

On the 0/1 Knapsack Polytope

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Abstract

Given a set N of items and a capacity $b \in \mathbb{N}$, and let N_j be the set of items with weight j , $1 \leq j \leq b$. The 0/1 knapsack polytope is the convex hull of all 0/1 vectors that satisfy the inequality

$$\sum_{j=1}^b \sum_{i \in N_j} jx_i \leq b.$$

In this paper we first present a complete linear description of the 0/1 knapsack polytope for two special cases: (a) $N_j = \emptyset$ for all $1 < j \leq \lfloor \frac{b}{2} \rfloor$ and (b) $N_j = \emptyset$ for all $1 < j \leq \lfloor \frac{b}{3} \rfloor$ and $N_j = \emptyset$ for all $j \geq \lfloor \frac{b}{2} \rfloor + 1$. It turns out that the inequalities that are needed for the complete description of these special polytopes are derived by means of some “reduction principle”. This principle is then generalized to yield valid and in many cases facet defining inequalities for the general 0/1 knapsack polytope. The separation problem for this class of inequalities can be solved in pseudo polynomial time via dynamic programming techniques.

Keywords: complete description, facets, knapsack polytope, knapsack problem, pseudo polynomial time, separation

1 Introduction and Notation

Since the early seventies, many researchers have investigated the polyhedral structure of the 0/1 knapsack problem. In particular, two reasons have nourished this

development: one is the increasing number of interesting applications that – at least as a subproblem – involve the single knapsack problem; the other is the discovery of beautiful concepts and results associated with minimal covers, $(1, k)$ -configurations or the lifting and complementing of variables.

Most of the polyhedral studies presented so far involve two basic and general objects: minimal covers (see for instance, [B75], [HJP75], [W75]) and $(1, k)$ -configurations (cf. [P80]). Let N be a subset of items, let b denote the knapsack capacity and suppose, every item $i \in N$ has a weight $W(i) > 0$. A set $S \subseteq N$ is a *cover* if $\sum_{i \in S} W(i) > b$ holds. The cover is *minimal*, if in addition $\sum_{i \in S \setminus \{s\}} W(i) \leq b$ for all $s \in S$. Let $N' \subseteq N$ be some nonempty subset of items and let $z \in N \setminus N'$. The set $N' \cup \{z\}$ is called a $(1, k)$ -*configuration*, if

$$\sum_{i \in N'} W(i) \leq b;$$

$K \cup \{z\}$ is a minimal cover for all $K \subset N'$ with $|K| = k$.

When it was shown that (simultaneous) lifting (see [P75], [Z74]) and complementing (see [W75]) of minimal cover inequalities yields all the facets of the 0/1 knapsack polytope (cf. [BZ78], [BZ84]), a theoretical machinery became available to attack knapsack problems from a polyhedral point of view.

In fact, since [CJP83] several papers have been written that are based on this polyhedral theory for the 0/1 knapsack problem and are meant to turn the theory into an algorithmic tool for the solution of practical problems (see for instance [RW87], [FMW93]). Moreover, the last decade has brought a wide range of interesting applications such as production planning problems ([RW87]), airline scheduling problems ([HP93]), vehicle routing problems (see for instance [Po93]), certain clustering and graph partitioning problems ([FMSWW94]) or subproblems that arise within the design of electronic circuits or the design of mainframe computers ([We92], [FGKKMW93], [F93]), in which the 0/1 knapsack problem is involved as a crucial subproblem. In particular, the polyhedral structure of the single knapsack problem is either inherited by the polytope associated with the more complex problem or there is a (non-trivial) way how to transform knapsack inequalities associated with a subproblem to inequalities for the original problem.

Computational experiments by several researchers have revealed that inequalities based on minimal covers and $(1, k)$ -configurations in combination with lifting and complementing are often not sufficient for obtaining satisfactory bounds for the optimum value of a knapsack problem. One possible explanation is that, in general, minimal covers and $(1, k)$ -configurations are proper subsets of the given set of items. Hence, the corresponding minimal cover- or $(1, k)$ -configuration in-

equality is facet defining only for the polytope associated with this subset. In order to obtain a facet of the original polytope we need to determine coefficients for the variables that correspond to items not in this subset (lifting, complementing). However, the final inequality depends on the order in which such coefficients are computed ([P75]). Moreover, we can usually not compute such coefficients explicitly. Thus, when starting with a minimal cover or $(1, k)$ -configuration inequality, we do not have much information about what the lifted inequality will look like. In particular, one important question for this approach is: what are the “good” objects to start with? In this paper we present families of inequalities that involve the whole set of items. It turns out that under mild assumptions such inequalities define facets of the given knapsack polytope.

The paper is organized as follows. In Section 2 we introduce, for $i \in \{1, \dots, \lfloor \frac{b}{2} \rfloor\}$, the knapsack problem \mathcal{K}_i , where $N_j = \emptyset$ for all $1 < j \leq \lfloor \frac{b}{i+1} \rfloor$ and $N_j = \emptyset$ for all $j \geq \lfloor \frac{b}{i} \rfloor + 1$. We show that in the two special cases $i = 1$ and $i = 2$ a complete description of the associated knapsack polyhedron is obtained. For $i = 2$ the proof is quite long and technical and thus is left to the Appendix. It turns out that the inequalities that are needed for the complete description of these special polytopes are derived by means of some “reduction principle”. This principle is generalized in Section 3 to yield valid and under certain conditions facet defining inequalities for the general 0/1 knapsack polytope. For this class of inequalities we present an algorithm that solves the corresponding separation problem in pseudo polynomial time. This is the topic of Section 4.

Notation

Given a set N of items, a capacity $b \in \mathbb{N}$ and let N_j be the subset of items in N with weight j ($j = 1, \dots, b$). The 0/1 knapsack polytope denoted by P is the convex hull of all 0/1 vectors that satisfy the *knapsack inequality* $\sum_{j=1}^b \sum_{i \in N_j} j x_i \leq b$. For every $i \in N$ we denote by $W(i)$ the weight of i , i.e., $W(i) = j$ if $i \in N_j$. We say F a face of the polytope P induced by the inequality $c^T x \leq \gamma$, if $F = \{x \in P \mid c^T x = \gamma\}$. Every $x \in F$ is also called a *root* of $c^T x \leq \gamma$. The inequalities $x_i \leq 1$, $i \in N$ and $x_i \geq 0$, $i \in N$ are called *trivial*. Given $b \in N$ and $i \in N$, $1 \leq i \leq \lfloor \frac{b}{2} \rfloor$. We denote the interval $[\lfloor \frac{b}{i+1} \rfloor + 1, \lfloor \frac{b}{i} \rfloor]$ by B^i . For real numbers τ_j , $j = 1, \dots, n$ we define $\sum_{j=v}^w \tau_j := 0$ if $v > w$. Finally, for $I \subseteq \{1, \dots, n\}$ we use the notation $\tau(I) := \sum_{i \in I} \tau_i$ with $\tau(\emptyset) = 0$.

2 A Complete Description of P^1 and P^2

In this section we deal with the special 0/1 knapsack polytopes P^i ($i \in \{1, \dots, \lfloor \frac{b}{2} \rfloor\}$) defined as the convex hull of 0/1 vectors that satisfy the inequality

$$\sum_{i \in N_1} x_i + \sum_{j=\lfloor \frac{b}{i+1} \rfloor + 1}^{\lfloor \frac{b}{i} \rfloor} \sum_{i \in N_j} jx_i \leq b.$$

Of course, for fixed i the problem of optimizing an objective function over P^i can be solved in polynomial time, because every feasible solution contains no more than i different items that belong to $\bigcup_{j=\lfloor \frac{b}{i+1} \rfloor + 1}^{\lfloor \frac{b}{i} \rfloor} N_j$. Hence, by enumeration the optimal solution can be found in polynomial time. Our discussions will purely concentrate on the cases $i = 1, 2$ and we show that a complete description of the corresponding polytopes can be derived. It turns out that the complete description of P^2 is already quite complicated and involves several types of inequalities which – up to our knowledge – have not been presented in this structured form in literature before. We first give an example.

Example 2.1. Consider the knapsack polytope defined as the convex hull of all 0/1 vectors that satisfy the inequality

$$x_1 + x_2 + x_3 + x_4 + 4x_5 + 4x_6 + 5x_7 + 5x_8 \leq 11.$$

Here B^2 consists of the numbers 4 and 5 and every item has a weight of 1, 4 or 5. A complete inequality description (checked by a program developed in [C91]) is given by the trivial inequalities $x_i \geq 0$, $x_i \leq 1$, $i = 1, \dots, 8$ and the following system of inequalities:

$$\begin{array}{lll} (1) & & +x_5 + x_6 + x_7 + x_8 \leq 2 \\ (2) & & +x_3 + x_4 \quad +x_7 + x_8 \leq 3 \\ (3) & +x_2 & +x_4 \quad +x_7 + x_8 \leq 3 \\ (4) & +x_2 + x_3 & +x_7 + x_8 \leq 3 \\ (5) & +x_1 & +x_4 \quad +x_7 + x_8 \leq 3 \\ (6) & +x_1 & +x_3 \quad +x_7 + x_8 \leq 3 \\ (7) & +x_1 + x_2 & +x_7 + x_8 \leq 3 \\ (8) & & +x_2 + x_3 + x_4 \quad +x_6 + x_7 + x_8 \leq 4 \\ (9) & & +x_2 + x_3 + x_4 + x_5 \quad +x_7 + x_8 \leq 4 \\ (10) & +x_1 & +x_3 + x_4 \quad +x_6 + x_7 + x_8 \leq 4 \\ (11) & +x_1 & +x_3 + x_4 + x_5 \quad +x_7 + x_8 \leq 4 \\ (12) & +x_1 + x_2 & +x_4 \quad +x_6 + x_7 + x_8 \leq 4 \\ (13) & +x_1 + x_2 & +x_4 + x_5 \quad +x_7 + x_8 \leq 4 \\ (14) & +x_1 + x_2 + x_3 & +x_6 + x_7 + x_8 \leq 4 \end{array}$$

$$\begin{aligned}
(15) \quad & +x_1 + x_2 + x_3 \quad +x_5 \quad +x_7 + x_8 \leq 4 \\
(16) \quad & \quad +x_2 + x_3 + x_4 + x_5 \quad +x_6 \quad +2x_7 + 2x_8 \leq 5 \\
(17) \quad & +x_1 \quad +x_3 + x_4 + x_5 \quad +x_6 \quad +2x_7 + 2x_8 \leq 5 \\
(18) \quad & +x_1 + x_2 \quad +x_4 + x_5 \quad +x_6 \quad +2x_7 + 2x_8 \leq 5 \\
(19) \quad & +x_1 + x_2 + x_3 \quad +x_5 \quad +x_6 \quad +2x_7 + 2x_8 \leq 5 \\
(20) \quad & +x_1 + x_2 + x_3 + x_4 + x_5 \quad +x_6 \quad +x_7 \quad +x_8 \leq 5 \\
(21) \quad & +x_1 + x_2 + x_3 + x_4 + x_5 \quad +2x_6 + 2x_7 + 2x_8 \leq 6 \\
(22) \quad & +x_1 + x_2 + x_3 + x_4 + 2x_5 + x_6 \quad +2x_7 + 2x_8 \leq 6 \\
(23) \quad & +x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 + 3x_7 + 3x_8 \leq 7
\end{aligned}$$

We now introduce three types of inequalities that are needed to describe the polytope P^2 . We prove that these inequalities are valid for P or P^2 . About when these inequalities define facets of P will be reported in Section 3.

Proposition 2.2.

(i) Let $T_1 \subseteq N_1$, $|T_1| < b$ and set $r := b - |T_1|$. The inequality

$$\sum_{i \in T_1} x_i + \sum_{j \geq r+1} \sum_{i \in N_j} (j-r)x_i \leq |T_1|$$

is valid for P (see [Le93]).

(ii) Let $T_1 \subseteq N_1$, let $i_0 \in N_{j_0}$, $2 \leq j_0$ such that $|T_1| + j_0 < b$ and define the rest capacity $r := b - |T_1| - j_0$. For every integer $\psi \in [0, r]$ such that $j_0 - \psi > 0$, the inequality

$$\begin{aligned}
& \sum_{i \in T_1} x_i + (j_0 - \psi)x_{i_0} + \sum_{j=r+1}^{j_0+r-\psi} \sum_{i \in N_j \setminus \{i_0\}} (j-r)x_i + \\
& \sum_{j=j_0+r-\psi+1}^{j_0+r} \sum_{i \in N_j} (j_0 - \psi)x_i + \sum_{j>j_0+r} \sum_{i \in N_j} (j-r-\psi)x_i \leq |T_1| + j_0 - \psi
\end{aligned}$$

is valid for P .

Proof.

(ii) Let $x \in P$ be given and denote the above inequality by $c^T x \leq \gamma$. We distinguish several cases.

Suppose, $x_{i_0} = 1$. If $x_i = 0$ for all $i \in N_j$, $j \geq r+1$, $i \neq i_0$, then the inequality is certainly valid. If $x_i = 1$ for some $i \in N_j$, $j \geq r+1$, $i \neq i_0$, then no more than $b - j_0 - j < |T_1|$ units in the knapsack are still available. Since $c_v \leq W(v)$ for all

items $v \in N$ and since $c_{i_0} + c_i + b - j - j_0 \leq b - r - \psi = |T_1| + j_0 - \psi$ holds, we have $c^T x \leq \gamma$.

Moreover, the coefficients of the items with weight in the range of $[j_0, j_0 + r]$ are less or equal than c_{i_0} . Hence, the same arguments as above apply to the case when $x_i = 1$ for $i \in N_j$, $j \in [j_0, j_0 + r]$.

In case $x_i = 1$ with $W(i) > j_0 + r$, then no more than $b - W(i) < |T_1|$ units in the knapsack are still available. Moreover, $c_v \leq W(v)$ for all items $v \in N$ and $c_i + b - W(i) = b - r - \psi = |T_1| + j_0 - \psi$.

The crucial case is when there exist $i' \in N_{j'}$, $i'' \in N_{j''}$, $r + 1 \leq j' \leq j''$, $i' \neq i''$ with $x_{i'} = x_{i''} = 1$, but $x_{i_0} = 0$. Then $c_{i'} + c_{i''} \leq j' + j'' - 2r$ holds. Moreover, $c_v \leq W(v)$ for all items $v \in N$ and $c_{i'} + c_{i''} + b - j' - j'' \leq j' + j'' - 2r + b - j' - j'' = b - r - r = |T_1| + j_0 - r \leq |T_1| + j_0 - \psi$. This completes the proof. \blacksquare

The inequalities defined in Proposition 2.2 (i) have been presented in [Le93] and can be viewed as special lifted cover inequalities if $N_{r+1} \neq \emptyset$ ([B75], [HJP75], [W75]). If $N_{r+1} = \emptyset$, but $N_j \neq \emptyset$ for some $j > r + 1$, these inequalities are lifted $(1, k)$ -configuration inequalities ([P80]).

For the inequalities defined in (2.2) (ii) the vector $\sum_{i \in T_1} e_i + e_{i_0}$ is a root. Moreover, the coefficient of item $i_0 \in N_{j_0}$ is defined as the corresponding weight decreased by some value $\psi \in [0, r]$. This guarantees that if we replace a subset T of T_1 and the item i_0 by two items $i' \in N_{j'}$ and $i'' \in N_{j''}$ such that $|T| + j_0 + r = j' + j''$, then $(j' - r) + (j'' - r) = |T| + (j_0 - r) \leq |T| + (j_0 - \psi)$. Nevertheless, there are inequalities $d^T x \leq \delta$ that define facets of P^2 such that $\delta = |T_1| + d_{i_0}$ for some $T_1 \subseteq N_1$, $i_0 \in N_{j_0} \setminus N_1$ and $d_{i_0} < j_0 - r$ where $r = b - |T_1| - j_0$.

Example 2.3. Consider the knapsack inequality

$$\sum_{i=1}^{100} x_i + \sum_{i=101}^{120} i x_i + 120 x_{121} + \sum_{i=122}^{151} (i-1) x_i \leq 300.$$

The inequality

$$\begin{aligned} & \sum_{i=51}^{100} x_i + \sum_{i=101}^{110} (i-100) x_i + \sum_{i=111}^{115} 10 x_i + \sum_{i=116}^{120} (i-105) x_i + \\ & \sum_{i=121}^{131} 15 x_i + \sum_{i=132}^{136} (i-1-115) x_i + \sum_{i=137}^{141} 20 x_i + \sum_{i=142}^{151} (i-1-120) x_i \leq 80 \end{aligned}$$

defines a facet of the corresponding polytope (checked by hand). Moreover, the vector x defined via $x_{151} = 1$ and $x_i = 1$ for all $i = 51, \dots, 100$ satisfies this

inequality at equation. Hence $r = 100$, but the coefficient of item 151 is 30 which is obviously smaller than $150 - r$. On the other hand, there exist two pairs of items $(s_1, t_1), (s_2, t_2)$, $s_1 = 110, t_1 = 136, s_2 = 120, t_2 = 121$ such that the vectors $x^1 = \sum_{i=51}^{100} e_i + e_{s_1} + e_{t_1}$ and $x^2 = \sum_{i=51}^{100} e_i + e_{s_2} + e_{t_2}$ satisfy the above inequality at equation. Moreover, summing the weight of the items s_1 and t_1 gives a weight of 245 which is smaller than the weight of item i_0 plus r ($j_0 + r = 250$). The same is true for the sum of weights of the items s_2 and t_2 (summing the weights of items s_2 and t_2 yields a weight of 240 which is by 10 smaller than $j_0 + r$). Therefore, items with weight in the range of $[121, 130]$ have a coefficient whose value is equal to $c_{s_2} = c_{t_2}$ and items with weight in the range of $[111, 115]$ or $[136, 140]$ have a coefficient with value c_{s_1} or c_{t_1} , respectively. In fact, there is some principle in generating this inequality that we want to explain now.

Recursion Principle (\star). Let $T_1 \subseteq N_1$ and $i_0 \in N_{j_0}, 2 \leq j_0$ such that $|T_1| + j_0 < b$ and set $r := b - |T_1| - j_0$. We choose pairs of items $(s_1, t_1), \dots, (s_k, t_k)$, $s_i, t_i \in N \setminus (N_1 \cup \{i_0\})$, $r + 1 \leq W(s_i) \leq W(t_i) < j_0$ and set $l_i := \max\{0, j_0 + r - W(s_i) - W(t_i)\}$, i.e., l_i is the value by which the sum of the weights of items s_i and t_i is smaller than $j_0 + r$. Moreover, we assume that $W(s_i) + l_i < W(s_{i+1})$ for $i = 1, \dots, k-1$, $W(t_i) + l_i < W(t_{i-1})$ for $i = 2, \dots, k$ and $j_0 > W(t_1) + l_1$. Finally, set $l_0 := 0$. Under these assumptions we define, for every $\mu_k \in [0, \min\{l_k, W(t_k) - W(s_k)\}]$, an inequality $c^T x \leq \gamma$ with $\gamma = |T_1| + c_{i_0}$ as follows.

For $i \in N_1$ we set $c_i := 1$ if $i \in T_1$. We set $c_i := 0$ if $W(i) \leq r$ and if $i \notin T_1$. For $i \in N_j$ with $r + 1 \leq j \leq W(s_1)$ we set $c_i := j - r$. Recursively we determine for $i \in N_j$, $u = 1, \dots, k-1$ the coefficient by setting $c_i := c_{s_u}$ if $W(s_u) \leq j \leq W(s_u) + l_u$ and $c_i := c_{s_u} + (j - W(s_u) - l_u)$ if $W(s_u) + l_u + 1 \leq j \leq W(s_{u+1})$. Similarly for $i \in N_j \setminus \{t_k\}$ we define $c_i := c_{s_k}$ if $W(s_k) \leq j \leq W(s_k) + l_k$ and we define $c_i := c_{s_k} + (j - W(s_k) - l_k)$, if $W(s_k) + l_k \leq j \leq W(t_k) + l_k - \mu_k$. We set $c_{t_k} := c_{s_k} + (W(t_k) - W(s_k) - \mu_k)$ and we define $c_i := c_{t_k}$ if $i \in N_j$ with $W(t_k) + l_k - \mu_k + 1 \leq j \leq W(t_k) + l_k$. Accordingly we determine for $i \in N_j$ the coefficient by setting $c_i := c_{t_u} + (j - W(t_u) - l_u)$ if $W(t_u) + l_u + 1 \leq j \leq W(t_{u-1})$ and $c_i := c_{t_{u-1}}$ if $W(t_{u-1}) \leq j \leq W(t_{u-1}) + l_{u-1}$ ($u = k, \dots, 2$). Finally, we set $c_i := c_{t_1} + (j - W(t_1) - l_1)$ if $W(t_1) + l_1 + 1 \leq j \leq j_0$, we set $c_i := c_{i_0}$ if $j_0 \leq j \leq j_0 + r$ and we set $c_i := c_{i_0} + (j - j_0 - r)$ if $j_0 + r \leq j$.

This yields explicitly the inequality

$$\begin{aligned}
& \sum_{i \in T_1} x_i + \sum_{j \leq W(s_1)} \sum_{i \in N_j} (j - r) x_i & + \\
& \sum_{u=1}^{k-1} \left(\sum_{j=W(s_u)+1}^{W(s_u)+l_u} \sum_{i \in N_j} (W(s_u) - r - \sum_{v=1}^{u-1} l_v) x_i \right) & + \\
& \sum_{j=W(s_u)+l_u+1}^{W(s_u+1)} \sum_{i \in N_j} (j - r - \sum_{v=1}^u l_v) x_i & + \\
& \sum_{j=W(s_k)+1}^{W(s_k)+l_k} \sum_{i \in N_j \setminus \{t_k\}} (W(s_k) - r - \sum_{v=1}^{k-1} l_v) x_i & + \\
& \sum_{j=W(s_k)+l_k+1}^{W(t_k)+l_k-\mu_k} \sum_{i \in N_j \setminus \{t_k\}} (j - r - \sum_{v=1}^k l_v) x_i & + \\
& (W(t_k) - r - \sum_{v=1}^{k-1} l_v - \mu_k) x_{t_k} & + \\
& \sum_{j=W(t_k)+l_k-\mu_k+1}^{W(t_k)+l_k} \sum_{i \in N_j} (W(t_k) - \mu_k - r - \sum_{v=1}^{k-1} l_v) x_i & + \\
& \sum_{u=2}^k \left(\sum_{j=W(t_u)+1}^{W(t_u)+l_u} \sum_{i \in N_j} (j - r - \sum_{v=1}^k l_v - \mu_k - \sum_{v=u}^{k-1} l_v) x_i \right) & + \\
& \sum_{j=W(t_{u-1})+1}^{W(t_{u-1})+l_{u-1}} \sum_{i \in N_j} (W(t_{u-1}) - r - \sum_{v=1}^k l_v - \mu_k - \sum_{v=u}^{k-1} l_v) x_i & + \\
& \sum_{j=W(t_1)+1}^{j_0} \sum_{i \in N_j} (j - r - \mu_k - l_k - 2 \sum_{v=1}^{k-1} l_v) x_i & + \\
& \sum_{j=j_0+r}^{j_0+r} \sum_{i \in N_j} (j_0 - r - \mu_k - l_k - 2 \sum_{v=1}^{k-1} l_v) x_i & + \\
& \sum_{j>j_0+r} \sum_{i \in N_j} (j - 2r - \mu_k - l_k - 2 \sum_{v=1}^{k-1} l_v) x_i & \leq \\
& |T_1| + (j_0 - r - \mu_k - l_k - 2 \sum_{v=1}^{k-1} l_v).
\end{aligned}$$

Proposition 2.4. The inequalities defined via the Recursion Principle (\star) are valid for P^2 (note that they are not necessarily valid for P).

Proof. Let $x \in P^2$ be given and denote the above inequality by $c^T x \leq \gamma$. If $x_i = 1$ for some i with $W(i) \geq j_0$, then by using similar arguments as in the proof of Proposition 2.2 we obtain $c^T x \leq \gamma$. The crucial case is when there exist $i' \in N_{j'}$, $i'' \in N_{j''}$, $r+1 \leq j' \leq j''$ with $x_{i'} = x_{i''} = 1$. We distinguish two cases:

(1) $W(s_{u-1}) + l_{u-1} \leq j' \leq W(s_u)$ for some $u \in \{2, \dots, u\}$ or $j' \leq W(s_1)$ ($u = 1$).

(1.1) If $W(t_u) \leq j'' \leq W(t_u) + l_u$, we obtain

$$\begin{aligned}
c_{i'} + c_{i''} & \leq j' - r - \sum_{v=1}^{u-1} l_v + W(t_u) - r - \sum_{v=1}^k l_v - \mu_k - \sum_{v=u+1}^{k-1} l_v \\
& = W(s_u) + W(t_u) + l_u - 2r - 2 \sum_{v=1}^{k-1} l_v - l_k - \mu_k + (j' - W(s_u)).
\end{aligned}$$

If $W(s_u) + W(t_u) \leq j_0 + r$, then $W(s_u) + W(t_u) + l_u = j_0 + r$ and we obtain $c^T x \leq \gamma$. Otherwise, $W(s_u) + W(t_u) > j_0 + r$ and by definition, $l_u = 0$. In this case we obtain $c_{i'} + c_{i''} \leq j' + j'' - 2r - 2 \sum_{v=1}^{k-1} l_v - l_k - \mu_k$. If $j' + j'' > j_0 + r$, then there exists $T \subseteq T_1$ with $|T| = (j' + j'' - j_0 - r)$ and $x_v = 0$ for all $v \in T$. Hence, $c^T x \leq \gamma$. Accordingly, $c^T x \leq \gamma$, if $j' + j'' \leq j_0 - r$.

(1.2) If $W(t_u) + l_u \leq j''$, we obtain

$$\begin{aligned}
c_{i'} + c_{i''} & \leq j' - r - \sum_{v=1}^{u-1} l_v + j'' - r - \sum_{v=1}^k l_v - \mu_k - \sum_{v=u}^{k-1} l_v \\
& = j' + j'' - 2r - 2 \sum_{v=1}^{k-1} l_v - l_k - \mu_k.
\end{aligned}$$

If $j' + j'' > j_0 + r$, then there exists $T \subseteq T_1$ with $|T| = (j' + j'' - j_0 - r)$ and $x_v = 0$ for all $v \in T$. Hence, $c^T x \leq \gamma$. Accordingly, $c^T x \leq \gamma$, if $j' + j'' \leq j_0 - r$.

(1.3) If $j'' < W(t_u)$, then the first subcase is $c_{i''} = c_{t_u} - (W(t_u) - j'')$ and hence, $c_{i'} + c_{i''} \leq W(s_u) + W(t_u) + l_u - 2r - 2\sum_{v=1}^{k-1} l_v - l_k - \mu_k + (j' + j'' - W(s_u) - W(t_u))$. Then by using exactly the same arguments as in the first subcase we obtain $c^T x \leq \gamma$. Otherwise, $c_{i''} > c_{t_u} - (W(t_u) - j'')$. Then there exists $v > u$ such that $j'' < W(t_v) + l_v$ and $W(s_v) + W(t_v) < j_0 + r$. Moreover, $c_{i'} < c_{s_v}$ and $c_{i''} \leq c_{t_v}$ and $|T_1| + c_{s_v} + c_{t_v} = \gamma$. This implies $c^T x \leq \gamma$ in this case.

(2) $W(s_u) < j' \leq W(s_u) + l_u$. This case is similar to case 1, so that we omit the details.

(3) $j' > W(s_k) + l_k$. Then, $c_{i'} \leq j' - r - \sum_{v=1}^k l_v$ if $i' \neq t_k$. Moreover, $j'' \geq j'$ and hence, $c_{i''} \leq j'' - r - \sum_{v=1}^k l_v$ if $i'' \neq t_k$. Since $c_{t_k} = W(t_k) - r - \sum_{v=1}^{k-1} l_v - \mu_k$, we obtain

$$\begin{aligned} c_{i'} + c_{i''} &\leq j' + j'' - 2r - 2\sum_{v=1}^{k-1} l_v - l_k - \mu_k \\ &= (j' + j'' - j_0 - r) + j_0 - r - 2\sum_{v=1}^{k-1} l_v - l_k - \mu_k. \end{aligned}$$

Hence, $c^T x \leq \gamma$, if $j' + j'' \leq j_0 + r$. Otherwise, there exists $T \subseteq T_1$ with $|T| = (j' + j'' - j_0 - r)$ and $x_v = 0$ for all $v \in T$. Therefore, $c^T x \leq \gamma$ in this case, too. \blacksquare

Theorem 2.5. The inequalities defined in Propositions 2.2 and 2.4 together with the inequalities

$$\begin{aligned} -x_i &\leq 0, \quad i \in N \\ x_i &\leq 1, \quad i \in N \setminus N_1 \\ \sum_{j \geq \lfloor \frac{b}{3} \rfloor + 1}^{\lfloor \frac{b}{2} \rfloor} \sum_{i \in N_j} x_i &\leq 2, \end{aligned}$$

and the knapsack inequality completely describe the polytope P^2 .

Proof. This proof is left to the Appendix. \blacksquare

Corollary 2.6. The system of inequalities

$$\begin{aligned} -x_i &\leq 0, \quad i \in N \\ \sum_{i \in T_1} x_i + \sum_{j > b - |T_1|} \sum_{i \in N_j} (j - b + |T_1|) x_i &\leq |T_1|, \quad T_1 \subseteq N_1 \\ \sum_{j \geq \lfloor \frac{b}{2} \rfloor + 1} \sum_{i \in N_j} x_i &\leq 1 \end{aligned}$$

and the knapsack inequality completely describes the polytope P^1 .

Proof. The proof follows quite easily from the proof of Theorem 2.5. We briefly repeat the main steps.

Let F be a non-trivial facet of P^1 that is induced by the inequality $c^T x \leq \gamma$ that is not the knapsack inequality. We define $T_1 := \{i \in N_1 \mid c_i > 0\}$ and w.l.o.g we assume that $T_1 = \{1, \dots, k\}$ and $c_1 \geq c_2 \geq \dots \geq c_k$. We distinguish the following cases:

Case 1: $k = 0$. In this case it is easy to see that F is induced by the inequality $\sum_{j=\lfloor \frac{b}{2} \rfloor + 1}^b \sum_{i \in N_j} x_i \leq 1$.

Case 2: $\gamma < \sum_{v=1}^k c_v$. Then, $k > b$ and consequently, every $x \in F$ satisfies $\sum_{i \in N_1} x_i + \sum_{j \in B^1} \sum_{i \in N_j} j x_i = b$.

Case 3: $\gamma = \sum_{v=1}^k c_v$. Since $F \not\subseteq \{x \in P^1 \mid \sum_{i \in N_1} x_i + \sum_{j \in B^1} \sum_{i \in N_j} j x_i = b\}$, we conclude that $k < b$ and we set $r := b - k$. Moreover, we define $x^0 := \sum_{v=1}^k e_v$. Since $\gamma = \sum_{v=1}^k c_v = c^T x^0$ the relation $c_i = 0$ for all $i \in N_j$ with $j \leq b - k$ holds.

Now let $i \in N_j$, $j \geq b - k + 1$ be given. Since $x := x^0 - \sum_{v=k-(j-r)+1}^k e_v + e_i$ is feasible, we obtain

$$c_i \leq c_k + c_{k-1} + \dots + c_{k-(j-r)+1}.$$

On the other hand, there exists a root x' with $x'_i = 1$. Since $j \geq b - k + 1$, there exists $T \subseteq T_1$, $|T| = j - r$ with $x_t = 0$ for all $t \in T$. This yields

$$c_i \geq c_k + c_{k-1} + \dots + c_{k-(j-r)+1}$$

and hence $c_i = c_k + c_{k-1} + \dots + c_{k-(j-r)+1}$. Now it follows that the inequality $c^T x \leq \gamma$ is of the type $\sum_{i \in T_1} x_i + \sum_{j \geq b-k} \sum_{i \in N_j} (j - (b - k)) x_i \leq k$.

Case 4: $\gamma > c(T_1)$. We suppose $F \not\subseteq \{x \in P^1 \mid \sum_{i \in N_1} x_i + \sum_{j \in B^1} \sum_{i \in N_j} j x_i = b\}$. Hence, there exists a root x^0 with $\sum_{i \in N_1} x_i^0 + \sum_{j \in B^1} \sum_{i \in N_j} j x_i^0 < b$. This root satisfies the condition $x_t^0 = 1$ for all $t \in T_1$. Moreover, $\gamma > c(T_1)$ implies that there exists $i_0 \in N_{j_0}$ such that $x_{i_0}^0 = 1$, i.e., $\gamma = c(T_1) + c_{i_0}$. This further implies that every root $x \in F$ satisfies the equation $\sum_{j=\lfloor \frac{b}{2} \rfloor + 1}^b \sum_{i \in N_j} x_i = 1$, and hence does not define a facet with $T_1 \neq \emptyset$. This proves the statement. \blacksquare

Remark 2.7. In case $b = 3$, the weights of the items are 1,2 or 3. Since in this case $B^1 = [2, 3]$, a complete description of the associated knapsack polytopes is

given by Corollary 2.6.

3 Valid Inequalities for the general case

Based on the complete description of P^1 and P^2 we now generalize some of the inequalities that came up such that they apply to the general 0/1 knapsack polytope.

Among the inequalities that were introduced in Section 2 (see Proposition 2.2) were the following ones.

(3.1) Let $T_1 \subseteq N_1$, $|T_1| < b$. The inequality

$$\sum_{i \in T_1} x_i + \sum_{j \geq b - |T_1| + 1} \sum_{i \in N_j} (j - b + |T_1|) x_i \leq |T_1|$$

is valid for P .

(3.2) Let $T_1 \subseteq N_1$, $i_0 \in N_{j_0}$, $2 \leq j_0$, $|T_1| + j_0 < b$ and set $r := b - |T_1| - j_0$. For every integer $\psi \in [0, r]$ with $j_0 - \psi > 0$ the inequality

$$\sum_{i \in T_1} x_i + (j_0 - \psi) x_{i_0} + \sum_{j=r+1}^{j_0+r-\psi} \sum_{i \in N_j \setminus \{i_0\}} (j - r) x_i + \sum_{j=j_0+r-\psi+1}^{j_0+r} \sum_{i \in N_j} (j_0 - \psi) x_i + \sum_{j > j_0+r} \sum_{i \in N_j} (j - r - \psi) x_i \leq |T_1| + j_0 - \psi$$

is valid for P .

One way to generalize the inequalities (3.1) and (3.2) is not only to consider a subset $T_1 \subseteq N_1$ and possibly one additional item i_0 , but a set $T_1 \subseteq N_1$ and a set $I = \{i_1, \dots, i_k\}$ of items. More precisely,

(3.3) Let $T_1 \subseteq N_1$ and $I = \{i_1, \dots, i_k\} \subseteq N \setminus N_1$ such that $i_v \in N_{j_v}$ for $v = 1, \dots, k$, $2 \leq j_1 \leq \dots \leq j_k$ and $|T_1| + \sum_{v=1}^k j_v < b$. We define $r := b - |T_1| - \sum_{v=1}^k j_v$. For every integer $\psi \in [0, r]$ such that $j_k - \psi > 0$ the inequality

$$\sum_{i \in T_1} x_i + \sum_{v=1}^{k-1} j_v x_{i_v} + (j_k - \psi) x_{i_k} + \sum_{j=r+1}^{j_k+r-\psi} \sum_{i \in N_j \setminus I} (j - r) x_i +$$

$$\sum_{j=j_k+r-\psi+1}^{j_k+r} \sum_{i \in N_j} (j_k - \psi)x_i + \sum_{j > j_k+r} \sum_{i \in N_j} (j - r - \psi)x_i \leq |T_1| + \sum_{v=1}^k j_v - \psi$$

is valid for P . The set $T_1 \cup I$ is called *starting set* and the parameter ψ *reduction parameter*, because the coefficient of i_k is equal to its weight reduced by ψ .

A proof of this statement is analogous to the proof of Proposition (2.2) (ii) and is omitted here.

The next question that we want to deal with is under which conditions the inequalities (3.1) – (3.3) define facets of P . Before doing so let us illustrate these inequalities on some example.

Example 3.4. We consider the knapsack inequality in 0/1 variables

$$x_1 + x_2 + x_3 + x_4 + x_5 + 3x_6 + 4x_7 + 6x_8 + 7x_9 + 9x_{10} + 10x_{11} \leq 13.$$

- (i) The inequality $x_1 + x_2 + x_3 + x_4 + x_5 + x_{10} + 2x_{11} \leq 5$ is of the type (3.1), since $r = 13 - 5 = 8$. It defines a facet of the corresponding knapsack polytope. Similarly, the inequality $x_1 + x_2 + x_3 + x_4 + x_{11} \leq 4$ is of the type (3.1) and facet defining.
- (ii) By setting $T_1 = \{1, 2, 3, 4\}$, $i_0 = 7$ we obtain $r = 13 - 4 - 4 = 5$. If we choose $\psi = 3$, we obtain the inequality

$$x_1 + x_2 + x_3 + x_4 + x_7 + x_8 + x_9 + x_{10} + 2x_{11} \leq 5.$$

Setting $\psi = 2$ yields the inequality

$$x_1 + x_2 + x_3 + x_4 + 2x_7 + x_8 + 2x_9 + 2x_{10} + 3x_{11} \leq 6.$$

Both inequalities are of type (3.2) and define facets of the corresponding knapsack polytope.

- (iii) Define $T_1 := \{1, 2, 3\}$, $i_1 := 6$, $i_2 = 7$ and set $\psi := 1$. Thus, $r = 13 - 3 - 3 - 4 = 3$. The inequality

$$x_1 + x_2 + x_3 + 3x_6 + 3x_7 + 3x_8 + 3x_9 + 5x_{10} + 6x_{11} \leq 9$$

is of type (3.3) and defines a facet of the corresponding knapsack polytope.

If we set $\psi = 0$, we obtain the facet-defining inequality

$$x_1 + x_2 + x_3 + 3x_6 + 4x_7 + 3x_8 + 4x_9 + 6x_{10} + 7x_{11} \leq 10.$$

The property that the above inequalities define facets of the given polytope was checked by a program developed in [C91].

For the knapsack polytope considered in Example 3.4 there are many more facet defining inequalities of the types (3.2) and (3.3). In principle, for every subset $T_1 \subseteq N_1$ and subset I of $N \setminus N_1$ such that $|T_1| + \sum_{i \in I} W(i) < b$ we can generate a series of inequalities of types (3.1) or (3.2) or (3.3), which differ according to the choice of the parameter ψ . However, not all of these inequalities define facets. This issue is discussed now.

Proposition 3.5.

- (i) The inequalities (3.1) with $T_1 \neq \emptyset$ define non-trivial facets of P if and only if $N_j \neq \emptyset$ for some $j > r$.
- (ii) The inequalities (3.2) with $T_1 \neq \emptyset$ and $0 < \psi < r$ define facets of P if and only if $N_{j_0+r-\psi} \neq \emptyset$.
- (iii) The inequalities (3.2) with $T_1 \neq \emptyset$ and $\psi = r$ define facets of P if and only if $N_{j_0} \setminus \{i_0\} \neq \emptyset$ or there exist numbers j_1 and j_2 and items $i_1 \in N_{j_1}$, $i_2 \in N_{j_2}$ with the properties $i_1 \neq i_2$, $r + 1 \leq j_1 \leq j_2 < j_0$ and $j_1 + j_2 \geq j_0 + r$.
- (iv) The inequalities (3.2) with $T_1 \neq \emptyset$ and $\psi = 0$ define facets of P if and only if $N_j \neq \emptyset$ for some $j \geq j_0 + r$.

Proof. First note that the inequalities are valid. This has been pointed out above. Moreover, statement (i) has been shown in [Le93].

(ii) Let $\emptyset \neq T_1 \subseteq N_1$, $i_0 \in N_{j_0}$, $j_0 \geq 2$ and $|T_1| + j_0 < b$. Moreover, we set $r = b - |T_1| - j_0$. Suppose $N_{j_0+r-\psi} = \emptyset$ and let $d^T x \leq \delta$ denote the inequality of type (3.2). Since $N_{j_0+r-\psi} = \emptyset$, every root $x \in F_d := \{x \in P \mid d^T x = \delta\}$ either satisfies ($x_i = 1$ for some $i \in N_j$, $r + 1 \leq j < j_0 + r - \psi$ and $x_{i_0} = 1$) or ($x_{i_0} = 1$ and $x_i = 0$ for all $i \in N_j \setminus \{i_0\}$, $j \geq r + 1$) or ($x_{i'} = 1$ for some $i' \in N_j$, $j > j_0 + r - \psi$ and $x_i = 0$ for all $i \in N_j \setminus \{i'\}$, $r + 1 \leq j$). Hence, every $x \in F_d$ satisfies the equation $\sum_{j > j_0+r-\psi} x_i + x_{i_0} = 1$ and consequently, $\dim(F_d) \leq |N| - 2$. Thus, F_d does not define a facet of P .

Conversely, if $N_{j_0+r-\psi} \neq \emptyset$, the inequality $d^T x \leq \delta$ defines a facet F_d of P for the following reasons.

Suppose, $c^T x \leq \gamma$ is a facet defining inequality of P such that $F_d \subseteq \{x \in P \mid c^T x = \gamma\}$. First note that for every $i \in N_j$, $j \leq r$ the vectors $x^0 := \sum_{v \in T_1} e_v + e_{i_0}$ and $x^0 + e_i$ are elements of F_d . Thus, $c_i = 0$ in this case.

Let $i \in N_{j_0+r-\psi}$ and consider the vectors x^0 , $x^1 = x^0 - e_{i_0} + e_i$ and $x^2 = x^0 + e_i - \sum_{v \in T} e_v$, where $T \subseteq T_1$, $|T| = j_0 - \psi$. As $d^T x^0 = d^T x^1 = d^T x^2$, we obtain $c_i = c_{i_0}$. Moreover, since T can be chosen arbitrarily, we conclude $c_{i'} = c_{i''}$ for all $i', i'' \in T_1$. Finally, this implies $c_i = c_{i_0} = (j_0 - \psi)c_{i'}$, where $i' \in T_1$.

Similary, for $i \in N_j$, $r + 1 \leq j \leq j_0 + r - \psi$ we define the feasible solutions $x^i = x^0 + e_i - \sum_{v \in T} e_v$, where $T \subseteq T_1$, $|T| = j - r$. Since $d^T x^0 = d^T x^i$ we conclude $c_i = (j - r)c_{i'}$ where $i' \in T_1$.

For $i \in N_j$, $j_0 + r - \psi < j < j_0 + r$ we have that the vectors x^0 and $x^0 - e_{i_0} + e_i$ are roots of F_d . This implies $c_i = c_{i_0}$.

Finally, for an item $i \in N_j$ with $j = j_0 + r + t$, $t \geq 1$ we obtain $c_i = c_{i_0} + tc_{i'}$ where $i' \in T_1$, because the vector $x^0 - e_{i_0} - \sum_{v \in T} e_v + e_i$ ($T \subseteq T_1$, $|T| = t$) belongs to the face F_d .

This shows that $c^T x \leq \gamma$ and $d^T x \leq \delta$ are equal up to multiplication by a scalar, which completes the proof of (ii).

The claims stated in (iii) and (iv) can be shown by using similar arguments as in the proof of (ii). ■

Now let $d^T x \leq \delta$ be an inequality of type (3.3) and suppose $\delta = |T_1| + \sum_{v=1}^k j_v$ for some set $\emptyset \neq T_1 \subseteq N_1$ and k items i_1, \dots, i_k , $i_v \in N_{j_v}$. In case $k \geq 2$ it is not easy anymore to derive necessary and sufficient conditions under which $d^T x \leq \delta$ defines a facet of P . In particular, we do not see how to treat several cases analogously and hence end up with a very long and painful proof. However, in one case we can easily decide that the inequality $d^T x \leq \delta$ of type (3.3) defines a facet of P .

Proposition 3.6. Let $\emptyset \neq T_1 \subseteq N_1$ and $I = \{i_1, \dots, i_k\} \subseteq N \setminus N_1$ such that $i_v \in N_{j_v}$ for $v = 1, \dots, k$, $2 \leq j_1 \leq \dots \leq j_k$ and $|T_1| + \sum_{v=1}^k j_v < b$. We define r as the rest capacity, i.e., $r := b - |T_1| - \sum_{v=1}^k j_v$. For every integer $\psi \in [0, \min\{r, j_k - j_{k-1}\}]$ such that $j_k - \psi \leq |T_1|$, the inequality

$$\sum_{i \in T_1} x_i + \sum_{v=1}^{k-1} j_v x_{i_v} + (j_k - \psi)x_{i_k} + \sum_{j=r+1}^{j_k+r-\psi} \sum_{i \in N_j \setminus I} (j-r)x_i + \sum_{j=j_k+r-\psi+1}^{j_k+r} \sum_{i \in N_j} (j_k - \psi)x_i + \sum_{j > j_k+r} \sum_{i \in N_j} (j-r-\psi)x_i \leq |T_1| + \sum_{v=1}^k j_v - \psi$$

defines a facet of P if $N_{j_v+r} \neq \emptyset$ for all $v = 1, \dots, k-1$ and $N_{j_k+r-\psi} \setminus \{i_k\} \neq \emptyset$.

Proof. We briefly sketch the proof of Proposition 3.6. Let $d^T x \leq \delta$ denote the above inequality and let x^0 denote the vector $\sum_{v \in T_1} e_v + \sum_{v=1}^k e_{i_v}$. Moreover, we assume that $c^T x \leq \gamma$ is a facet defining inequality of P such that $F_d := \{x \in P \mid d^T x = \delta\} \subseteq \{x \in P \mid c^T x = \gamma\}$.

As $j_k - \psi \leq |T_1|$, the vector $x = \sum_{v \in T} e_v + \sum_{v=1}^k e_{i_v} + e_i$ with $i \in N_j \setminus I$, $r+1 \leq j \leq j_k + r - \psi$, $T \subseteq T_1$, $|T| = j - r$ is an element of F_d . Hence, $c_i = c(T)$.

Moreover, $c_{i'} = c_{i''}$ for all $i', i'' \in T_1$, because T can be chosen arbitrarily. This further implies $c_i = (j - r)c_{i'}$ where $i' \in T_1$ for all $i \in N_j$, $r+1 \leq j \leq j_k + r - \psi$, $i \notin I$. Also, $c_i = c_{i_0}$ for all $i \in N_j$, $j_k + r - \psi + 1 \leq j \leq j_k + r$ and $c_i = c_{i_0} + t c_{i'}$ for all $i \in N_{j_k + r + t}$, $i' \in T_1$ can be shown by applying the same arguments as given in the proof of Proposition 3.5.

Due to the conditions $N_{j_v + r} \neq \emptyset$ for all $v = 1, \dots, k-1$ we obtain that $c_{i_v} = c_i$ for $i \in N_{j_v + r}$, because $x^v := x^0 - e_{i_v} + e_i$ is a root of F_d . Finally, the condition $N_{j_k + r - \psi} \neq \emptyset$ guarantees that $x := x^0 - e_{i_k} + e_i$ satisfies $d^T x = \delta$ where $i \in N_{j_k + r - \psi}$. Consequently, $c_i = c_{i_k}$.

To conclude that $c^T x \leq \gamma$ and $d^T x \leq \delta$ define the same face, we just notice that for all $i \in N_j \setminus I$, $2 \leq j \leq r$ the vector $x^0 + e_i$ is a root of F_d which shows $c_i = 0$. ■

Of course, there are many more possibilities to obtain valid inequalities by extending this “reduction principle”. Namely, in the inequalities of type (3.3) an item of $I \setminus \{i_k\}$ has a coefficient whose value is equal to the original weight of this item. In principle, the inequality of type (3.3) can be extended to the case where we choose for every $i \in I$ a reduction coefficient $\psi_i \in [0, r]$ independently and define an inequality $c^T x \leq \gamma$ by setting $c_i = W(i) - \psi_i$ for $i \in I$ and $c_i = 1$ for $i \in T_1$. In this case, it is no longer obvious to determine the coefficients for the remaining items explicitly, because for every $i \in N_j \setminus (N_1 \cup I)$, $j \geq r+1$ we need to solve the following integer programming problem

$$(\star) \quad \min \left\{ \sum_{i \in T_1 \cup I} c_i z_i \mid z_i \in \{0, 1\}, i \in T_1 \cup I, \sum_{i \in T_1 \cup I} W(i) z_i \geq j - r \right\}.$$

Under certain assumptions this problem can be solved explicitly. For instance, if we choose a subset $I = \{i_1, \dots, i_k\}$, $i_v \in N_{j_v}$, $r+1 \leq j_1 \leq \dots \leq j_k$, with the property that the weight of the items in I lie in the range $[i_1, i_1 + r]$ and if we choose a reduction parameter $\psi_k \in [0, i_1 + r - i_k]$ and set $\psi_t = j_t + \psi_k - j_k$ for $t = 1, \dots, k-1$, then one can solve problem (\star) for all $i \in N \setminus (I \cup T_1)$ explicitly.

One might also think of extending an inequality of type (3.3) in a similar way as

was done in Proposition 2.4 for the special case where $|I| = 1$. We claim that in principle this is possible, but a formal proof would require many technical details (see Proposition 2.4) that we do not want to go into here.

Rather let us end this section by posing the question: How can we extend an inequality of type (3.3) to the case when $T_1 \not\subseteq N_1$, but is chosen arbitrarily? Certainly, if $T_1 \subseteq N \setminus N_1$, this question is of no interest for the polytopes P^s , because the only non-trivial facet defining inequality $d^T x \leq \delta$ for P^s that is not the knapsack inequality and satisfies $d_v = 0$ for $v \in N_1$ is the one $\sum_{j \in B^s} \sum_{i \in N_j} x_i \leq s$. However, for the 0/1 knapsack polytope in general, there are many inequalities $d^T x \leq \gamma$ that define facets and have the property that $d_i = 0$ for all $i \in N_1$.

Example 3.7. Consider the knapsack polytope defined as convex hull of all 0/1 vectors that satisfy the constraint

$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 + 2x_6 + 3x_7 + 4x_8 + 5x_9 + 7x_{10} + 10x_{11} \leq 15.$$

For instance, the inequalities

- (1) $x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + 2x_{11} \leq 4$,
- (2) $x_5 + x_6 + x_7 + 2x_8 + x_9 + x_{10} + 3x_{11} \leq 5$,
- (3) $x_1 + x_2 + x_4 + x_5 + 2x_8 + x_{10} + 3x_{11} \leq 6$,
- (4) $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + 2x_7 + x_8 + x_9 + 2x_{10} + 4x_{11} \leq 8$,

define facets of the corresponding polyhedron (checked by a program developed in [C91]).

We now show how the inequalities (1) – (4) in Example 3.7 can be obtained by generalizing the inequalities of type (3.3). Here, the idea is the following. Choose a subset $\emptyset \neq T_1 = \{t_1, \dots, t_l\} \subseteq N$, $W(t_1) \leq \dots \leq W(t_l)$ and some item $i_0 \in N_{j_0} \subseteq N \setminus (N_1 \cup T_1)$ such that $j_0 + \sum_{v=1}^l W(t_v) < b$, $j_0 > j_l$ and $j_l \leq r$ where $r = b - j_0 - \sum_{u=1}^l W(t_u)$. Under these assumptions an inequality $d^T x \leq \delta$ is defined by setting $d_i = 0$ for all $i \in N_1$ and $d_i = 1$ for all $i \in T_1$. For the items $i \in N \setminus (N_1 \cup T_1)$ we compute two numbers: the first one is its “new” weight which is the old weight relative to the weights of the items in T_1 . The second number is the “new” (weight minus rest capacity r) which is the original weight minus the rest capacity relative to the weights of the items in T_1 . The coefficient of an item is its “new” (weight minus rest capacity) if its original weight is smaller than j_0 . For the item i_0 we choose some coefficient in the range of its “new” (weight minus rest capacity) and its new weight”. To compute the coefficients of the remaining

items we apply the same principle as we did in Proposition 2.2 (ii) based on their “new” weights.

More precisely, we obtain the following proposition.

Proposition 3.8. Let $\emptyset \neq T_1 = \{t_1, \dots, t_l\} \subseteq N$ and $i_0 \in N_{j_0} \subseteq N \setminus T_1$ such that $\sum_{v=1}^l W(t_v) + j_0 < b$, $W(t_1) \leq \dots \leq W(t_l) < j_0$ and $W(t_l) \leq r$ where $r = b - \sum_{v=1}^l W(t_v) - j_0$ denotes the rest capacity. We set $\phi(j) := 0$ if j is a negative integer and for every integer $0 \leq j \leq \sum_{v=1}^l W(t_v)$ we define $\phi(j)$ as the value j relative to the weight of the items in T_1 . In formulas,

$$\phi(j) = \min\{u \geq 0 \mid \sum_{v=l-u+1}^l W(t_v) \geq j\}.$$

We choose some integer $\psi \in [0, \phi(j_0) - \phi(j_0 - r) - 1]$, we require $\sum_{v=1}^l W(t_v) \geq \phi(j_0) - \psi$ and define $j_1 = \min\{j \geq r + 1 \mid \phi(j - r) = \phi(j_0) - \psi\}$ (note that since $W(t_l) \leq r$, then $\phi(j_0) - \phi(j_0 - r) \geq 1$ and $j_1 \in [j_0, j_0 + r]$). Finally, in case $\psi = 0$ we require that $\phi(j - j_0 - r) + \phi(j_0) \leq \phi(j - r)$ for all $j > j_0 + r$ such that $N_j \neq \emptyset$ and $j - r \leq \sum_{v=1}^l W(t_v)$. Under these assumptions we define the inequality

$$\sum_{i \in T_1} x_i + (\phi(j_0) - \psi)x_{i_0} + \sum_{j=r+1}^{j_1-1} \sum_{i \in N_j \setminus \{i_0\}} \phi(j - r)x_i +$$

$$\sum_{j=j_1}^{j_0+r} \sum_{i \in N_j} (\phi(j_0) - \psi)x_i + \sum_{j>j_0+r} \sum_{i \in N_j} (\phi(j - j_0 - r) + \phi(j_0) - \psi)x_i \leq |T_1| + \phi(j_0) - \psi.$$

It is valid for P . If $N_{j_1} \setminus \{i_0\} \neq \emptyset$ and if there exists $i \in N_j \setminus \{i_0\}$ for some $r + 1 \leq j \leq j_1$ with $\sum_{v=1}^{\phi(j-r)} W(t_v) \geq j - r$, the inequality defines a facet of P .

Proof. We show that the inequality is valid for P .

First note that $\phi(a) + \phi(b) \leq \phi(a + b) + 1$ for all positive integers a, b such that $a + b \leq \sum_{v=1}^l W(t_v)$. Moreover, $W(t_l) \leq r$ and hence, $\phi(a - r) \leq \phi(a) - 1$ for all numbers $r + 1 \leq a \leq \sum_{v=1}^l W(t_v)$. Finally, it is not hard to show that $\phi(a) + \phi(b) \leq \phi(c) + \phi(d) + 1$, if $0 < a, b, c, d < \sum_{v=1}^l W(t_v)$ and $a + b = c + d > \sum_{v=1}^l W(t_v)$.

Let $x \in P$ be given and denote the above inequality by $c^T x \leq \gamma$. If $x_{i_0} = 1$ or if $x_i = 1$ for some $i \in N_j$, $j \geq j_1$, then $c^T x \leq \gamma$ can be shown by using similar arguments as in the proof of Proposition 2.2. Moreover, if $x_i = 1$ for some $i \in N_j \setminus \{i_0\}$, $j < j_1$ and $x_{i'} = 0$ for all $i' \in N \setminus (T_1 \cup \{i\})$, we have $c^T x \leq \gamma$, because $c_i \leq c_{i_0}$. The crucial case is when there exist $i' \in N_{j'}, i'' \in N_{j''}$, $i' \neq i''$, $r + 1 \leq j' \leq j'' \leq j_1$ with $x_{i'} = x_{i''} = 1$.

Let $I \subseteq N \setminus T_1$, $r+1 \leq W(u) \leq j_0+r-\psi$, $u \in I$ be the subset with $x_u = 1$ for all $u \in I$. If $W(I) \leq j_0+r$, then $c^T x \leq |T_1| + \sum_{u \in I} \phi(W(u) - r) \leq |T_1| + \phi(W(u) - |I|r) + |I| - 1 \leq |T_1| + \phi(j_0+r - |I|r) + |I| - 1 \leq |T_1| + \phi(j_0-r) - (|I|-2) + |I| - 1 \leq |T_1| + c_{i_0} = \gamma$.

If $W(I) > j_0+r$, then there exists $T \subseteq T_1$, $|T| \geq W(I) - j_0 - r$ such that $x_t = 0$ for all $t \in T$. As $|I| \geq 2$, we have $c^T x \leq |T_1| - |T| + \sum_{u \in I} \phi(W(u) - r) \leq |T_1| - |T| + \phi(j_0+r-2r) + 1 + \phi(W(I) - j_0 - r) \leq |T_1| - |T| + c_{i_0} + |T| = \gamma$.

Under the assumptions in Proposition 3.8 it is easy to show that the inequality defines a facet. ■

Remark. In Proposition 3.8 we consider only ψ -values such that $\psi < \phi(j_0) - \phi(j_0 - r)$. Indeed, if $\psi = \phi(j_0) - \phi(j_0 - r)$, the proof of Proposition 3.8 does not work anymore. In this case additional conditions must be satisfied that guarantee that the inequality is valid. Finding such conditions is not very hard, but we refrain from explaining the details.

Example 3.9. Consider the knapsack polytope introduced in Example 3.7. Setting $T_1 = \{5, 6, 7\}$ and $i_0 = 8$ yields $r = 4$ and $\phi(j_0) = 2$. If we choose $\psi = 0$, we obtain inequality 2 and if we choose $\psi = 1$, we obtain inequality (1).

Inequality (3) can be derived from Proposition 3.8 if we choose $T_1 = \{1, 2, 4, 5\}$, $i_0 = 8$ and $\psi = 0$.

Finally, setting $T_1 = \{1, 2, 3, 4, 5, 6\}$ and $i_0 = 7$ yields $r = 3$ and $\phi(j_0) = 2$. With $\psi = 0$ we obtain the inequality (4).

To conclude this section let us mention that Proposition 3.8 can in principle be generalized by replacing the single item i_0 by a set $I = \{i_1, \dots, i_k\}$ that satisfies certain conditions. Then, the definition of $\phi(j)$ must be replaced by some more complicated formula. The conditions under which the corresponding inequality is valid for P are quite technical and we refrain within this paper from explaining the details. Instead we give an example.

Example 3.10. Consider the knapsack polytope introduced in Example 3.7. For instance, the inequality

$$x_6 + x_7 + 2x_8 + 2x_9 + 3x_{10} + 4x_{11} \leq 6$$

defines a facet of this polytope (checked by a program developed in [C91]). By setting $T_1 = \{6, 7\}$ and $I = \{8, 9\}$ we obtain $r = 1$. Moreover, we choose $\psi = 0$. Let $c^T x \leq \gamma$ denote the above inequality. The weight of items 8 and 9 relative to the weight of the items in T_1 is 2 and hence, $c_8 = c_9 = 2$. The weight of item

10 is equal to 7. Since $r = 1$ and $7 - r = 6 \geq 5 + 3 = W(7) + W(9)$, we obtain $\phi(7 - r) = c_7 + c_9 = 3 = c_{10}$. Finally, the weight of item 11 is equal to 10. Since the weight of the items in I is 4 and 5, respectively, and since $10 - r = 4 + 5$, we obtain $c_{11} = c_8 + c_9 = 4$. This explains the above inequality.

4 The Separation Problem

Consider the separation problem for the inequalities of type (3.3). i.e., given a fractional point $y_i, i \in N$; we are looking for a set $T_1 \subseteq N_1$ and $I = \{i_1, \dots, i_k\} \subseteq N \setminus N_1$ such that $i_v \in N_{j_v}$ for $v = 1, \dots, k, 2 \leq j_1 \leq \dots \leq j_k$ and $|T_1| + \sum_{v=1}^k j_v < b$. We define $r := b - |T_1| - \sum_{v=1}^k j_v$. For every integer $\psi \in [0, r], j_k - \psi > 0$, the inequality

$$\sum_{i \in T_1} x_i + \sum_{v=1}^{k-1} j_v x_{i_v} + (j_k - \psi) x_{i_k} + \sum_{j=r+1}^{j_k+r-\psi} \sum_{i \in N_j \setminus I} (j - r) x_i + \sum_{j=j_k+r-\psi+1}^{j_k+r} \sum_{i \in N_j} (j_k - \psi) x_i + \sum_{j>j_k+r} \sum_{i \in N_j} (j - r - \psi) x_i \leq |T_1| + \sum_{v=1}^k j_v - \psi$$

is valid for P .

Given T_1 and I and denote for $\psi \in [0, r], j_k - \psi > 0$, the corresponding inequality of type (3.3) by $c(\psi)^T x \leq \gamma(\psi)$. Then we can determine a value ψ^* in polynomial time such that $c(\psi^*)^T y - \gamma(\psi^*) = \max\{c(\psi)^T y - \gamma(\psi) \mid \psi \in [0, r], j_k - \psi > 0, N_{j_0+r-\psi} \neq \emptyset\}$ by applying the subsequent algorithm.

- Determine $N^\psi := \{0 \leq \psi \leq r \mid N_{j_k+r-\psi} \neq \emptyset, j_k - \psi > 0\}$.
- Determine $\psi^* \in N^\psi$ such that $\sum_{j=j_k+r-\psi^*+1}^{j_k+r} \sum_{i \in N_j} (j - (j_k + r - \psi^*)) y_i + \sum_{j>j_k+r} \sum_{i \in N_j} \psi^* y_i + \psi^* y_{i_k} - \psi^* = \min_{\psi \in N^\psi} \left\{ \sum_{j=j_k+r-\psi+1}^{j_k+r} \sum_{i \in N_j} (j - (j_k + r - \psi)) y_i + \sum_{j>j_k+r} \sum_{i \in N_j} \psi y_i + \psi y_{i_k} - \psi \right\}$.
- Return ψ^* .

This procedure works correct, because, for given T_1 and I two inequalities of type (3.3) with different values of ψ differ only in the right hand side, the coefficient of item i_k and the coefficients for the items with weight greater than $j_k + r - \psi$. The coefficients for the items with weight smaller than j_k are identical. Since $|N^\psi| \leq |N|$, this algorithm runs in time that is polynomial in $|N|$. If we do not

restrict the set N^ψ to values of ψ with $N_{j_k+r-\psi} \neq \emptyset$ the worst case time of this algorithm is pseudo polynomial.

In fact, the output of the above algorithm is not dependent on the set T_1 and I , but only on the sum of the weights of the items in $T_1 \cup I$. This indicates that the separation problem for the inequalities of type (3.3) can be solved in pseudo polynomial time and space complexity via a dynamic programming approach (see [Be54]). Indeed, this is true and we briefly explain the idea now.

Given an item i_k with weight j_k . Then the sum of the weights of items in $T_1 \cup I \setminus \{i_k\}$ lies in the range of $[1, b - j_k - 1]$. For every value of $d \in [1, b - j_k - 1]$ we need to determine the “best” ψ and the “best” set $T_1 \cup I \setminus \{i_k\}$ with sum of the weights equal to d (in this context “best” means to find objects so that the left hand side minus the right hand side of the inequality to be determined is maximized). However, computing the best ψ and the best set $T_1 \cup I \setminus \{i_k\}$ can be performed independently, since the items in $T_1 \cup I \setminus \{i_k\}$ have a weight which is smaller than j_k and changing ψ has an impact on the value of the coefficients of items only if the corresponding weight exceeds j_k . Second, all items with weight $j \leq j_k$ have a coefficient of $\max\{0, j - r\}$, except for the elements in $T_1 \cup I$. For an item $i \in T_1 \cup I \setminus \{i_k\}$ with weight j , its coefficient is j . Hence, by solving the knapsack problem $\max\{\sum_{i \in \bigcup_{j=1}^{j_k} N_j \setminus \{i_k\}} \min\{r, W(i)\} y_i z_i \mid \sum_{i \in \bigcup_{j=1}^{j_k} N_j \setminus \{i_k\}} W(i) z_i = d, z_i \in \{0, 1\}, i \in \bigcup_{j=1}^{j_k} N_j \setminus \{i_k\}\}$ and setting $T_1 := \{i \in N_1 \mid z_i = 1\}$ and $I := \{i \in N \setminus (N_1 \cup \{i_k\}) \mid z_i = 1\} \cup \{i_k\}$, we find the best sets $T_1 \subseteq N_1$ and $I \subseteq N \setminus N_1$ such that $|T_1| + W(I) = d + j_k$. Explicitly, the algorithm works as follows.

For all $i_k \in N_{j_k}$, $2 \leq j_k \leq b - 1$ perform the following steps:

For all $d = 1, d \leq b - j_k - 1$ perform the following steps:

- Set $r = b - j_k - d$ and initialize $N^\psi := \{0 \leq \psi \leq r \mid j_k - \psi > 0\}$.
- Determine $\psi^* \in N^\psi$ such that $\sum_{j=j_k+r-\psi^*+1}^{j_k+r} \sum_{i \in N_j} (j - (j_k + r - \psi^*)) y_i + \sum_{j > j_k+r} \sum_{i \in N_j} \psi^* y_i + \psi^* y_{i_k} - \psi^* = \min_{\psi \in N^\psi} \{\sum_{j=j_k+r-\psi+1}^{j_k+r} \sum_{i \in N_j} (j - (j_k + r - \psi)) y_i + \sum_{j > j_k+r} \sum_{i \in N_j} \psi y_i + \psi^* y_{i_k} - \psi\}$.
- Determine the optimum solution to the following knapsack problem

$$\max\left\{ \sum_{i \in \bigcup_{j=1}^{j_k} N_j \setminus \{i_k\}} \min\{r, W(i)\} y_i z_i \mid \sum_{i \in \bigcup_{j=1}^{j_k} N_j \setminus \{i_k\}} W(i) z_i = d, \right.$$

$$\left. z_i \in \{0, 1\}, i \in \bigcup_{j=1}^{j_k} N_j \setminus \{i_k\} \right\}.$$

- Set $T_1 := \{i \in N_1 \mid z_i = 1\}$ and $I := \{i \in N \setminus (N_1 \cup \{i_k\}) \mid z_i = 1\} \cup \{i_k\}$.
- Let $c^T x \leq \gamma$ be the inequality of type (3.3) with starting set $T_1 \cup I$ and reduction parameter ψ^* and set $g(d + j_k) = c^T y - \gamma$.

Determine $g(d^*) = \max\{g(d) \mid d \in \{j_k + 1, \dots, b - 1\}\}$. The corresponding inequality is the one with maximal slack subject to $i_k \in I$.

Since the knapsack problem can be solved in pseudo polynomial time and since d is bounded by the knapsack capacity b we obtain a pseudo polynomial running and space complexity of this algorithm. Of course, for large problem instances this (exact) method is not practical anymore. Nevertheless, instead of solving for every d the knapsack problem exactly, one could restrict the number of d -values heuristically and apply some primal heuristic for the knapsack problem.

Conclusions

Our complete description of the 0/1 knapsack polytope when the weight of the items is equal to one or lies in the range of $[\lfloor \frac{b}{3} \rfloor + 1, \lfloor \frac{b}{2} \rfloor]$ seems to be a further step to understand the richness of possibilities according to which knapsack inequalities can be derived. It would be interesting to find extensions to more general cases. For instance, given a complete description of the polytopes

$$\text{conv}\{\sum_{i \in N_1} x_i + \sum_{j=\lfloor \frac{b}{u+1} \rfloor + 1}^{\lfloor \frac{b}{u} \rfloor} \sum_{i \in N_j} j x_i \leq b \mid x_i \in \{0, 1\}, i \in N_j\} \text{ and}$$

$$\text{conv}\{\sum_{i \in N_1} x_i + \sum_{j=\lfloor \frac{b}{v+1} \rfloor + 1}^{\lfloor \frac{b}{v} \rfloor} \sum_{i \in N_j} j x_i \leq b \mid x_i \in \{0, 1\}, i \in N_j\} \text{ for } u \neq v.$$

Can one derive a complete description for the polytope

$$\text{conv}\{\sum_{i \in N_1} x_i + \sum_{j=\lfloor \frac{b}{u+1} \rfloor + 1}^{\lfloor \frac{b}{u} \rfloor} \sum_{i \in N_j} j x_i + \sum_{j=\lfloor \frac{b}{v+1} \rfloor + 1}^{\lfloor \frac{b}{v} \rfloor} \sum_{i \in N_j} j x_i \leq b \mid x_i \in \{0, 1\}, i \in N_j\}?$$

To our knowledge, the principle of generating a series of valid inequalities for the 0/1 knapsack polytope that differ by a parameter by which the weight of a particular item is decreased has not been introduced before. In particular, these inequalities are not classified whether they involve the concept of covers or $(1, k)$ -configurations, but this property is given a priori by the starting set and the

reduction parameter. We have also shown that these inequalities can be separated in pseudo polynomial time by applying dynamic programming techniques.

Of course, finding good and efficient heuristic strategies for separating such inequalities certainly requires further efforts and testing, but at least the idea of using such inequalities computationally seems to be worth trying.

Appendix

Proof of Theorem 2.5. Let F be a non-trivial facet of P^2 that is induced by the inequality $c^T x \leq \gamma$ that is not the knapsack inequality. We define $T_1 := \{i \in N_1 \mid c_i > 0\}$ and w.l.o.g we assume that $T_1 = \{1, \dots, k\}$ and $c_1 \geq c_2 \geq \dots \geq c_k$. We distinguish the following cases:

Case 1: $k = 0$. In this case it is easy to check that F is induced by the inequality $\sum_{j=\lfloor \frac{b}{3} \rfloor + 1}^{\lfloor \frac{b}{2} \rfloor - 1} \sum_{i \in N_j} x_i \leq 2$, since F is non-trivial.

Case 2: $\gamma < \sum_{v=1}^k c_v$. Then, $k > b$ and consequently, every $x \in F$ satisfies $\sum_{i \in N_1} x_i + \sum_{j \in B^2} \sum_{i \in N_j} j x_i = b$.

Case 3: $\gamma = \sum_{v=1}^k c_v$. Since $c^T x \leq \gamma$ is not the knapsack inequality, we conclude that $k < b$ and we set $r := b - k$. Moreover, we define $x^0 := \sum_{v=1}^k e_v$. Since $\gamma = \sum_{v=1}^k c_v = c^T x^0$ the relation $c_i = 0$ for all $i \in N_j$ with $2 \leq j \leq r$ holds.

Let $i \in N_j$, $j \geq r + 1$ be given. Since $x := x^0 - \sum_{v=k-(j-r)+1}^k e_v + e_i$ is feasible, we obtain

$$c_i \leq c_k + c_{k-1} + \dots + c_{k-(j-r)+1}.$$

On the other hand, there exists a root x' with $x'_i = 1$. Since $j \geq r + 1$, there exists $T \subseteq T_1$, $|T| = j - r$ with $x'_t = 0$ for all $t \in T$. This yields

$$c_i \geq c_k + c_{k-1} + \dots + c_{k-(j-r)+1}$$

and hence $c_i = c_k + c_{k-1} + \dots + c_{k-(j-r)+1}$. Now it follows that the inequality $c^T x \leq \gamma$ is of the type $\sum_{i \in T_1} x_i + \sum_{j > b-k} \sum_{i \in N_j} (j - (b - k)) x_i \leq k$.

Case 4: $\gamma > c(T_1)$. Since $F \not\subseteq \{x \in P^2 \mid \sum_{i \in N_1} x_i + \sum_{j \in B^2} \sum_{i \in N_j} j x_i = b\}$, there exists a root x^0 with $\sum_{i \in N_1} x_i^0 + \sum_{j \in B^2} \sum_{i \in N_j} j x_i^0 < b$. This root satisfies the

condition $x_t^0 = 1$ for all $t \in T_1$. Now, we can assume that there exists $i_0 \in N_{j_0}$, $j_0 \geq 2$ such that $x_{i_0}^0 = 1$ and $\gamma = c(T_1) + c_{i_0}$. Otherwise, $\gamma \neq c(T_1) + c_i$ holds for all $i \in N_j$, $j \geq 2$ and $x_{i'}^0 = 1$, $x_{i''}^0 = 1$ for some $i' \in N_{j'}$ and $i'' \in N_{j''}$, $2 \leq j' \leq j''$. Since $F \not\subseteq \{y \in P^2 \mid \sum_{j \in B^2} \sum_{i \in N_j} y_i = 2\}$, there exists a root $y \in F$ such that $y_{i'''} = 1$ for some $i''' \in N_{j'''}$, $j''' \geq 2$, $i''' \neq i'$, $i''' \neq i''$, $y_i = 0$ for all $i \in N_j$, $i \neq i'''$. However, $W(i''') < W(i') + W(i'')$ and consequently, $y_t = 1$ for all $t \in T_1$. This yields $\gamma = c(T_1) + c_{i'''}$, a contradiction.

Hence, we can assume that $\gamma = c(T_1) + c_{i_0}$ holds. We can also assume that $c_i < c_{i_0}$ for all $i \in N_j$ with $j < j_0$ and we can assume that $c_i \leq c_{i_0}$ for all $i \in N_j$ with $j_0 \leq j \leq j_0 + r$, where $r := b - j_0 - |T_1|$. In the following $x^0 := e_{i_0} + \sum_{v=1}^k e_v$ denotes the corresponding root. We can derive several relations.

First note that for every $i \in N_j$, $j \in B^2$ the relation $j_0 + j \leq b = j_0 + k + r$ holds. Hence, $k + r \geq j$ holds.

Due to this observation we have that for every $i \in N_j \setminus \{i_0\}$, $j \in B^2$ the vector $x^0 - \sum_{v=k-(j-r)+1}^k e_v + e_i$ is feasible. This yields the inequality

$$c_i \leq \sum_{v=k-(j-r)+1}^k c_v.$$

For the item i_0 we now show $c_{i_0} \leq \sum_{v=k-j_0+1}^k c_v$. Suppose, $c_{i_0} > \sum_{v=k-j_0+1}^k c_v$. As $c_i \leq \sum_{v=k-(j-r)+1}^k c_v$ for all $i \in N_j \setminus \{i_0\}$, $j \in B^2$, we claim that F is the face induced by the inequality $x_{i_0} \leq 1$. Assume that x is a root of F such that $x_{i_0} = 0$. If there exists $i \in N_j$, $j > j_0 + r$ with $x_i = 1$, then by exchanging i with i_0 and an appropriate subset of T_1 we obtain $c_{i_0} \leq \sum_{v=k-j_0+1}^k c_v$, a contradiction. Similarly, if there exists $i \in N_j$, $j_0 \leq j \leq j_0 + r$ with $x_i = 1$, then $c_i \leq \sum_{v=k-(j-r)+1}^k c_v$ holds. By exchanging i with i_0 this would also contradict the assumption $c_{i_0} > \sum_{v=k-j_0+1}^k c_v$. Otherwise, there exist two items $i \in N_j$, $i' \in N_{j'}$, $r+1 \leq j \leq j' \leq j_0$ with $x_i = x_{i'} = 1$. If $j' + j \leq j_0 + r$, then by exchanging i and i' with i_0 we obtain $c_{i_0} \leq \sum_{v=k-j_0-r+1}^k c_v$, a contradiction. Otherwise, $j' + j > j_0 + r$ and there exists $T \subseteq T_1$ with $x_t = 0$ for all $t \in T$. Exchanging i and i' with i_0 and T again yields $c_{i_0} \leq \sum_{v=k-j_0-r+1}^k c_v$, a contradiction. This shows that the case $c_{i_0} > \sum_{v=k-j_0+1}^k c_v$ is not possible.

Now suppose that $N_j \neq \emptyset$ for some $j > j_0 + r$, $j = j_0 + r + t$ ($t \geq 1$) say, let $i \in N_j$ be some item and let x be root such that $x_i = 1$. Since $x^0 - e_{i_0} - \sum_{v=k-t+1}^k e_v + e_i$ is a feasible point in P^2 , we know $c_i \leq c_{i_0} + \sum_{v=k-t+1}^k c_v$. We distinguish two cases:

(i) $x_{i_0} = 0$. Then, the vector $x - e_i + e_{i_0} + \sum_{v=k-t+1}^k e_v$ is a feasible point

as well. Thus, $c_i \geq c_{i_0} + \sum_{v=k-t+1}^k c_v$ and together with the above relation, $c_i = c_{i_0} + \sum_{v=k-t+1}^k c_v$.

(ii) $x_{i_0} = 1$. In this case the vector $x - e_i + \sum_{v=k-(t+j_0)+1}^k e_v$ is a feasible point. This gives $c_i \geq \sum_{v=k-(t+j_0)+1}^k c_v$ and together with the above relation, $c_i = \sum_{v=k-(t+j_0)+1}^k c_v$. Finally, $x - e_{i_0} + \sum_{v=k-(j_0+t)+1}^{k-t} e_v$ is feasible. Therefore, $c_{i_0} \geq \sum_{v=k-(t+j_0)+1}^{v=k-t} c_v = c_i - \sum_{v=k-t+1}^k c_v$. Together with $c_{i_0} \leq \sum_{v=k-j_0-r+1}^k c_v$ we obtain $c_i = c_{i_0} + \sum_{v=k-t+1}^k c_v$.

Hence, for every $i \in N_j$, $j > j_0 + r$ we obtain the relation

$$c_i = c_{i_0} + \sum_{v=k-(j-j_0-r)+1}^k c_v.$$

Based on the value of the coefficient c_{i_0} we distinguish two cases.

(1) The first case is $\sum_{v=k-(j_0-r)+1}^k c_v \leq c_{i_0} \leq \sum_{v=k-j_0+1}^k c_v$. Let $j_s := \min\{j \mid \sum_{v=k-(j-r)+1}^k c_v \geq c_{i_0}\}$. Since $j_0 \leq j_s \leq j_0 + r$, we immediately obtain $c_i = c_{i_0}$ for all $i \in N_j$ with $j_s \leq j \leq j_0 + r$ and $c_i < c_{i_0}$ for all $i \in N_j$ with $j_0 \leq j < j_s$.

Let $i \in N_j \setminus \{i_0\}$, $2 \leq j < j_s$ be given. If $c_i < \sum_{v=k-(j-r)+1}^k c_v$ holds and if x is a root with $x_i = 1$, then there exists an item $i' \neq i$, $i' \in N_{j'}$, $2 \leq j' < j_s$ such that $x_{i'} = 1$. Set $d := \max\{0, j + j' - j_0 - r\}$. Obviously, $j + j' > j_0$, so that $x - e_i - e_{i'} + e_{i_0} + \sum_{v=k-d+1}^k e_v$ is feasible and consequently, $c_{i_0} + \sum_{v=k-d+1}^k c_v \leq c_i + c_{i'} < \sum_{v=k-(j-r)+1}^k c_v + \sum_{v=k-(j'-r)+1}^k c_v \leq \sum_{v=k-(j_0-r)+1}^k c_v + \sum_{v=k-d+1}^k c_v \leq c_{i_0} + \sum_{v=k-d+1}^k c_v$, a contradiction. Thus, $c_i = \sum_{v=k-(j-r)+1}^k c_v$ for all $i \in N_j$, $2 \leq j \leq j_s$. Now it is easy to see that F is the face induced by the inequality

$$\begin{aligned} & \sum_{i \in T_1} x_i + \sum_{r+1 \leq j \leq j_s-1} \sum_{i \in N_j \setminus \{i_0\}} (j-r)x_i + (j_s-r)x_{i_0} + \\ & \sum_{j_s \leq j \leq j_0+r} \sum_{i \in N_j} (j_s-r)x_i + \sum_{j \geq j_0+r} \sum_{i \in N_j} (j_s+j-j_0-r)x_i \leq k + j_s - r, \end{aligned}$$

which completes the first case.

(2) Finally, we have that $\sum_{v=k-(j_0-r)+1}^k c_v > c_{i_0}$. In this case it is easy to see that $c_i = c_{i_0}$ for all $i \in N_j$, $j_0 \leq j \leq j_0 + r$. Moreover, we observe the following.

(O1) There exists an item $i^* \in N_{j^*} \setminus \{i_0\}$, $r+1 \leq j^* < j_0$ and a root $x \in F$ such that $x_{i^*} = 1$ and $x_{i_0} = 1$. For if not then every vector $x \in F$ also satisfies $\sum_{r+1 \leq j < j_0} \sum_{i \in N_j} x_i + \sum_{j \geq j_0} \sum_{i \in N_j} 2x_i = 2$ and hence, $\dim(F) \leq |N| - 2$, a contradiction.

(O2) Second, we observe that for every root $x \in F$ with $x_i = 1$, $i \in N_j$, $r + 1 \leq j < j_0$ there exists some other item $i' \in N_{j'}$, $r + 1 \leq j' < j_0$ such that $x_{i'} = 1$. Moreover, $j + j' > j_0$ and hence, $c_{i_0} = c_i + c_{i'}$ if $j + j' \leq j_0 + r$. Otherwise, the relation $c_i + c_{i'} = c_{i_0} + \sum_{v=k-(j+j'-j_0-r)+1}^k c_v$ must be satisfied, since both, x and x^0 are roots.

For every item $i \in N_j$ with $r + 1 \leq j \leq j_0 - 1$ and every root x such that $x_i = 1$, there exists some other item $i' \in N_{j'}$, $r + 1 \leq j \leq j_0$ with $x_{i'} = 1$. Let $r + 1 \leq j_s \leq j_0 - 1$ be the minimum number such that there exist items $s \in N_{j_s}$ and $t \in N_{j_t}$, $j_s \leq j_t \leq j_0$ with $j_s + j_t = \min\{j + j' \mid x \text{ is a root with } x_i = x_{i'} = 1, i \in N_j \setminus \{i_0\}, i' \in N_{j'} \setminus \{i_0\}, r + 1 \leq j \leq j_0 - 1, j \leq j'\}$.

We now show that $j_s + j_t < j_0 + r$.

Suppose, $j_s + j_t \geq j_0 + r$. Then F is contained in the face induced by the inequality $d^T x \leq \delta$ which is the following

$$\begin{aligned} \sum_{i \in T_1} x_i + \sum_{r+1 \leq j \leq j_0} \sum_{i \in N_j} (j-r)x_i + \sum_{j_0 \leq j \leq j_0+r} \sum_{i \in N_j \setminus \{i_0\}} (j_0-r)x_i + \\ \sum_{j \geq j_0+r} \sum_{i \in N_j} (j-2r)x_i \leq |T_1| + j_0 - r \end{aligned}$$

for the following reason: If $x \in F$ is a root with $x_i = x_{i'} = 1$, $i \in N_j$, $i' \in N_{j'}$, $r + 1 \leq j \leq j_0 - 1$, $j \leq j' \leq j_0$, then $j + j' \geq j_s + j_t > j_0 + r$ and hence, $d^T x = \delta$. In all other cases it is obvious that $x \in F$ implies $d^T x = \delta$. However, $d_{i_0} = j_0 - r$, which contradicts the assumption of case (2), $\sum_{v=k-(j_0-r)+1}^k c_v > c_{i_0}$.

Thus, $j_s + j_t < j_0 + r$ and we set $l := j_0 + r - j_s - j_t$. First of all we have several implications.

As $j_s + j_t < j_0 + r$, we obtain $c_i \leq c_s$ for all $i \in N_j \setminus \{t\}$ with $j_s \leq j \leq j_s + l$. Similarly, $c_i \leq c_t$ for all $i \in N_j$ with $j_t \leq j \leq j_t + l$. Moreover, if $i \in N_j \setminus \{t\}$ is an item with $j = j_s + l + v$, $v \geq 1$, then $c_i \leq c_s + \sum_{u=k-v+1}^k c_u$. Accordingly, for $i \in N_j$ with $j = j_t + l + v$, $v \geq 1$, we obtain $c_i \leq c_t + \sum_{u=k-v+1}^k c_u$.

Again we distinguish two cases.

Suppose, there does not exist an item $i \in N_{j_s-x}$ with $r + 1 \leq j_s - x \leq j_s$ such that $c_i = \sum_{v=k-(j_s-x-r)+1}^k c_v$. Then every $i' \in N_{j'} \setminus \{t\}$, $r + 1 \leq j'$ satisfies $c_{i'} < \sum_{v=k-(j'-r)+1}^k c_v$. This implies, $t = i^*$ (see (O1)).

Our first case is $c_t = \sum_{u=k-(j_t-r)+1}^k c_u$.

Now we claim that F is contained in the face induced by the inequality

$$\begin{aligned} & \sum_{i \in T_1} x_i + \sum_{j=r+1}^{j_s} \sum_{i \in N_j \setminus \{t\}} (j-r)x_i + \sum_{j=j_s+1}^{j_s+l} \sum_{i \in N_j \setminus \{t\}} (j_s-r)x_i + \\ & (j_t-r)x_t + \sum_{j=j_s+l+1}^{j_0} \sum_{i \in N_j \setminus \{t\}} (j-r-l)x_i + \sum_{j=j_0+1}^{j_0+r} \sum_{i \in N_j} (j_0-r-l)x_i + \\ & \sum_{j \geq j_0+r} \sum_{i \in N_j} (j-l-2r)x_i \leq |T_1| + j_0 - r - l \end{aligned}$$

and we briefly explain why. For ease of notation let us denote this inequality by $d^T x \leq \delta$.

Let x be a root with $x_i = x_{i'} = 1$ for some $i \in N_j, i' \in N_{j'}, r+1 \leq j \leq j'$. If $j < j_s, j = j_s - x$ say, then $j' \geq j_t + l + x$, because otherwise by exchanging i and a subset $T \subseteq T_1, |T| = j_t - j - (j_0 + r - j_t - j_s)$ with t would give the relation $c_t < c_k + \dots + c_{k-(j_t-r)+1}$, a contradiction. Thus, $j' \geq j_t + l + x$ and in this case $d^T x = \delta$ holds. If $j_s < j \leq j_t - 1$ and $i' \neq t$, then by exchanging i and a subset $T \subseteq T_1, |T| = j_t - j$ with t would give the relation $c_t < c_k + \dots + c_{k-(j_t-r)+1}$, because $c_i < \sum_{v=k-(j-r)+1}^k c_v$. Hence this case can be excluded. In all other cases it is not difficult to see that $x \in F$ implies $d^T x = \delta$.

The remaining case is $c_t < \sum_{u=k-(j_t-r)+1}^k c_u$. Here, we know that there exists an item $i \in N_{j_s-x}, r+1 \leq j_s - x \leq j_s$ such that $c_i = \sum_{v=k-(j_s-x-r)+1}^k c_v$. Let x_1 be minimal such that there exists $i \in N_{j_s-x_1}, r+1 \leq j_s - x_1 \leq j_s$ with $c_i = \sum_{v=k-(j_s-x_1-r)+1}^k c_v$. Let us denote this item i by s_1 .

Now it is easy to see that $c_i = \sum_{v=k-(j_s-y-r)+1}^k c_v$ for all $i \in N_{j_s-y}, y \geq x_1, W(s_1) - y \geq r+1$. Moreover, let t_1 be an item with minimal weight greater or equal than $W(s_1)$ such that there exists a root $x \in F$ with $x_{s_1} = x_{t_1} = 1$. We define $l_1 = \max\{0, j_0 + r - W(s_1) - W(t_1)\}$. Note that if $W(s_1) < W(s)$ we further know that $W(t_1) > W(t) + l$. Now the following relations can be derived:

$$\begin{aligned} c_i & \leq c_{s_1}, i \in N_j \setminus \{t_1\}, W(s_1) \leq j \leq W(s_1) + l_1 \\ c_i & \leq c_{s_1} + \sum_{v=k-(j-W(s_1)-l_1)+1}^k c_v, i \in N_j \setminus \{t_1\}, j > W(s_1) + l_1 \\ c_i & \leq c_{t_1}, i \in N_j, W(t_1) \leq j \leq W(t_1) + l_1 \\ c_i & \leq c_{t_1} + \sum_{v=k-(j-W(t_1)-l_1)+1}^k c_v, i \in N_j, j > W(t_1) + l_1 \end{aligned}$$

Moreover, it is easy to see that $c_i = c_{s_1}$ for all $i \in N_j, W(s_1) \leq j \leq W(s_1) + l_1$ and $c_i = c_{t_1} + \sum_{v=k-(j-W(t_1)-l_1)+1}^k c_v$ for all $i \in N_j, j > W(t_1) + l_1$.

Now let $s_2, W(t_1) > W(s_2) > W(s_1) + l_1$ be an item with minimal weight such that there exists an item $i, W(s_2) < W(i) < W(t_1), W(i) + W(s_2) < j_0 + r$ and a root $x \in F$ with $x_{s_2} = x_i = 1$. Note that s_2 and i exist, since s and t satisfy these properties. Let t_2 be an item with minimal weight greater or equal than $W(s_2)$ that satisfies the above properties, i.e., $W(s_2) < W(t_2) < W(t_1), W(t_2) + W(s_2) < j_0 + r$ and there exists a root $x \in F$ with $x_{s_2} = x_{t_2} = 1$. We define $l_2 := j_0 + r - W(s_2) - W(t_2)$. Due to the choice of t_1 we conclude that $W(t_1) \geq W(t_2) + l_2$. Moreover, we can derive similar relations as pointed out above. In particular, one can show $c_i = c_{s_2}$ for all $i \in N_j \setminus \{t_2\}, W(s_2) \leq j \leq W(s_2) + l_2$ and $c_i = c_{t_2} + \sum_{v=k-(j-W(t_2)-l_2)+1}^k c_v$ for all $i \in N_j, j > W(t_2) + l_2$.

Iterating this argument shows that there exist pairs of items $(s_1, t_1), \dots, (s_\alpha, t_\alpha)$ that satisfy the following properties:

- $W(s_i) \leq W(t_i)$ with $W(s_i) + W(t_i) < j_0 + r$. We set $l_i := \max\{0, j_0 + r - W(s_i) - W(t_i)\}$.
- $W(s_i) + l_i < W(s_{i+1})$ for $i = 1, \dots, \alpha - 1$ and $W(t_i) + l_i < W(t_{i-1})$ for $i = 2, \dots, \alpha$.
- There exists a root $x \in F$ with $x_{s_i} = x_{t_i} = 1$.
- For every item $i \in N_j \setminus \cup_{u=1}^\alpha \{s_u, t_u\}$, $W(s_i) + l_i < j < j_0$ or $j < W(s_1)$ there does not exist a root x with $x_i = x_{i'} = 1$ for some $i' \in N_{j'}, j' \geq j$ such that $j + j' < j_0 + r$.

Now it is easy to see that the face F is contained in the face induced by some inequality

$$\begin{aligned}
& \sum_{i \in T_1} x_i + \sum_{j \leq W(s_1)} \sum_{i \in N_j} (j - r) x_i & + \\
& \sum_{u=1}^{\alpha-1} \left(\sum_{j=W(s_u)+1}^{W(s_u)+l_u} \sum_{i \in N_j} (W(s_u) - r - \sum_{v=1}^{u-1} l_v) x_i \right) & + \\
& \sum_{j=W(s_u)+l_u+1}^{W(s_{u+1})} \sum_{i \in N_j} (j - r - \sum_{v=1}^u l_v) x_i & + \\
& \sum_{j=W(s_\alpha)+1}^{W(s_\alpha)+l_\alpha} \sum_{i \in N_j \setminus \{t_\alpha\}} (W(s_\alpha) - r - \sum_{v=1}^{\alpha-1} l_v) x_i & + \\
& \sum_{j=W(s_\alpha)+l_\alpha+1}^{W(t_\alpha)+l_\alpha-\mu_\alpha} \sum_{i \in N_j \setminus \{t_\alpha\}} (j - r - \sum_{v=1}^\alpha l_v) x_i & + \\
& (W(t_\alpha) - r - \sum_{v=1}^{\alpha-1} l_v - \mu_\alpha) x_{t_\alpha} & + \\
& \sum_{j=W(t_\alpha)+l_\alpha-\mu_\alpha+1}^{W(t_\alpha)+l_\alpha} \sum_{i \in N_j} (W(t_\alpha) - \mu_\alpha - r - \sum_{v=1}^{\alpha-1} l_v) x_i & + \\
& \sum_{u=2}^\alpha \left(\sum_{j=W(t_u)+1}^{W(t_{u-1})} \sum_{i \in N_j} (j - r - \sum_{v=1}^\alpha l_v - \mu_\alpha - \sum_{v=u}^{\alpha-1} l_v) x_i \right) & + \\
& \sum_{j=W(t_{u-1})+1}^{W(t_{u-1})+l_{u-1}} \sum_{i \in N_j} (W(t_{u-1}) - r - \sum_{v=1}^\alpha l_v - \mu_\alpha - \sum_{v=u}^{\alpha-1} l_v) x_i & + \\
& \sum_{j=W(t_1)+1}^{j_0} \sum_{i \in N_j} (j - r - \mu_\alpha - l_\alpha - 2 \sum_{v=1}^{\alpha-1} l_v) x_i & + \\
& \sum_{j=j_0+1}^{j_0+r} \sum_{i \in N_j} (j_0 - r - \mu_\alpha - l_\alpha - 2 \sum_{v=1}^{\alpha-1} l_v) x_i & + \\
& \sum_{j>j_0+r} \sum_{i \in N_j} (j - 2r - \mu_\alpha - l_\alpha - 2 \sum_{v=1}^{\alpha-1} l_v) x_i & \leq \\
& |T_1| + (j_0 - r - \mu_\alpha - l_\alpha - 2 \sum_{v=1}^{\alpha-1} l_v). &
\end{aligned}$$

where $\mu_\alpha \in [0, \min\{l_\alpha, W(t_\alpha) - W(s_\alpha)\}]$. This finally completes the proof. ■

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