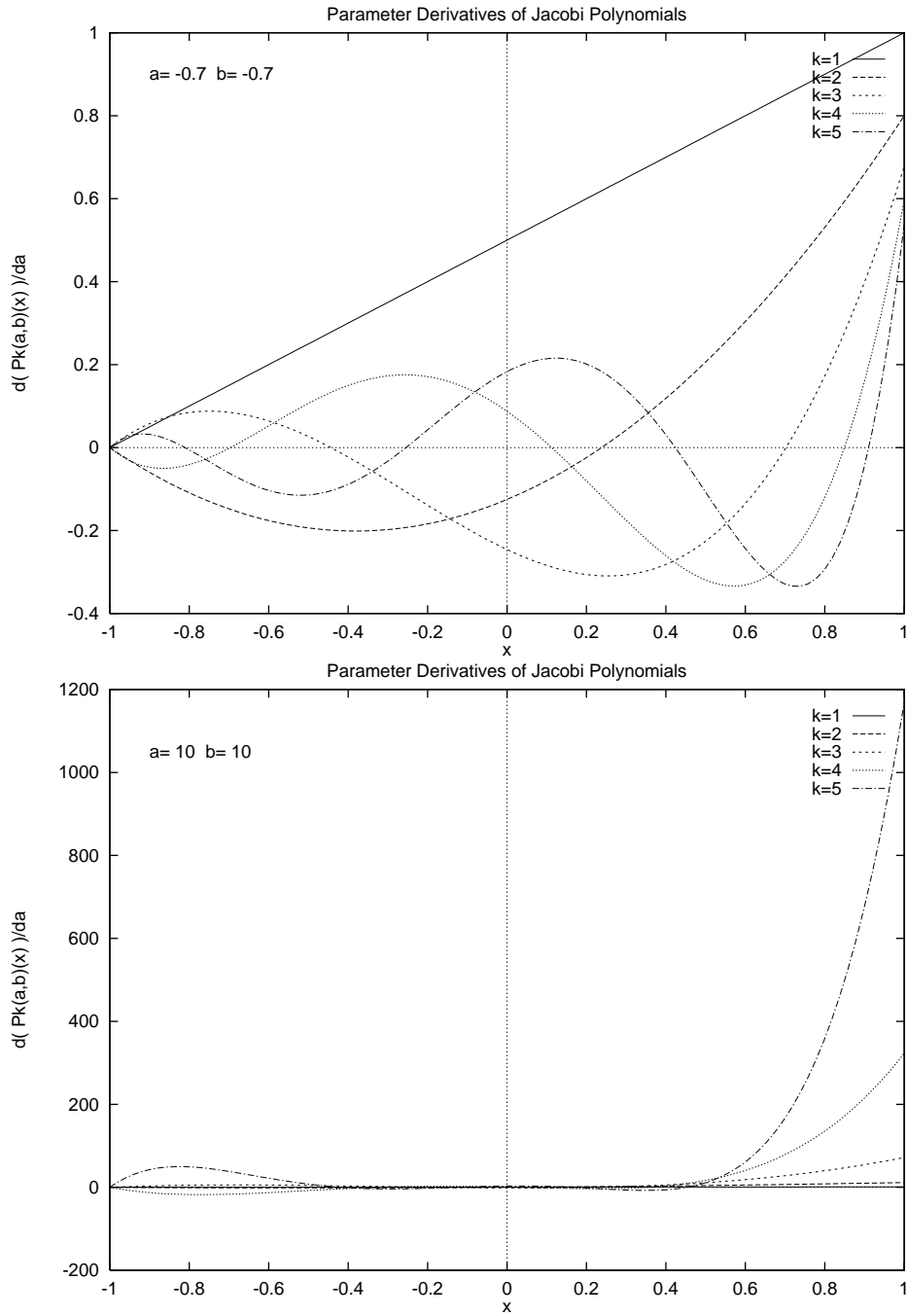


FIG. 1. Plots of  $f_k^{\alpha,\beta}(x) = \partial_\alpha P_k^{\alpha,\beta}(x)$  for  $\alpha=\beta=-0.7$  and  $\alpha=\beta=10$ .