Konrad-Zuse-Zentrum
für Informationstechnik Berlin

Takustraße 7
D-14195 Berlin-Dahlem
Germany

STEPHEN S. ANCO [1], S. ALI [2], THOMAS WOLF [3]

Exact solutions of nonlinear partial differential equations by the method of group foliation reduction

[1]Department of Mathematics, Brock University, St.Catharines, Canada
[2]School of Electrical Engineering and Computer Sciences, National University of Sciences and Technology, Islamabad, Pakistan
[3]Department of Mathematics, Brock University, St.Catharines, Canada and ZIB Fellow

ZIB-Report 11-25 (June 2011)

EXACT SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS BY THE METHOD OF GROUP FOLIATION REDUCTION

STEPHEN C. ANCO, S. ALI, AND THOMAS WOLF

Abstract. A novel symmetry method for finding exact solutions to nonlinear PDEs is illustrated by applying it to a semilinear reaction-diffusion equation in multi-dimensions. The method is based on group foliation reduction and employs a separation ansatz to solve an equivalent first-order group foliation system whose independent and dependent variables respectively consist of the invariants and differential invariants of a given one-dimensional group of point symmetries for the reaction-diffusion equation. With this method, solutions of the reaction-diffusion equation are obtained in an explicit form, including group-invariant similarity solutions and travelling-wave solutions, as well as dynamically interesting solutions that are not invariant under any of the point symmetries admitted by this equation.

1. Introduction

The construction of group foliations using admitted point symmetry groups for nonlinear partial differential equations (PDEs) is originally due to Lie and Vessiot and was revived in its modern form by Ovsiannikov [1]. In general a group foliation converts a given nonlinear PDE into an equivalent first-order PDE system, called the group-resolving equations, whose independent and dependent variables respectively consist of the invariants and differential invariants of a given one-dimensional group of point symmetry transformations. Each solution of the group-resolving equations geometrically corresponds to an explicit one-parameter family of exact solutions of the original nonlinear PDE, such that the family is closed under the given one-dimensional symmetry group acting in the solution space of the PDE.

Because a group foliation contains all solutions of the given nonlinear PDE, ansatzes or differential-algebraic constraints must be used to reduce the group-resolving equations into an overdetermined system for the purpose of obtaining explicit solutions. Compared with classical symmetry reduction [2, 3], a main difficulty to-date has been how to find effective, systematic ansatzes that lead to useful reductions.

An important step toward overcoming this difficulty has been taken in recent work [4, 5] on finding exact solutions to semilinear wave equations and heat equations with power nonlinearities. Specifically, this work demonstrates that the group-resolving equations for such nonlinear PDEs have solutions arising from a simple separation ansatz in terms of the group-invariant variables. Through this ansatz, many explicit solutions to the nonlinear PDE are easily found, whose form would not be readily obvious just by trying simple direct ansatzes using the original independent and dependent variables in the nonlinear PDE, or by simply writing down the form for classical group-invariant solutions. In particular, some of these solutions are not invariant under any of the point symmetries of the nonlinear PDE and thus fall completely outside of classical symmetry reduction

1991 *Mathematics Subject Classification.* 35K58;35C06;35A25;58J70;34C14.

Key words and phrases. semilinear heat equation, similarity reduction, exact solutions, group foliation, symmetry .

(while others coincide with explicit group-invariant solutions). Most importantly for applications, many of the solutions also have interesting analytical properties related to critical dynamics, blow-up behaviour, asymptotic behaviour and attractors.

We will illustrate this group-foliation reduction method by applying it to obtain explicit exact solutions for the semilinear radial reaction-diffusion equation

$$u_t = u_{rr} + (n-1)r^{-1}u_r + (p - ku^q)u, \quad k = \pm 1, \quad p = \text{const.} \tag{1}$$

for $u(t,r)$, with a nonlinearity power $q \neq 0, -1$, where r denotes the radial coordinate in $n > 1$ dimensions or the half-line coordinate in $n = 1$ dimension. The symmetry structure of this reaction-diffusion equation is given by [6]

$$\text{time translation} \quad \mathbf{X}_1 = \partial/\partial t \quad \text{for all } n, q, p, \tag{2}$$

$$\text{scaling} \quad \mathbf{X}_2 = 2t\partial/\partial t + r\partial/\partial r - (2/q)u\partial/\partial u \quad \text{only for } p = 0, \tag{3}$$

$$\text{space translation} \quad \mathbf{X}_3 = \partial/\partial r \quad \text{only for } n = 1, \tag{4}$$

where $\mathbf{X}$ is the infinitesimal generator of a one-parameter group of point transformations acting on (t, r, u). For constructing a group foliation, it is natural to use the time translation generator (2), since this is the only point symmetry admitted for all cases of the parameters $n \geq 1$, $q \neq 0$, p.

In Sec. 2, we first set up the system of group-resolving equations given by the time-translation symmetry (2) for the reaction-diffusion equation (1), which uses the invariants and differential invariants of the symmetry generator $\mathbf{X}_1$ as the independent and dependent variables in the system. We next state the form required for solutions of the group-resolving system to correspond to group-invariant solutions of the reaction-diffusion equation (1) with respect to the point symmetries generated by $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$.

In Sec. 3, we explain the separation ansatz for directly reducing the system of group-resolving equations. This reduction yields an overdetermined system of differential-algebraic equations which can be readily solved by computer algebra. We present the explicit solutions of these equations and then we derive the resulting exact solutions of the reaction-diffusion equation. These solutions include explicit similarity solutions in the case $p = 0$, and explicit travelling wave solutions in addition to an explicit non-invariant solution in the case $n = 1$.

In Sec. 4, we show how the success of the reduction ansatz can be understood equivalently as constructing partially-invariant subspaces for a nonlinear operator that arises in a natural way from the structure of the group-resolving equations. This important observation puts our method on a wider mathematical foundation within the general theory of invariant subspaces developed by Galaktionov [7].

Finally, we make some general concluding remarks in Sec. 5.

2. Group-resolving equations and symmetries

To proceed with setting up the time-translation group foliation for the reaction-diffusion equation (1), we first write down the invariants (in terms of t, r, u)

$$x = r, \quad v = u, \tag{5}$$

satisfying $\mathbf{X}_1 x = \mathbf{X}_1 v = 0$, and the differential invariants (in terms of u_t, u_r)

$$G = u_t, \quad H = u_r, \tag{6}$$

satisfying $\mathbf{X}_1^{(1)} G = \mathbf{X}_1^{(1)} H = 0$ where $\mathbf{X}_1^{(1)}$ is the first-order prolongation of the generator (2). Here x and v are mutually independent, while G and H are related by equality of mixed r, t derivatives on u_t and u_r, which gives

$$D_r G = D_t H \tag{7}$$

where D_r, D_t denote total derivatives with respect to r, t. In addition, v, G, H are related through the reaction-diffusion equation (1) by

$$G - r^{1-n} D_r (r^{n-1} H) = (p - kv^q)v. \tag{8}$$

Now we put $G = G(x, v)$, $H = H(x, v)$ into equations (7) and (8) and use equation (5) combined with the chain rule to arrive at a first-order PDE system

$$G_x + HG_v - GH_v = 0, \tag{9}$$

$$G - (n-1)H/x - H_x - HH_v = (p - kv^q)v, \tag{10}$$

with independent variables x, v, and dependent variables G, H. These PDEs are called the *time-translation-group resolving system* for the reaction-diffusion equation (1).

The respective solution spaces of equation (1) and system (9)–(10) are related by a group-invariant mapping that is defined through the invariants (5) and differential invariants (6).

Lemma 1. *Solutions* $(G(x, v), H(x, v))$ *of the time-translation-group resolving system (9)–(10) are in one-to-one correspondence with one-parameter families of solutions* $u(t, r, c)$ *of the reaction-diffusion equation (1) satisfying the translation-invariance property*

$$u(t + \epsilon, r, c) = u(t, r, \tilde{c}(\epsilon, c)) \tag{11}$$

where $\tilde{c}(0, c) = c$ *in terms of an arbitrary constant* c *and parameter* ϵ*, such that*

$$u_t = G(r, u), \quad u_r = H(r, u) \tag{12}$$

constitutes a consistent pair of parametric first-order ODEs whose integration constant is c.

We now examine the relationship between the symmetry structure of the reaction-diffusion equation (1) and the symmetry structure inherited by the time-translation-group resolving system (9)–(10).

Firstly, through the identifications defined by the variables (5)–(6), the prolongation of any point symmetry generator $\mathbf{X} = a_1\mathbf{X}_1 + a_2\mathbf{X}_2 + a_3\mathbf{X}_3$ of equation (1) has a natural projection to a point symmetry generator $\mathbf{Y} = a_2\mathbf{X}_2^{(1)} + a_3\mathbf{X}_3^{(1)}$ modulo $\mathbf{X}_1^{(1)}$ of system (9)–(10). The time-translation $\mathbf{X}_1$ thus gets annihilated by this projection, i.e. $\mathbf{Y}_1 = 0$, while the scaling $\mathbf{X}_2$ and the space-translation $\mathbf{X}_3$ respectively project to

$$\mathbf{Y}_2 = x\partial/\partial x - (2/q)v\partial/\partial v - 2(1 + 1/q)G\partial/\partial G - (1 + 2/q)H\partial/\partial H \quad \text{when } p = 0, \tag{13}$$

and

$$\mathbf{Y}_3 = \partial/\partial x \quad \text{when } n = 1. \tag{14}$$

Secondly, with respect to these inherited symmetries (13) and (14), the system (9)–(10) has a reduction to ODEs yielding solutions where (G, H) is invariant respectively under scalings

$$x \to \lambda x, \quad v \to \lambda^{-2/q} v \quad \text{when } p = 0, \tag{15}$$

and under translations

$$x \to x + \epsilon \quad \text{when } n = 1. \tag{16}$$

Thus, translation-invariant solutions have the form

$$(G, H) = (g(v), h(v)) \tag{17}$$

satisfying the ODE system

$$(h/g)' = 0, \quad g - hh' = (p - kv^q)v. \tag{18}$$

Scaling-invariant solutions have the form

$$(G, H) = (x^{-2-2/q}g(V), x^{-1-2/q}h(V)), \quad V = vx^{2/q} \tag{19}$$

satisfying the ODE system

$$((h + 2V/q)/g)' = -2/g, \quad g + (2 - n + 2/q)h - (h + 2V/q)h' = -kV^{q+1}. \tag{20}$$

Integration of the parametric ODEs (12) for such solutions (17)–(18) and (19)–(20) leads to the following two correspondence results.

Lemma 2. *In the case $n = 1$, there is a one-to-one correspondence between solutions of the translation-group resolving system (9)–(10) with the invariant form (17) and one-parameter families of travelling-wave solutions of the reaction-diffusion equation (1) given by the group-invariant form $u = f(\xi)$ where, modulo time-translations $t \to t + c$, the variable $\xi = r - t/a$ is an invariant of the translation symmetry $\mathbf{X} = a\partial/\partial t + \partial/\partial r = a\mathbf{X}_1 + \mathbf{X}_3$ in terms of some constant a (determined by ODE system (18)).*

Lemma 3. *In the case $p = 0$, there is a one-to-one correspondence between solutions of the translation-group resolving system (9)–(10) with the invariant form (19) and one-parameter families of similarity solutions of the reaction-diffusion equation (1) given by the group-invariant form $u = r^{-2/q}f(\xi)$ where, modulo time-translations $t \to t + c$, the variable $\xi = t/r^2$ is an invariant of the scaling symmetry $\mathbf{X} = 2t\partial/\partial t + r\partial/\partial r - (2/q)u\partial/\partial u = \mathbf{X}_2$.*

Furthermore, in all cases, static solutions $u(r)$ of the reaction-diffusion equation correspond to solutions of the translation-group resolving system with $G = 0$. Hereafter we will be interested only in solutions with $G \neq 0$, corresponding to dynamical solutions of the reaction-diffusion equation.

3. Main results

To find explicit solutions of the group foliation system (9)–(10) for $(G(x, v), H(x, v))$, we will make use of the same general homogeneity features utilized in Refs. [4, 5]. First, the non-derivative terms $(p - ku^q)u$ in the reaction-diffusion equation (1) appear only as an inhomogeneous term in equation (10). Second, in both equations (9) and (10) the linear terms involve no derivatives with respect to v. Third, the nonlinear terms in the homogeneous equation (9) have the skew-symmetric form $HG_v - GH_v$, while HH_v is the only nonlinear term appearing in the non-homogeneous equation (10). Based on these features, this system can be expected to have solutions given by the separable power form

$$G = g_1(x)v + g_2(x)v^a, \quad H = h_1(x)v + h_2(x)v^a, \quad a \neq 1. \tag{21}$$

For such a separation ansatz (21), the linear terms G_x, G, H/x, H_x in equations (9) and (10) will contain the same powers v, v^a that appear in both G and H, and moreover the nonlinear term $HG_v - GH_v$ in the homogeneous equation (9) will produce only the power v^a due to the identities $v^a(v)_v - v(v^a)_v = (a-1)v^a$ and $v(v)_v - v(v)_v = v^a(v^a)_v - v^a(v^a)_v = 0$. Thus, equation (9) can be satisfied by having the coefficients of v and v^a separately vanish. Similarly the nonlinear term HH_v in the non-homogeneous equation (10) will only yield the powers v, v^a, v^{2a-1}. Since we have $a \neq 1$ and $q \neq 0$, equation (10) can be satisfied by again having the coefficients of v and v^a separately vanish and by also having the term containing v^{2a-1} balance the inhomogeneous term kv^{q+1}. In this fashion we find that equations (9) and (10) reduce to an overdetermined system of 5 algebraic-differential equations for $g_1(x)$, $g_2(x)$, $h_1(x)$, $h_2(x)$, together with the relation $a = 1+q/2$. This system can be solved by a systematic integrability analysis. We have carried out this analysis using the computer algebra program Crack [8], which contains a wide repertoire of modules for reduction of differential order and polynomial degree, splittings with respect to the independent variable(s), eliminations, substitutions, factorizations, integrations, and length-shortening of equations, among others.

Proposition 1. *For $q \neq 0, -1$, the separation ansatz (21) yields altogether five solutions of the translation-group resolving system (9)–(10) with $G \neq 0$:*

$$
\begin{aligned}
&G = \pm(4-n)\Big(\frac{n-2}{n-3}k\Big)^{1/2} v^{(n-3)/(n-2)}/x, \\
&H = \pm\Big(\frac{n-2}{n-3}k\Big)^{1/2} v^{(n-3)/(n-2)} + (2-n)v/x \\
&p = 0, \quad n \neq 2,3, \quad q = 2/(2-n); && (22) \\
&G = \frac{q+4}{q+2}\big(pv + (pk)^{1/2} v^{1+q/2}\big), \\
&H = \pm\Big(\Big(\frac{2p}{q+2}\Big)^{1/2} v + \Big(\frac{2k}{q+2}\Big)^{1/2} v^{1+q/2}\Big), \\
&n = 1, \quad q \neq -2; && (23) \\
&G = \frac{q+4}{q+2}\big(pv - (pk)^{1/2} v^{1+q/2}\big), \\
&H = \pm\Big(\Big(\frac{2p}{q+2}\Big)^{1/2} v - \Big(\frac{2k}{q+2}\Big)^{1/2} v^{1+q/2}\Big), \\
&n = 1, \quad q \neq -2; && (24) \\
&G = \frac{3}{2}\left(pv \pm (pk)^{1/2} \tanh\big((p/2)^{1/2}(x+c_1)\big) v^2\right), \\
&H = \pm(k/2)^{1/2} v^2 + (p/2)^{1/2} \tanh\big((p/2)^{1/2}(x+c_1)\big) v \\
&n = 1, \quad q = 2, \quad p > 0; && (25) \\
&G = \frac{3}{2}\big(pv \mp (-pk)^{1/2} \tan\big((-p/2)^{1/2}(x+c_1)\big) v^2\big), \\
&H = \big(\pm (k/2)^{1/2} v^2 - (-p/2)^{1/2} \tan\big((-p/2)^{1/2}(x+c_1)\big) v\big), \\
&n = 1, \quad q = 2, \quad p < 0. && (26)
\end{aligned}
$$

Solution (22) satisfies the scaling-invariance reduction (19); solutions (23) and (24) satisfy the translation-invariance reduction (17).

Remark 1. *A shift of the arbitrary constant $c_1 \to c_1 \pm i\pi/(2p)^{1/2}$ in solution (25) and $c_1 \to c_1 \pm \pi/(-2p)^{1/2}$ in solution (26) respectively yields the two further solutions*

$$
\begin{aligned}
&G = \frac{3}{2}\left(pv \pm (pk)^{1/2} \coth\big((p/2)^{1/2}(x+c_1)\big) v^2\right), \\
&H = \pm(k/2)^{1/2} v^2 + (p/2)^{1/2} \coth\big((p/2)^{1/2}(x+c_1)\big) v, \\
&n = 1, \quad q = 2, \quad p > 0, && (27)
\end{aligned}
$$

and

$$\begin{aligned}
&G = \frac{3}{2}\big(pv \pm (-pk)^{1/2}\cot\big((-p/2)^{1/2}(x+c_1)\big)v^2\big),\\
&H = \big(\pm (k/2)^{1/2}v^2 + (-p/2)^{1/2}\cot\big((-p/2)^{1/2}(x+c_1)\big)v\big),\\
&n=1, \quad q=2, \quad p<0,
\end{aligned}\tag{28}$$

as obtained through the trigonometric identities

$$\tanh(\theta \pm i\pi/2) = \coth(\theta), \quad \tan(\theta \pm \pi/2) = -\cot(\theta). \tag{29}$$

The form of the separation ansatz (21) can be naturally generalized to include additional powers

$$G = g_1(x)v + \sum_{i=1}^{N} g_{1+i}(x)v^{a_i}, \quad H = h_1(x)v + \sum_{i=1}^{N} h_{1+i}(x)v^{a_i}, \tag{30}$$

where

$$a_i \neq 1, \quad a_i - a_j \neq 0. \tag{31}$$

This multi-term ansatz (for $N \geq 2$) leads to a considerably more complicated analysis compared to the previous two-term ansatz (with $N = 1$). Specifically, the homogeneous equation (9) now contains the powers $v^{a_i+a_j-1}$ in addition to v, v^{a_i}, while the non-homogeneous equation (10) contains the further powers v^{2a_i-1}, and v^{q+1}. To determine the exponents in these powers, a systematic examination of the possible balances (which rapidly increase in number with N) is necessary. We carry out this analysis by computer algebra, again using the program Crack [8]. In particular, for any fixed $N \geq 1$, Crack can be run automatically by setting up a priority list of modules to be tried recursively with case splittings given a high priority, in order to split equations (9) and (10) under the ansatz (30)–(31) and in each case solve the resulting overdetermined system of algebraic-differential equations for g_i and h_i, $i = 1, 2, \ldots, N$. Importantly, the modules in Crack are able to organize the case distinctions in an efficient way that lessens a combinatorial explosion by using algebraic conditions (equalities and inequalities) arising in the steps of solving the equations to keep track of whether a possible balance of two powers of v generates a new case (or subcase) which has to be solved or not. Running Crack, we find that the system (9)–(10), (30)–(31) has solutions when $N = 2$ but not when $N = 3$. (Based on this outcome we have not tried to investigate the system when $N > 3$.)

Proposition 2. *For $q \neq 0, -1$, and $N \leq 3$, the separation ansatz (30) yields two additional solutions of the translation-group resolving system (9)–(10) with $G \neq 0$:*

$$\begin{aligned}
&G = 3(pv + (k^2p)^{1/3} + (kp^2)^{1/3}v^{1/2}),\\
&H = -(2/p)^{1/2}(pv + (k^2p)^{1/3} + (kp^2)^{1/3}v^{1/2}),\\
&n=1, \quad q=-3/2, \quad p>0;
\end{aligned}\tag{32}$$

$$\begin{aligned}
&G = 4(pv + (-pk)^{1/2}v^{-1/3} + (-4p^3k)^{1/4}v^{1/3}),\\
&H = -(3/p)^{1/2}(pv + (-pk)^{1/2}v^{-1/3} + (-4p^3k)^{1/4}v^{1/3}),\\
&n=1, \quad q=-8/3, \quad p>0.
\end{aligned}\tag{33}$$

Both solutions satisfy the translation-invariance reduction (17).

We now obtain explicit solutions $u(t,r)$ of the reaction-diffusion equation (1) from the solutions $(G(x,v), H(x,v))$ of its translation-group resolving system (9)–(10) by integrating the corresponding pair of parametric first-order ODEs (12). This integration yields a one-parameter solution family $u(t,r,c)$ which is closed under the action of the group of time-translations $t \to t+\epsilon$.

Theorem 1. *The semilinear reaction-diffusion equation (1) has the following exact solutions arising from the explicit solutions of its translation-group resolving system found in Propositions 1 and 2 (and Remark 1):*

$$u = \left(\pm \left(\frac{k}{(n-3)(n-2)} \right)^{1/2} \left(\frac{r}{2} - \frac{(n-4)(t+c)}{r} \right) \right)^{n-2},$$
$$n \neq 2,3, \quad q = 2/(2-n), \quad p = 0; \tag{34}$$

$$u = \left(-(k/p)^{1/2} + \exp\left(\mp q \left(\frac{p}{2(q+2)} \right)^{1/2} \left(r \pm (q+4) \left(\frac{p}{2(q+2)} \right)^{1/2} (t+c) \right) \right) \right)^{-2/q},$$
$$n = 1, \quad q \neq -2, \quad p \neq 0; \tag{35}$$

$$u = \left((k/p)^{1/2} + \exp\left(\mp q \left(\frac{p}{2(q+2)} \right)^{1/2} \left(r \pm (q+4) \left(\frac{p}{2(q+2)} \right)^{1/2} (t+c) \right) \right) \right)^{-2/q},$$
$$n = 1, \quad q \neq -2, \quad p \neq 0; \tag{36}$$

$$\begin{aligned} &u = f(r - 3(p/2)^{1/2}(t+c)) \quad \text{where} \\ &\quad \ln(pf(\xi) + (kp^2)^{1/3} f(\xi)^{1/2} + (k^2 p)^{1/3}) - (2/\sqrt{3}) \arctan((2(p/k)^{1/3} f(\xi)^{1/2} + 1)/\sqrt{3}) \\ &\qquad = -(2p)^{1/2}\xi, \\ &n = 1, \quad q = -3/2, \quad p > 0; \end{aligned} \tag{37}$$

$$\begin{aligned} &u = f(r - 4(p/3)^{1/2}(t+c)) \quad \text{where} \\ &\quad \ln(pf(\xi)^{4/3} + (-pk)^{1/2} + (-4p^3 k)^{1/4} f(\xi)^{2/3}) - 2\arctan((-4p/k)^{1/4} f(\xi)^{2/3} + 1) \\ &\qquad = -4(p/3)^{1/2}\xi, \\ &n = 1, \quad q = -8/3, \quad p > 0; \end{aligned} \tag{38}$$

$$u = \frac{\cosh((p/2)^{1/2}(r+c_1))}{(k/p)^{1/2} \sinh((p/2)^{1/2}(r+c_1)) \pm \exp(-3p(t+c)/2)},$$
$$n = 1, \quad q = 2, \quad p > 0; \tag{39}$$

$$u = \frac{\sinh((p/2)^{1/2}(r+c_1))}{(k/p)^{1/2} \cosh((p/2)^{1/2}(r+c_1)) \pm \exp(-3p(t+c)/2)},$$
$$n = 1, \quad q = 2, \quad p > 0; \tag{40}$$

$$u = \frac{\cos((-p/2)^{1/2}(r+c_1))}{(-k/p)^{1/2} \sin((-p/2)^{1/2}(r+c_1)) \pm \exp(-3p(t+c)/2)},$$
$$n = 1, \quad q = 2, \quad p < 0; \tag{41}$$

$$u = \frac{\sin((-p/2)^{1/2}(r+c_1))}{(-k/p)^{1/2}\cos((-p/2)^{1/2}(r+c_1)) \mp \exp(-3p(t+c)/2)},$$
$$n = 1, \quad q = 2, \quad p < 0, \tag{42}$$

where c, c_1 are arbitrary constants.

Remark 2. *Solutions (39) and (40) are related by shifts of the arbitrary constants $c_1 \to c_1 \pm i\pi/(2p)^{1/2}$ and $c \to c \mp i\pi/(3p)$ through the trigonometric identities*

$$\sinh(\theta \pm i\pi/2) = \pm i\cosh(\theta), \quad \cosh(\theta \pm i\pi/2) = \pm i\sinh(\theta), \tag{43}$$

while solutions (41) and (42) are related similarly through the trigonometric identities

$$\sin(\theta \pm \pi/2) = \pm\cos(\theta), \quad \cos(\theta \pm \pi/2) = \mp\sin(\theta). \tag{44}$$

Modulo time-translations, solution (34) is a similarity solution, since it has the form shown in Lemma 3. This solution has been obtained in our previous work [5] using a two-term separation ansatz to solve the group-resolving equations given by the scaling symmetry (3) for the reaction-diffusion equation (1) in the case $p = 0$.

Solutions (35) to (38) are travelling-wave solutions with the form shown in Lemma 2 (where the constant a is given by the wave speed). The two solutions (35) and (36) were first obtained in Ref. [11] through standard symmetry reduction. To the best knowledge of the authors, the other solutions (37) and (38) are new.

In contrast, solutions (39) to (42) are non-invariant solutions, as they contain both t and r whereas the symmetry group of the reaction-diffusion equation (1) in the case $p \neq 0$ and $n = 1$ is generated by both time-translation and space-translation symmetries (2) and (4). These solutions have been found previously in Ref. [9, 10] using Bluman and Cole's nonclassical method.

4. Invariance properties of the reduction ansatz

As we will now explain, the success of the separation ansatzes (21) and (30) for reducing the system of group-foliation equations (9) and (10) has a mathematically natural interpretation within the theory of invariant subspaces [7].

We start by writing the group-foliation equations in an evolutionary operator form

$$\begin{pmatrix} G \\ H \end{pmatrix}_x = \Phi\left(\begin{pmatrix} G \\ H \end{pmatrix}\right) = \begin{pmatrix} 0 \\ (kv^q - p)v \end{pmatrix} + \begin{pmatrix} 0 \\ G + (1-n)H/x \end{pmatrix} + \begin{pmatrix} GH_v - HG_v \\ -HH_v \end{pmatrix} \tag{45}$$

where Φ defines a nonlinear operator acting on the pair of variables (G, H). Then we view the ansatz (21) as defining a linear space of functions with a separable power form

$$\begin{pmatrix} G \\ H \end{pmatrix} = \begin{pmatrix} g_1 \\ h_1 \end{pmatrix} v + \begin{pmatrix} g_2 \\ h_2 \end{pmatrix} v^a, \quad a \neq 1 \tag{46}$$

where the coefficients depend only on x. This linear space (46) is not invariant under Φ because

$$\begin{aligned} &\Phi\left(\begin{pmatrix} g_1 \\ h_1 \end{pmatrix} v + \begin{pmatrix} g_2 \\ h_2 \end{pmatrix} v^a\right) = \\ &\qquad \begin{pmatrix} 0 \\ g_1 + (1-n)h_1/x - h_1^2 - p \end{pmatrix} v + \begin{pmatrix} (a-1)(h_2 g_1 - h_1 g_2) \\ g_2 + (1-n)h_2/x - (a+1)h_1 h_2 \end{pmatrix} v^a \\ &\qquad\quad + \begin{pmatrix} 0 \\ -a h_2^2 \end{pmatrix} v^{2a-1} + \begin{pmatrix} 0 \\ k \end{pmatrix} v^{q+1} \end{aligned} \tag{47}$$

produces terms that have additional powers v^{2a-1}, v^{q+1}.

However, the operator Φ will preserve a set of functions contained in the linear space (46) if the power a and the coefficients g_1, g_2, h_1, h_2 satisfy the conditions

$$2a-1 = q+1, \quad -ah_2^2 = k. \tag{48}$$

In this sense [7] the linear space will then be partially-invariant under Φ. As a consequence of this invariance property, the evolution equation (45) for (G, H) reduces to an overdetermined system of differential-algebraic equations which we can solve for g_1, g_2, h_1, as discussed in Sec. 2.

The separable power ansatz (21) thus can be viewed as explicitly constructing a partially-invariant linear subspace for the group-foliation operator Φ. A similar discussion applies to the multi-term ansatz (30).

It is worth pointing out that apart from the mathematical interpretation given to the ansatzes (21) and (30), the theory of (partially) invariant linear subspaces does not provide any general constructive method or algorithm for finding a successful ansatz. So the main aspect of the present work is to demonstrate the effectiveness of a general separation ansatz for solving group-foliation equations. We also emphasize that this approach is much more effective than if an analogous separation method were to be applied either directly for solving the given reaction-diffusion PDE or instead for seeking a (partially) invariant linear subspace for that PDE itself.

5. Concluding remarks

In general the method of group foliation reduction using a separation ansatz as illustrated in this paper is able to yield exact solutions to nonlinear 2nd order PDEs with power nonlinearities.

This method works with any admitted group of point (or contact) symmetries and gives a systematic reduction of the group foliation equations into an overdetermined system consisting of algebraic equations and 1st order differential equations that can be derived and in most cases solved by means of computer algebra (e.g. using the program Crack [8]). In particular, for a given nonlinear 2nd order PDE having two independent variables, solutions are produced in an explicit form, whereas standard symmetry reduction only gives a 2nd order ODE that still has to be solved to find group-invariant solutions explicitly and in general this step can be quite difficult.

Moreover, because the group foliation equations contain all solutions of the given nonlinear PDE, the method can yield solutions that are not invariant under any of the point (or contact) symmetries admitted by the nonlinear PDE.

It is straightforward to extend this method to higher-order PDEs with power nonlinearities. As well, it should be possible to apply the same method to PDEs having more general forms of nonlinearities by utilizing a general separation of variables ansatz with respect to all of the independent variables in the group foliation equations. In general, the success of such an ansatz can be interpreted mathematically [7] as constructing a (partially) invariant linear subspace for a nonlinear operator coming from the structure of the group foliation equations.

Related work using a similar group-foliation method applied to nonlinear diffusion equations appears in Ref. [12]. Group foliation equations were first used successfully in Refs. [13, 14, 15] for obtaining exact solutions to nonlinear PDEs by a different method that is applicable when the group of point symmetries of a given PDE is infinite-dimensional, compared to the example of a finite-dimensional symmetry group considered both in Ref. [4, 5] and in the present work.

6. Acknowledgement

S. Anco and T. Wolf are each supported by an NSERC research grant.

S. Ali thanks the Mathematics Department of Brock University for support during the period of a research visit when this paper was written.

Computations were partly performed on computers of the Sharcnet consortium (www.sharcnet.ca).

The referees and the editor are thanked for valuable comments which have improved this paper.

References

[1] L.V. Ovsiannikov, *Group Analysis of Differential Equations* (New York, Academic) 1982.

[2] P.J. Olver, *Applications of Lie Groups to Differential Equations* (Springer, New York) 1986.

[3] G. Bluman and S.C. Anco, *Symmetry and Integration Methods for Differential Equations* (Springer, New York) 2002.

[4] S. Anco and S. Liu, Exact solutions of semilinear radial wave equations in n dimensions, J. Math. Anal. Appl. 297 (2004) 317–342.

[5] S. Anco, S. Ali, and T. Wolf, Symmetry analysis and exact solutions of semilinear heat flow in multi-dimensions, J. Math. Anal. Appl., (2011) to appear.

[6] V.A. Dorodnitsyn, On invariant solutions of the equation of nonlinear heat conduction with a source, USSR Comp. Math. Math. Phys. 22 (1982), 115–122.

[7] V.A. Galaktionov and S.R. Svirshchevskii, *Exact Solutions and Invariant Subspaces of Nonlinear Partial Differential Equations in Mechanics and Physics* (Chapman & Hall/CRC, Boca Raton) 2007.

[8] T. Wolf, Applications of CRACK in the Classification of Integrable Systems, in CRM Proceedings and Lecture Notes, 37 (2004) 283–300. http://lie.math.brocku.ca/crack/demo/

[9] P. A. Clarkson and E. L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equations, Physica D 70 (1993), 250-288.

[10] D. J. Arrigo, J. M. Hill and P. Broadbridge, Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source, IMA J. Appl. Math. 52 (1994), 1–24.

[11] K. Vijayakumar, On the integrability and exact solutions of the nonlinear diffusion equation with a nonlinear source, J. Austral. Math. Soc. Ser. B 39 (1998), 513–517.

[12] C. Qu and S.-L. Zhang, Group foliation method and functional separation of variables to nonlinear diffusion equations, Chin. Phys. Lett. 22, No. 7 (2005) 1563–1566.

[13] S. Golovin, Applications of the differential invariants of infinite dimensional groups in hydrodynamics, Commun. Nonlinear Sci. Numer. Simul. 9 (2004), 35–51.

[14] Y. Nutku and M.B. Sheftel, Differential invariants and group foliation for the complex Monge-Amp'ere equation, J. Phys. A 34 (2001) 137–156.

[15] M.B. Sheftel, Method of group foliation and non-invariant solutions of partial differential equations, Eur. Phys. J. B 29 (2002) 203–206.

Department of Mathematics, Brock University, St. Catharines, ON L2S 3A1 Canada
E-mail address: `sanco@brocku.ca`

School of Electrical Engineering and Computer Sciences, National University of Sciences and Technology, H-12 Campus, Islamabad 44000, Pakistan
E-mail address: `sajid_ali@mail.com`

Department of Mathematics, Brock University, St. Catharines, ON L2S 3A1 Canada
E-mail address: `twolf@brocku.ca`