Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

STEPHEN S. ANCO [1], S. ALI [2], THOMAS WOLF [3]

Symmetry analysis and exact solutions of semilinear heat flow in multi-dimensions

[1]Department of Mathematics, Brock University, St.Catha rines, Canada
[2]School of Electrical Engineering and Computer Sciences, National University of Sciences and Technology, Islamabad, Pakistan
[3]Department of Mathematics, Brock University, St.Catharines, Canada an d ZIB Fellow

ZIB-Report 11-26 (June 2011)

SYMMETRY ANALYSIS AND EXACT SOLUTIONS OF SEMILINEAR HEAT FLOW IN MULTI-DIMENSIONS

STEPHEN C. ANCO, S. ALI, AND THOMAS WOLF

Abstract. A symmetry group method is used to obtain exact solutions for a semilinear radial heat equation in $n > 1$ dimensions with a general power nonlinearity. The method involves an ansatz technique to solve an equivalent first-order PDE system of similarity variables given by group foliations of this heat equation, using its admitted group of scaling symmetries. This technique yields explicit similarity solutions as well as other explicit solutions of a more general (non-similarity) form having interesting analytical behavior connected with blow up and dispersion. In contrast, standard similarity reduction of this heat equation gives a semilinear ODE that cannot be explicitly solved by familiar integration techniques such as point symmetry reduction or integrating factors.

1. Introduction

In the study of nonlinear partial differential equations (PDEs), similarity solutions are important for the understanding of asymptotic behaviour and attractors, critical dynamics, and blow-up behaviour. Such solutions are characterized by a scaling homogeneous form arising from invariance of a PDE under a point symmetry group of scaling transformations that act on the independent and dependent variables in the PDE [1, 2].

For scaling invariant PDEs that have only two independent variables, similarity solutions satisfy an ordinary differential equation (ODE) formulated in terms of the invariants of the scaling transformations. However, this ODE can often be very difficult to solve explicitly, and as a consequence, special ansatzes or ad hoc techniques may be necessary in order to obtain any solutions in an explicit form. The same difficulties occur more generally in trying to find explicit group-invariant solutions to nonlinear PDEs with other types of point symmetry groups.

An interesting example is the semilinear radial heat equation

$$u_t = u_{rr} + (n-1)r^{-1}u_r + k|u|^q u, \quad k = \pm 1 \tag{1}$$

for $u(t, r)$, which has the scaling symmetry group

$$t \to \lambda^2 t, \quad r \to \lambda r, \quad u \to \lambda^{-2/q} u, \tag{2}$$

where r denotes the radial coordinate in $n > 1$ dimensions. This equation describes radial heat flow with a nonlinear heat source/sink term depending on a power $q \neq 0$. The coefficient k of this term determines the stability of solutions to the initial-value problem. In particular, for $k = -1$ all smooth solutions $u(t, r)$ asymptotically approach a similarity form $u = t^{-1/q}U(r/\sqrt{t})$ exhibiting global dispersive behaviour $u \to 0$ as $t \to \infty$ for all $r \geq 0$, while for $k = 1$ some solutions $u(t, r)$ exhibit a blow-up behaviour $u \to \infty$ given by a similarity form $u = (T-t)^{-1/q}U(r/\sqrt{T-t})$ as $t \to T < \infty$ [3, 4]. In both cases, U satisfies a nonlinear ODE

$$U'' + ((n-1)\xi^{-1} - \tfrac{1}{2}k\xi)U' - \tfrac{1}{2}qkU + kU|U|^q = 0 \tag{3}$$

1991 *Mathematics Subject Classification.* 35K58;35C06;35A25;58J70;34C14.

Key words and phrases. semilinear heat equation, similarity reduction, exact solutions, group foliation, symmetry .

with

$$\xi = \begin{cases} r/\sqrt{t}, & k=-1 \\ r/\sqrt{T-t}, & k=1 \end{cases} \tag{4}$$

which arises from the scaling invariance (2). This ODE cannot be explicitly solved by standard integration techniques [2] such as symmetry reduction or integrating factors when $n \neq 1$ and $q \neq 0$. (More specifically, ODE (3) has no point symmetries $X = \eta(\xi,U)\partial/\partial U + \zeta(\xi,U)\partial/\partial\xi$ and no quadratic first integrals $\Psi = A(\xi,U) + B(\xi,U)U' + C(\xi,U)U'^2$, as established by solving the standard determining equations [1, 2] for X and Ψ.) As a result, few exact solutions $U(\xi)$ other than the explicit constant solution $U = (q/2)^{1/q}$ are apparently known to-date.

In this paper we will obtain explicit exact solutions for the heat equation (1) by applying an alternative similarity method developed in previous work [5] on finding exact solutions to a semilinear radial wave equation with a power nonlinearity. The method uses the group foliation equations [6] associated with a given point symmetry of a nonlinear PDE. These equations consist of an equivalent first-order PDE system whose independent and dependent variables are respectively given by the invariants and differential invariants of the point symmetry transformation. In the case of a PDE with power nonlinearities, the form of the resulting group-foliation system allows explicit solutions to be found by a systematic separation technique in terms of the group-invariant variables. Each solution of the system geometrically corresponds to an explicit one-parameter family of exact solutions of the original nonlinear PDE, such that the family is closed under the given symmetry group acting in the solution space of the PDE.

In Sec. 2, we set up the group foliation system given by the scaling symmetry (2) for the heat equation (1) and explain the separation technique that we use to find explicit solutions of this system. The resulting exact solutions of the heat equation are summarized in Sec. 3. These solutions include explicit similarity solutions as well as other solutions whose form is not scaling homogeneous, and we discuss their analytical features of interest pertaining to blow-up and dispersion. Finally, we make some concluding remarks in Sec. 4.

Related work using a similar method applied to nonlinear diffusion equations appears in Ref. [7]. Group foliation equations were first used successfully in Refs. [8, 9, 10] for obtaining exact solutions to nonlinear PDEs by a different method that is applicable when the group of point symmetries of a given PDE is infinite-dimensional, compared to the example of a finite-dimensional symmetry group considered both in Ref. [5] and in the present work.

2. Symmetries and group foliation

For the purpose of symmetry analysis and finding exact solutions, it is easier to work with a slightly modified form of the heat equation (1):

$$u_t = u_{rr} + (n-1)r^{-1}u_r + ku^{q+1}, \quad k = \text{const.}, \quad q \neq 0. \tag{5}$$

In $n > 1$ dimensions, this heat equation (5) admits only two point symmetries:

$$\text{time translation} \quad \mathbf{X}_{\text{trans.}} = \partial/\partial t \quad \text{for all } q, \tag{6}$$

$$\text{scaling} \quad \mathbf{X}_{\text{scal.}} = 2t\partial/\partial t + r\partial/\partial r - (2/q)u\partial/\partial u \quad \text{for all } q \neq 0, \tag{7}$$

where $\mathbf{X}_{.}$ is the infinitesimal generator of a one-parameter group of point transformations acting on (t, r, u). There are no special powers or dimensions for which any extra point symmetries exist for equation (5), as found by a direct analysis of the symmetry determining equations.

To proceed with setting up the group foliation equations using the scaling point symmetry, we first write down the invariants and differential invariants determined by the generator (7). The simplest invariants in terms of t, r, u are given by

$$x = t/r^2, \quad v = u/r^p, \tag{8}$$

satisfying $\mathbf{X}_{\text{scal.}}x = \mathbf{X}_{\text{scal.}}v = 0$ with

$$p = -2/q. \tag{9}$$

A convenient choice of differential invariants satisfying $\mathbf{X}^{(1)}_{\text{scal.}}G = \mathbf{X}^{(1)}_{\text{scal.}}H = 0$ for $G(t, r, u_t)$ and $H(t, r, u_r)$ consists of

$$G = r^{2-p}u_t, \quad H = r^{1-p}u_r, \tag{10}$$

where $\mathbf{X}^{(1)}_{\text{scal.}} = \mathbf{X}_{\text{scal.}} - (2+2/q)u_t\partial/\partial u_t - (1+2/q)u_r\partial/\partial u_r$ is the first-order prolongation of the generator (7). Here x and v are mutually independent, while G and H are related by equality of mixed r, t derivatives on u_t and u_r, which gives

$$D_r(r^{p-2}G) = D_t(r^{p-1}H) \tag{11}$$

where D_r, D_t denote total derivatives with respect to r, t. Furthermore, v, G, H are related through the heat equation (5) by

$$r^{p-2}G - D_r(r^{p-1}H) = r^{p-2}((n-1)H + kv^{q+1}). \tag{12}$$

Now we put $G = G(x, v)$, $H = H(x, v)$ into equations (11) and (12) and use equation (8) combined with the chain rule to arrive at a first-order PDE system

$$(p-2)G - pvG_v - 2xG_x - H_x + HG_v - GH_v = 0, \tag{13}$$

$$G - (p+n-2)H + pvH_v + 2xH_x - HH_v = kv^{q+1}, \tag{14}$$

with independent variables x, v, and dependent variables G, H. These PDEs are called the *scaling-group resolving system* for the heat equation (5).

The respective solution spaces of equation (5) and system (13)–(14) are related by a group-invariant mapping that is defined through the invariants (8) and differential invariants (10). In particular, the map $(G, H) \to u$ is given by integration of a consistent pair of parametric first-order ODEs

$$u_t = r^{p-2}G(t/r^2, u/r^p), \quad u_r = r^{p-1}H(t/r^2, u/r^p) \tag{15}$$

whose general solution will involve a single arbitrary constant. The inverse map $u \to (G, H)$ can be derived in the same way as shown in Ref. [5] for the wave equation, which gives the following correspondence result.

Lemma 1. *Solutions $(G(x, v), H(x, v))$ of the scaling-group resolving system (13)–(14) are in one-to-one correspondence with one-parameter families of solutions $u(t, r, c)$ of the heat equation (5) satisfying the scaling-invariance property*

$$\lambda^{-p}u(\lambda^2 t, \lambda r, c) = u(t, r, \tilde{c}(\lambda, c)) \tag{16}$$

where $\tilde{c}(1, c) = c$ in terms of an arbitrary constant c.

This correspondence leads to an explicit characterization of similarity solutions of the heat equation (5) in terms of a condition on solutions of the scaling-group resolving system (13)–(14). Consider any one-parameter family of solutions

$$u(t, r) = r^p v, \quad v = V(x, c), \tag{17}$$

having a scaling-homogeneous form, where

$$4x^2V'' - (1 + (2p+n-4)2x)V' + p(p+n-2)V + kV^{q+1} = 0 \tag{18}$$

is the ODE given by reduction of PDE (5). ¿From relation (10) we have

$$G(x, V(x,c)) = V'(x,c), \quad H(x, V(x,c)) = pV(x,c) - 2xV'(x,c). \tag{19}$$

Next we eliminate c in terms of x and v by using the implicit function theorem on $V(x,c) - v = 0$ to express $c = C(x,v)$. Substitution of this expression into equation (19) yields

$$H + 2xG = pv \tag{20}$$

where $G = V'(x, C(x,v))$, $H = pv - 2xV'(x, C(x,v))$ are some functions of x, v. The relation (20) is easily verified to satisfy PDE (13). In addition, PDE (14) simplifies to

$$-4x^2(G_x + GG_v) + (1 + (2p+n-4)2x)G = p(p+n-2)v + kv^{q+1}. \tag{21}$$

We then see that the characteristic ODEs for solving this first-order PDE are precisely

$$dv/dx = G, \quad -4x^2 dG/dx + (1 + (2p+n-4)2x)G = p(p+n-2)v + kv^{q+1}, \tag{22}$$

which are satisfied due to equations (18) and (19). Hence, we have established the following result.

Lemma 2. *There is a one-to-one correspondence between one-parameter families of similarity solutions (17) of heat equation (5) and solutions of the scaling-group resolving system (13)–(14) that satisfy the similarity relation (20).*

We now note that, under the mapping (15), static solutions $u(r)$ of the heat equation correspond to solutions of the scaling-group resolving system with $G = 0$. Consequently, hereafter we will be interested only in solutions such that $G \neq 0$, corresponding to dynamical solutions of the heat equation.

To find explicit solutions of the PDE system (13)–(14) for $G(x,v), H(x,v)$, we will exploit its following general features. First, the power nonlinearity ku^{q+1} in the heat equation appears only as an inhomogeneous term kv^{q+1} in the PDE (14). Second, in both PDEs (13) and (14) the linear terms that involve v derivatives have the scaling homogeneous form vG_v and vH_v with respect to v. Third, the nonlinear terms in the homogeneous PDE (13) have the skew-symmetric form $HG_v - GH_v$, while HH_v is the only nonlinear term appearing in the non-homogeneous PDE (14). These features suggest that this PDE system can be expected to have solutions given by the separable power form

$$G = g_1(x)v^a + g_2(x)v, \quad H = h_1(x)v^a + h_2(x)v, \quad a \neq 1. \tag{23}$$

For such an ansatz, we readily see that the linear derivative terms G_x, H_x, vG_v, vH_v in each PDE (13) and (14) will contain the same powers v, v^a that appear in both G and H, and moreover the nonlinear term $HG_v - GH_v$ in the homogeneous PDE (13) will produce only the power v^a due to the identities $v^a(v)_v - v(v^a)_v = (a-1)v^a$ and $v(v)_v - v(v)_v = v^a(v^a)_v - v^a(v^a)_v = 0$. Similarly we see that the nonlinear term HH_v in the non-homogeneous PDE (14) will only yield the powers v, v^a, v^{2a-1}. Since we have $a \neq 1$ and $q \neq 0$, the inhomogeneous term kv^{q+1} must therefore balance one of the powers v^{2a-1} or v^a.

In the case when we balance $q + 1 = a$, the terms containing $v^a = v^{q+1}$ and v^{2a-1} in PDE (14) immediately yield

$$h_1 = 0, \quad g_1 = k. \tag{24}$$

Then the terms containing v^a in PDE (13) reduce to

$$h_2 = 0 \tag{25}$$

which leads to a simplification of the remaining terms in both PDEs, yielding

$$xg_2' = g_2 = 0. \tag{26}$$

Thus, for this case, the ansatz (23) gives a one-term solution for G with $H = 0$.

In the other case, balancing $2a - 1 = q + 1$, we get

$$a = 1 + \frac{q}{2}. \tag{27}$$

The terms containing v, v^a, $v^{2a-1} = v^{q+1}$ in the PDEs (13) and (14) then yield

$$h_1' + 2xg_1' + (1 - \frac{q}{2}h_2)g_1 + \frac{q}{2}h_1 g_2 = 0 \tag{28a}$$
$$h_2' + 2xg_2' + 2g_2 = 0 \tag{28b}$$
$$2xh_1' - (n - 1 + \frac{q+4}{2}h_2)h_1 + g_1 = 0 \tag{28c}$$
$$2xh_2' - {h_2}^2 - (n-2)h_2 + g_2 = 0 \tag{28d}$$
$$(1 + \frac{q}{2}){h_1}^2 + k = 0. \tag{28e}$$

Through equations (28e), (28c) and (28d), we obtain

$$h_1 = \pm\sqrt{\frac{-2k}{q+2}} \tag{29}$$
$$g_1 = \pm\sqrt{\frac{-2k}{q+2}}(n - 1 + \frac{q+4}{2}h_2) \tag{30}$$
$$g_2 = -2xh_2' + (n - 2 + h_2)h_2 \tag{31}$$

and thereby we find that equations (28a) and (28b) reduce to an overdetermined system of nonlinear ODEs

$$4x^2h_2' - 2x{h_2}^2 - (1 + 2(n-2)x)h_2 = c = \text{const.} \tag{32a}$$
$$4xh_2' - \frac{q(2+q)}{4}{h_2}^2 + 2h_2 + n - 1 = 0 \tag{32b}$$

where c is an arbitrary integration constant. The ODE system (32) can be solved by a systematic integrability analysis, which we have carried out using computer algebra (discussed in more detail in Sec. 2.1). The results of the analysis give three two-term solutions with $G \neq 0$.

Proposition 1. *For $n \neq 1$, ansatz (23) yields altogether four solutions of the scaling-group resolving system (13)–(14) with $G \neq 0$:*

$$G = kv^{q+1}, \quad H = 0, \quad q \neq -1 \tag{33}$$

$$G = \pm(4-n)\sqrt{\frac{-k(n-2)}{n-3}}v^{(n-3)/(n-2)}, \quad H = \frac{1}{4-n}G + (2-n)v, \tag{34}$$
$$q = \frac{2}{2-n} \neq -1, \quad n \neq 2,3,4$$

$$G = \frac{3v}{3x+1} \pm \frac{3\sqrt{k}}{2v}, \quad H = \frac{2}{3}G - \frac{v}{2}, \quad q = -4, \quad n = 5/2 \tag{35}$$

$$G = \frac{3v\left(1 \pm \sqrt{-2kv}\right)}{3x+1}, \quad H = \frac{1}{6}(3x+1)G - \frac{3x-1}{3x+1}v, \quad q = 2, \quad n = 5/2 \tag{36}$$

None of these solutions satisfy the similarity relation (20).

Motivated by the success of the ansatz (23), we now consider a more general three-term ansatz

$$G = g_1(x)v^a + g_2(x)v^b + g_3(x)v, \quad H = h_1(x)v^a + h_2(x)v^b + h_3(x)v \tag{37}$$

where

$$a \neq b, \quad a \neq 1, \quad b \neq 1. \tag{38}$$

This ansatz leads to a more complicated analysis compared to the previous two-term ansatz. Specifically, the homogeneous PDE (13) now contains the power v^{a+b-1} in addition to v, v^a, v^b, while the non-homogeneous PDE (14) contains the further powers $v^{2a-1}, v^{2b-1}, v^{q+1}$. We determine the exponents in these powers by a systematic examination of all possible balances.

Firstly, since $q \neq 0$ in PDE (14), v^{q+1} must balance one of v^a, v^{2a-1}, v^{a+b-1}. (Note, by the symmetry $a \leftrightarrow b$ in the ansatz (37), the other possibilities v^b, v^{2b-1} for balancing v^{q+1} are redundant.) Secondly, v^{2a-1} can balance only v^{q+1} or v^b due to conditions (38), and otherwise if v^{2a-1} is unbalanced then its coefficient $a{h_1}^2$ must vanish. Likewise v^{2b-1} can balance only v^{q+1} or v^a, and otherwise its coefficient $b{h_2}^2$ must vanish. In a similar way, either v^{a+b-1} balances v^{q+1} or v, and otherwise if v^{a+b-1} is unbalanced then its coefficient $(a+b)h_1h_2$ vanishes. Finally, in PDE (13), v^{a+b-1} can balance only v, and otherwise the factor $g_1h_2 - g_2h_1$ must vanish in the coefficient of v^{a+b-1}.

Several cases arise from examining all of these different possibilities. After eliminating all trivial cases that lead to $h_2 = g_2 = 0$ (whereby the ansatz (37) just reduces to the previously considered two-term case (23)), we find the following non-trivial cases to consider:

$$q = a = 2, \quad b = 0; \tag{39}$$
$$q = -3/2, \quad a = 0, \quad b = -1/2; \tag{40}$$
$$q = -2/3, \quad a = -b = -1/3. \tag{41}$$

For case (39), the PDEs (13) and (14) yield

$$h_1' + 2xg_1' + (1 - h_3)g_1 + h_1 g_3 = 0 \tag{42a}$$
$$2xh_1' - (n - 1 + 3h_3)h_1 + g_1 = 0 \tag{42b}$$
$$h_3' + 2xg_3' + 2g_2 h_1 - 2g_1 h_2 + 2g_3 = 0 \tag{42c}$$
$$2xh_3' - {h_3}^2 - 2h_1 h_2 + (2 - n)h_3 + g_3 = 0 \tag{42d}$$
$$h_2' + 2xg_2' + (3 + h_3)g_2 - g_3 h_2 = 0 \tag{42e}$$
$$2xh_2' - h_2 h_3 + (3 - n)h_2 + g_2 = 0 \tag{42f}$$
$$2{h_1}^2 + k = 0. \tag{42g}$$

¿From equations (42g), (42f), (42d), (42b), we have

$$h_1 = \pm\sqrt{\frac{-k}{2}} \tag{43}$$
$$g_2 = -2xh_2' + (n - 3 + h_3)h_2 \tag{44}$$
$$g_3 = -2xh_3' + (n - 2 + h_3)h_3 \pm \sqrt{-2k}h_2 \tag{45}$$
$$g_1 = \pm\sqrt{\frac{-k}{2}}(n - 1 + 3h_3) \tag{46}$$

and then equation (42a) gives

$$h_2 = \pm\frac{1}{\sqrt{-2k}}(4xh_3' - 2{h_3}^2 + 2h_3 + n - 1) \tag{47}$$

The remaining equations (42c) and (42e) become, respectively,

$$4x^2h_3'' - (12xh_3 + 2(n - 6)x + 1)h_3' + 4{h_3}^3 - 6{h_3}^2 + 2(3 - 2n)h_3 = 0 \tag{48a}$$
$$\begin{aligned} &4x^3h_3''' - x(4xh_3 + 2(n - 13)x + 1)h_3'' + (6x{h_3}^2 + (2(n - 12)x + 1)h_3 + 9(4 - n)x - 3/2)h_3' \\ &\quad - 12x^2{h_3'}^2 - \tfrac{1}{2}(2{h_3}^2 - 2h_3 + 1 - n)({h_3}^2 - 2h_3 + 5 - 2n) = 0 \end{aligned} \tag{48b}$$

which is an overdetermined system of two nonlinear ODEs for $h_3(x)$. We solve this system (48) by an integrability analysis using computer algebra. This yields one solution with $G = 0$, plus two solutions with $G \neq 0$ which are summarized in Proposition 2.

In a similar way, each of the cases (40) and (41) leads to an overdetermined system of four nonlinear ODEs for $h_2(x), h_3(x)$. For both cases the results of an integrability analysis yield only solutions with $G = 0$.

Thus we have the following result.

Proposition 2. *For $n \neq 1$, ansatz (37) yields two additional solutions of the scaling-group resolving system (13)–(14) with $G \neq 0$:*

$$G = \pm\frac{3}{4}\sqrt{-2k}\left(v \pm \frac{1}{\sqrt{-2k}}\right)^2, \quad H = \frac{2}{3}G + v \pm \frac{2}{\sqrt{-2k}}, \quad q = 2, \quad n = 5/2 \tag{49}$$
$$G = \pm\frac{15}{4}\sqrt{-2k}\left(v \mp \frac{1}{\sqrt{-2k}}\right)^2, \quad H = \frac{2}{15}G + v \mp \frac{2}{\sqrt{-2k}}, \quad q = 2, \quad n = 5/2 \tag{50}$$

Neither of these solutions satisfies the similarity relation (20).

2.1. **Computational remarks.**

The integrability analysis of the previous ODE systems is non-trivial due to the degree of nonlinearity of the ODEs and the algebraic complexity of the coefficients in addition to the appearance of parameters in each system.

For the first ODE system (32), the integrability analysis consists of the following main steps. We eliminate h_2' to get a single algebraic equation, which is quadratic in h_2. By differentiating this equation and using it to eliminate h_2' from either of the original ODEs in the system, we obtain a second algebraic equation, which is cubic in h_2. The coefficients in each algebraic equation are expressions in terms of the independent variable x and the parameters q, n. We next use cross-multiplication repeatedly to eliminate the highest-degree monomial terms in both of the algebraic equations until one equation no longer contains h_2 while the other equation is linear in h_2. At each algebraic elimination step, we must note that a case distinction will arise if the coefficient of a highest-degree monomial vanishes for some values of q or n. For each case, once the final algebraic equations have been obtained, we solve the equation without h_2 by splitting it with respect to x, which will yield conditions on the parameters q, n, and we then solve the linear equation for h_2 subject to these conditions (if any).

The integrability analysis for the second ODE system (48) is the same except that we must first use differentiation combined with cross-multiplication to eliminate h_3''' and h_3'', thus reducing the differential order of the system down to first-order, where the coefficients are expressions in terms of the independent variable x and the single parameter n. We may then proceed as before by using algebraic elimination to reduce this system to a linear equation that can be solved for h_3 and an equation that does not contain h_3 and thereby determines n.

A similar integrability analysis applies to the two other ODE systems, each of which requires solving four ODEs that contain two dependent variables h_2, h_3 and their derivatives up to second order, in addition to the independent variable x and the single parameter n.

Because of the complexity of the algebraic expressions and the number of case distinctions that arise in these analyses, it is very difficult for an automatic computer algebra program to fully classify and find all solutions. (For example, the Maple program RiffSimp running on a workstation for several days was unable to complete the full computation for any of the second, third, and fourth systems.)

To overcome these difficulties, we have used the interactive package Crack [11] which has a wide repertoire of techniques available, including eliminations, substitutions, integrations, length-shortening of equations, and factorizations, among others. Using Crack, the complete solution of the integrability analysis was obtained in about 50 interactive steps taking 2 seconds in total for the system (32), and about 400 interactive steps taking 7 minutes in total for the system (48), while less than 100 steps taking under 1 minute in total were needed for each of the other two systems [12].

3. Exact solutions

To obtain explicit solutions $u(t, r)$ of the heat equation (5) from solutions $(G(x, v), H(x, v))$ of its scaling-group resolving system (13)–(14), we integrate the corresponding pair of parametric first-order ODEs (10). The integration yields a one-parameter solution family $u(t, r, c)$ which is closed under the action of the group of scaling transformations (2).

Theorem 1. *The semilinear heat equation (5) has the following exact solutions arising from the explicit solutions of its scaling-group resolving system found in Propositions 1 and 2:*

$$u = (-kq(t+c))^{-1/q}, \quad q \neq 0 \tag{51}$$

$$u = \left(\pm\sqrt{\frac{-k}{(n-2)(n-3)}}\left(\frac{r}{2} - \frac{(n-4)(t+c)}{r}\right)\right)^{n-2}, \quad q = \frac{2}{2-n} \neq -1, \quad n \neq 2,3,4 \tag{52}$$

$$u = \left(\pm\sqrt{k}\left(1 + c(3t+r^2)\right)\left(\frac{3t}{r} + r\right)\right)^{1/2}, \quad q = -4, \quad n = 5/2 \tag{53}$$

$$u = \pm\frac{5(3t+r^2)}{(r(15t+r^2) + c\sqrt{r})\sqrt{-2k}}, \quad q = 2, \quad n = 5/2 \tag{54}$$

$$u = \pm\frac{3(t+c-r^2)}{r(3(t+c)+r^2)\sqrt{-2k}}, \quad q = 2, \quad n = 5/2 \tag{55}$$

$$u = \pm\frac{5(3(t+c)+r^2)}{r(15(t+c)+r^2)\sqrt{-2k}}, \quad q = 2, \quad n = 5/2 \tag{56}$$

where c is an arbitrary constant.

Modulo time-translations $t \to t - c$, solutions (51), (52), (55), (56) are similarity solutions since their form with parameter $c = 0$ is preserved under scaling transformations (2) on r, t, u. In contrast, the parameter c in solutions (53) and (54) cannot be removed by time-translations, and consequently the form of these solutions is not scaling-homogeneous since c gets scaled under the transformations (2) on r, t, u. Thus, with respect to the action of the full group of point symmetries generated by time-translations and scalings for the heat equation (5), the solutions (53) and (54) for $c \neq 0$ yield two-dimensional orbits of non-similarity solutions given by

$$u = \left(\pm\sqrt{k}\left(1 + c(3(t+\tilde{c})+r^2)\right)\left(\frac{3(t+\tilde{c})}{r} + r\right)\right)^{1/2}, \quad q = -4, \quad n = 5/2 \tag{57}$$

$$u = \pm\frac{5(3(t+\tilde{c})+r^2)}{(r(15(t+\tilde{c})+r^2) + c\sqrt{r})\sqrt{-2k}}, \quad q = 2, \quad n = 5/2 \tag{58}$$

whereas the solutions (51), (52), (55), (56) represent one-dimensional time-translation orbits of similarity solutions

$$u = (-kq)^{-1/q} r^{-2/q} (t/r^2)^{-1/q}, \quad q \neq 0 \tag{59}$$

$$u = \left(\frac{(n-2)(n-3)}{-4k}\right)^{1-n/2} r^{n-2}\left(\pm(1 - 2(n-4)t/r^2)\right)^{n-2}, \quad q = \frac{2}{2-n} \neq -1, \quad n \neq 2,3,4 \tag{60}$$

$$u = \pm(-2k)^{-1/2} r^{-1}\frac{3(t/r^2 - 1)}{3t/r^2 + 1}, \quad q = 2, \quad n = 5/2 \tag{61}$$

$$u = \pm(-2k)^{-1/2} r^{-1}\frac{5(3t/r^2 + 1)}{15t/r^2 + 1}, \quad q = 2, \quad n = 5/2. \tag{62}$$

Note that solution (56) is a special case of solution (58) given by $c = 0$. We remark that solution (58) previously has been obtained in work [13] on nonlinear diffusion equations through use of Bluman and Cole's nonclassical method.

Among all these solutions, the ones (51) and (52) that exist for integer values of n describe radial heat flow in $\mathbb{R}^n$. In section Sec. 3.1 we will discuss their analytical features related to blow-up and dispersion for $n \neq 1$. The remaining solutions (55), (56), (57), (58) that exist only for a non-integer value $n = 5/2$ have a different interpretation describing heat flow in the plane $\mathbb{R}^2$ with a point-source of radial heat flux at the origin, as we will show later in section Sec. 3.2. We will also discuss this interpretation for the solution (52) in the case of non-integer values of n.

Before proceeding, we observe that the heat equation (5) for all values of n can be written in the form of a gradient flow

$$u_t = -\delta E/\delta u \tag{63}$$

using the "energy" integral

$$E = \int_0^\infty \left(\frac{1}{2}u_r^2 - kf(u)\right) r^{n-1}dr \tag{64}$$

with

$$f(u) = \begin{cases} \dfrac{1}{q+2}u^{q+2}, & q \neq -2, \\ \ln|u|, & q = -2. \end{cases} \tag{65}$$

Here $\delta/\delta u$ denotes the usual variational derivative (i.e. Euler operator) with respect to u. This integral (64) obeys the equation

$$\frac{dE}{dt} = -\lim_{r\to 0}(r^{n-1}u_r u_t) - \int_0^\infty u_t^2 r^{n-1}dr \tag{66}$$

for any solution $u(t,r)$ with sufficient asymptotic decay for large r. If the "energy flux" $\lim\limits_{r\to 0}(r^{n-1}u_r u_t)$ of a solution $u(t,r)$ is non-negative then equation (66) shows that E is a decreasing function of t. As a result, for solutions $u(t,r)$ that also are non-negative, both terms in E will be non-negative if $k < 0$ and $q \neq -2$, so then E decreases to zero as $t \to \infty$. In this case $u(t,r)$ will have dispersive behaviour such that $u \to 0$ and $u_r \to 0$ for all $r \geq 0$ as $t \to \infty$. If instead $k > 0$ or $q = -2$, then the two terms in E will have opposite signs, in which case E may decrease without bound, allowing $u(t,r)$ to have blow-up behaviour such that $u \to \infty$ or $u_r \to \infty$ as $t \to T$ for some $T < \infty$.

3.1. **Behavior of solutions for $n = 2, 3$, etc.**

Similarity solution (51) is spatially homogeneous and has no restriction on the sign of k. It thus represents the general solution of the ODE $u_t = ku^{q+1}$ with an arbitrary nonlinearity power $q+1 \neq 1$. For $q > 0$ the behaviour of u is determined by the sign of k. In the case $k < 0$, $u = (|k|q)^{-1/q}/(t+c)^{1/q} \to 0$ is dispersive as $t \to \infty$, whereas in the case $k > 0$, $u = (kq)^{-1/q}/(T-t)^{1/q} \to \infty$ has a blow-up for $t \to T$ with $0 < T = -c < \infty$.

In contrast, similarity solution (52) is restricted to the special nonlinearity power $q+1 = (n-4)(n-2) \geq 1/3$ and requires $k < 0$ and $n \geq 5$. In this solution, for all $t > 0$, u is singular at $r = 0$ and is unbounded as $r \to \infty$. Interestingly, u vanishes on the space-time parabola given by $t + c = r^2/(2n-8) \geq 0$. As a consequence we can modify the solution to have better analytical behaviour by using this parabola (with parameter $c = 0$) as a cutoff such that

$$u(t,r) = \begin{cases} 0, & r \geq \alpha\sqrt{t} \\ \beta r^{2-n}\left(t - (r/\alpha)^2\right)^{n-2}, & 0 \leq r \leq \alpha\sqrt{t} \end{cases} \tag{67}$$

(see figure 1) with

$$\alpha = \sqrt{2(n-4)}, \quad \beta = \left(\frac{(n-2)(n-3)}{|k|(n-4)^2}\right)^{1-n/2}. \tag{68}$$

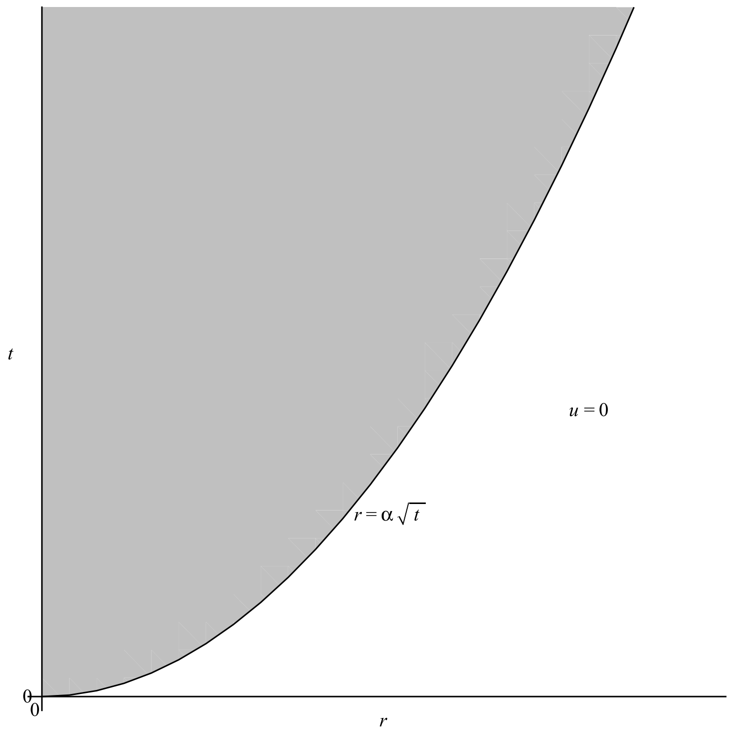

FIGURE 1. Space-time graph for solution (67).

At all space-time points away from $r = 0$, this modified similarity solution (67) is continuous and has continuous partial derivatives of order $n-3 \geq 2$, so thus it satisfies the heat equation (5) in the classical sense (i.e. u is at least C^1 in t and C^2 in r) on the spatial domain $r > 0$. At $r = 0$, u remains singular for all $t > 0$, but since $r^{n-1}u = \beta r t^{n-2} + O(r^2)$ is non-singular as $r \to 0$, u is n-dimensionally integrable on the whole spatial domain $r \geq 0$ (see figure 2).

A physical interpretation of similarity solution (67) can be seen by considering the radial heat flux equation

$$\frac{dH}{dt} = S + F \tag{69}$$

satisfied by the radial heat integral

$$H = \int_0^\infty u r^{n-1} dr \tag{70}$$

which is a measure of the total amount of heat in $u(t, r)$, where

$$F = \lim_{r\to 0}(-r^{n-1}u_r) \tag{71}$$

defines the outward radial heat flux at the origin, and

$$S = k\int_0^\infty u^{q+1} r^{n-1} dr \tag{72}$$

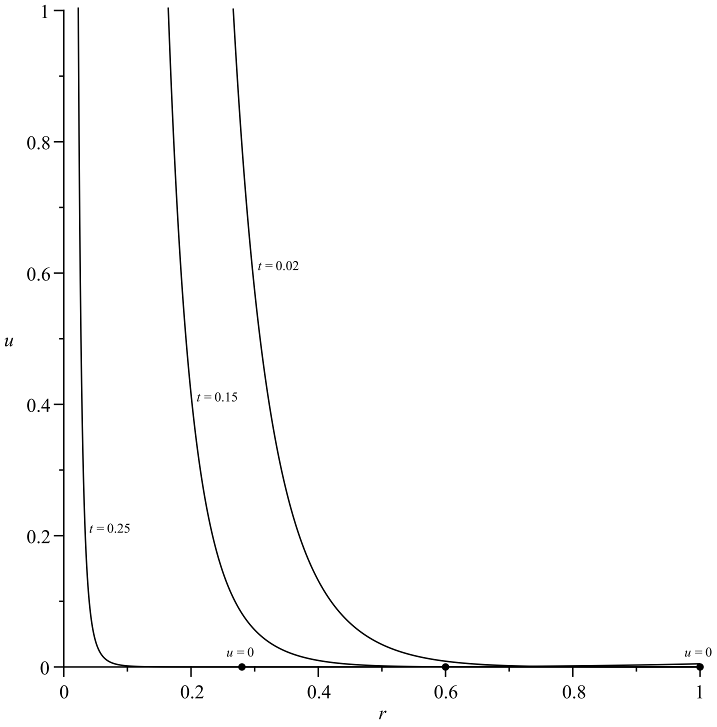

FIGURE 2. Graph of solution (67) with $n = 6$ ($q + 1 = 1/2$).

gives the net amount of heating or cooling produced by the nonlinear source/sink term in the heat equation (5). For the solution (67) these quantities are given by

$$\int_0^{\alpha\sqrt{t}} \beta \left(t - (r/\alpha)^2\right)^{n-2} r dr = \frac{n-4}{n-1}\beta t^{n-1} = H > 0, \tag{73}$$

$$\lim_{r\to 0} \left(-(2-n)\beta \left(t - (r/\alpha)^2\right)^{n-2} + O(r)\right) = (n-2)\beta t^{n-2} = F > 0, \tag{74}$$

$$\int_0^{\alpha\sqrt{t}} \beta^{(n-4)/(n-2)} \left(t - (r/\alpha)^2\right)^{n-4} r^3 dr = -2\beta t^{n-2} = S < 0. \tag{75}$$

Hence u has a positive amount of heat (73) that increases with t due to an increasing, positive radial outward heat flux at $r = 0$. Since $n \geq 5$, this flux (74) is greater than the net cooling (75) caused by the nonlinear sink term. Therefore this similarity solution (67) physically describes the dispersion of heat produced by an outward radial heat flux at the origin in $\mathbb{R}^n$, with some heat absorbed by a nonlinear heat sink proportional to $u^{(n-4)/(n-2)}$ at all points in $\mathbb{R}^n$. The dispersion has the behaviour of a radial temperature front at $r = \alpha\sqrt{t}$ that moves outward with speed $dr/dt = \alpha/(2\sqrt{t})$ for all $t > 0$.

Interestingly, for long times, u increases without bound in t at any spatial point $r > 0$ inside the temperature front. This non-dispersive temporal behaviour is related to u having an infinite "energy" (64), i.e. $E = +\infty$, so that the flux equation (66) is not well defined, allowing $u \to \infty$ as $t \to \infty$ despite the "energy" integral being formally positive due to $k < 0$.

3.2. **Behavior of solutions for** $n \neq 2, 3$, **etc.**

The heat equation (5) can be written in a different form

$$u_t = u_{rr} + (1-\nu)r^{-1}u_r + ku^{q+1}, \quad \nu = \text{const.}, \quad k = \text{const.} \tag{76}$$

in terms of a parameter $\nu = 2-n$ which applies to non-integer values of n. This equation (76) describes radial heat flow in $\mathbb{R}^2$ with an extra source/sink term given by $\nu u_r/r$ [14]. To interpret this term physically, we consider the 2-dimensional radial heat integral

$$H = \int_0^\infty u r dr \tag{77}$$

satisfying the radial flux equation

$$\frac{dH}{dt} = S + F + \nu \lim_{r\to 0} u \tag{78}$$

where

$$F = \lim_{r\to 0}(-ru_r) \tag{79}$$

defines the outward radial heat flux at the origin, and

$$S = k\int_0^\infty u^{q+1} r dr \tag{80}$$

gives the net amount of heating or cooling caused by the nonlinear source/sink term in the heat equation (76). The flux equation (78) shows that, for the respective cases $\nu > 0$ or $\nu < 0$, the term $-\nu u_r/r$ has the interpretation of a heating or cooling point-source at the origin in $\mathbb{R}^2$. Thus, solutions $u(t,r)$ will physically describe radial heat flow arising from a point source in a thin layer, with heat also produced or absorbed at all points in the layer due to a nonlinear source/sink term ku^{q+1}.

Note that the 2-dimensional heat equation (76) retains the form of a gradient flow (63) with $n = 2-\nu$.

Consider similarity solution (52) with $n = 2-\nu$:

$$u(t,r) = \beta r^\nu \left(\pm(\alpha(t+c)+r^2)\right)^{-\nu}, \quad q = -2/\nu, \quad \nu \neq 0,-1,-2 \tag{81}$$

where

$$\alpha = 2(\nu+2), \quad \beta = (-4\nu(\nu+1)/k)^{\nu/2}, \tag{82}$$

which requires $k > 0$ if $0 > \nu > -1$ and $k < 0$ if $\nu < -1$ or $\nu > 0$. The behaviour of this solution depends essentially on the separate signs of α and ν.

For $\alpha < 0$, (81) can be modified similarly to (67) by putting a cutoff on u at the space-time parabola where $u = 0$. Then

$$u(t,r) = \begin{cases} 0, & r \geq \sqrt{t/|\alpha|} \\ \beta r^\nu \left(|\alpha|t - r^2\right)^{-\nu}, & 0 \leq r \leq \sqrt{t/|\alpha|} \end{cases} \tag{83}$$

gives a classical solution of the heat equation (76) on the spatial domain $r > 0$ (i.e. u belongs to $C^1(\mathbb{R}^+)$ in t and $C^2(\mathbb{R}^+)$ in r), with $\nu < -2$ and $k < 0$. However, at $r = 0$, u is singular such that, for all $t > 0$, $H = +\infty$ and $F + \nu\lim_{r\to 0} u = 2(\nu/\gamma)t^{-\nu-1}\lim_{r\to 0} r^{\nu+1} = +\infty$ since $\nu + 1 < -1$. The modified similarity solution (83) therefore has the physical interpretation of heat dispersion produced by an infinite net outward radial heat flux at the origin in $\mathbb{R}^2$, with a radial temperature front located at $r = \sqrt{t/|\alpha|}$ for $t > 0$.

For $\alpha > 0$, (81) is smooth and positive on the spatial domain $r > 0$. It thus gives a $C^\infty(\mathbb{R}\times\mathbb{R}^+/\{0\})$ solution of the heat equation (76) with the asymptotic behaviour

$$u(t,r) = \beta r^\nu(\alpha t + r^2)^{-\nu} = \begin{cases} r^\nu(\beta\alpha^{-\nu}t^{-\nu} + O(r)), & r \to 0 \\ r^{-\nu}(\beta + O(1/r)), & r \to \infty \end{cases} \tag{84}$$

(with $c = 0$) where $\nu > -2$. In the least interesting case when $0 > \nu > -2$, u is unbounded for large r, whereby it has an infinite amount of heat $H = +\infty$ and "energy"

$E = +\infty$. In contrast, when $\nu > 0$, u decays to 0 for large r and vanishes at $r = 0$ for $t > 0$. As a result, in this case the "energy" of u for $t > 0$ is given by

$$E = \frac{1}{2^{3\nu/2}3}\nu(\nu^2 + 2\nu + 6)(\nu + 2)^{(1-\nu)/2}\frac{\Gamma(1 + \nu/2)\Gamma(3\nu/2)}{\Gamma(2 + 2\nu)}\beta t^{-(1+3\nu)/2} \tag{85}$$

which is finite and positive due to $k < 0$ for $\nu > 0$. In particular, E decreases to 0 as $t \to \infty$ in accordance with the flux equation (66). Moreover, provided $\nu > 2$, u has a finite amount of heat

$$H = \frac{4\sqrt{\pi}}{2^{3\nu/2}3}(\nu + 2)^{(1-\nu)/2}(\nu - 2)^{-1}\frac{\Gamma(1 + \nu/2)}{\Gamma((1 + \nu)/2)}\beta t^{1-\nu/2} \tag{86}$$

which decreases to 0 as $t \to \infty$. For $t > 0$ the corresponding heat flux quantities in this case are given by

$$S = -\frac{\nu - 2}{2}H/t < 0, \tag{87}$$

$$F = 0, \quad \nu \lim_{r\to 0} u = 0. \tag{88}$$

Therefore, in the most physically interesting case $\nu > 2$, the similarity solution (84) describes an initial monopole-like circular heat distribution in $\mathbb{R}^2$ (see figure 3) producing

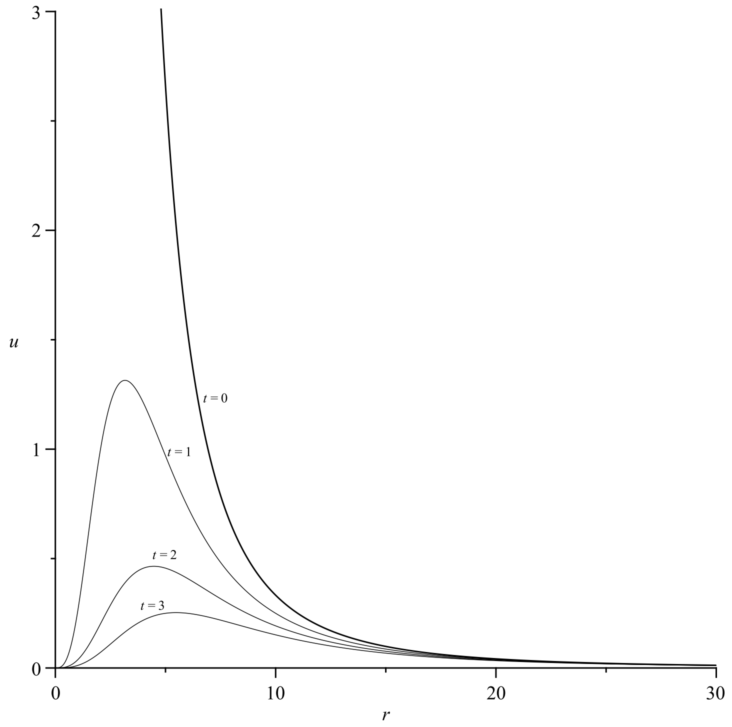

FIGURE 3. Graph of solution (84) with $\nu = 3$.

a smooth positive dispersive radial heat flow with some heat absorbed by a nonlinear heat sink proportional to $u^{1-2/\nu}$ at all points in $\mathbb{R}^2$. In particular, this solution has no point-source or heat flux at the origin for all times $t > 0$.

Similarity solutions (55) and (56) have $q + 1 = 3$, $\nu = -1/2$, and $k < 0$. In both solutions, for all t, u is singular at $r = 0$ and has slow radial decay such that $u = O(1/r)$ for large r. Consequently, the amount of heat in u is $H = \pm\infty$. In particular, at $t = 0$, u is given by a $1/r$ heat distribution for $r \geq 0$, while for $t \to \infty$ at any spatial point

$r > 0$, $u = r^{-1}(\pm(1/\sqrt{-2k}) + O(1/t))$ approaches a constant multiple of the same $1/r$ heat distribution. This non-dispersive temporal behaviour occurs because u has "energy" $E = +\infty$, which does not decrease with t.

Non-similarity solution (54) also has $q + 1 = 3$, $\nu = -1/2$, and $k < 0$. Compared with the previous similarity solutions (55) and (56), it exhibits the same long-time behaviour $u = r^{-1}(\pm(1/\sqrt{-2k}) + O(1/t))$ as $t \to \infty$ at any spatial point $r > 0$. It also exhibits the same radial decay $u = O(1/r)$ as $r \to \infty$ for all t, whereby $H = \pm\infty$ and $E = +\infty$. However, near $r = 0$, (54) is less singular than (55) and (56), such that $u = r^{-1/2}(\pm(15/c\sqrt{-2k})t + O(r))$. As a consequence the heat flux quantities for (54) have the properties $F + \nu \lim_{r\to 0} u = 0$ and $0 < -S < \infty$. Moreover, at $t = 0$, (54) reduces to the heat distribution $u = \pm(5/\sqrt{-2k})(r + c/r^{3/2})^{-1}$ which is C^1 for $r \geq 0$ if $c > 0$ and which vanishes at both $r = 0$ and $r = \infty$ (see figure 4). This non-similarity solution (54)

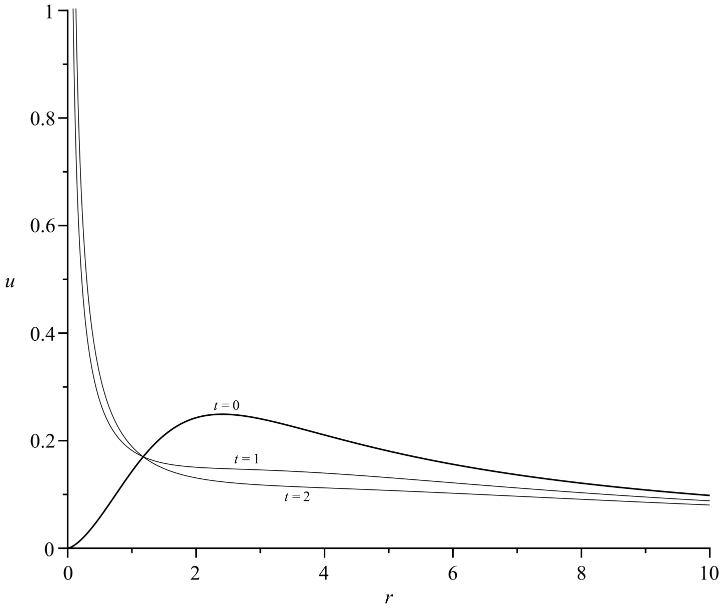

FIGURE 4. Graph of solution (54).

therefore has the physical interpretation of an initial circularly peaked heat distribution which vanishes both at the origin and spatial infinity in $\mathbb{R}^2$, producing a radial heat flow that is singular at the origin and has a $1/r$ decay for large radius, with a finite amount of heat absorbed by a nonlinear heat sink proportional to u^3 at all points in $\mathbb{R}^2$. The singularity in the heat distribution u at the origin corresponds to an infinite cooling point-source plus an infinite outward heat flux, whose net contribution to the heat flow for all times $t > 0$ is zero.

Finally, non-similarity solution (57) has $q + 1 = -3$, $\nu = -1/2$, and $k > 0$. In this solution, for all t, u is unbounded as $r \to \infty$ and has a cusp at the space-time parabolas $t + \tilde{c} = -r^2/3$ and $t + \tilde{c} = -(r^2 + 1/c)/3$ where u_r blows up and u vanishes. We can modify the solution by putting a cutoff on u at both parabolas so that

$$u(t,r) = \begin{cases} 0, & t - \beta \leq -r^2/3 \text{ or } t - \alpha \geq -r^2/3 \\ r^{-1/2}\sqrt{\gamma(3(\alpha - t) - r^2)(3(t-\beta) + r^2)}, & t - \beta \geq -r^2/3 \text{ and } t - \alpha \leq -r^2/3 \end{cases} \tag{89}$$

(see figure 5) where

$$\gamma = \frac{\sqrt{k}}{3(\alpha - \beta)} \tag{90}$$

and $\alpha = -\tilde{c} > \beta = -(\tilde{c}+1/3c) > 0$ with $c > 0$. Then the modified solution (89) is C^∞ in r and t at all space-time points other than $r = 0$, $\alpha - t = r^2/3$ and $\beta - t = r^2/3$. Near $r = 0$, for $\beta < t < \alpha$, $u = r^{-1/2}\left(3\sqrt{\gamma(\alpha-t)(t-\beta)} + O(r)\right)$ is singular (see figure 6), while near the inner and outer cusps, $u = (\sqrt{3(\alpha-t)} - r)^{1/2}\left(\sqrt{6\gamma(\alpha-\beta)} + O(\sqrt{3(\alpha-t)} - r)\right)$ for $t < \alpha$ and $u = (r - \sqrt{3(\beta-t)})^{1/2}\left(\sqrt{6\gamma(\alpha-\beta)} + O(r - \sqrt{3(\beta-t)})\right)$ for $t < \beta$ each have square-root behaviour in r (see figure 7). ¿From these properties, (89) can be checked to satisfy the heat equation (76) in a weak sense (i.e. u is only C^0 in t and r) on the spatial domain $r \geq 0$. In particular, u has a finite, non-negative amount of heat

$$H = \begin{cases} 0, & t > \alpha \\ \sqrt{\gamma}\displaystyle\int_0^{\sqrt{3(\alpha-t)}} (3(\alpha-t) - r^2)^{1/2}(3(t-\beta) + r^2)^{1/2} r^{1/2} dr & \\ \quad = \dfrac{\pi}{16} k^{1/4}(3(\alpha-\beta))^{3/2}(3(\alpha-t))^{-1/4}{}_2F_1(1/4, 3/2; 3; (\alpha-\beta)/(\alpha-t)), & \beta < t < \alpha \\ \sqrt{\gamma}\displaystyle\int_{\sqrt{3(\beta-t)}}^{\sqrt{3(\alpha-t)}} (3(\alpha-t) - r^2)^{1/2}(3(t-\beta) + r^2)^{1/2} r^{1/2} dr & \\ \quad = \dfrac{2}{5}\sqrt{\dfrac{2}{\pi}}\Gamma(3/4)^2 k^{1/4}(3(\alpha-t))^{5/4}{}_2F_1(-1/2, 3/2; 9/4; (\alpha-t)/(\alpha-\beta)), & t < \beta \end{cases} \tag{91}$$

so thus u is in $L^1(\mathbb{R}^+)$ for all $t \geq 0$. The corresponding heat flux equation is given by

$$\frac{dH}{dt} = \begin{cases} 0, & t > \alpha \\ F - \lim\limits_{r\to 0} u/2 - F_{\text{outer}} + S & \\ \quad = -\dfrac{3\pi}{8} k^{1/4}(3(\alpha-\beta))^{1/2}(3(\alpha-t))^{-1/4}\big({}_2F_1(1/4, 1/2; 2; (\alpha-\beta)/(\alpha-t)) & \\ \qquad -{}_2F_1(1/4, 3/2; 2; (\alpha-\beta)/(\alpha-t))\big), & \beta < t < \alpha \\ -F_{\text{inner}} + F_{\text{outer}} + S & \\ \quad = -\dfrac{3}{5}\sqrt{\dfrac{2}{\pi}}\Gamma(3/4)^2 k^{1/4}(3(\alpha-t))^{1/4}\big(\dfrac{5}{2}\ {}_2F_1(-1/2, 3/2; 5/4; (\alpha-t)/(\alpha-\beta)) & \\ \qquad -\dfrac{\alpha-t}{\alpha-\beta}\ {}_2F_1(1/2, 3/2; 9/4; (\alpha-t)/(\alpha-\beta))\big), & t < \beta \end{cases} \tag{92}$$

where

$$F - \lim_{r\to 0} u/2 = \lim_{r\to 0}(ru_r - u/2) = 0, \tag{93}$$

$$F_{\text{inner}} = \lim_{r\to\sqrt{3(\beta-t)}} ru_r = +\infty, \tag{94}$$

$$F_{\text{outer}} = \lim_{r\to\sqrt{3(\alpha-t)}} ru_r = -\infty, \tag{95}$$

$$S = \begin{cases} k\gamma^{-3/2}\displaystyle\int_0^{\sqrt{3(\alpha-t)}} (3(\alpha-t)-r^2)^{-3/2}(3(t-\beta)+r^2)^{-3/2}r^{-1/2}dr, & \beta < t < \alpha \\ k\gamma^{-3/2}\displaystyle\int_{\sqrt{3(\beta-t)}}^{\sqrt{3(\alpha-t)}} (3(\alpha-t)-r^2)^{-3/2}(3(t-\beta)+r^2)^{-3/2}r^{-1/2}dr, & t < \beta. \end{cases} \tag{96}$$

Here the quantities F_{inner} and F_{outer} are defined to be inward/outward heat fluxes arising from the inner and outer cusps, respectively. These fluxes act as cooling sources that cancel the singular contributions coming from the endpoints in the nonlinear heating source S in the flux equation (92). Their net effect gives a finite cooling rate, $dH/dt < 0$. Hence H decreases from its initial value $H(0) \neq 0$ at $t = 0$ to $H(\alpha) = 0$ at $t = \alpha$.

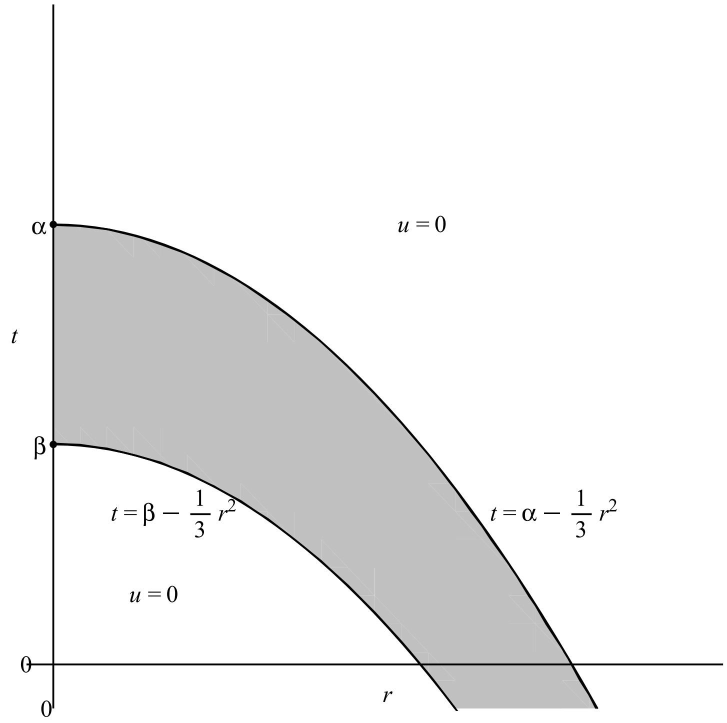

Figure 5. Space-time graph of solution (89).

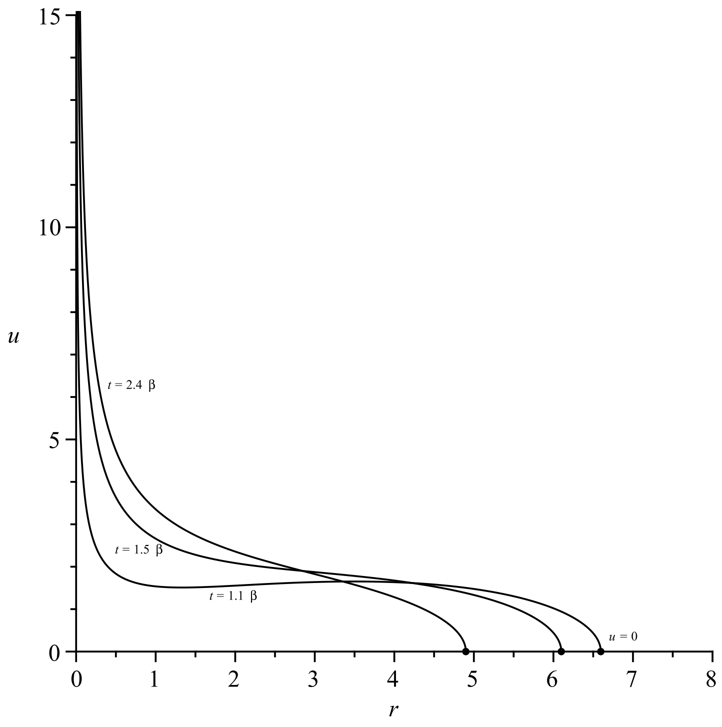

FIGURE 6. Graph of solution (89) with $\beta = 5,\ \alpha = 20$.

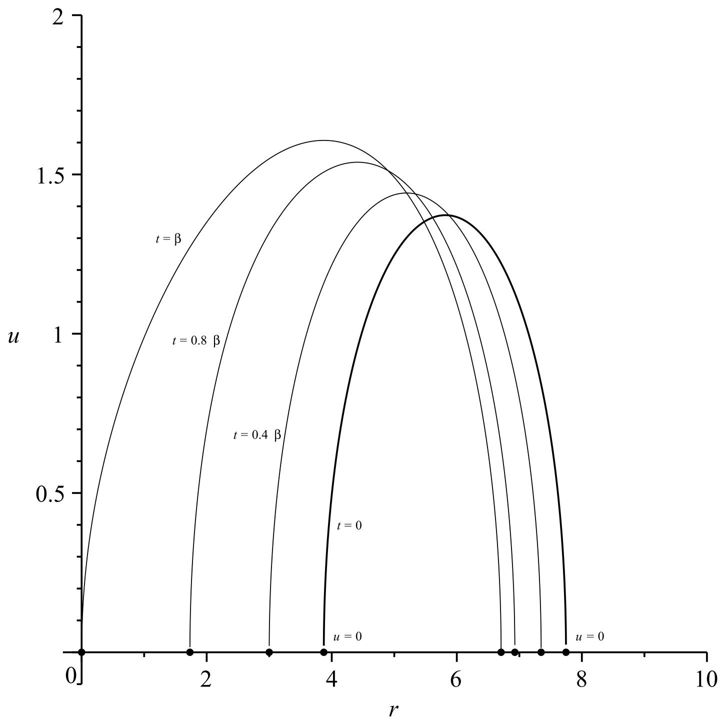

FIGURE 7. Graph of solution (89) with $\beta = 5$.

As a result, non-similarity solution (57) physically describes an initial ring-shaped heat distribution in $\mathbb{R}^2$ with radial temperature fronts at $r = \sqrt{3(\beta - t)}$ and $r = \sqrt{3(\alpha - t)}$ where $0 < \beta < \alpha$. These fronts behave as heat flux sinks and move radially toward the origin at speeds $dr/dt = -\sqrt{3/(\beta - t)}$ and $dr/dt = -\sqrt{3/(\alpha - t)}$. At time $t = \beta$ the inner front reaches the origin and produces an infinite cooling point-source plus a compensating infinite outward heat flux that both persist until time $t = \alpha$ when the outer front reaches the origin. For all times $0 \le t \le \alpha$, the heat absorbed at the temperature fronts exceeds the heat produced by the nonlinear source term u^{-3} at all points in $\mathbb{R}^2$ inside the fronts, whereby the total amount of heat decreases to zero in a finite time $t = \alpha$, with the heat distribution being given by $u = 0$ at all points $r \ge 0$ for times $t \ge \alpha$.

4. Concluding remarks

As main results in this paper, analytically interesting exact solutions have been obtained for a multi-dimensional semilinear heat equation (1) via a separation technique applied to the group foliation equations associated with the group of scaling symmetries (2) admitted by this equation. The solutions consist of explicit similarity solutions as well as other explicit solutions of a more general (non-similarity) form.

In general our method provides a highly effective alternative to standard similarity reduction for finding exact solutions to nonlinear PDEs with a group of scaling symmetries. Firstly, this method is an algorithmic refinement of the basic approach developed for the semilinear wave equation in [5], which leads to systematic reductions of the group foliation equations into overdetermined systems of ODEs that can be derived and solved by means of computer algebra. In particular, these ODE systems are tractable to solve using the computer algebra package Crack [11].

Secondly, the method is able to yield exact similarity solutions in an explicit form, whereas standard similarity reduction only gives an ODE that still has to be solved to find solutions explicitly and in general this step can be quite difficult. Indeed, the resulting similarity ODE (3) for the semilinear heat equation (1) cannot be solved by standard integration techniques such as symmetry reduction or integrating factors.

Thirdly, because the group foliation equations contain all solutions of the given nonlinear PDE, our method can yield non-similarity solutions that are also not invariant under any other (non-scaling) point symmetries admitted by the nonlinear PDE.

We can apply the same method more generally to nonlinear PDEs without scaling symmetries by utilizing the group foliation equations associated with any admitted one-dimensional group of point symmetries of the given nonlinear PDE and by adapting the separation technique to the specific form of the non-derivative terms that appear in the given group foliation equations. The algorithmic aspects of these steps will be the same as in the similarity case we have presented in this paper, since every one-dimensional group of point symmetries can be equivalently expressed as a group of scalings under an appropriate change of independent and dependent variables (i.e. by an invertible point transformation).

For future work, we plan to present a full comparison between the present group foliation method and standard symmetry reduction as applied to many typical linear and nonlinear PDEs of interest, e.g. linear heat and wave equations; semilinear diffusion and telegraph equations; integrable semilinear evolution equations such as the Korteweg de Vries and Boussinesq equations.

5. Acknowledgement

S. Anco and T. Wolf are each supported by an NSERC research grant.

S. Ali thanks the Mathematics Department of Brock University for support during the period of a research visit when this paper was written.

Computations were partly performed on computers of the Sharcnet consortium (www.sharcnet.ca).

The referees are thanked for valuable comments which have improved this paper.

References

[1] P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York) 1986.

[2] G. Bluman and S.C. Anco, *Symmetry and Integration Methods for Differential Equations* (Springer, New York) 2002.

[3] J. Velazquez, in *Recent Advances in partial differential equations (El Escorial, 1992)*, RAM Res. Appl. Math. 30 (1994) 131–145.

[4] A.A. Samarskii, V.A. Galaktionov, S.P. Kurdyumov, A.P. Mikailov, *Blow-up in Quasilinear Parabolic Equations* (Walter de Gruyter) 1994.

[5] S. Anco and S. Liu, J. Math. Anal. Appl. 297 (2004) 317–342.

[6] L.V. Ovsiannikov, *Group Analysis of Differential Equations* (New York, Academic) 1982.

[7] C. Qu and S.-L. Zhang, Chin. Phys. Lett. 22, No. 7 (2005) 1563–1566.

[8] S. Golovin, Commun. Nonlinear Sci. Numer. Simul. 9 (2004), 35–51.

[9] Y. Nutku and M.B. Sheftel, J. Phys. A 34 (2001) 137–156.

[10] M.B. Sheftel, Eur. Phys. J. B 29 (2002) 203–206.

[11] T. Wolf, in CRM Proceedings and Lecture Notes, 37 (2004) 283–300. http://lie.math.brocku.ca/crack/demo/

[12] Input and output files for all of the runs, and a detailed annotation for one run, are provided at the webpage http://lie.math.brocku.ca/twolf/papers/AnSaWo2010/.

[13] O.O. Vaneeva, R.O. Popovych, C. Sophocleous, Acta Appl. Math. 106 (2009) 1–46; ibid., Proc. of 4th Int. Workshop "Group Analysis of Differential Equations and Integrable Systems" (2008, Protaras, Cyprus) (2009) 191–209; arXiv:0904.3424.
[14] I. Rubinstein and L. Rubinstein, *Partial Differential Equations in Classical Mathematical Physics* (Cambridge University Press) 1998.

Department of Mathematics, Brock University, St. Catharines, ON L2S 3A1 Canada
E-mail address: sanco@brocku.ca

School of Electrical Engineering and Computer Sciences, National University of Sciences and Technology, H-12 Campus, Islamabad 44000, Pakistan
E-mail address: sajid_ali@mail.com

Department of Mathematics, Brock University, St. Catharines, ON L2S 3A1 Canada
E-mail address: twolf@brocku.ca