

Numerical Continuation of Periodic Orbits with Symmetry

Claudia Wulff ^{*}, Andreas Hohmann [‡], Peter Deuffhard [‡]

^{*}Freie Universität Berlin, Institut für Mathematik I, Arnimallee 2–6,
D-14195 Berlin-Dahlem, Germany

[‡]Konrad-Zuse-Zentrum für Informationstechnik Berlin, Heilbronner Strasse 10,
D-10711 Berlin-Wilmersdorf, Germany

Abstract

We consider periodic orbits of autonomous parameter dependent ODE's. Starting from a shooting algorithm for the numerical computation of periodic orbits via an adaptive Poincaré-section we develop a pathfollowing algorithm for periodic solutions based on a tangential continuation method with implicit reparametrization. For ODE's equivariant w.r.t. a finite group we show that spatial as well as spatio-temporal symmetries of periodic orbits can be exploited within the (multiple) shooting context. We describe how turning points, period doubling bifurcations and Hopf points along the branch of periodic solutions can be handled. Furthermore equivariant Hopf points and generic secondary bifurcations of periodic orbits with \mathbf{Z}_m -symmetry are treated. We tested the code with standard examples, e.g., the period doubling cascade in the Lorenz equations. To show the efficiency of the described methods we also used the program for an application from electronics, a ring oscillator with n inverters. In this example the exploitation of symmetry reduces the amount of work for the continuation of periodic orbits from $\mathcal{O}(n^2)$ to $\mathcal{O}(n)$.

Contents

| | |
|-----------------------------------------------------|-----------|
| Introduction | 1 |
| 1 Computation of Single Periodic Orbits | 2 |
| 2 Continuation of Periodic Orbits | 5 |
| 3 Exploitation of Symmetry | 9 |
| 4 Computation of Bifurcations | 11 |
| 4.1 Generic Bifurcations Without Symmetry | 11 |
| 4.2 Symmetry-breaking Bifurcations | 15 |
| 5 Examples | 18 |
| References | 26 |

Introduction

Periodic processes play an important role in science and technology, e.g. in electronics, chemistry and also in biochemistry and biology. Often these processes are modelled by ordinary differential equations which depend on parameters. It is then interesting to examine the behaviour of the periodic processes in dependence of the parameters and to compute changes of the dynamics, i.e. bifurcations. In many cases the considered problems possess certain symmetries which can be exploited effectively during the computer simulation.

The aim of this paper is to compute periodic solutions of a parameter-dependent dynamical system

$$x' = f(x, \lambda), \quad f : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n, \quad \lambda \in \mathbf{R}$$

with autonomous right-hand-side f and to continue the periodic solutions w.r.t. the parameter λ . Thereby possible symmetries are to be exploited and bifurcations to be computed. To achieve this aim we first need an algorithm for the computation of single periodic orbits of ODE's $x' = f(x)$. For this we employ the multiple shooting algorithm presented in Deuffhard [3], which we briefly recall in Section 1.

Section 2 deals with the numerical continuation of periodic orbits. The tangential continuation method with implicit reparametrization for stationary solutions of parameter-dependent ODE's as described in Deuffhard, Fiedler, Kunkel [9] is extended to the periodic case.

Of particular interest for our investigations is the case when the underlying ODE is equivariant w.r.t. some symmetry group $\Gamma \subset O(n)$, i.e.

$$\gamma f(x) = f(\gamma x) \quad \forall \gamma \in \Gamma.$$

Section 3 is concerned with the exploitation of spatial and spatio-temporal symmetries of periodic orbits.

In Section 4 bifurcations are treated, first bifurcations without symmetry and then symmetry breaking bifurcations. Both, in Section 3 and in Section 4 numerical techniques are generalized which have been developed by Gatermann, Hohmann [7] for the exploitation of symmetry and the computation of symmetry breaking bifurcations of stationary solutions. A different method for the numerical continuation of periodic solutions with symmetry based on Fourier expansions has been presented by Dellnitz [1], [2].

Finally, in Section 5, examples are presented to illustrate the performance of the developed algorithmic tools.

The results and examples to be presented can be found in more details in the diploma thesis of Wulff [15].

1 Computation of Single Periodic Orbits

In this section we briefly recollect the algorithm for the computation of periodic orbits of autonomous ODE's

$$x' = f(x), \quad f : \mathbf{R}^n \rightarrow \mathbf{R}^n, \quad \Phi_t(\cdot) \text{ flow of } f, \quad (1.1)$$

which has been introduced in [3]. If $x(t) = \Phi_t(x_0)$ is a periodic solution of (1.1) then any time-shifted solution $x(t + t_0)$, $t_0 \in \mathbf{R}$, is also a periodic solution, because the system (1.1) is autonomous. All these solutions determine the same periodic orbit $C_{x(t)}$. In order to avoid this uninteresting non-uniqueness a well-known analytical technique is to fix a Poincaré-section S_{x_0} which is an $(n - 1)$ -dimensional affine hyperplane *transversal* to the periodic orbit $C_{x(t)}$ at the point x_0 . We will use the Poincaré-section orthogonal to the orbit

$$S_{x_0} = x_0 + S'_{x_0} \quad \text{where} \quad S'_{x_0} = \langle f(x_0) \rangle^\perp.$$

Then x_0 is a fixed point of the Poincaré-map (first return map) $\Pi_{x_0} : S_{x_0} \rightarrow S_{x_0}$. If $(D_x \Pi_{x_0}(x)|_{x=x_0} - I)$ is regular, the fixed point x_0 of Π_{x_0} is locally unique.

The most popular way of computing periodic orbits involves fixing the Poincaré-section in advance by prescribing the value of one component $x_i = \text{const}$ of the solution, so that the Poincaré-section is parallel to some coordinate-plane. The fixed point equation

$$\Pi_{x_0}(x) - x = 0,$$

which is an $(n - 1, n - 1)$ -system, is then solved by means of a Newton method. However, since the periodic solution is not known beforehand it is not really clear which plane to choose as a Poincaré-section. So it could and does happen that the periodic solution which has to be computed does not intersect the a priori fixed Poincaré-section. In this case the Newton method will not converge.

In [3] a different method for computing periodic orbits has been presented. It is based on an *adaptive Poincaré-section*, which is approximately *orthogonal* to the periodic orbit. A point x on the periodic solution together with its period T are computed. They satisfy the underdetermined equation

$$F(x, T) = \Phi_T(x) - x = 0, \quad F : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n.$$

The Jacobian $F'(x, T)$ of $F(x, T)$ is given by

$$F'(x, T) = [-I + D_x \Phi_T(x), f(\Phi_T(x))]. \quad (1.2)$$

The equation $F(y) = 0$, where now $y = (x, T)$, is solved by an underdetermined Gauss-Newton-method:

$$\begin{aligned}\Delta y^k &= -F'(y^k)^+ F(y^k), \\ y^{k+1} &= y^k + \Delta y^k.\end{aligned}$$

Here $F'(y^k)^+$ denotes the Moore-Penrose pseudo-inverse of $F'(y^k)$. Remember that for $A \in \text{Mat}(m, n)$, $m \leq n$, $\text{rank } A = m$, \mathcal{N} kernel of A , $x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $x = A^+ b$ is defined by

$$Ax = b, \quad x \perp \mathcal{N}.$$

The kernel vector t of $F'(y)$ at the solution point $y = (x, T)$ is the tangent $(f(x), 0)$ to the trajectory (disregarding normalization). Since for the kernel vectors $t^k = (t_x^k, t_T^k)$ of $F'(y^k)$ we have $t^k \rightarrow t$ as $k \rightarrow \infty$, the Gauss-Newton-iterates $x^{k+1} = x^k + \Delta x^k$ are in the adaptive Poincaré-section $x^k + \langle t_x^k \rangle^\perp \approx x + \langle f(x) \rangle^\perp$.

The Gauss-Newton-method converges to a solution if $F'(x, T)$ is regular in the solution point (x, T) , or equivalently, if $(D_z \Pi_x(z)|_{z=x} - I)$ is regular. This condition does not depend on the computed point x on the periodic orbit $C_{x(t)}$, since for all $x_0 \in C_{x(t)}$ the matrix $D_z \Pi_{x_0}(z)|_{z=x_0}$ has the same eigenvalues. Furthermore, we have the following uniqueness theorem.

Theorem 1 *Let $x(t)$ be a T -periodic solution and let $F'(x_0, T)$ be regular on some point x_0 on the periodic solution. Then there is a neighborhood of the periodic orbit $C_{x(t)}$ where there is no other periodic solution with period near T .*

Proof. The proof (see [15]) is based on the Gronwall Lemma.

In order to include unstable periodic solutions we use the just described algorithm in the *multiple* shooting context: we compute k points on a periodic orbit by solving the underdetermined equation

$$F(x_1, \dots, x_k, T) = 0, \quad F : \mathbf{R}^{kn} \times \mathbf{R} \rightarrow \mathbf{R}^{kn},$$

where $0 = s_1 < \dots < s_{k+1} = 1$ is a partition of the unit interval, $\Delta s_i = s_{i+1} - s_i$ for $i = 1, \dots, k$, and

$$F_i(x_1, \dots, x_k, T) = \begin{cases} \Phi_{\Delta s_i T}(x_i) - x_{i+1} & \text{for } i = 1, \dots, k-1, \\ \Phi_{\Delta s_k T}(x_k) - x_1 & \text{for } i = k. \end{cases}$$

The linear systems which arise in the Gauss-Newton-method are of the form $Jy = b$, where $y = (x, T) \in \mathbf{R}^{nk+1}$, $x = (x_1, \dots, x_k)$, $b = (b_1, \dots, b_k)$,

$$J = F'(x, T) = \begin{pmatrix} G_1 & -I & & & & g_1 \\ & G_2 & -I & & & g_2 \\ & & \ddots & \ddots & & \vdots \\ & & & G_{k-1} & -I & g_{k-1} \\ -I & & & & G_k & g_k \end{pmatrix}.$$

$G_i = D_x \Phi_{\Delta s_i T}(x_i)$ are the Wronskian matrices and

$$g_i = D_T F_i(x, T) = D_T \Phi_{\Delta s_i T}(x_i) = \Delta s_i f(\Phi_{\Delta s_i T}(x_i))$$

are the derivatives w.r.t. the period T .

Let $J = [H, g]$, where H is an (nk, nk) -matrix, then we have

$$Jy = b \Leftrightarrow [H, g] \begin{pmatrix} x \\ T \end{pmatrix} = b \Leftrightarrow Hx = b - gT,$$

so we can use the well-known Gaussian block elimination for H , to solve these linear systems which yields the following algorithm:

a) Compute the condensed vectors

$$\begin{aligned} b_c &:= C(b) = b_k + G_k b_{k-1} + \dots + G_k \dots G_2 b_1, \\ g_c &:= C(g) = g_k + G_k g_{k-1} + \dots + G_k \dots G_2 g_1, \end{aligned}$$

and the condensed matrix $[E_c, g_c]$ mit $E_c := -I + G_k \dots G_1$.

b) Compute a solution of the condensed system $[E_c, g_c] \begin{pmatrix} x_1 \\ T \end{pmatrix} = b_c$, e.g. $\begin{pmatrix} x_1 \\ T \end{pmatrix} = [E_c, g_c]^+ b_c$, using QR-decomposition.

c) Compute x via the explicit recursion

$$x_i = G_{i-1} x_{i-1} - b_{i-1} + g_{i-1} T \quad \text{for } i = 2, \dots, k.$$

We have now obtained a solution $y = J^- b$ where J^- is an outer inverse of J . In order to use the Moore-Penrose-pseudo inverse of J we have to add one more step:

d) Compute the tangent $t = (t_x, t_T)$ of J , where $t_x = (t_{x_1}, t_{x_2}, \dots, t_{x_k})$. Starting from a tangent of the condensed system $\begin{pmatrix} t_{x_1} \\ t_T \end{pmatrix}$

$$[E_c, g_c] \begin{pmatrix} t_{x_1} \\ t_T \end{pmatrix} = 0, \quad \|(t_{x_1}, t_T)\| = 1,$$

we obtain a tangent t of the whole system by

$$t_i = G_{i-1}t_{i-1} + g_{i-1}t_T \quad \text{for } i = 2, \dots, k$$

and normalization.

In the end we project $y \rightarrow y - \langle y, t \rangle t$.

An easy computation shows that in a solution point we have

$$[E_c, g_c] = [-I + D_x \Phi_T(x_1), f(\Phi_T(x_1))], \quad (1.3)$$

so the condensed matrix $[E_c, g_c]$ equals the Jacobian of the single shooting approach (1.2) in the point x_1 which is closely related to the derivative of the Poincaré-map at x_1 . This relationship between Poincaré-map on the one hand and the shooting equations on the other hand will play an important role in the computation of bifurcations, see section 4.

The Jacobian J is regular iff $[E_c, g_c]$ is regular. In this case theorem 1 holds and the Gauss-Newton-method converges.

It is important to use adaptive methods for the computation of the flow $x \rightarrow \Phi_t(x)$ and the Wronskians $D_x \Phi_t(x)$ which are needed for the evaluation of F resp. F' ; we use extrapolation codes [4]. We have realized a global inexact Gauss-Newton method (see [6]), and in the local case, when we have good guess values x and T at hand, we can choose between ordinary, simplified and Quasi-Gauss-Newton-method, for details see [15]. The corresponding program package PERIOD is written in C.

2 Continuation of Periodic Orbits

In this section we show how the pathfollowing method for stationary solutions described in [9] can be extended to the case of periodic solutions. We are concerned with the parameter-dependent dynamical system

$$x' = f(x, \lambda), \quad f : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n, \quad \lambda \in \Lambda = \mathbf{R}. \quad (2.4)$$

We first consider stationary solutions

$$f(y) = 0, \quad f : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n, \quad y = (x, \lambda). \quad (2.5)$$

If $\bar{y} = (\bar{x}, \bar{\lambda})$ is a stationary solution and $f_y(y)$ is regular at \bar{y} then (2.5) locally defines a solution branch. We apply the tangential continuation method based on implicit reparametrization presented in [9] to compute this solution branch. By writing $y = (x, \lambda)$ we want to express that the parameter λ

does not play any extraordinary role so that turning points can be treated easily. The pathfollowing algorithm works as follows: if a solution \bar{y} is given a new guess point y is computed by setting $\hat{y} = \bar{y} + st(\bar{y})$ where $t(y)$ is the normalized kernel vector of $f_y(y)$ and s is a suitably chosen stepsize. Then an underdetermined Gauss-Newton-method as in section 1 is used for the iteration from the guess value \hat{y} back to the solution path. The stepsize control is described in [9].

In the case of periodic solutions we want to compute fixed points of the parameter dependent Poincaré-map $\Pi : S \times \mathbf{R} \rightarrow S$

$$\Pi(x, \lambda) = x \quad \Leftrightarrow \quad z(x, \lambda) := \Pi(x, \lambda) - x = 0. \quad (2.6)$$

Since the Poincaré section S is $(n - 1)$ -dimensional we have a parameter dependent nonlinear equation

$$z(x, \lambda) = 0, \quad z : \mathbf{R}^{n-1} \times \mathbf{R} \rightarrow \mathbf{R}^{n-1},$$

and we can in principle apply the above described continuation method to this equation. The continuation tangent in a solution point (x, λ) is simply the kernel vector $t_{z'}$ of $z'(x, \lambda)$. But in the numerical realization of this idea we want to use the adaptive Poincaré-section of the last paragraph. So we again introduce the period as a new variable and solve (in the single shooting approach) the underdetermined equation

$$F : \mathbf{R}^{n+2} \rightarrow \mathbf{R}^n, \quad F(x, T, \lambda) = \Phi_T(x, \lambda) - x = 0 \quad (2.7)$$

with a Gauss-Newton-procedure. The kernel $\mathcal{N}(F')$ of F' is this time two-dimensional. In a solution point (x, λ) one kernel vector of F' is the tangent to the periodic orbit

$$(f(x, \lambda), 0, 0) \in \mathcal{N}(F'(x, \lambda)).$$

We want to determine the continuation tangent t^{cont} in such a way that it corresponds to the theoretical tangent vector $t_{z'}$. First the continuation tangent has to be in the kernel of F' , and second the continuation tangent should lie in the Poincaré-section S . Since we choose the Poincaré-section orthogonal to the orbit, this leads to the conditions

$$t^{\text{cont}} \in \mathcal{N}(F'), \quad t^{\text{cont}} \perp (f(x, \lambda), 0, 0).$$

If the Jacobian in the solution point (x, T, λ)

$$F'(x, T, \lambda) = [E, g, p] := [D_x \Phi_T(x, \lambda) - I, f(x, \lambda), D_\lambda \Phi_T(x, \lambda)] \quad (2.8)$$

is regular then the Gauss-Newton-procedure converges. This condition does not depend on the computed point x on the periodic orbit $C_{x(t)}$ (for a proof see [15]) and if it is satisfied then we have the following uniqueness theorem (analogously to Theorem 1):

Theorem 2 *Let $(x(t), T, \lambda)$ be a periodic solution of (2.4) and let $F'(x_0, T, \lambda)$ be regular on some point x_0 on the periodic orbit $C_{x(t)}$. Then the branch of periodic solutions given by (2.6) is locally unique in the following sense: there is an $\epsilon > 0$ such that every periodic solution $(\tilde{x}(t), \tilde{T}, \tilde{\lambda})$ with*

$$\text{dist}(C_{\tilde{x}(t)}, C_{x(t)}) < \epsilon, \quad |T - \tilde{T}| < \epsilon, \quad |\tilde{\lambda} - \lambda| < \epsilon$$

lies on the path of periodic solutions defined by (2.6) .

In the multiple shooting approach we solve the parameter dependent equation

$$F(x_1, \dots, x_k, T, \lambda) = 0, \quad F : \mathbf{R}^{nk} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}^{nk},$$

and nearly everything carries over from section 1, we just have one more column in the Jacobian consisting of the parameter derivatives

$$P_i = D_\lambda \Phi_{\Delta s_i T}(x_i, \lambda), \quad i = 1, \dots, k,$$

and so, to solve the linear equations $Jy = b$, $y = (x_1, \dots, x_k, T, \lambda)$, we have to compute an additional condensed vector (point a) in the Gaussian block elimination algorithm) namely

$$p^c := C(P) = P_k + G_k P_{k-1} + \dots + G_k \dots G_2 P_1,$$

the condensed matrix is of the form $[E_c, g_c, p_c]$, the recursion (point c)) has to be modified to

$$x_i = G_{i-1} x_{i-1} - b_{i-1} + g_{i-1} T + P_{i-1} \lambda \quad \text{for } i = 2, \dots, k,$$

and in point d) we compute an orthonormal basis of the 2-dimensional kernel of J and project the preliminary solutions $y = J^{-1}b$ onto the orthogonal complement of this kernel.

The corresponding program is PERCON (PERiodic CONTinuation).

As can be seen from the Gaussian block elimination, J has full rank, if the condensed matrix $[E_c, g_c, p_c]$ has full rank. A simple computation shows that the matrix $[E_c, g_c, p_c]$ equals the Jacobian of the single shooting approach (2.8). Thus the Gauss-Newton-method of the multiple shooting approach converges under the same conditions to a solution as the Gauss-Newton-method of the single-shooting method, namely if the assumptions of theorem 2 are satisfied.

Remark. For periodic solutions of Hamiltonian systems

$$\begin{aligned} p' &= -H_q(p, q) \\ q' &= H_p(p, q) \end{aligned} \tag{2.9}$$

the derivative of the Poincaré-map $D_x\Pi$ has an eigenvalue 1 because of conservation of energy, so that the Jacobian F' of the shooting method of the previous section is singular. More precisely, since $H(x) = H(\Phi_T(x))$ for all $(p, q) = x \in \mathbf{R}^n$, $T \in \mathbf{R}$, we have

$$\begin{aligned} D_{x_i}H(x) &= \sum_{j=1}^n D_{x_j}H(\Phi_T(x))D_{x_i}(\Phi_T(x))_j \\ \implies H_x(x) &= (D_x\Phi_T(x))^T H_x(\Phi_T(x)). \end{aligned}$$

Thus, in a periodic solution characterized by (x, T) we get

$$(D_x\Phi_T(x) - I)^T H_x(x) = 0.$$

Furthermore

$$D_T H(\Phi_T(x)) = 0 \implies \langle H_x(\Phi_T(x)), f(\Phi_T(x)) \rangle = 0,$$

therefore $F'(x, T)^T H_x(x) = 0$ in a solution point (x, T) of the single-shooting equation $F(x, T) = 0$, and $F'(x, T)$ is singular. Periodic orbits of Hamiltonian systems can be computed by adding dissipation so that (2.9) becomes a one-parameter family, see [5]:

$$\begin{aligned} p' &= -H_q(p, q) + \lambda H_p(p, q) \\ q' &= H_p(p, q) + \lambda H_q(p, q). \end{aligned}$$

For a solution $(p(t), q(t))$ we have

$$\begin{aligned} \frac{d}{dt}H(p(t), q(t)) &= H_p(p(t), q(t))p'(t) + H_q(p(t), q(t))q'(t) \\ &= \lambda(H_p(p(t), q(t))^2 + H_q(p(t), q(t))^2). \end{aligned}$$

So if $(p(t), q(t))$ is non-stationary then $H(p(t), q(t))$ is strictly monotone in t if $\lambda \neq 0$. Since for a periodic solution $(p(t), q(t), T, \lambda)$ we have $H(p(0), q(0)) = H(p(T), q(T))$ it follows that $\lambda = 0$. The Jacobian $F'(x, T, \lambda)|_{\lambda=0}$ given by (2.8) is no longer singular since in a solution point $(x, T, 0)$

$$\begin{aligned} \langle H_x(x), D_\lambda\Phi_T(x, \lambda)|_{\lambda=0} \rangle &= D_\lambda H(\Phi_T(x, \lambda))|_{\lambda=0} \\ &= \int_0^T (H_p(\Phi_t(x, 0))^2 + H_q(\Phi_t(x, 0))^2) dt \neq 0, \end{aligned}$$

and thus $F'(x, T, 0)^T H_x(x) = \langle H_x(x), D_\lambda\Phi_T(x, 0) \rangle \neq 0$. So by adding the artificial parameter λ in the equations (2.9) and using PERCON we can follow the branches of periodic orbits parametrized by energy which occur in Hamiltonian systems.

3 Exploitation of Symmetry

In this section we show how to exploit spatial and spatio-temporal symmetries of periodic solutions in the multiple shooting context.

Let G be a finite group and $\vartheta : G \rightarrow O(n)$ an orthogonal, faithful representation of G , that is, ϑ is a homomorphism of G into the group of orthogonal matrices $O(n)$ and ϑ is injective. Let $\Gamma = \vartheta(G)$ and let f be Γ -equivariant:

$$f(\gamma x) = \gamma f(x) \quad \forall x \in \mathbf{R}^n, \gamma \in \Gamma.$$

If $x(t)$ is a solution of the dynamical system then also $\gamma x(t)$ is a solution. The solution $\gamma x(t)$ is called conjugate to $x(t)$.

We first consider stationary solutions $x \in \mathbf{R}^n$ with $f(x) = 0$. $\gamma \in \Gamma$ is called a symmetry of x if $\gamma x = x$; the set of all symmetries of x (isotropy group of x) is given by $K = \{\gamma \in \Gamma \mid \gamma x = x\}$. It can be seen easily that the fixed point space of K

$$\text{Fix}(K) = \{x \in \mathbf{R}^n \mid \gamma x = x \forall \gamma \in K\}$$

is invariant under f , thus we can restrict the nonlinear equation $f(x) = 0$ onto the fixed point space $\text{Fix}(K)$ which has a lower dimension $n_{\text{red}} \leq n$, and so we obtain a symmetry-reduced system $f_{\text{red}} : \mathbf{R}^{n_{\text{red}}} \rightarrow \mathbf{R}^{n_{\text{red}}}$ which can be computed symbolically (see Gatermann, Hohmann [7]).

The spatial symmetries K of periodic solutions $x(t)$ are those group elements $\gamma \in \Gamma$ which leave each point on the periodic orbit invariant:

$$K := \Gamma_{x(t)} = \{\gamma \in \Gamma \mid \gamma x(t) = x(t) \forall t\}.$$

Since the flow Φ_t is also Γ -equivariant the set of spatial symmetries K of a periodic solution $x(t)$ does not depend on the time t . In addition to spatial symmetries there are also spatio-temporal symmetries which leave the periodic orbit $C := C_{x(t)}$ invariant as a whole but not pointwise. The spatio-temporal symmetries are given by

$$H := \{\gamma \in \Gamma \mid \gamma C = C\}.$$

Each $h \in H$ corresponds to a phase shift $\Theta(h)T$ of the T -periodic solution $x(t)$:

$$h \in H \Rightarrow hx(t) = x(t + \Theta(h)T), \text{ where } \Theta(h) \in \mathbf{R}/\mathbf{Z}.$$

Thereby $\Theta : H \rightarrow \mathbf{R}/\mathbf{Z}$ is a group homomorphism with the spatial symmetries K as kernel and

$$H/K \cong \mathbf{Z}_l, \quad l \in \mathbf{N},$$

see [11]. The spatial symmetries of periodic solutions can be exploited in the same way as in the case of stationary solutions, namely by restriction onto the fixed point space $\text{Fix}(K)$ of the spatial symmetries, i.e., by using a symmetry reduced system $f_{\text{red}} : \text{Fix}(K) \rightarrow \text{Fix}(K)$. The remaining spatio-temporal symmetries are a finite cyclic group $H = \mathbf{Z}_l$ and can be exploited in the following way: let $h \in H$ be that element in H that corresponds to the smallest possible non-zero phase-shift T/l :

$$x\left(t + \frac{T}{l}\right) = h x(t) \quad \forall t. \quad (3.10)$$

Then each point on the periodic orbit satisfies the underdetermined equation

$$F : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}^n, \quad F(x, T) = h^{-1} \Phi_{\frac{T}{l}}(x) - x = 0, \quad (3.11)$$

which can be solved by a Gauss-Newton-method. Thereby it suffices to compute the flow and the Wronskian matrix only up to time $\frac{T}{l}$ instead of T , which is a remarkable reduction of the computational cost in the case of high spatio-temporal symmetries.

The same idea was used in the Poincaré-map-context by Fiedler [5] in order to classify symmetry-breaking bifurcations of periodic orbits, see also section 4.2.

In the multiple-shooting-approach we solve the underdetermined equation

$$F(x_1, \dots, x_k, T) = 0, \quad F : \mathbf{R}^{kn} \times \mathbf{R} \rightarrow \mathbf{R}^{kn},$$

where $0 = s_1 < \dots < s_{k+1} = \frac{1}{l}$, $\Delta s_i = s_{i+1} - s_i$ for $i = 1, \dots, k$, and

$$F_i(x_1, \dots, x_k, T) = \begin{cases} \Phi_{\Delta s_i T}(x_i) - x_{i+1} & \text{for } i = 1, \dots, k-1, \\ h^{-1} \Phi_{\Delta s_k T}(x_k) - x_1 & \text{for } i = k. \end{cases}$$

The extension to parameter-dependent systems is trivial.

Remark. In the case of reversible dynamical systems there is an $R \in GL(n)$ with $R^2 = I$ and

$$f(Rx) = -Rf(x) \quad \forall x \in \mathbf{R}^n.$$

If $x(t)$ is a solution then also $Rx(-t)$ is a solution. A solution orbit C is called reversible if $RC = C$. A reversible periodic orbit $(x(t), T)$ satisfies (after an appropriate time-shift)

$$x(0), x(T/2) \in \text{Fix}(R),$$

where $\text{Fix}(R) = \{z \in \mathbf{R}^n \mid Rz = z\}$. Let $\dim \text{Fix}(R) = d$ and let (after some linear transformation) $\text{Fix}(R)$ be spanned by the first d unit vectors e_1, \dots, e_d .

Then reversible periodic orbits can be computed in the single shooting context by solving

$$F : \mathbf{R}^d \times \mathbf{R} \rightarrow \mathbf{R}^{n-d}, F(z_1, \dots, z_d, T) = \begin{pmatrix} (\Phi_{T/2}(z_1, \dots, z_d, 0, \dots, 0))_{d+1} \\ \vdots \\ (\Phi_{T/2}(z_1, \dots, z_d, 0, \dots, 0))_n \end{pmatrix} = 0.$$

If $p := 2d - n + 1 = 1$, that is, $n = 2d$, then we have a path of reversible periodic orbits which can be computed using PERCON.

The extension to the multiple shooting context is straightforward.

4 Computation of Bifurcations

In this section we show how generic bifurcations of periodic orbits on a branch of periodic solutions can be treated. This implies the detection and computation of bifurcation points and the computation of the start-off directions for the solution branches. In the case of symmetry we only have to follow non-conjugate branches. Moreover, we have to distinguish between two types of symmetry-breaking bifurcations: there are bifurcations which lead to a super group of the symmetry group of the original solution, which means, that the bifurcating solutions possess more symmetry, and we have symmetry-breaking bifurcations which lead to a subgroup of the symmetry group of the original solution.

We extend the techniques described in [7] for the numerical computation of symmetry-breaking bifurcations of stationary solutions to the case of periodic solutions. In particular, the symmetry-monitoring functions which change sign at the bifurcation point and are used for the detection of symmetry-breaking bifurcations are related to those used in [7]. We start with generic bifurcations without symmetry where the ODE is not assumed to be equivariant.

4.1 Generic Bifurcations Without Symmetry

First we consider turning-points of periodic orbits. Necessary for passing a turning point is a change of sign of the λ -component t_λ^{cont} of the continuation tangent t^{cont} of the periodic solution. If a turning point has been detected, it can be computed by Hermite interpolation exactly in the same way as in the case of stationary solutions, see [9].

We now deal with the Hopf bifurcation and the period doubling bifurcation which can be considered as symmetry-breaking bifurcations w.r.t. the group

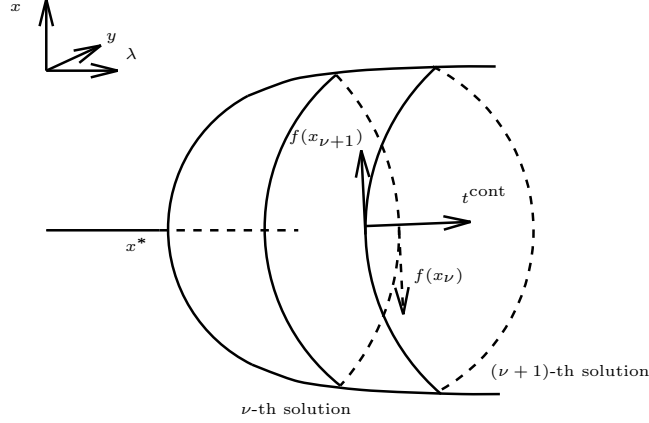


Figure 1: Hopf Bifurcation

S^1 which operates on a T -periodic solution $x(t)$ by

$$\theta x(t) = x(t + \theta T), \quad \theta \in S^1 = \mathbf{R}/\mathbf{Z}.$$

Hopf Bifurcation

A Hopf bifurcation point is a stationary solution (x^*, λ^*) , for which the Jacobian $f'(x^*, \lambda^*)$ has exactly one pair of purely imaginary eigenvalues $\pm\omega i$ and no other eigenvalues on the imaginary axis. Let $\mu(\lambda) \pm \rho(\lambda)i$ be the path of this pair of eigenvalues along the branch of stationary solutions such that $\mu(\lambda^*) = 0$ and $\rho(\lambda^*) = \omega$. Then we furthermore demand the transversality condition

$$\frac{\partial \mu}{\partial \lambda}(\lambda^*) \neq 0$$

to hold. Under these assumptions a unique branch of periodic solutions is emanating from the stationary solution with small amplitude and period $T \approx 2\pi/\omega$. This surface of periodic solutions is tangential to the real eigenspace N_ω of $\pm\omega i$ and generically agrees (after a smooth coordinate change) to second order with the paraboloid $\lambda - \lambda^* = C(x^2 + y^2)$ [Hopf, 1942].

We can consider the Hopf point (x^*, λ^*) as an S^1 -invariant $2\pi/\omega$ -periodic solution

$$x^*(t) \equiv x^* \quad \forall t \implies \theta x^*(t) = x^*(t) \quad \forall t.$$

Then the Hopf bifurcation is an S^1 -symmetry breaking bifurcation.

If a pathfollowing algorithm for periodic solutions runs through a Hopf point the continuation direction changes its sign and the same path of periodic orbits is computed again. Therefore Hopf points which occur during the

pathfollowing of periodic orbits should be detected so that the pathfollowing routine can be stopped at the Hopf point.

Let $y^{(\nu)}$ denote the ν -th computed solution point where $y = (x, T, \lambda)$, $x = (x_1, \dots, x_k)$. If there is a Hopf point between $y^{(\nu-1)}$ and $y^{(\nu)}$, the vectors $f(x_i)$ which are the infinitesimal generators of the S^1 -symmetry in the point x_i go through zero. Thus $f(x_i)$ is our symmetry monitoring function in this case. If the angle between the vectors $f(x_i)$ of the $(\nu - 1)$ -th and the ν -th computed solution point is greater than 90 degree, i.e., if

$$\frac{\langle f(x_i^\nu), f(x_i^{\nu-1}) \rangle}{\|f(x_i^\nu)\| \|f(x_i^{\nu-1})\|} < 0,$$

then a Hopf point between the $(\nu - 1)$ -th and the ν -th solution point can be detected.

If a Hopf point along a path of periodic orbits is detected, it can be computed via the extended system of Jepson [8] where mean values of the $(\nu - 1)$ -th and the ν -th solution point and their periods can be used as guess values for the Hopf point and its imaginary eigenvalue $\omega = 2\pi/T$.

If we are given a Hopf point together with its purely imaginary eigenvalue $i\omega$, we can use as a start-off direction for the emanating periodic orbits any vector t in the plane N_ω . Then we get starting values for the periodic solutions by setting

$$x_{\text{start}} = x^* + s t, \quad \lambda_{\text{start}} = \lambda^*, \quad T_{\text{start}} = 2\pi/\omega.$$

Thereby $s > 0$ is the initial stepsize. This method only works because we are using a continuation method with implicit reparametrization. In the case of fixed parametrization the method would compute the Hopf point again.

Period Doubling Bifurcation

A point (x^*, λ^*) characterizing a periodic solution with period T^* is a period doubling bifurcation point if the Jacobian of the Poincaré-map $D_x \Pi$ has a single eigenvalue -1 in (x^*, λ^*) with eigenvector y_0 and if this is the only eigenvalue on the complex unit circle. Furthermore the path $\mu(\lambda)$ of eigenvalues on the solution branch of $\Pi(x, \lambda) = x$ with $\mu(\lambda^*) = -1$ has to satisfy the transversality condition

$$\frac{\partial \mu}{\partial \lambda}(\lambda^*) \neq 0. \tag{4.12}$$

Under these assumptions (x^*, λ^*) is a pitchfork bifurcation point of the map $\mathcal{Z}(x, \lambda) = \Pi(\Pi(x, \lambda), \lambda) - x = 0$, see [10]. The vector $t = (y_0, 0)$ is the

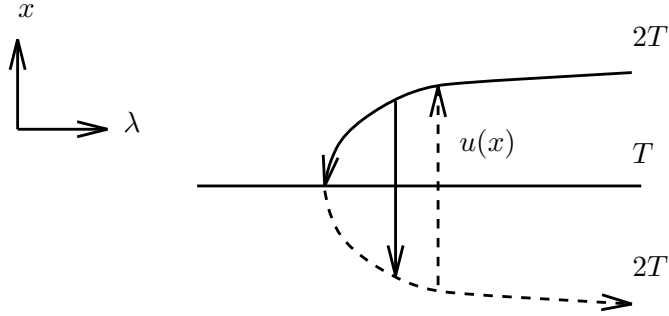


Figure 2: Detection of Period Halvings

tangent vector of the bifurcating branch in (x^*, λ^*) . The branching solutions have approximately twice the period of the original periodic solution. \mathcal{Z} is \mathbf{Z}_2 -equivariant where the nonlinear \mathbf{Z}_2 -action is given by the Poincaré-map $(x, \lambda) \rightarrow (\Pi(x), \lambda)$. So a period doubling bifurcation is a \mathbf{Z}_2 -symmetry-breaking bifurcation of \mathcal{Z} . If we consider the T -periodic solutions on the original branch as \tilde{T} -periodic, where $\tilde{T} := 2T$, the original branch has temporal \mathbf{Z}_2 -symmetry

$$\theta x(t) = x(t + \theta \tilde{T}) = x(t) \quad \text{for } \theta = 1/2,$$

and the branching solutions are not \mathbf{Z}_2 -symmetric.

Period doublings can be detected by a change of sign of $\det(D_x \Phi_T(x, \lambda) + I)$ which occurs due to the transversality condition (4.12). The matrix $D_x \Phi_T(x, \lambda)$ is computed in the single-shooting approach to obtain $F'(x, T)$ (see equation 1.2) and also in the multiple shooting approach in the computation of the condensed matrix (see equation 1.3).

We will now consider transitions through period doubling bifurcations from the branch of solutions with doubled period which we will call *period halvings*. They can be detected as follows: for a solution point $y = (x_1, \dots, x_k, T, \lambda)$ we compute

$$u(x) := \Phi_{\frac{T}{2}}(x_1, \lambda) - x_1.$$

One multiple-shooting-node is set to $s_j = 1/2$ so that no additional initial value problem has to be solved. If there is a period-halving-point between two consecutively computed periodic solutions $y^{(\nu)}$ and $y^{(\nu+1)}$ the vector u goes through zero. Therefore a period-halving point can be detected by the following condition on the angles:

$$\frac{\langle u(y^{(\nu)}), u(y^{(\nu+1)}) \rangle}{\|u(y^{(\nu)})\| \|u(y^{(\nu+1)})\|} < 0.$$

If a period doubling point has been detected it can be computed by use of linear interpolation and a Gauss-Newton-procedure to iterate back to the solution path (very similar to the computation of turning points). Alternatively also extended systems for x^* , T^* , λ^* and y_0 could be used.

We now come to the computation of the start-off directions. The continuation tangent for the T -periodic branch is just the usual continuation tangent. For the computation of the start-off direction of the $2T$ -periodic solutions we first consider the single shooting approach: we want the start-off direction to be orthogonal to the T^* -periodic orbit, so that it lies in the Poincaré-section. In the Poincaré-section $S \times \Lambda$ the tangent should be the vector $(y_0, 0)$ where y_0 is the eigenvector to the eigenvalue -1 , see above. This can be achieved by computing the normalized kernel vector \tilde{t} of $D_x \Phi_T(x, \lambda) + I$ and projecting it onto the orthogonal complement of the tangent $f(x^*)$ to the periodic orbit

$$t_x = \tilde{t} - \frac{\langle \tilde{t}, f(x^*) \rangle}{\langle f(x^*), f(x^*) \rangle} f(x^*).$$

Then $(t_x, 0, 0)$ is the start-off direction for the $2T$ -periodic solutions. In phase space the $2T$ -periodic solutions are approximately lying on a Moebius-band in the middle of which is the original T -periodic solution. The start-off direction is tangential to the Moebius-band and orthogonal to the original solution.

The adaption to the multiple-shooting-approach is straight forward.

4.2 Symmetry-breaking Bifurcations

In this section we deal with equivariant Hopf bifurcation and generic secondary bifurcations of periodic orbits with \mathbf{Z}_m -symmetry. The right hand side f of the ODE is again assumed to be Γ -equivariant under a finite group $\Gamma \subset O(n)$.

Equivariant Hopf Bifurcation

We are starting from a stationary solution (x^*, λ^*) , for which the Jacobian $f'(x^*, \lambda^*)$ has one pair $\pm \omega i$ of Γ -simple eigenvalues and no other eigenvalues on the imaginary axis (for a definition of a Γ -simple eigenvalue see [11]). As in the case of the standard Hopf bifurcation we assume for the path $\rho(\lambda) \pm \mu(\lambda)i$ of this pair of eigenvalues the transversality condition

$$\frac{\partial \mu}{\partial \lambda}(\lambda^*) \neq 0.$$

Furthermore we define the operation of $\Gamma \times S^1$ on a T -periodic solution $x(t)$ by

$$(\gamma, \theta)x(t) = \gamma^{-1}x(t + \theta T) \quad \text{for } (\gamma, \theta) \in \Gamma \times S^1,$$

and the operation of $\Gamma \times S^1$ on the real eigenspace N_ω of $\pm\omega i$ by

$$(\gamma, \theta)z = \gamma^{-1}e^{\theta J T}z, \quad (\gamma, \theta) \in \Gamma \times S^1, \quad z \in N_\omega. \quad (4.13)$$

Thereby $J : N_\omega \rightarrow N_\omega$ is the restriction of $f_x(x^*, \lambda^*)$ on N_ω and $T = 2\pi/\omega$.

Theorem 3 (*Equivariant Hopf Theorem*) *Let the above conditions be satisfied. If then for a subgroup $\Sigma \subset \Gamma \times S^1$ the fixed point space*

$$N_\omega^\Sigma := \{z \in N_\omega : (\sigma, \phi)z = z \forall (\sigma, \phi) \in \Sigma\}$$

satisfies the condition

$$\dim N_\omega^\Sigma = 2, \quad (4.14)$$

then there is a unique branch of small-amplitude periodic solutions with minimal periods T near $2\pi/\omega$ and Σ as symmetry group.

For a proof see [11].

The equivariant Hopf theorem provides the spatial and spatio-temporal symmetries Σ of the periodic orbits and the planes N_ω^Σ from which the start-off directions for the emanating periodic orbits can be chosen. Therefore we can apply the numerical methods of the standard Hopf bifurcation to treat also the equivariant Hopf bifurcation.

Bifurcations of Periodic Orbits with \mathbf{Z}_m -Symmetry

In this section we describe how the generic secondary bifurcations of periodic orbits with \mathbf{Z}_m -symmetry, which have been classified by Fiedler [5], can be treated numerically. We only deal with bifurcations of periodic orbits into other periodic orbits (not torus bifurcations).

We consider Γ -equivariant ODE's where $\Gamma = \vartheta(G)$ and $G = \mathbf{Z}_m$ is a finite cyclic group. We are starting from a periodic solution $x(t)$ with period T , spatial symmetry K , spatio-temporal symmetry H such that $H/K = \mathbf{Z}_l$, homomorphism Θ and $h \in H$ satisfying $\Theta(h) = \frac{1}{l}$. For $x \in C_{x(t)}$ define the Poincaré-section as usual by $S = x + \langle f(x) \rangle^\perp$.

To examine bifurcations of symmetric periodic orbits the reduced Poincaré-map

$$\Pi_{\text{red}} = h^{-1}\hat{\Pi}, \quad \hat{\Pi} : S \rightarrow hS$$

is used where $\hat{\Pi}$ maps points of S into the points where the positive semi-flow through x first hits hS . The fixed point equation $\Pi_{\text{red}}(x) = x$ is then

analogous to equation (3.11) which is used for the numerical computation of symmetric periodic orbits.

The relationship between Π and Π_{red} is given by

$$\Pi = h^l \Pi_{\text{red}}^l.$$

Using this relationship one can reduce the considered bifurcations of periodic orbits to bifurcations of the reduced Poincaré-map Π_{red} which arise from an eigenvalue ± 1 of the Jacobian $D_x \Pi_{\text{red}}$.

It can be seen that the reduced Poincaré-map is K -equivariant. Therefore we have to distinguish between bifurcations with and without breaking of the spatial symmetry.

In the second case the symmetry group K is acting trivially on the eigenspace to the eigenvalue 1 respectively -1 , in the first case it is generically acting as multiplication by -1 on the eigenspace of the critical eigenvalue (since these two irreducible representations are the only absolutely irreducible representations of cyclic groups and generically eigenspaces corresponding to real eigenvalues are absolutely irreducible representation spaces, see [11]).

Generic bifurcations of Π_{red} without breaking of spatial symmetry are turning points and period doublings (flips); generic bifurcations of Π_{red} with breaking of spatial symmetry are pitchfork bifurcations called flops (eigenvalue 1) and period doublings (eigenvalue -1) called flip-flops. These bifurcations of Π_{red} lead to turning points or to pitchfork bifurcations respectively period doubling bifurcations of Π depending on h and whether l is odd or even, for details see [5].

As an example we consider the flip pitchfork bifurcation. By restriction onto the fixed point space $\text{Fix}(K)$ the spatial symmetry can be exploited so that we can assume

$$K = \{\text{id}\}, \quad H = \mathbf{Z}_l.$$

Let l be even. If then Π_{red} undergoes a flip bifurcation which means that $D_x \Pi_{\text{red}}$ has an eigenvalue -1 we can use the relationship

$$\Pi = h^l \Pi_{\text{red}}^l = \Pi_{\text{red}}^l \quad \text{since } h^l = \text{id}$$

to conclude that the derivative of the Poincaré-map Π has an eigenvalue 1. The branching solution is a fixed point \tilde{x} of

$$\Pi_{\text{red}}^2(\tilde{x}) = \tilde{x}.$$

From this equation we get

$$\Pi(\tilde{x}) = (\Pi_{\text{red}}^2)^{l/2}(\tilde{x}) = \tilde{x},$$

so the branching solutions have approximately the same period but their symmetry $\tilde{H} = \mathbf{Z}_{l/2}$ has been halved, $\tilde{h} = h^2$, $\tilde{\Theta}(\tilde{h}) = 2/l$, and the Poincaré-map Π undergoes a pitchfork bifurcation.

Since all these secondary bifurcations of periodic orbits with underlying symmetry group \mathbf{Z}_m are generated by flips, flops or flip-flops of the reduced Poincaré-map they can be treated numerically with the methods for the period doubling bifurcation described in section 4.1, for details see [15].

5 Examples

In this section we illustrate the described methods by three examples two of which are the notorious test problems, whereas one is of real technical interest in electronics. All computations were done in ANSI-C and on a SUN Sparc Workstation IPX.

Example 1: Brusselator with 4 Cells

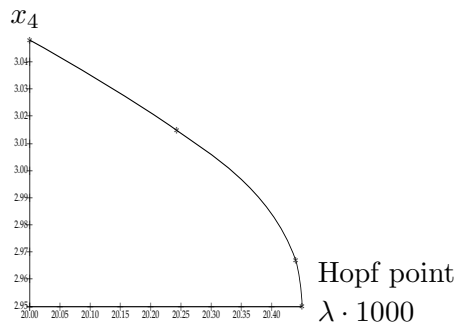


Figure 3: Pathfollowing of Σ -symmetric solutions

We consider four identical cells in which the same reaction takes place and which are coupled by diffusion. The equations for the 4-Cell-Brusselator are the following ($j = 1, 3, 5, 7$, $x_i := x_{i-8}$ for $i > 8$)

$$\begin{aligned} x'_j &= A - (B + 1)x_j + x_j^2 x_{j+1} + \lambda(-3x_j + x_{j+2} + x_{j+4} + x_{j+6}), \\ x'_{j+1} &= Bx_j - x_j^2 x_{j+1} + 10\lambda(-3x_{j+1} + x_{j+3} + x_{j+5} + x_{j+7}), \end{aligned}$$

where $A = 2.0$, $B = 5.9$. The problem is invariant w.r.t. permutations of the cells, thus the equations are S_4 -equivariant. We denote the elements of $\Gamma = S_4$ by γ_{ijkl} , e.g.,

$$\gamma_{3124} = \begin{pmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I. \end{pmatrix}, \quad I \in \text{Mat}(2, 2).$$

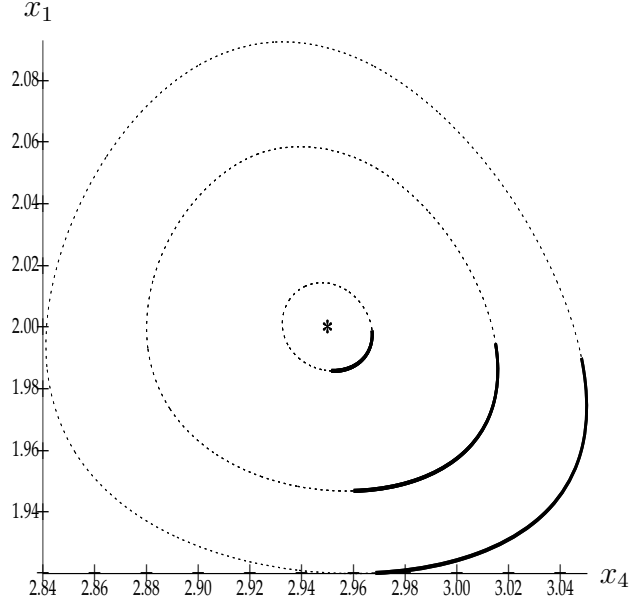


Figure 4: 4-cell-Brusselator: Σ -symmetric solutions

From [1] we took the following equivariant Hopf point

$$(x_0, \lambda_0) = (2.0, 2.95, 2.0, 2.95, 2.0, 2.95, 2.0, 2.95, 0.02045).$$

The corresponding imaginary eigenvalue ωi has the value $\omega = 0.62058$ and belongs to an absolutely irreducible 3-dimensional representation. The real eigenspace N_ω of ωi is of the form $N_\omega = W_1 \oplus W_2$ with

$$\begin{aligned} W_1 &= \{(y_1, 0, y_2, 0, y_3, 0, -y_1 - y_2 - y_3, 0) \mid y_1, y_2, y_3 \in \mathbf{R}\}, \\ W_2 &= \{(0, y_1, 0, y_2, 0, y_3, 0, -y_1 - y_2 - y_3) \mid y_1, y_2, y_3 \in \mathbf{R}\}. \end{aligned}$$

Using the equivariant Hopf Theorem we can now compute the symmetries and initial planes of the emanating periodic solutions. As an example we consider the symmetry group

$$\Sigma = \{\text{id}, (\gamma_{3142}, \frac{3}{4}), (\gamma_{4321}, \frac{1}{2}), (\gamma_{2413}, \frac{1}{4})\},$$

this means

$$H = \mathbf{Z}_4, K = \{id\}.$$

The result is shown in figure 3 and 4 (for computational details, e.g., the number of f-evaluations which was needed etc. see [15]).

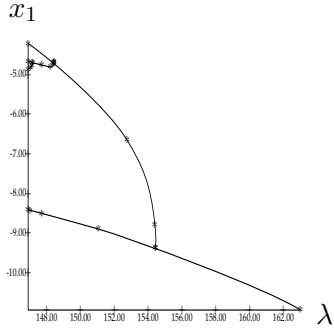


Figure 5: Flip bifurcations in the Lorenz equation

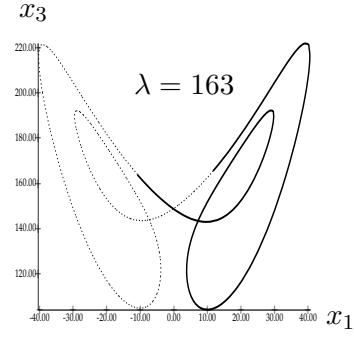


Figure 6: Symmetric solution of the Lorenz equation

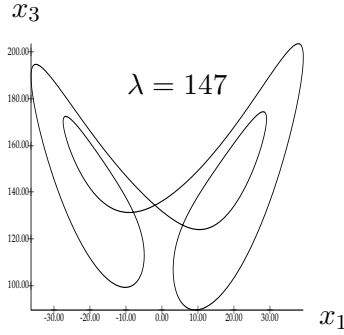


Figure 7: Unsymmetric solutions of the Lorenz equation

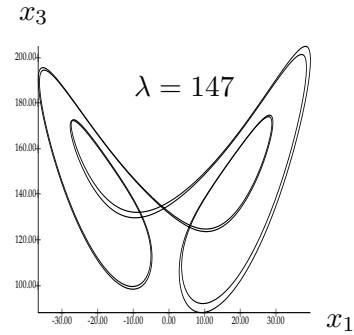


Figure 8: Lorenz equation, solution with doubled period

Example 2: Lorenz Equation

The Lorenz equations are given by

$$\begin{aligned} x_1' &= -\sigma x_1 + \sigma x_2, \\ x_2' &= -x_1 x_3 + \lambda x_1 - x_2 \\ x_3' &= x_1 x_2 - b x_3. \end{aligned}$$

The problem is \mathbf{Z}_2 -equivariant where the \mathbf{Z}_2 -action is given by $(x_1, x_2, x_3) \rightarrow (-x_1, -x_2, x_3)$. We want to compute the bifurcation diagram for the parameters $\sigma = 10$, $b = 8/3$ and λ between 147 and 163. We are starting from the symmetric solution

$$x_{\text{start}} = (-8.4168, 32.22, 145.26), \quad T_{\text{start}} = 1.2041, \quad \lambda_{\text{start}} = 147.$$

According to [14] there is a flip-pitchfork-bifurcation at $\lambda \approx 154$ and a flip-doubling-cascade at $\lambda \approx 147$. The symmetric solution is shown in figure 6, the unsymmetric periodic solution which bifurcates at $\lambda = 154.435$ is shown in figure 7. The solution with twice the period which bifurcates at $\lambda = 148.43$

is shown in figure 8. As can be seen from figure 5 where the bifurcation diagram is shown also a solution with fourth the period was computed. By shifting the left parameter boundary downwards we could compute periodic solutions up to the 32-nd of the original period.

Example 3: Electronic Ring Oscillator

In this section we are concerned with the simulation of electrical circuits consisting of n MOSFET-inverters (c.f. Kampowsky, Rentrop, Schmidt [12]). Such a circuit can be modelled in the following way: let U_i be the voltage at the i -th node, then using Kirchhoff's law we get the differential equations

$$f, \tilde{f} : \mathbf{R}^n \rightarrow \mathbf{R}^n, \quad U' = f(U), \quad f(U) = -C^{-1}\tilde{f}(U)$$

for the vector $U = (U_1, \dots, U_n)$ of the voltages. Thereby C is the capacity matrix

$$C = \begin{pmatrix} 2C_p + C & -C_p & & -C_p \\ -C_p & 2C_p + C & -C_p & \\ & \ddots & \ddots & \\ -C_p & & -C_p & 2C_p + C \end{pmatrix},$$

and the function f is given by

$$\tilde{f}(U) = \begin{pmatrix} 1/R(U_1 - U_{op}) + g(U_n, U_1, U_0) \\ 1/R(U_2 - U_{op}) + g(U_1, U_2, U_0) \\ \vdots \\ 1/R(U_n - U_{op}) + g(U_{n-1}, U_n, U_0) \end{pmatrix},$$

where

$$g(U_G, U_D, U_S) = K \max\{(U_G - U_S - U_T), 0\}^2 - K \max\{(U_G - U_D - U_T), 0\}^2.$$

K , U_T , U_0 , U_{op} , R and the capacities C and C_p are technical parameters, which have (after appropriate scaling) the values $K = 0.2$, $U_T = 1$, $U_0 = 0$, $U_{op} = 5$, $R = 5$, $C = 0.21$, $C_p = 5 \cdot 10^{-3}$. From [12] we got the initial values of a periodic solution, namely $U_1 = 4.2087$, $U_2 = 0.917$, $U_3 = 2.6534$, $U_4 = 4.2762$, $U_5 = 0.6928$, $T = 6.298$. This periodic solution is shown in figure 10. Furthermore there is a stationary solution given by $U_i(t) \equiv 2.56155$, $i = 1, \dots, 5$. Our aim is now the numerical simulation of large electrical circuits, i.e., the computation of periodic orbits for large n .

Obviously the function $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is \mathbf{Z}_n -equivariant where the generating element g of $\mathbf{Z}_n = \{\text{id}, g, \dots, g^{n-1}\}$ is acting on \mathbf{R}^n by shifting the

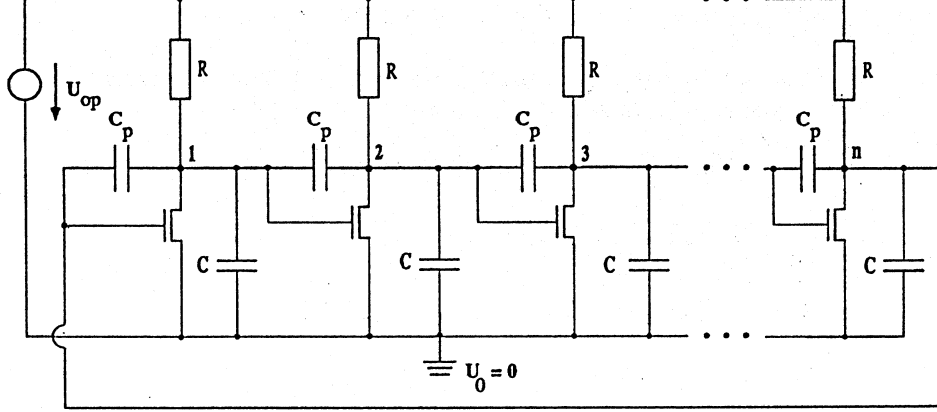


Figure 9: Circuit of ring oscillator

components to the right $(x_1, \dots, x_n) \rightarrow (x_2, \dots, x_n, x_1)$. This representation ϑ of \mathbf{Z}_n is called the regular representation of \mathbf{Z}_n , see [13]. Using PERIOD we checked that the above periodic solution (with $n = 5$) has spatio-temporal symmetry $H = \mathbf{Z}_5$. Since the stationary solution is \mathbf{Z}_n -invariant the periodic solution might have branched from the stationary solution via an equivariant Hopf bifurcation. So we introduce U_{op} as a continuation parameter and examine the stationary \mathbf{Z}_n -invariant solutions $U = (u, \dots, u) \in \mathbf{R}^n$, $u \in \mathbf{R}$, which are given by the quadratic equation

$$\begin{aligned} 1/R(u - U_{op}) + g(u, u, U_0) &= 0 \\ \iff 1/R(u - U_{op}) + K(\max(u - U_T, 0))^2 &= 0. \end{aligned}$$

In $U = (u, \dots, u)$ the Jacobian $f_U(U)$ is \mathbf{Z}_n -symmetric:

$$\gamma f_U(U) = f_U(U) \gamma \quad \forall \gamma \in \Gamma = \vartheta(\mathbf{Z}_n). \quad (5.15)$$

From elementary representation theory it is known that in a regular representation each irreducible representation ϑ_i is contained n_i -times, where n_i is the dimension of the representation ϑ_i (see [13]). Finite cyclic groups \mathbf{Z}_n only have one-dimensional irreducible representations (in complex vector spaces) which are defined by

$$\vartheta_j(g) = e^{\frac{2\pi i j}{n}}, \quad j = 0, \dots, n-1 \quad (5.16)$$

(see [13]). Let W_j denote the irreducible subspace of \mathbf{R}^n belonging to ϑ_j . Then W_j , $j = 0, \dots, n-1$, is spanned by the vector

$$v_j = \left(1, e^{\frac{2\pi i j}{n}}, e^{\frac{2\pi i 2j}{n}}, \dots, e^{\frac{2\pi i (n-1)j}{n}}\right).$$

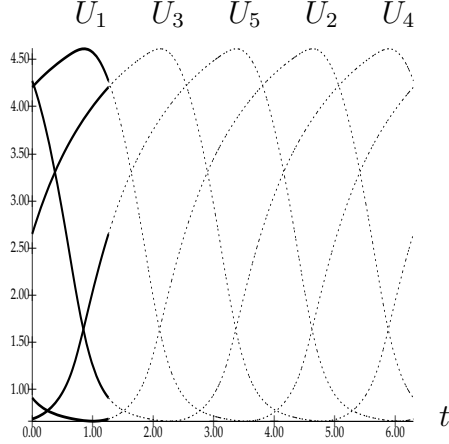


Figure 10: Periodic solution for 5 inverters

Using (5.15) we conclude that the Jacobian $f_U(U)$ can be diagonalised. Thereby v_j is an eigenvector to the eigenvalue

$$\lambda_j = \frac{1/R + 2K \max(u - U_T, 0) e^{\frac{2\pi i(n-1)j}{n}}}{C + (2 - 2 \cos \frac{2\pi j}{n}) C_p}, \quad (5.17)$$

as can be computed easily. Setting $\Re(\lambda_j) = 0$ we get simple expressions for the stationary solution $u = u^{(j)}$ and the corresponding parameter $U_{op}^{(j)}$ for which an equivariant Hopf bifurcation can occur, and also for the frequency ω_j of the bifurcating periodic solutions. Using the equivariant Hopf theorem we conclude that every bifurcating periodic solution has the symmetry $H = \mathbf{Z}_n$. But we also have to determine the homomorphism Θ . For this we have to know the operation of $\Gamma \times S^1$ auf N_ω , which is given by equation (4.13) from the equivariant Hopf theorem. For simplicity we only consider the one-dimensional complex eigenspace $W_j = \langle v_j \rangle$. $\Gamma \times S^1$ operates on W_j in the following way

$$(\gamma, \theta)v_j = \vartheta_j(\gamma)^{-1} e^{\lambda_j T \theta} v_j = \vartheta_j(\gamma)^{-1} e^{2\pi i \theta} v_j, \quad (5.18)$$

where $\gamma \in \Gamma$, $\theta \in S^1 \equiv \mathbf{R}/\mathbf{Z}$. We can characterize Θ by $\Theta(g)$. It is $(g, \Theta(g))v_j = v_j$, thus from (5.18) and (5.16) we get

$$\Theta(g) = j/n.$$

Let r be the g.c.d. of n and j , $n_{\text{red}} := n/r$. Then K is generated by $g^{n_{\text{red}}}$ and the fixed point space $\text{Fix}(K)$ contains all vectors in \mathbf{R}^n which consist of r equal sections of length n_{red} . Therefore n_{red} is the dimension of the reduced system.

Table I: Amount of work for the computation of the circuits depending on n

| | with symmetry | without symmetry |
|----------------------------------|---------------|------------------|
| flow $\Phi_t(x)$ | $O(n)$ | $C n$ |
| Wronskian matrix $D_x \Phi_t(x)$ | $O(n)$ | $C n^2$ |
| linear algebra | $O(n)$ | $O(n^3)$ |

After restriction onto the fixed point space $\text{Fix}(K) = \mathbf{R}^{n_{\text{red}}}$ we obtain $H = \mathbf{Z}_{n_{\text{red}}}$, $K = \{\text{id}\}$. Let $g \equiv gK$ be the generating element of $\mathbf{Z}_{n_{\text{red}}}$, then g is again acting on $\mathbf{R}^{n_{\text{red}}}$ by shifting the components to the right. The representation $\vartheta_{j_{\text{red}}}$ with $j_{\text{red}} := j/r$ is the irreducible representation of $\mathbf{Z}_{n_{\text{red}}}$ on $\mathbf{R}^{n_{\text{red}}}$ which corresponds to ϑ_j . Thus

$$E_{\text{red}} = \langle \Re(v_{j_{\text{red}}}), \Im(v_{j_{\text{red}}}) \rangle \subset \mathbf{R}^{n_{\text{red}}}$$

is the initial plane for the periodic solutions in the reduced coordinates. Finally we have to determine the element $h \in H$ with the smallest phase shift $\Theta(h)$. Let $h = g^k$, where k is a natural number between 1 and $n_{\text{red}} - 1$. Then h is given by the equation $\Theta(h) = 1/n_{\text{red}}$, i.e.,

$$\vartheta_{j_{\text{red}}}(h)^{-1} e^{2\pi i/n_{\text{red}}} = 1 \Leftrightarrow j_{\text{red}} k \equiv 1 \pmod{n_{\text{red}}}.$$

By the just described method we can compute initial values for periodic solutions for arbitrary n , and by exploiting symmetry we can reduce the amount of work effectively: if the considered periodic solution has large spatial symmetry then the ODE is reduced to a system with small dimension n_{red} . In this case the amount of work does not depend on the dimension n . For example, in the case $n = 1000$, there is a periodic solution with $K = \mathbf{Z}_{200}$, $n_{\text{red}} = 5$, which corresponds to the above computed solution with $n = 5$.

In the worst case the periodic solution does not have any spatial symmetry, but only spatio-temporal symmetry. In this case only the integration interval can be reduced by exploitation of symmetry. Let C be the number of integration steps for the full integration interval $[0, T]$. If the symmetry is not exploited, $C n$ multiplications are needed for the computation of the flow; exploiting the symmetry we can reduce the number of integration steps from C to C/n , since the integration interval is only $[0, T/n]$. Since the derivatives $f_x(x)$ have sparse-structure (they contain only $\mathcal{O}(n)$ non-zero components), $C n^2$ multiplications are needed for the computation of the Wronskian matrices if the symmetry is not exploited; by exploitation of symmetry the amount of work reduces to $\mathcal{O}(n)$. For the solution of the linear equations which arise

in the Gauss-Newton method $\mathcal{O}(n^3)$ multiplications are needed if the symmetry is not exploited. If the symmetry is exploited then for large n only few steps of integration are necessary for the computation of the Wronskian matrices, e.g., in the case $n = 100$ the integration interval has the order of 10^{-2} and only one integration step is needed. Therefore the Wronskian matrices are sparse, and the amount of work for the linear algebra reduces to $\mathcal{O}(n)$. Altogether we obtain a remarkable reduction of the computational cost by exploitation of symmetry, cf. table I.

Acknowledgement

The authors are pleased to thank Karin Gatermann for her support concerning the computation of symmetry-breaking bifurcations of periodic orbits and Prof. Peter Rentrop for making Example 3 available to us at an early stage of this work.

References

- [1] M. Dellnitz. A Computational Method and Pathfollowing for Periodic Solutions with Symmetry. In D. Roose, B. de Dier, A. Spence, editor, *Continuation and Bifurcations: Numerical Techniques and Applications*, volume 313, pages 153–167. NATO ASI SERIES C, 1990.
- [2] M. Dellnitz. Computational bifurcation of periodic solutions in systems with symmetry. *IMA Journal of Numerical Analysis*, 12:429–455, 1992.
- [3] P. Deuffhard. Computation of Periodic Solutions of Nonlinear ODE's. *BIT*, 24:456–466, 1984.
- [4] P. Deuffhard. Recent Progress in Extrapolation Methods for Ordinary Differential Equations. *SIAM Rev.*, 27:505–535, 1985.
- [5] B. Fiedler. *Global Bifurcation of Periodic Solutions with Symmetry*. Lecture Notes in Mathematics. Springer Verlag, Berlin, Heidelberg, New York, 1988.
- [6] A. Hohmann. *Inexact Gauss Newton Methods for Parameter Dependent Nonlinear Problems*. PhD thesis, Freie Universität Berlin, 1993.
- [7] K. Gatermann, A. Hohmann. Symbolic Exploitation of Symmetry in Numerical Pathfollowing. *IMPACT Comput. Sci. Engrg.*, 3:330–365, 1991.
- [8] A. D. Jepson. *Numerical Hopf Bifurcation*. PhD thesis, California Institute of Technology, 1981.
- [9] P. Deuffhard, B. Fiedler, P. Kunkel. Efficient Numerical Pathfollowing beyond Critical Points. *SIAM J. Numer. Anal.*, 18:949–987, 1987.
- [10] M. G. Crandall, P. H. Rabinowitz. Bifurcation from Simple Eigenvalues. *J. Func. Anal.*, 8:321–340, 1971.
- [11] M. Golubitsky, I. Stewart, D. Schaeffer. *Singularities and Groups in Bifurcation Theory*, volume 2. Springer-Verlag, Berlin, Heidelberg, New York, 1988.
- [12] W. Kampowsky, P. Rentrop, W. Schmidt. Classification and Numerical Simulation of Electric Circuits. *Surveys on Mathematics for Industry*, 2(1):23–65, 1992.

- [13] J.-P. Serre. *Linear Representations of Finite Groups*. Springer-Verlag, New York, Heidelberg, Berlin, 1977.
- [14] C. Sparrow. *The Lorenz Equations: Bifurcations, Chaos and Strange Attractors*. Springer Verlag, New York, 1982.
- [15] C. Wulff. Numerische Pfadverfolgung von periodischen Lösungen mit Symmetrie. Diplomarbeit, Freie Universität Berlin, 1993.