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A Posteriori Error Estimates for Elliptic Problems in Two and Three Space Dimensions

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Abstract. Let $u \in H$ be the exact solution of a given self–adjoint elliptic boundary value problem, which is approximated by some $\tilde{u} \in \mathcal{S}$, $\mathcal{S}$ being a suitable finite element space. Efficient and reliable a posteriori estimates of the error $\| u - \tilde{u} \|$, measuring the (local) quality of $\tilde{u}$, play a crucial role in termination criteria and in the adaptive refinement of the underlying mesh. A well–known class of error estimates can be derived systematically by localizing the discretized defect problem using domain decomposition techniques. In the present paper, we provide a guideline for the theoretical analysis of such error estimates. We further clarify the relation to other concepts. Our analysis leads to new error estimates, which are specially suited to three space dimensions. The theoretical results are illustrated by numerical computations.

Chapter 1

Introduction

Assume that the solution space H of a given selfadjoint elliptic problem is approximated by a suitable subspace $\mathcal{S} \subset H$ and that we have computed an approximation $\tilde{u} \in \mathcal{S}$ of the exact solution $u \in H$. We are interested in efficient and reliable estimates of the corresponding error $\| u - \tilde{u} \|$, measuring the (local) quality of the approximation $\tilde{u}$. Among the variety of different concepts (see for example the bibliographies included in the monographs of Johnson [13], Szabo and Babuška [17] or Zienkiewicz and Taylor [21]) we frequently recover the following two major steps

- Discretize the defect problem with respect to an enlarged space $\mathcal{Q} \subset H$.
- Localize the discrete defect problem by domain decomposition.

For example, the discretization of the defect problem played a prominent role in the paper of Bank and Weiser [5], while meanwhile standard techniques of domain decomposition were developed in the pioneering work of Babuška and Rheinboldt [2]. To our knowledge, the *explicit* hierarchical preconditioning of the discretized defect problem first appeared in a paper of Deuflhard, Leinen and Yserentant [8]. This construction principle has been extended successfully from selfadjoint elliptic equations to a variety of other problems (c.f. for example [4, 6, 7, 14, 12]).

However, this recent work concentrates on most simple finite element spaces $\mathcal{S}$ and $\mathcal{Q}$, where the proofs of reliability and efficiency of the resulting error estimates are immediate. In the present paper, we intend to provide a guideline for the analysis of more complicated situations. Using finite elements of higher order as a model example, it becomes clear, where to branch off in other special cases. We further clarify the relation to other residual based error estimates, resulting from apparently different concepts. By the way, this unification leads to a better understanding of previous results. As a further outcome of our theoretical considerations, we explain why error estimation is more difficult in three than in two space dimensions and introduce so–called *hybrid error estimates* to remedy those problems.

The paper is organized as follows.

In the next section, we consider the discretization of the defect equation. It turns out that we obtain efficient and reliable error estimates, if and only if $\mathcal{Q}$ satisfies a saturation assumption (C0). This result also gives some insight in the principal limitations of a posteriori error estimation.

The application of domain decomposition to the discrete defect equation is considered in Section 3. Without striving for utmost generality, we restrict our considerations to affine finite elements (c.f. condition (C1)) and we assume that the splitting of the enlarged space $\mathcal{Q}$ into the original space $\mathcal{S}$ and an extension $\mathcal{V}$ is induced by the interpolation operator (c.f. condition (C2)). We emphasize that the main result stated in Theorem 4.1 can be extended to any other splitting of $\mathcal{Q}$, which is *stable* in the sense of Oswald [15]. This indicates how to proceed in the case of non–affine elements, playing a crucial role for higher order problems. Of course, the interpolation operator can be replaced by other stable (quasi) projections, which may be of some importance in connection with h–p methods.

In Section 4, we reformulate the well–known Babuška–Miller estimate [3] for two-dimensional problems in terms of so–called *hierarchical p extensions* $\mathcal{V}$, taking advantage of related work by Verfürth [18, 19]. This may be a typical example, how locally equivalent error estimates can be formulated in quite different ways.

It is shown in the final section that Babuška–Miller type estimates and hierarchical p extensions do not coincide any more in three space dimensions. This gives rise to hybrid error estimates, which may be regarded as a union of the two original concepts. The numerical properties are compared in the case of a model example, showing that the hybrid estimates perform better than their single components.

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Chapter 2

Discrete Defect Problems

Let Ω be a bounded, polygonal (polyhedral) domain in the Euclidean space $\mathbb{R}^d$, $d = 2, 3$. For simplicity, we consider the variational problem

$$u \in H_0^1(\Omega): \quad a(u, v) = (f, v), \;\; v \in H_0^1(\Omega), \tag{2.1}$$

with the bilinear form $a(\cdot,\cdot)$ given by

$$a(v, w) = \sum_{i,j=1}^{d} \int_\Omega a_{ij} \partial_i v \partial_j w \; dx, \;\; v, w \in H_0^1(\Omega),$$

and

$$(f, v) = \int_\Omega f v \; dx, \quad f, v \in L^2(\Omega),$$

denoting the usual scalar product in $L^2(\Omega)$. We assume that $a_{ij} \in L^\infty(\Omega)$, satisfying $a_{ij}(x) = a_{ji}(x)$, $i, j = 1, \ldots, d$, and

$$\alpha_0 |\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x) \xi_i \xi_j \leq \alpha_1 |\xi|^2, \; \xi \in \mathbb{R}^d, \; 0 < \alpha_0 \leq \alpha_1, \tag{2.2}$$

for almost all $x \in \Omega$. More general boundary conditions may be incorporated in the usual way.

We will make frequent use of the energy norm $\| v \| = a(v, v)^{1/2}$ of $v \in H_0^1(\Omega)$ and of the equivalent (semi) norm $|v|_1 = (\sum_{i=1}^{d} \int_\Omega (\partial_i v)^2 \; dx)^{1/2}$.

With conforming finite element methods in mind, we approximate the solution space $H_0^1(\Omega)$ by a suitable finite dimensional subspace $\mathcal{S} \subset H_0^1(\Omega)$. The corresponding approximation $u_\mathcal{S} \in H_0^1(\Omega)$ of the exact solution $u \in H_0^1(\Omega)$ is the unique solution of the discrete variational problem

$$u_\mathcal{S} \in \mathcal{S}: \quad a(u_\mathcal{S}, v) = (f, v), \;\; v \in \mathcal{S}. \tag{2.3}$$

In most practical calculations, only a further approximation $\tilde{u} \in \mathcal{S}$ of $u_\mathcal{S}$ is available. For example, $\tilde{u}$ may result from the iterative solution of (2.3). In the remainder of this paper, we concentrate on estimates of the *total error* $\| u - \tilde{u} \|$, measuring the quality of the overall approximation of u. Here, the *algebraic error* $\| u_\mathcal{S} - \tilde{u} \|$ may interfere with the *discretization error* $\| u - u_\mathcal{S} \|$. Estimates, which provide upper and lower bounds for the total error, are called *reliable* and *efficient*, respectively. Of course, reliability is more important than efficiency, but unfortunately it turns out to be more difficult to obtain.

For given $\tilde{u} \in \mathcal{S}$, the desired defect $d = u - \tilde{u} \in H_0^1(\Omega)$ is the solution of

$$d \in H_0^1(\Omega): \quad a(d, v) = r_{\tilde{u}}(v), \;\; v \in H_0^1(\Omega), \tag{2.4}$$

where the right–hand side

$$r_{\tilde{u}}(v) = (f, v) - a(\tilde{u}, v), \;\; v \in H_0^1(\Omega),$$

denotes the residual of $\tilde{u}$. To discretize the continuous defect problem (2.4), we introduce an enlarged subspace $\mathcal{Q} \subset H_0^1(\Omega)$,

$$\mathcal{Q} = \mathcal{S} + \mathcal{V},$$

by adding the finite dimensional subspace $\mathcal{V} \subset H_0^1(\Omega)$ to the given space $\mathcal{S}$. The resulting *discrete defect problem*

$$d_{\mathcal{Q}} \in \mathcal{Q}: \quad a(d_{\mathcal{Q}}, v) = r_{\tilde{u}}(v), \;\; v \in \mathcal{Q}, \tag{2.5}$$

provides the approximation $d_{\mathcal{Q}}$ of the exact defect d. Under certain conditions, the discrete error $\| d_{\mathcal{Q}} \|$ will turn out to be a reliable and efficient estimate of the total error.

Observe that the discrete defect $d_{\mathcal{Q}} \in \mathcal{Q}$ can be rewritten as

$$d_{\mathcal{Q}} = u_{\mathcal{Q}} - \tilde{u},$$

where $u_{\mathcal{Q}} \in \mathcal{Q}$ is the solution of the extended problem

$$u_{\mathcal{Q}} \in \mathcal{Q}: \quad a(u_{\mathcal{Q}}, v) = (f, v), \;\; v \in \mathcal{Q}. \tag{2.6}$$

On the other hand, we can utilize the Ritz projection $P_{\mathcal{Q}} : H_0^1(\Omega) \to \mathcal{Q}$, defined by

$$P_{\mathcal{Q}} w \in \mathcal{Q}: \quad a(P_{\mathcal{Q}} w, v) = a(w, v), \;\; v \in \mathcal{Q}, \;\; w \in H_0^1(\Omega),$$

to see that $d_{\mathcal{Q}} = P_{\mathcal{Q}} d$ is just the orthogonal projection of $d \in H_0^1(\Omega)$ to $\mathcal{Q}$. As orthogonal projections have unit norm, we have the following lower bound.

Proposition 2.1 *The discrete defect $d_{\mathcal{Q}} = u_{\mathcal{Q}} - \tilde{u}$ satisfies*

$$\| u_{\mathcal{Q}} - \tilde{u} \| \le \| u - \tilde{u} \| . \tag{2.7}$$

To derive an upper bound, we have to utilize a *saturation assumption*

$$\text{(C0)} \qquad \| u - u_{\mathcal{Q}} \| \le \beta \, \| u - u_{\mathcal{S}} \|, \quad \beta < 1.$$

Obviously, (C0) states that the *larger* space $\mathcal{Q} \supset \mathcal{S}$ must lead to a *better* approximation $u_{\mathcal{Q}} \neq u_{\mathcal{S}}$.

Theorem 2.1 *The saturation assumption (C0) is equivalent to each of the following upper estimates*

$$\| u - \tilde{u} \| \le (1 - \beta^2)^{-1/2} \, \| u_{\mathcal{Q}} - \tilde{u} \|, \qquad \forall \tilde{u} \in \mathcal{S}, \tag{2.8}$$

and

$$\| u - u_{\mathcal{S}} \| \le (1 - \beta^2)^{-1/2} \, \| u_{\mathcal{Q}} - u_{\mathcal{S}} \| . \tag{2.9}$$

Proof. We first show that (C0) implies (2.8). Let $\tilde{u} \in \mathcal{S}$. Exploiting the orthogonality

$$a(u_{\mathcal{Q}} - u, \tilde{u}) = a(u_{\mathcal{Q}} - u, u_{\mathcal{Q}}) = 0,$$

resulting from (2.6), we obtain by elementary calculations and (C0) that

$$\| u_{\mathcal{Q}} - \tilde{u} \|^2 = \| u - \tilde{u} \|^2 - \| u - u_{\mathcal{Q}} \|^2 \geq (1 - \beta^2) \| u - \tilde{u} \|^2 .$$

It is clear that (2.8) implies (2.9). To show that (C0) follows from (2.9), we calculate

$$\|u - u_{\mathcal{S}}\|^2 = \|u - u_{\mathcal{Q}}\|^2 + \|u_{\mathcal{Q}} - u_{\mathcal{S}}\|^2 \geq \|u - u_{\mathcal{Q}}\|^2 + (1 - \beta^2)\|u - u_{\mathcal{S}}\|^2,$$

providing the saturation (C0). ∎

Consider a sequence of spaces $\mathcal{S}_l$, and extensions $\mathcal{V}_l$, $l = 0, 1, \ldots$. If the corresponding enlarged spaces $\mathcal{Q}_l$ allow for approximations $u_{\mathcal{Q}_l}$ of *higher order*, then the resulting error estimates are *asymptotically exact*. However, it is not known *a priori*, at which index l the asymptotic behavior starts.

In fact, for fixed $\mathcal{S}$ and any finite dimensional extension $\mathcal{V}$, we can find nontrivial right–hand sides, such that the saturation assumption (C0) is violated.

Proposition 2.2 *Assume that the subspace $\mathcal{L} \subset L^2(\Omega)$ satisfies*

$$dim\ \mathcal{L} > dim\ \mathcal{V}. \tag{2.10}$$

Then there is at least one non–vanishing right–hand side $f \in \mathcal{L}$ so that the discrete problems (2.3) and (2.6) have the same solutions $u_{\mathcal{Q}} = u_{\mathcal{S}}$.

Proof. Consider the defect operator

$$D : \mathcal{L} \to \mathcal{S}^{\perp}, \quad f \mapsto u_{\mathcal{Q}} - u_{\mathcal{S}},$$

where $\mathcal{S}^{\perp}$ denotes the (energy) orthogonal complement of $\mathcal{S}$ in $\mathcal{Q}$. Because of

$$\dim \mathcal{L} > \dim \mathcal{V} \geq \dim \mathcal{S}^{\perp}$$

the operator D cannot be one-to-one. Therefore exists the asserted nontrivial element f in the kernel of D. ∎

In view of Theorem 2.1 and Proposition 2.2, the subspace $\mathcal{V}$ has to be well–suited to the considered data in order to give upper bounds of the total error. In this sense, the reliability of a posteriori error estimates is still based on certain a priori information.

Chapter 3

Local Defect Problems

Let $\mathcal{T}$ be a partition of Ω in triangular (tetrahedral) elements. The sets of interior edges (and triangular faces) of the elements $T \in \mathcal{T}$ are called $\mathcal{E}$ (and $\mathcal{F}$), respectively. Denoting by h_T and ρ_T the diameters of the circumscribed and of the inscribed ball of an element $T \in \mathcal{T}$, the *shape regularity* σ of $\mathcal{T}$ is an upper bound of the aspect ratio h_T/ρ_T for all $T \in \mathcal{T}$. Finally, let the partition $\mathcal{T}$ be conforming in the sense that the intersection of two different elements of $\mathcal{T}$ is either a common vertex, a common edge, (a common triangular face) or is empty.

We approximate the solution space $H_0^1(\Omega)$ by the space $\mathcal{S}^p$ of conforming finite elements of p–th order with respect to the triangulation $\mathcal{T}$,

$$\mathcal{S}^p = \{v \in H_0^1(\Omega) \mid v|_T \in \Pi^p(T),\ T \in \mathcal{T}\}, \tag{3.1}$$

where $\Pi^p(T)$ stands for the polynomials of order not greater than p on T. Recall that a function $v \in \mathcal{S}^p$ is characterized by the values $v(P)$ in the (Lagrangian) nodal points $P \in \mathcal{N}^p$. The restriction $v|_T \in \Pi^p(T)$ to an element $T \in \mathcal{T}$ is determined by the values of v in $\mathcal{N}_T^p = \mathcal{N}^p \cap T$. We will frequently omit the superscript p in the sequel.

As in the preceding section, we consider the enlarged space $\mathcal{Q} \subset H_0^1(\Omega)$, resulting from the extension of the given finite element space $\mathcal{S}$ by a suitable space $\mathcal{V} \subset H_0^1(\Omega)$. To prepare the assumption (C1) on $\mathcal{V}$, a subset Ψ of $H_0^1(\Omega)$ is called *locally affine*, if for each element $T \in \mathcal{T}$ the set of non–vanishing restrictions $\Psi_T = \{\psi|_T \mid \psi \in \Psi, \psi|_T \not\equiv 0\}$ can be identified with a finite set $\Psi_{\hat{T}}$ of linearly independent shape functions on a fixed reference triangle (tetrahedron) $\hat{T}$ via the transformation

$$\psi \circ \phi_T = \hat{\psi} \in \Psi_{\hat{T}}, \quad \psi \in \Psi_T. \tag{3.2}$$

Here, the affine transformation ϕ_T maps the reference element $\hat{T}$ one-to-one onto T. The following assumption will be crucial for the remainder of this section.

(C1) The extension $\mathcal{V}$ has a locally affine basis Ψ.

As a consequence of (C1), all non–vanishing restrictions $\psi|_T$, $\psi \in \Psi$, are linearly independent on $T \in \mathcal{T}$. Note that the treatment of elliptic problems of higher order, as discretized by conforming, but non–affine finite elements (as for example the Argyris element), may give rise to suitable generalizations of the condition (C1).

Assuming that $\mathcal{Q}$ consists of continuous functions v, we define the interpolation operator $\mathcal{I} : \mathcal{Q} \to \mathcal{S}$ by

$$\mathcal{I}v \in \mathcal{S} : \quad \mathcal{I}v(P) = v(P), \;\; P \in \mathcal{N}, \tag{3.3}$$

playing an important role in the following condition.

(C2) The extension $\mathcal{V}$ consists of continuous functions and provides a direct splitting of $\mathcal{Q} = \mathcal{S} \oplus \mathcal{V}$ such that

$$\mathcal{S} = \mathcal{I}\mathcal{Q}, \qquad \mathcal{V} = (\mathrm{id} - \mathcal{I})\mathcal{Q}.$$

It may be useful (for example in the framework of h–p methods) to modify the condition (C2) by replacing the interpolation $\mathcal{I}$ by different (quasi)–projections. In this case the proof of corresponding stability estimates (c.f. Lemma 3.1) becomes less local (and more complicated).

Extensions $\mathcal{V}$ will be frequently defined via suitable shape functions $\Psi_{\hat{T}}$, vanishing on the nodal points $\mathcal{N}_{\hat{T}} = \phi_T^{-1}\mathcal{N}_T$ of the reference element $\hat{T}$. In this way, the assumptions (C1) and (C2) are clearly satisfied. Note that this approach covers the extension of $\mathcal{S}$ by uniform h, p and h–p refinement.

Example 3.1: We consider the case of piecewise linear finite elements in three space dimensions. Then, the set $\mathcal{N}^1$ of nodal points coincides with the interior vertices of the elements $T \in \mathcal{T}$ and the following products of the barycentric coordinates $\lambda_0, \ldots, \lambda_3$ on $\hat{T}$ clearly vanish in $P \in \mathcal{N}_{\hat{T}}$:

$$\hat{\psi}_{\hat{E}} = \lambda_{P_0}\lambda_{P_1}, \quad \hat{E} = (P_0, P_1),$$

$$\hat{\psi}_{\hat{F}} = \lambda_{P_0}\lambda_{P_1}\lambda_{P_2}, \quad \hat{F} = (P_0, P_1, P_2),$$

$$\hat{\psi}_{\hat{T}} = \lambda_{P_0}\lambda_{P_1}\lambda_{P_2}\lambda_{P_3}, \quad \hat{T} = (P_0, P_1, P_2, P_3).$$

Here, $\hat{E}$ and $\hat{F}$ run through all edges and faces of $\hat{T}$. The resulting basis functions ψ_E, ψ_F and ψ_T are called *quadratic, cubic* and *quartic bubbles* on the edges $E \in \mathcal{E}$, the triangular faces $F \in \mathcal{F}$ and the tetrahedra $T \in \mathcal{T}$, respectively. Note that the extension of $\mathcal{S}^1$ by the quadratic bubbles $\mathcal{V}^2 = \{\psi_E, \; E \in \mathcal{E}\}$ is producing the piecewise quadratic finite element space $\mathcal{S}^2 = \mathcal{S}^1 \oplus \mathcal{V}^2$.

The splitting

$$\mathcal{Q} = \mathcal{S} \oplus \bigoplus_{\psi \in \Psi} V_\psi, \;\; V_\psi = \mathrm{span}\{\psi\}, \tag{3.4}$$

gives rise to the following *local defect problems*

$$d_{\mathcal{S}} \in \mathcal{S} : \quad a(d_{\mathcal{S}}, v) = r_{\tilde{u}}(v), \;\; v \in \mathcal{S}, \tag{3.5}$$

and

$$d_\psi \in V_\psi : \quad a(d_\psi, v) = r_{\tilde{u}}(v), \;\; v \in V_\psi. \tag{3.6}$$

Solving (3.5) and (3.6) instead of the discrete defect equation (2.5), we replace the bilinear form $a(\cdot,\cdot)$ on the enlarged space $\mathcal{Q}$ by the preconditioner induced by the splitting (3.4). The corresponding error estimate

$$\| u_{\mathcal{Q}} - \tilde{u} \|^2 \approx \eta_{\mathcal{S}} + \sum_{\psi\in\Psi} \eta_\psi \tag{3.7}$$

consists of the *algebraic error*

$$\eta_{\mathcal{S}} = \| d_{\mathcal{S}} \|^2 = \| u_{\mathcal{S}} - \tilde{u} \|^2 \tag{3.8}$$

and the *scaled residuals*

$$\eta_\psi = \| d_\psi \|^2 = r_{\tilde{u}}(\psi)^2 / a(\psi,\psi), \quad \psi \in \Psi. \tag{3.9}$$

In the remainder of this section, we will show that under the assumptions (C1) and (C2), the estimate (3.7) provides lower and upper bounds for the discrete error $\| u_{\mathcal{Q}} - \tilde{u} \|$. Recall that the relation of the discrete error to the desired true error has been treated in the preceding section.

We will make frequent use of the local (semi)–norm $|\cdot|_{1,T}$, where the integration is carried out only over the element $T \in \mathcal{T}$.

We will further utilize the reference spaces $\mathcal{S}_{\hat{T}} = \Pi^p(\hat{T})$ and $\mathcal{V}_{\hat{T}} = \text{span}\ \Psi_{\hat{T}}$. The extended reference space $\mathcal{Q}_{\hat{T}} = \mathcal{S}_{\hat{T}} \oplus \mathcal{V}_{\hat{T}}$ is spanned by the form functions on the reference element $\hat{T}$.

Throughout this paper $a \preceq b$, $a \succeq b$ and $a \asymp b$ stands for $a \le Cb$, $a \ge cb$ and $cb \le a \le Cb$. The constants c, C only depend on the degree p of the finite element space $\mathcal{S}$, the shape regularity σ of $\mathcal{T}$, the ellipticity of the continuous problem and the form functions $\Psi_{\hat{T}}$ generating the extension $\mathcal{V}$.

Lemma 3.1 *Assume that the extension $\mathcal{V}$ satisfies the conditions (C1), (C2). Then the interpolation operator $\mathcal{I} : \mathcal{Q} \to \mathcal{S}$ is stable in the sense that*

$$\| \mathcal{I} v \| \preceq \| v \|, \quad v \in \mathcal{Q}. \tag{3.10}$$

Proof. We consider the interpolation $\mathcal{I}v|_T$ of some fixed $v \in \mathcal{Q}$ on an arbitrary fixed element $T \in \mathcal{T}$. Using the affine transformation, we obtain

$$|v|^2_{1,T} \asymp h_T^{-2}\ \text{meas}(T)\ |v \circ \phi_T|^2_{1,\hat{T}}. \tag{3.11}$$

It is easily seen that the operator $\hat{\mathcal{I}} : \mathcal{Q}_{\hat{T}} \to \mathcal{S}_{\hat{T}}$, interpolating in the nodal points $\mathcal{N}_{\hat{T}}$, satisfies the equation

$$(\mathcal{I}v) \circ \phi_T = \hat{\mathcal{I}}\hat{v}, \quad \hat{v} = v \circ \phi_T. \tag{3.12}$$

As $\hat{\mathcal{I}}$ is reproducing constants on $\hat{T}$ and the reference space $\mathcal{Q}_{\hat{T}}$ is finite–dimensional, we obtain

$$|\hat{\mathcal{I}}\hat{v}|_{1,\hat{T}} \preceq |\hat{v}|_{1,\hat{T}}. \tag{3.13}$$

Now we only have to insert (3.12) in (3.11) and to apply (3.13) in order to show

$$|\mathcal{I}v|_{1,T}^2 \preceq |v|_{1,T}^2. \tag{3.14}$$

Summing up over all $T \in \mathcal{T}$ and exploiting the ellipticity (2.2) gives the assertion. ■

The following Lemma is an immediate consequence of Lemma 3.1.

Lemma 3.2 *Assume that the extension $\mathcal{V}$ satisfies the conditions (C1), (C2). Then the equivalence*

$$\| v \|^2 \asymp \| v_{\mathcal{S}} \|^2 + \| v_{\mathcal{V}} \|^2, \quad v \in \mathcal{Q}, \tag{3.15}$$

holds, where $v = v_{\mathcal{S}} + v_{\mathcal{V}}$ is uniquely decomposed in $v_{\mathcal{S}} \in \mathcal{S}$ and $v_{\mathcal{V}} \in \mathcal{V}$.

Proof. Let $v \in \mathcal{Q}$. By condition (C2), we have $v_{\mathcal{S}} = \mathcal{I}v$, and $v_{\mathcal{V}} = v - \mathcal{I}v$ so that the lower estimate

$$\| \mathcal{I}v \|^2 + \| v - \mathcal{I}v \|^2 \preceq \| v \|^2$$

follows from Lemma 3.1. On the other hand the triangle inequality and the Cauchy–Schwarz inequality yield

$$\| v_{\mathcal{S}} + v_{\mathcal{V}} \|^2 \leq (\| v_{\mathcal{S}} \| + \| v_{\mathcal{V}} \|)^2 \leq 2(\| v_{\mathcal{S}} \|^2 + \| v_{\mathcal{V}} \|^2).$$

This completes the proof. ■

The further splitting of the extension $\mathcal{V}$ is considered in the next lemma.

Lemma 3.3 *Assume that the extension $\mathcal{V}$ satisfies the conditions (C1), (C2). Then the equivalence*

$$\| v \|^2 \asymp \sum_{\psi \in \Psi} \| v_\psi \|^2, \quad v \in \mathcal{V}, \tag{3.16}$$

holds, where $v = \sum_{\psi \in \Psi} v_\psi$ is uniquely decomposed in $v_\psi \in V_\psi$, $\psi \in \Psi$.

Proof. Let some fixed $v \in \mathcal{V}$ be decomposed according to

$$v = \sum_{\psi \in \Psi} v_\psi, \quad v_\psi \in V_\psi. \tag{3.17}$$

We consider v on an arbitrary fixed element $T \in \mathcal{T}$, using the transformed function $\hat{v} = v \circ \phi_T \in \mathcal{V}_{\hat{T}}$ on the reference triangle $\hat{T}$. Due to condition (C1), the transformation of the decomposition (3.17) takes the form

$$\hat{v} = \sum_{\hat{\psi} \in \Psi_{\hat{T}}} v_{\hat{\psi}}, \tag{3.18}$$

where $v_{\hat{\psi}} = v_\psi \circ \phi_T$, $\psi \in \Psi_T$. As a consequence of condition (C2), all functions in the reference space $\mathcal{V}_{\hat{T}}$ vanish in the nodes $\mathcal{N}_{\hat{T}}$ so that $|\cdot|_{1,\hat{T}}$ is a norm on $\mathcal{V}_{\hat{T}}$. As all norms on a finite dimensional space are equivalent, we have

$$|\hat{v}|_{1,\hat{T}}^2 \asymp \sum_{\hat{\psi} \in \Psi_{\hat{T}}} |v_{\hat{\psi}}|_{1,\hat{T}}^2. \tag{3.19}$$

In view of the shape regularity of $\mathcal{T}$, this equivalence transforms to

$$|v|_{1,T}^2 \asymp \sum_{\psi \in \Psi} |v_\psi|_{1,T}^2. \tag{3.20}$$

Summing up over $T \in \mathcal{T}$ and exploiting the ellipticity (2.2) gives the assertion. ■

Following the proof of Lemma 3.2, the upper bound $C = \dim V_{\hat{T}}$ in (3.16) can be alternatively shown by the Cauchy–Schwarz inequality.

Now we are ready to state the main result of this section.

Theorem 3.1 *Assume that the extension $\mathcal{V}$ satisfies the conditions (C1) and (C2). Then the algebraic error $\eta_{\mathcal{S}}$ and the local contributions η_ψ, $\psi \in \Psi$, provide lower and upper bounds for the discrete error,*

$$\| u_{\mathcal{Q}} - \tilde{u} \|^2 \asymp \eta_{\mathcal{S}} + \sum_{\psi \in \Psi} \eta_\psi. \tag{3.21}$$

Proof. Lemma 3.2 and 3.3 jointly state for the direct splitting

$$v = v_{\mathcal{S}} + \sum_{\psi \in \Psi} v_\psi \tag{3.22}$$

of $v \in \mathcal{Q}$ into $v_{\mathcal{S}} \in \mathcal{S}$ and $v_\psi \in V_\psi$ that the norm equivalence

$$\| v \|^2 \asymp \| v_{\mathcal{S}} \|^2 + \sum_{\psi \in \Psi} \| v_\psi \|^2$$

holds. Since (3.22) is the *only* additive splitting of v into elements of the spaces $\mathcal{S}$ and V_ψ, $\psi \in \Psi$, standard arguments from domain decomposition as condensed in Lemma 3.1 of [20] give for $v \in \mathcal{Q}$ the norm equivalence

$$\| v \|^2 \asymp \| P_{\mathcal{S}} v \|^2 + \sum_{\psi \in \Psi} \| P_\psi v \|^2 .$$

Here, $P_{\mathcal{S}} : H_0^1(\Omega) \to \mathcal{S}$ and $P_\psi : H_0^1(\Omega) \to V_\psi$ denote the Ritz projections

$$a(P_{\mathcal{S}} v, w) = a(v, w), \quad w \in \mathcal{S}$$

and

$$a(P_\psi v, \psi) = a(v, \psi).$$

Applying that result to the difference $v = u_{\mathcal{Q}} - \tilde{u}$, we obtain the assertion.

■

Recall from the preceding section that $\| u_{\mathcal{Q}} - \tilde{u} \|$ is equivalent to the true error $\| u - \tilde{u} \|$, if the saturation assumption (C0) is fulfilled.

Estimates of the algebraic error $\eta_{\mathcal{S}}$ may be derived by preconditioning of the algebraic defect equation (3.8) (see Deuflhard, Leinen and Yserentant [8], Bornemann, Erdmann and Kornhuber [7]) or by arguments based on the particular linear solver, which is used (e.g., for the cg-method Deuflhard [9]).

If the exact finite element solution $u_{\mathcal{S}} = \tilde{u}$ is available, the remaining contributions $\eta_{\mathcal{V}} = \sum_{\psi \in \Psi} \eta_\psi$ provide an estimate of the discretization error $\| u - u_{\mathcal{S}} \|$. Assuming that $\tilde{u} \approx u_{\mathcal{S}}$, $\eta_{\mathcal{V}}$ is frequently used to judge the quality of the underlying discretization. Following Babuška and Rheinboldt [2], the local contributions η_ψ are used as local error indicators in an adaptive refinement process. More precisely, all elements T, which are contained in the support of $\psi \in \Psi$, are marked for refinement, if η_ψ exceeds a certain threshold θ. See [6] for further information.

Of course, the results of Theorem 3.1 carry over to the case that $\mathcal{V}$ is split into larger subspaces spanned by more than one basis function $\psi \in \Psi$. In this way, error estimates of Babuška–Rheinboldt–type can be obtained. On the other hand, the complete decomposition (3.4) is closely related to the approach of Babuška and Miller [3], as will turn out in the following section.

Chapter 4

On p Extensions in 2 Space Dimensions

The spaces $\mathcal{V}$ with the property

$$\mathcal{S}^{p+1} = \mathcal{S}^{p} + \mathcal{V} \tag{4.1}$$

are called *p extensions* of $\mathcal{S}^p$. As the extended space $\mathcal{Q} = \mathcal{S}^{p+1}$ provides approximations of higher order, the saturation assumption (C0) is clearly satisfied, if the given data are sufficiently regular and the triangulation $\mathcal{T}$ is fine enough. Moreover, *hierarchical p extensions* $\mathcal{V}^{p+1}$ with the properties (C1) and (C2) can be obtained in a straightforward way.

In this section, we will concentrate on hierarchical p extensions in the case of two space dimensions $d = 2$. In particular, we give a reinterpretation of the local contributions η_ψ, $\psi \in \Psi$, in terms of jumps of the normal fluxes and local consistency errors. This reinterpretation allows the illustration and extension of recent results of Verfürth [18] and motivates the choice of certain non–standard extensions in the 3–D case, which will be considered in the final section. We will frequently omit the superscripts p, $p+1$ in the sequel.

Assume that the conditions (C0), (C1) and (C2) are fulfilled and that the exact finite element solution $u_{\mathcal{S}}$ of the discrete problem (2.3) is known. Then it follows from the Theorems 2.1 and 3.1 that the solutions d_ψ, $\psi \in \Psi$, of the local defect problems (3.6) provide the efficient and reliable estimate $\eta_{\mathcal{V}} = \sum_{\psi\in\Psi} \eta_\psi$ of the discretization error $\| u - u_{\mathcal{S}} \|^2$. In the piecewise linear case $p = 1$, this error estimate has been introduced by Deuflhard, Leinen and Yserentant [8].

The *local consistency error R_T of $u_{\mathcal{S}}$ on $T \in \mathcal{T}$* is defined by

$$R_T = f + \sum_{i,j=1}^{2} \partial_i \left(a_{ij} \partial_j u_{\mathcal{S}} \right), \quad T \in \mathcal{T}.$$

Utilizing additionally the *jumps R_E of the normal fluxes of $u_{\mathcal{S}}$ across interior edges $E \in \mathcal{E}$*,

$$R_E = -\left[\sum_{i,j=1}^{2} a_{ij} n_i \partial_j u_{\mathcal{S}} \right]_E, \quad E \in \mathcal{E},$$

$n = (n_1, n_2)$ being a unit normal to E, we introduce the local error indicators η_T^{BM}, $T \in \mathcal{T}$, (c.f. Babuška and Miller [3])

$$\eta_T^{BM} = h_T^2 \|R_T\|_{0,T}^2 + \sum_{E=E_1,E_2,E_3} \tfrac{1}{2} h_E \|R_E\|_{0,E}^2, \quad T = (E_1, E_2, E_3) \in \mathcal{T}. \tag{4.2}$$

Here h_E denotes the length of the edge $E \in \mathcal{E}$ and we made use of the local L^2–norms $\|v\|_{0,E}$, $\|v\|_{0,T}$ induced by the corresponding scalar products $(\cdot,\cdot)_E$ and $(\cdot,\cdot)_T$, respectively.

It was shown by Verfürth [18] for the Poisson equation that

$$\| u - u_{\mathcal{S}} \|^2 \asymp \sum_{T\in\mathcal{T}} \eta_T^{BM} \tag{4.3}$$

holds for piecewise constant data $f \in \mathcal{C}$,

$$\mathcal{C} = \{v \in L^2(\Omega) \mid v|_T = \text{constant},\ T \in \mathcal{T}\}.$$

Denoting by $\omega_E = T_1(E) \cup T_2(E)$ the union of the two triangles $T_1(E), T_2(E)$ with the common interior edge $E \in \mathcal{E}$ and

$$R_{\omega_E}(x) = R_{T_i(E)}(x), \quad x \in \text{ int } T_i(E), \ \ i = 1, 2,$$

the residual $r(v) = (f, v) - a(u_{\mathcal{S}}, v)$ can be rewritten as

$$r(v) = (R_{\omega_E}, v)_{\omega_E} + (R_E, v)_E, \ \ v \in \mathcal{Q}, \ \ \text{supp } v \subset \omega_E. \tag{4.4}$$

Based on the representation (4.4), we will show that the local contributions η_ψ, $\psi \in \Psi$, and η_T^{BM}, $T \in \mathcal{T}$, are locally equivalent for $p > 1$. Note that (4.4) can also be used for an efficient implementation of η_ψ.

In view of (4.4), we introduce the subsets $\Psi_E = \{\psi \in \Psi \mid \text{supp } \psi \subset \omega_E\}$ of Ψ and the corresponding subspaces $\mathcal{V}_E = \text{span } \{\psi \in \Psi_E\}$, $E \in \mathcal{E}$. As a consequence of condition (C1), a function $v \in \mathcal{V}$ is contained in $\mathcal{V}_E$, if and only if supp $v \subset \omega_E$. Exploiting that locally constant functions are not contained in $\mathcal{V}$, the inequalities

$$\|v\|_{0,E} \preceq h_E^{1/2} \| v \|_{\omega_E}, \ \ v \in \mathcal{V}, \tag{4.5}$$

and

$$\|v\|_{0,T} \asymp h_T \| v \|_T, \ \ v \in \mathcal{V}, \tag{4.6}$$

can be derived by the standard affine transformation technique. As usual, the subscripts ω_E, T indicate the corresponding localization of the energy norm.

We are ready to estimate η_ψ, $\psi \in \Psi_E$, in terms of R_{ω_E} and R_E.

Lemma 4.1 *Assume that the conditions (C1) and (C2) are satisfied and that $a_{ij} \in \mathcal{C}$, $i, j = 1, 2$. Then the estimates*

$$h_E\|R_E\|_{0,E}^2 - h_E^2\|R_{\omega_E}\|_{0,\omega_E}^2 \preceq \sum_{\psi\in\Psi_E} \eta_\psi \preceq h_E\|R_E\|_{0,E}^2 + h_E^2\|R_{\omega_E}\|_{0,\omega_E}^2 \tag{4.7}$$

hold uniformly for all $E \in \mathcal{E}$ and all right–hand sides $f \in \mathcal{C}$.

Proof. Consider some fixed $E \in \mathcal{E}$. We introduce the auxiliary problem

$$d_E \in \mathcal{V}_E : \quad a(d_E, v) = r(v), \quad v \in \mathcal{V}_E, \tag{4.8}$$

with respect to the subspace $\mathcal{V}_E$. Denoting $\eta_E = \| d_E \|^2$, the equivalence

$$\sum_{\psi \in \Psi_E} \eta_\psi \asymp \eta_E \tag{4.9}$$

follows from Lemma 3.3. Hence, it is sufficient to show (4.7) with $\sum_{\psi\in\Psi_E} \eta_\psi$ replaced by η_E.

Inserting $v = d_E$ in (4.8) and using the representation (4.4), the Cauchy–Schwarz inequality, the estimates (4.5), (4.6) and $h_E \asymp h_{T_i(E)}$, $i = 1, 2$, we obtain

$$\begin{aligned}\eta_E = r(d_E) &= (R_{\omega_E}, d_E)_{\omega_E} + (R_E, d_E)_E \\ &\leq \|R_{\omega_E}\|_{0,\omega_E} \|d_E\|_{0,\omega_E} + \|R_E\|_{0,E} \|d_E\|_{0,E} \preceq \\ &\preceq \eta_E^{1/2} \left(h_E \|R_{\omega_E}\|_{0,\omega_E} + h_E^{1/2} \|R_E\|_{0,E} \right).\end{aligned}$$

This gives the right estimate in (4.7).

To prove the left inequality in (4.7), we follow the arguments of Verfürth [18]. In particular, for given R_E, we construct a function $v_E \in \mathcal{Q}$ leading to a suitable test function $v_\mathcal{V} \in \mathcal{V}_E$, which can be used in (4.8). Here we will utilize the quadratic bubble function $\psi_E \in \mathcal{S}^2$, which is defined according to the 3–D analogue in Example 3.1. It is easily checked that ψ_E is non–negative on supp $\psi_E = \omega_E$ and

$$\|w\|_{0,E} \preceq \|w \psi_E^{1/2}\|_{0,E}, \quad \|\psi_E v\|_{0,\omega_E} \preceq \|v\|_{0,\omega_E} \tag{4.10}$$

holds for $w \in \Pi^{p-1}(E)$ and $v|_{T_{1,2}(E)} \in \Pi^{p-1}(T_{1,2}(E))$, respectively. Using constant extension on the reference triangle (c.f. Verfürth ([18], p. 7), we define an extension operator $P : C^0(E) \to C^0(\omega_E)$ such that

$$\|Pv\|_{0,\omega_E} \preceq h_E^{1/2} \|v\|_{0,E}, \quad v \in \Pi^{p-1}(E). \tag{4.11}$$

We finally set $v_E = \psi_E P(R_E)$. As the coefficients a_{ij} are piecewise constant, we have $R_E \in \Pi^{p-1}(E)$ so that $v_E \in \mathcal{Q}$. According to (4.10) and (4.11), v_E has the property

$$\|v_E\|_{0,\omega_E} \preceq h_E^{1/2} \|R_E\|_{0,E}. \tag{4.12}$$

Decomposing $v_E = v_\mathcal{S} + v_\mathcal{V}$ in $v_\mathcal{S} = \mathcal{I} v_E \in \mathcal{S}$ and $v_\mathcal{V} = v_E - \mathcal{I} v_E \in \mathcal{V}$, we clearly have $r(v_\mathcal{S}) = 0$, as $u_\mathcal{S}$ is the exact finite element solution. On the other hand, we conclude from supp $v_\mathcal{V} \subset \omega_E$ that $v_\mathcal{V} \in \mathcal{V}_E$. Hence, $v_\mathcal{V}$ is an admissible test function in the auxiliary problem (4.8). Moreover, it follows from the stability of the interpolation $\mathcal{I}$, as stated in Lemma 3.1, together with the (inverse) inequalities (4.6) and (4.12) that

$$\| v_\mathcal{V} \| \preceq \| v_E \| \preceq h_E^{-1} \|v_E\|_{0,\omega_E} \preceq h_E^{-1/2} \|R_E\|_{0,E}. \tag{4.13}$$

Using again (4.4), the Cauchy–Schwarz inequality and the estimates (4.13) and (4.12), the assertion follows from

$$\begin{aligned}\|R_E\|_{0,E}^2 \preceq \|R_E\psi_E^{1/2}\|_{0,E}^2 = (R_E, v_E)_E = r(v_{\mathcal{S}}) + r(v_{\mathcal{V}}) - (R_{\omega_E}, v_E)_{\omega_E} \\ = a(d_E, v_{\mathcal{V}}) - (R_{\omega_E}, v_E)_{\omega_E} \le \| d_E \| \, \| v_{\mathcal{V}} \| + \|R_{\omega_E}\|_{0,\omega_E}\|v_E\|_{0,\omega_E} \\ \preceq h_E^{-1/2}\|R_E\|_{0,E}\left(\eta_E^{1/2} + h_E\|R_{\omega_E}\|_{0,\omega_E}\right).\end{aligned}$$

■

To improve the sub-optimal lower bound in (4.7), we now derive additional estimates for the local consistency error R_T. For this reason, we introduce the subsets $\Psi_T = \{\psi \in \Psi \mid \text{supp}\ \psi \subset T\}$ of Ψ and the corresponding subspaces $\mathcal{V}_T = \text{span}\{\psi \in \Psi_T\}$, $T \in \mathcal{T}$. Due to (C1), we again get that a function $v \in \mathcal{V}$ is contained in $\mathcal{V}_T$, if and only if supp $v \subset T$. Note that Ψ_T is empty in the piecewise linear case $p = 1$.

Lemma 4.2 *Assume that the conditions (C1) and (C2) are satisfied, that $a_{ij} \in \mathcal{C}$, $i, j = 1, 2$, and $p > 1$. Then the equivalence*

$$\sum_{\psi\in\Psi_T} \eta_\psi \asymp h_T^2\|R_T\|_{0,T}^2 \tag{4.14}$$

holds uniformly for all $T \in \mathcal{T}$ and all right–hand sides $f \in \mathcal{C}$.

Proof. Consider some fixed $T \in \mathcal{T}$. Again, we utilize an auxiliary problem

$$d_T \in \mathcal{V}_T : \quad a(d_T, v) = r(v), \quad v \in \mathcal{V}_T, \tag{4.15}$$

denoting $\eta_T = \| d_T \|^2$. According to Lemma 3.3, we have

$$\eta_T \asymp \sum_{\psi\in\Psi_T} \eta_\psi. \tag{4.16}$$

Inserting $v = d_T$ in (4.15), the upper bound $\eta_T \preceq h_T^2\|R_T\|_{0,T}^2$ follows immediately from (4.4), the Cauchy–Schwarz inequality and (4.6).

To show the remaining estimate $h_T^2\|R_T\|_{0,T}^2 \preceq \eta_T$, we follow the lines of proof of Lemma 4.1, replacing ψ_E by the cubic bubble function $\psi_T \in \mathcal{S}^3$, also taken from Example 3.1. It is easily seen that

$$\|v\psi_T\|_{0,T} \preceq \|v\|_{0,T} \preceq \|v\psi_T^{1/2}\|_{0,T}, \quad v \in \Pi^{p-2}(T). \tag{4.17}$$

Due to the piecewise constant data, we have $R_T \in \Pi_{p-2}(T)$, so that $v_T = \psi_T R_T \in \mathcal{Q}$. Now $v_{\mathcal{S}} = \mathcal{I}v_T$ satisfies $r(v_{\mathcal{S}}) = 0$ and $v_{\mathcal{V}} = v_T - \mathcal{I}v_T \in \mathcal{V}_T$ is an admissible test function in (4.15). Using the stability of $\mathcal{I}$, the inverse estimate (4.6) and (4.17), we conclude

$$\| v_{\mathcal{V}} \| \preceq \| v_T \|_T \preceq h_T^{-1}\|v_T\|_{0,T} \preceq h_T^{-1}\|R_T\|_{0,T}$$

so that the assertion follows from

$$\|R_T\|_{0,T}^2 \preceq \|\psi_T^{1/2} R_T\|_{0,T}^2 =$$
$$= (R_T, v_T) = r(v_{\mathcal{S}}) + r(v_{\mathcal{V}}) = a(d_T, v_{\mathcal{V}}) \preceq h_T^{-1} \|R_T\|_{0,T}\, \eta_T^{1/2}.$$

∎

The following Theorem is an immediate consequence of Lemma 4.1 and 4.2.

Theorem 4.1 *Assume that the conditions (C1) and (C2) are satisfied, that* $a_{ij} \in \mathcal{C}$, $i, j = 1, 2$, *and* $p > 1$. *Then the local error indicators* η_ψ, $\psi \in \Psi$, *and* η_T^{BM}, $T \in \mathcal{T}$, *are equivalent in the sense that*

$$\sum_{\psi \in \Psi_E} \eta_\psi \preceq \eta_{T_1(E)}^{BM} + \eta_{T_2(E)}^{BM}, \qquad E \in \mathcal{E}, \tag{4.18}$$

and

$$\eta_T^{BM} \preceq \sum_{E=E_1,E_2,E_3} \sum_{\psi \in \Psi_E} \eta_\psi, \qquad T = (E_1, E_2, E_3) \in \mathcal{T}, \tag{4.19}$$

holds uniformly for all right–hand sides $f \in \mathcal{C}$.

Theorem 4.1 provides a reinterpretation of the indicators η_T^{BM} in terms of hierarchical p extensions. Recall that the indicators η_ψ are always scaled properly. On the other hand, it has been shown by Verfürth [18] that the estimate $\eta^{BM} = \sum_{T \in \mathcal{T}} \eta_T^{BM}$ of the discretization error is *robust* in the sense that the constants are independent of $f \in \mathcal{C}$. In view of Theorems 2.1, 3.1 and 4.1, this implies that hierarchical p extensions $\mathcal{V}^{p+1}$ saturate uniformly in $f \in \mathcal{C}$, if the given finite element space $\mathcal{S}^p$ is of order $p > 1$.

Let us take a closer look at the exceptional case $p = 1$. Then, the hierarchical extension $\mathcal{V}^2$ is spanned by the quadratic bubble functions ψ_E, $E \in \mathcal{E}$. Note that dim $\mathcal{V} \geq$ dim $\mathcal{C}$ so that the arguments in Section 2 would not contradict a corresponding uniform saturation property of $\mathcal{V}^2$. However, there are simple counterexamples with piecewise constant data, giving $u_{\mathcal{Q}} = u_{\mathcal{S}}$. In view of the proof of Lemma 4.2, we can increase the robustness of $\eta_{\mathcal{V}^2}$ by adding the cubic bubble functions $\mathcal{V}^{\mathcal{T}} = \text{span}\{\psi_T,\ T \in \mathcal{T}\}$ to $\mathcal{V}^2$, to obtain the larger extension $\mathcal{V}^{BM} = \mathcal{V}^2 \oplus \mathcal{V}^{\mathcal{T}}$. The resulting estimate $\eta_{\mathcal{V}^{BM}}$ is now locally equivalent to η^{BM}. However, the additional work caused by $\mathcal{V}^{\mathcal{T}}$ usually does not pay off in practice.

Roughly speaking, we found that the two presented concepts of hierarchical p extensions and of local jumps and consistency errors (almost) coincide in two space dimensions. The resulting error estimates thus combine higher order approximation with a certain robustness. The next section will show that the situation is different in three space dimensions.

Chapter 5

On p–Extensions in 3 Space Dimensions

We concentrate on the most simple case of piecewise linear finite elements. As in two space dimensions, the hierarchical p extension of $\mathcal{S}^1$ is given by the space of quadratic bubbles $\mathcal{V}^2 = \{\psi_E,\ E \in \mathcal{E}\}$. It is clear that $\mathcal{V}^2$ satisfies the conditions (C1) and (C2), stated in Section 3.

The straightforward extension of the Babuška–Miller–type indicators from two to three space dimensions has the form

$$\eta_T^{BM} = h_T^2 \|R_T\|_{0,T}^2 + \sum_{F=F_1,F_2,F_3,F_4} \tfrac{1}{2} h_F \|R_F\|_{0,F}^2, \quad T \in \mathcal{T}, \tag{5.1}$$

where R_T is the local consistency error of the exact finite element solution $u_{\mathcal{S}}$ in the interior of the tetrahedra,

$$R_T = f + \sum_{i,j=1}^{3} \partial_i \left(a_{ij} \partial_j u_{\mathcal{S}} \right), \quad T \in \mathcal{T},$$

and R_F denotes the jump of the normal flux of $u_{\mathcal{S}}$ across the triangular faces,

$$R_F = - \left[\sum_{i,j=1}^{3} a_{ij} n_i \partial_j u_{\mathcal{S}} \right]_F, \quad F \in \mathcal{F}.$$

Again, we can reformulate (5.1) in terms of a suitable extension $\mathcal{V}^{BM}$. For this reason, we define the spaces $\mathcal{V}^{\mathcal{F}} = \operatorname{span} \Psi^{\mathcal{F}}$ and $\mathcal{V}^{\mathcal{T}} = \operatorname{span} \Psi^{\mathcal{T}}$ spanned by the cubic bubbles $\Psi^{\mathcal{F}} = \{\psi_F,\ F \in \mathcal{F}\}$ and the quartic bubbles $\Psi^{\mathcal{T}} = \{\psi_T,\ T \in \mathcal{T}\}$, respectively. The resulting extension $\mathcal{V}^{BM} = \mathcal{V}^{\mathcal{F}} \oplus \mathcal{V}^{\mathcal{T}}$ clearly satisfies the conditions (C1) and (C2) and is producing the local error indicators η_ψ, $\psi \in \Psi^{BM} = \Psi^{\mathcal{F}} \cup \Psi^{\mathcal{T}}$, as described in Section 3.

For each triangular face $F \in \mathcal{F}$, the subset $\Psi_F^{BM} = \{\psi_F, \psi_{T_1(F)}, \psi_{T_2(F)}\}$ contains the three bubble functions in Ψ^{BM}, which vanish outside of the tetrahedra $T_1(F)$, $T_2(F)$ with the common face F. Now the following proposition can be established along the lines of the preceding section.

Proposition 5.1 *Assume that the coefficients a_{ij}, $i,j = 1,2,3$, are piecewise constant. Then the local error indicators η_ψ, $\psi \in \Psi^{BM}$, and η_T^{BM}, $T \in \mathcal{T}$, are equivalent in the sense that the estimates*

$$\sum_{\psi \in \Psi_F^{BM}} \eta_\psi \preceq \eta_{T_1(F)}^{BM} + \eta_{T_2(F)}^{BM}, \quad F \in \mathcal{F}, \tag{5.2}$$

and

$$\eta_T^{BM} \preceq \sum_{F=F_1,F_2,F_3,F_4} \sum_{\psi \in \Psi_F^{BM}} \eta_\psi, \quad T = (F_1, F_2, F_3, F_4) \in \mathcal{T}, \tag{5.3}$$

hold uniformly for all piecewise constant right–hand sides f.

Observe that the extension $\mathcal{V}^{BM}$, representing the local indicators η_T^{BM} of Babuška–Miller–type, is complementing the hierarchical p extension $\mathcal{V}^2$,

$$\mathcal{V}^2 \cap \mathcal{V}^{BM} = \{0\}. \tag{5.4}$$

This is different from the 2D–case, where we have shown $\mathcal{V}^2 \subset \mathcal{V}^{BM}$. Now it becomes clear, why hierarchical p extensions work slightly sub–optimal in three space dimensions (c.f. [7]). On the other hand, we can no longer expect that the performance of Babuška–Miller–type estimates (implicitly) takes advantage of higher order saturation.

To accumulate the good properties of p–extensions and Babuška–Miller–type estimates, we introduce the error estimates resulting from the *hybrid extensions* $\mathcal{V}^{\mathcal{EF}} = \mathcal{V}^2 \oplus \mathcal{V}^{\mathcal{F}}$ and $\mathcal{V}^{\mathcal{EFT}} = \mathcal{V}^{\mathcal{EF}} \oplus \mathcal{V}^{\mathcal{T}}$. Note that the extension $\mathcal{V}^{\mathcal{EFT}} = \mathcal{V}^2 \oplus \mathcal{V}^{BM}$ may be regarded as the direct sum of both concepts.

All four error estimates presented above will be compared in the following numerical example.

Example 5.1 As a model problem, we consider the Laplacian on the unit cube $\Omega = [0,1]^3$. The right–hand side f is given in such a way that

$$u(x) = u_0(x) \sum_{i=1}^{3} a_i \exp(-b_i |x - x^{(i)}|^2)$$

becomes the exact solution. Here, the function $u_0(x_1, x_2, x_3) = \frac{1}{2}\Pi_{i=1}^3 x_i(x_i - 1)$ provides the zero boundary conditions, while $a_1 = 100.$, $a_2 = 180.$, $a_3 = 120.$, $b_1 = 150.$, $b_2 = 50.$, $b_3 = 150$ and $x^{(1)} = (0.5, 0.5, 0.5)$, $x^{(2)} = (0.7, 0.6, 0.5)$, $x^{(3)} = (0.3, 0.6, 0.5)$ characterize the height, the slope and the location of the local extrema $x^{(i)}$, $i = 1, 2, 3$. Figure 5.1 shows the level curves of the solution at the cutting plane $x_3 = 0.5$. The figure has been generated with the help of the graphical environment GRAPE [16].

Starting with a very coarse initial partition $\mathcal{T}_0$ of Ω (with only one interior node), the continuous problem is discretized by piecewise linear finite elements with respect to a sequence of triangulations $\mathcal{T}_0, \mathcal{T}_1 \ldots, \mathcal{T}_l$. Each refinement level l corresponds to an adaptive cycle, involving assembly of the discrete problem, (iterative) solution, error estimation and possible refinement. The *refinement depth* j_l of a partition $\mathcal{T}_l$ is denoting the maximal number of subsequent refinements applied to an initial tetrahedron $T_0 \in \mathcal{T}_0$. In the present case of a uniform initial partition, the refinement depth characterizes the minimal stepsize of $\mathcal{T}_l$. On each refinement level, the discrete solution is computed up to an (unreasonable) high accuracy, to make sure that the algebraic error and the discretization error do not interfere. Then the discretization error is approximated by one of the four error estimates in question. The local contributions η_ψ are used as error indicators in the adaptive process. More precisely, the two tetrahedra $T_1(F)$, $T_2(F)$ are marked for

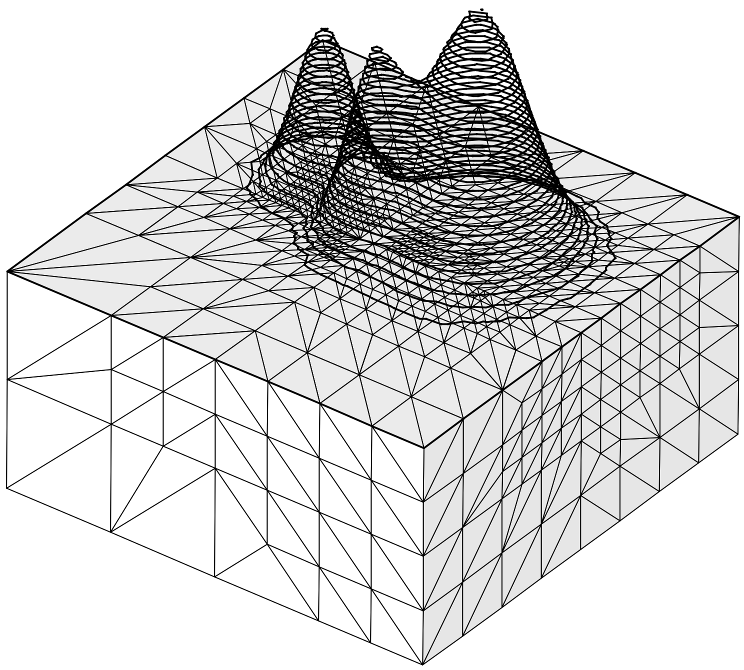

Figure 5.1: Solution u at $x_3 = 0.5$

refinement, if the corresponding sum $\sum_{\psi\in\Psi_F}\eta_\psi$ exceeds a certain threshold θ. Here, Ψ_F corresponds to the actual extension. The threshold θ is computed by extrapolation as proposed by Babuška and Rheinboldt [2] (see for example [6] for further information). As usual, the adaptive algorithm is stopped, if a certain fixed accuracy (and the related refinement depth) is reached.

The constant β,

$$\beta = \| u - u_{\mathcal{Q}} \| \,/\, \| u - u_{\mathcal{S}} \|,$$

is describing the saturation property (C0) of the extension $\mathcal{V}$. Figure 5.2 shows the development of β with increasing refinement and $\mathcal{V}$ running through the four spaces $\mathcal{V}^2$, $\mathcal{V}^{BM}$, $\mathcal{V}^{\mathcal{EF}}$ and $\mathcal{V}^{\mathcal{EFT}}$. The corresponding curves are denoted by P–EXT, BM, EF, and EFT, respectively. It is clearly visible that (C0) is satisfied by all extensions in question, but that $\mathcal{V}^{BM}$ is the only extension, which does not provide an approximation of higher order.

To illustrate the effect of localization, the following Figure 5.3 shows the ratio

$$\kappa_{prc} = \eta_{\mathcal{V}} / \| u_{\mathcal{Q}} - u_{\mathcal{S}} \|^2$$

as a function of the refinement level l. It comes out that the underlying preconditioning of the discrete defect equation (2.5) hardly affects the results, i.e. we have $\kappa_{prc} \approx 1$ in all four cases.

As a consequence, the effectivity index

$$\kappa_{eff} = \eta_{\mathcal{V}} / \| u - u_{\mathcal{S}} \|^2 = \kappa_{prc}(1 - \beta^2)$$

is closely related to the saturation constant β. This explains the poor performance of BM. Note that the additional extension of $\mathcal{V}^{\mathcal{EF}}$ by the quartic bubbles $\mathcal{V}^{\mathcal{T}}$ scarcely changes the results.

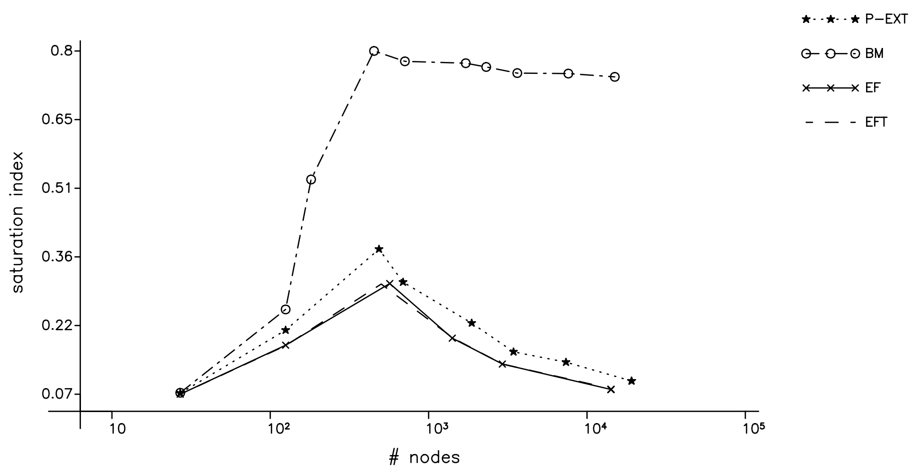

Figure 5.2: Saturation Property

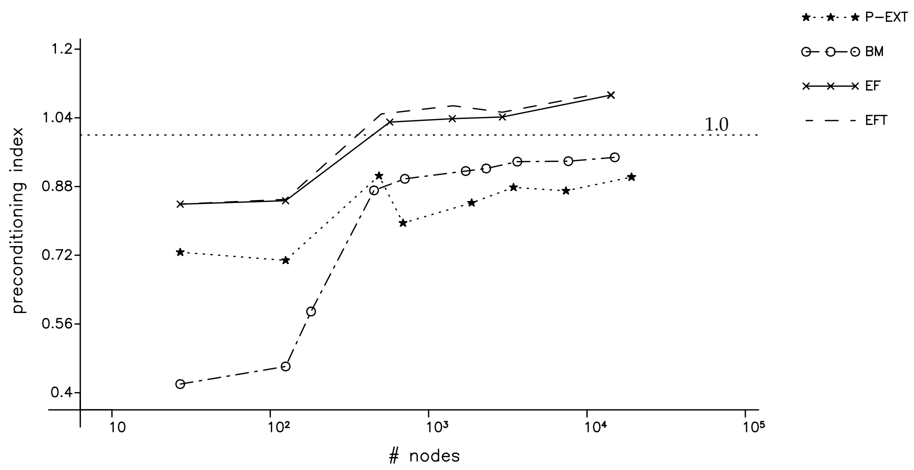

Figure 5.3: Effect of Preconditioning

Finally, Table 5.1 shows the complete approximation history for P–EXT, BM and the hybrid estimate EF. EFT is more expensive and again provides almost the same results as EF. If we require a certain fixed accuracy, all local error indicators in question are producing more or less the same (reasonable) mesh. Hence, we can compare their numerical efficiency by comparing the number of adaptive cycles, which are needed until this mesh is obtained. For example, the hybrid extension EF, provides the accuracy 2.15e-1 after only seven adaptive cycles. Two or three more adaptive cycles are needed by the canonical hierarchical extension P–EXT to obtain a comparable accu-

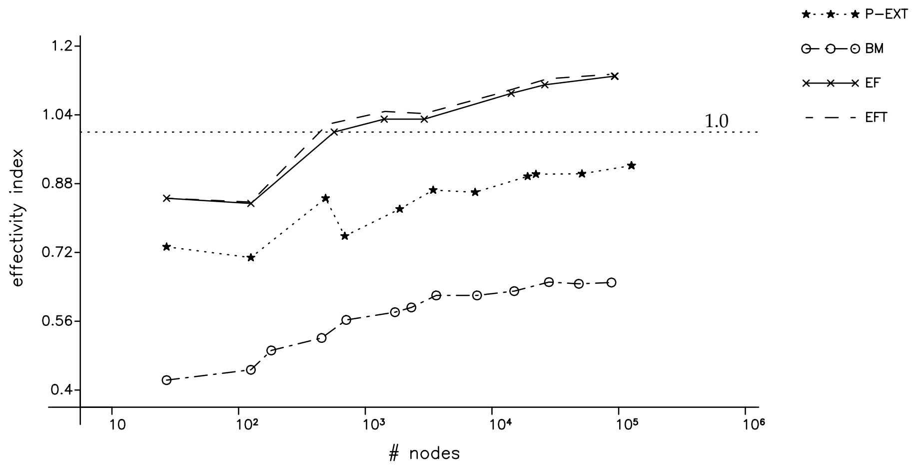

Figure 5.4: Effectivity Index

level	P-EXT			BM			EF		
	j	nodes	error	j	nodes	error	j	nodes	error
0	0	27	3.45e+1	0	27	3.45e+1	0	27	3.45e+1
1	1	125	7.49e+0	1	125	7.49e+0	1	125	7.49e+0
2	2	486	1.69e+0	2	181	3.15e+0	2	568	1.69e+0
3	3	689	1.29e+0	3	452	1.45e+0	3	1410	1.05e+0
4	4	1870	8.89e–1	4	708	1.19e+0	4	2922	7.40e–1
5	4	3435	6.90e–1	4	1715	8.85e–1	5	14179	4.20e–1
6	5	7367	5.36e–1	4	2310	7.96e–1	6	26115	3.29e–1
7	6	19119	3.70e–1	5	3630	6.63e–1	6	93084	2.15e–1
8	6	22224	3.47e–1	5	7625	5.21e–1			
9	6	51337	2.69e–1	5	14982	4.08e–1			
10	6	126278	1.93e–1	6	28149	3.21e–1			
11				6	48512	2.72e–1			
12				6	87713	2.20e–1			

Table 5.1: Approximation History

racy (and a comparable grid). Providing corresponding results not before 12 adaptive cycles, the performance of the Babuška–Miller–type extension BM is still much worse.

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